A NECESSARY AND SUFFICIENT CONDITION
FOR THE EXISTENCE OF NON-SINGULAR
G-VECTOR FIELDS ON G-MANIFOLDS

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(Received July 30, 1975)

1. Introduction

Throughout this paper G always denotes a compact Lie group and G-
manifolds are smooth manifolds with smooth G-actions. In this paper we
will give a necessary and sufficient condition for the existence of non-singular
G-vector fields on closed G-manifolds.

Let \( E \to X \) be a G-vector bundle. After choosing a G-invariant Riemannian
metric on \( E \), we denote by \( S(E) \to X \) the associated G-sphere bundle of \( E \).
We abbreviate continuous G-equivariant cross section of \( E \) to G-cross section
of \( E \).

Let \( M \) be a compact G-manifold, and \( s: M \to \tau(M) \) a G-cross section of the
tangent bundle \( \tau(M) \) of \( M \). \( s \) is called a non-singular G-vector field on \( M \), if \( s \) is
not zero at each point of \( M \). For a positive integer \( k \), by a G-k-field on \( M \) we
will mean \( k \) G-cross sections of \( \tau(M) \) which are linearly independent at each
point of \( M \).

For a closed subgroup \( H \) of \( G \) we set

\[ M_H = \{ x \in M | G_x = H \}, \]

where \( G_x \) is the isotropy group at \( x \). Let \( N(H) \) be the normalizer of \( H \) in \( G \).
Then \( M_H \) is an \( \Lambda(G) \)-manifold and also a free \( N(H)/H \)-manifold.

For a topological space \( X \) we define \( |\chi|(X) \) to be the sum \( \Sigma_x |\chi(Y)| \), where
\( \chi(Y) \) is the Euler characteristic of \( Y \) and \( Y \) runs over the connected components
of \( X \).

In this paper we will obtain the following results:

**Theorem 1.1.** Let \( M \) be a compact G-manifold. Let \( s: \partial M \to S(\tau(\partial M)) \) be
a G-cross section of \( S(\tau(\partial M)) \). Then \( s \) is extendible to a G-cross section of \( S(\tau(M)) \)
if and only if

\[ |\chi|(M_{G_x}N(G_x)) = 0, \quad \text{or} \quad \dim N(G_x) - \dim G_x > 0 \]

for all \( x \in M \).
From this theorem we immediately obtain the main result of this paper:

**Corollary 1.2.** A closed $G$-manifold $M$ admits a non-singular $G$-vector field if and only if

$$\left|\chi\right|(M_{G_x}|N(G_x)) = 0,$$

or

$$\dim N(G_x) - \dim G_x > 0$$

for all $x \in M$.

We also obtain the following corollary:

**Corollary 1.3.** Let $M$ be a compact $G$-manifold, $F$ the stationary point set of $M$, and $k$ a positive integer. We suppose that

$$\dim N(G_x) - \dim G_x \geq k$$

for all $x \in M - F$. Then $M$ admits a $G$-$k$-field if and only if $F$ admits a $k$-field.

*Note.* The existence of a non-singular continuous $G$-vector field on a compact $G$-manifold implies the existence of a non-singular smooth $G$-vector field. This fact is assured by the differentiable approximation theorem [3; (6.7)] and the usual process of averaging cross sections.

### 2. Preliminaries

(2-1) For a closed subgroup $H$ of $G$ let $(H)$ be the conjugacy class of $H$ in $G$. If $H$ is an isotropy group occurring on a $G$-manifold $M$, $(H)$ is called an isotropy type on $M$.

Set

$$M_H = \{x \in M | G_x = H\}$$

$$M_{(H)} = \{x \in M | (G_x) = (H)\}.$$

Then $M_H$ and $M_{(H)}$ are submanifolds of $M$, but, in general, not compact. If $(H)$ is a maximal isotropy type on a compact $G$-manifold $M$, $M_H$ and $M_{(H)}$ are compact.

(2-2) Let $\pi: E \rightarrow X$, $\pi': E' \rightarrow X'$ be $G$-fibre bundles, and $f: E \rightarrow E'$ a $G$-bundle map covering $f: X \rightarrow X'$. Let $s': X' \rightarrow E'$ be a $G$-cross section of $E'$. $s'$ induces a $G$-cross section

$$s: X \rightarrow f^*E' = \{(x, e) \in X \times E' | f(x) = \pi'(e)\}$$

of the induced $G$-fibre bundle $f^*E'$ which sends $x \in X$ to $(x, s'f(x)) \in X \times E'$. $E$ is canonically isomorphic to $f^*E'$. So $s$ induces a $G$-cross section of $E$. We denote this $G$-cross section by $f^*s'$, and call induced $G$-cross section from $s'$ by $f$.

(2-3) Recall known results:
Proposition 2.1 (Segal [2; Proposition 1.3]). Let $X$, $Y$ be $G$-spaces, and $X$ compact. If $f_0, f_1: X \to Y$ are $G$-homotopic $G$-maps, and $E \to Y$ is a $G$-vector bundle, then there is a $G$-bundle isomorphism

$$\phi: f_0^*E \cong f_1^*E.$$  

From this proposition we easily obtain the following. $I$ denotes the interval $[0,1]$ with trivial $G$-action.

Proposition 2.2. If $X$ is a compact $G$-space, then any $G$-vector bundle $E$ over $X \times I$ is isomorphic to $(E|X \times \{0\}) \times I$ as $G$-vector bundles.

Proposition 2.1 may be stated in a more precise form as the following: If $f_0, f_1$ are $G$-homotopic relative to a closed $G$-invariant subspace $A$ of $X$, and if we consider $f_0^*E, f_1^*E$ subspaces of $X \times E$, then the $G$-bundle isomorphism $\phi$ satisfies $\phi(x, e) = (x, e)$ for all $x \in A$ and $e \in E$.

From this fact we obtain

Proposition 2.3. Let $E_i \to X_i$, $i=1,2$, be $G$-vector bundles with $X_i$ compact. Let $f: X_1 \times I \to X_2$ be a $G$-homotopy which is constant on a closed $G$-invariant subspace $A$ of $X_1$. Let $f_0: E_1 \to E_2$ be a $G$-bundle map over $f_0 = f|X_1 \times \{0\}$. Then there is a $G$-bundle map $f: E_1 \times I \to E_2$ over $f$ which is a homotopy of $f_0$ and is constant on $E_1|A$.

Corollary 2.4. Let $E \to X$ be a $G$-vector bundle with $X$ compact, $A$ a closed $G$-invariant subspace of $X$, and $i: A \to X$ the inclusion map. Let $f: X \to A$ be a $G$-map such that $f$ is $G$-homotopic to the identity of $X$ relative to $A$. Then there is a $G$-bundle map $f: E \to E|A$ over $f$ which is the identity on $E|A$.

(2-4) The following result has been obtained by U. Koschorke in his paper [1; §1].

Proposition 2.5. Let $M$ be a compact manifold, and $s: \partial M \to S(\tau(\partial M))$ a cross section of $S(\tau(\partial M))$. Then $s$ is extendible to a cross section of $S(\tau(M))$ if and only if $|X|(M) = 0$.

3. Proof of Theorem 1.1

The proof will proceed by an induction for the number of isotropy types on $M$.

(3-1) In the first place, let $M$ be of one isotropy type, and let $(H)$ be the isotropy type on $M$.

"if" part: $M_H$ is a compact $N(H)$-manifold with boundary $M_H \cap \partial M$. Consider the sphere bundle
The $G$-cross section $s$ induces a cross section

$$s_1: \partial(M_H/N(H)) \rightarrow S(\tau(M_H))/N(H) \mid \partial(M_H/N(H)).$$

This is assured by the $G$-equivariancy of $s$ and the fact

$$\partial(M_H/N(H)) = \partial M_H/N(H) = M_H \cap \partial M/H(N(H)).$$

We may extend $s_1$ to a cross section of $S(\tau(M_H))/N(H)$ as follows. If

$$\dim N(H) - \dim H > 0,$$

then the dimension as a cell complex of $M_H/N(H) = M_H/(N(H)/H)$ is less than or equal to the dimension of fibre of $S(\tau(M_H))/N(H)$. Therefore the obstruction to extending $s_1$ over $M_H/N(H)$ vanishes. If

$$\dim N(H) - \dim H = 0,$$

then

$$|\chi|(M_H/N(H)) = 0$$

by the assumption, and then

$$S(\tau(M_H))/N(H) \cong S(\tau(M_H/N(H)))$$

for $N(H)/H$ is a finite group and $M_H/N(H) = M_H/(N(H)/H)$. The image of $s_1$ is in $S(\tau(\partial M_H))/N(H)$. Then $s_1$ can be extended over $M_H/N(H)$ by Proposition 2.5.

Let

$$s_2: M_H/N(H) \rightarrow S(\tau(M_H))/N(H)$$

be an extension of $s_1$. Let

$$\pi: S(\tau(M_H)) \rightarrow S(\tau(M_H))/N(H)$$

be the canonical projection, and let

$$\pi^*s_2: M_H \rightarrow S(\tau(M_H))$$

be the induced $N(H)$-cross section. $\pi^*s_2$ may be considered an $N(H)$-cross section of $S(\tau(M))/M_H$, since $S(\tau(M_H))$ is a subbundle of $S(\tau(M))/M_H$. Moreover $\pi^*s_2$ is an extension of $s \mid \partial M_H$. Since $M$ is of one isotropy type, the $G$-action

$$G \times (S(\tau(M))/M_H) \rightarrow S(\tau(M))$$

induces a $G$-bundle isomorphism
Then \( \pi^*s \) induces a \( G \)-cross section
\[
s: M \to S(\tau(M))
\]
which is an extension of \( s \).

"only if" part: Let
\[
t: M \to S(\tau(M))
\]
be an extension of \( s \). By the \( G \)-equivariancy of \( t \), \( t \) is restricted to an \( N(H) \)-cross section
\[
t: M_H \to S(\tau(M_H)).
\]
\( t \) induces a cross section
\[
t: M_H/N(H) \to S(\tau(M_H))/N(H).
\]
If
\[
dim N(H) - \dim H = 0,
\]
then
\[
S(\tau(M_H))/N(H) \cong S(\tau(M_H|N(H))),
\]
and
\[
t_2(\partial(M_H|N(H))) \subseteq S(\tau(\partial(M_H|N(H)))).
\]
Therefore
\[
dim N(H) - \dim H = 0
\]
implies
\[
|X|(M_H|N(H)) = 0
\]
by Proposition 2.5.

This completes the proof for the case in which \( M \) is of one isotropy type.

(3-2) Let the theorem be true for the case in which the number of isotropy types is \( k-1 \). Let \( M \) be a compact \( G \)-manifold with \( k \) isotropy types.

"if" part: Let \( (H) \) be a maximal isotropy type on \( M \). Then \( M_{(H)} \) is a compact \( G \)-submanifold of \( M \) with one isotropy type. From the preceding argument we obtain a \( G \)-cross section
\[
s: M_{(H)} \to S(\tau(M))|M_{(H)}
\]
such that the image of \( s \) is in \( S(\tau(M_{(H)})) \) and \( s|\partial M_{(H)} = s|\partial M_{(H)} \).

Let \( T(M_{(H)}) \) be a closed \( G \)-invariant tubular neighborhood of \( M_{(H)} \) in \( M \).
By Corollary 2.4 we obtain a $G$-bundle map
$$\pi: S(\tau(M)) \mid T(M_{CH}) \to S(\tau(M)) \mid M_{CH},$$
such that $\pi$ covers the canonical projection of $T(M_{CH})$ to $M_{CH}$ and $\pi$ is the identity on $S(\tau(M)) \mid M_{CH}$. $\pi$ induces a $G$-cross section
$$\pi^*s_i: T(M_{CH}) \to S(\tau(M)) \mid T(M_{CH})$$
from $s_i$ such that $\pi^*s_i \mid M_{CH} = s_i$.

To obtain a $G$-cross section
$$L_i = \partial M \cup T(M_{CH}) \to S(\tau(M)) \mid L_i,$$
extending $s$, we may apply Lemma 5.1 (stated in the last section) as follows. Apply $E \to X$, $A$, $B$, $D$, $S_A$ and $S_B$ in Lemma 5.1 to $\tau(M) \to M$, $T(M_{CH})$, $\partial M$, $\partial M_{CH}$, $\pi^*s_i$ and $s$, respectively. So, in this case,
$$C = A \cap B = T(M_{CH}) \cap \partial M = T(M_{CH}) \mid \partial M_{CH},$$
and this is compact. Moreover this is equivariantly deformable to $\partial M_{CH}$, and
$$\pi^*s_i \mid \partial M_{CH} = s \mid \partial M_{CH}.$$

Let $K$ be a closed $G$-invariant collar of $\partial M_{CH}$ in $M_{CH}$. By Proposition 2.2, $T(M_{CH}) \mid K$ has the desired property as $U$ in Lemma 5.1. Therefore we can apply Lemma 5.1 to this case. So we obtain a $G$-cross section
$$s_i: L_i \to S(\tau(M)) \mid L_i,$$
extending $s$.

Let $T^c(M_{CH})$ be the part of $T(M_{CH})$ corresponding to the open disc bundle. Set
$$L = M - T^c(M_{CH}).$$
Then $L$ is a compact $G$-manifold with corner. Smoothing the corner of $L$, let $L'$ be the resulting smooth $G$-manifold. Note that $L$ and $L'$ are the same topological space. Let
$$\varphi: \partial L' \times [0, 1] \to L'$$
be a $G$-invariant collar of $\partial L'$ in $L'$ such that
$$\varphi(\partial L' \times \{0\}) = \partial L'.$$
Identify $\partial L' \times [0, 1]$ with the image of $\varphi$. By Proposition 2.2 there is a $G$-bundle isomorphism
$$S(\tau(M)) \mid \partial L' \times [0, 1] \cong (S(\tau(M)) \mid \partial L') \times [0, 1].$$
$S(\tau(M))|\partial L'$ admits a $G$-cross section $s_3$ which is a restriction of $s_2$. From $s_3$, the above isomorphism induces a $G$-cross section

$$s_4: \partial L' \times [0, 1] \rightarrow S(\tau(M))|\partial L' \times [0, 1].$$

Set

$$L_2 = L' - \partial L' \times [0, 1].$$

$L_2$ is a compact $G$-manifold with $k-1$ isotropy types. $S(\tau(M))|L_2$ admits a $G$-cross section $s_5$ on $\partial L_2 = \partial L' \times \{1\}$ which is the restriction of $s_4$. $S(\tau(M))|L_2$ and $S(\tau(L_2))$ are identical, since the smoothing process of the corner of $L$ does not change the differentiable structure outside the corner. By careful consideration we see that the image of $s_5$ is in $S(\tau(3L_2))$. So, by the hypothesis of the induction, $s_5$ is extendible to a $G$-cross section

$$s_6: L_2 \rightarrow S(\tau(L_2)) = S(\tau(M))|L_2.$$

Then $s_5$, $s_4$, and $s_6$ give a $G$-cross section of $S(\tau(M))$ extending $s$.

"only if" part: It suffices to show that, for each isotropy type $(H)$ on $M$, if

$$\dim N(H) - \dim H = 0$$

then

$$|\chi|(M_H|N(H)) = 0.$$

Let $(H)$ be a maximal isotropy type on $M$. As in the "only if" part for the case of one isotropy type, we see that

$$\dim N(H) - \dim H = 0$$

implies

$$|\chi|(M_H|N(H)) = 0.$$

Let $(H')$ be another isotropy type on $M$. Consider $L'_{H'}$ where $L'$ is the compact $G$-manifold in the previous "if" part. Since $L'$ is of $k-1$ isotropy types,

$$\dim N(H') - \dim H' = 0$$

implies

$$|\chi|(L'_{H'}|N(H')) = 0$$

by the hypothesis of the induction. Furthermore $L'_{H'}$ is equivariantly homotopy equivalent to $M_{H'}$. Then

$$\dim N(H') - \dim H' = 0$$

implies

$$|\chi|(M_{H'}|N(H')) = 0.$$
Thus Theorem 1.1 is completely proved.

4. Proof of Corollary 1.3

If $M$ admits a $G$-$k$-field, then the $k$-field is tangent to $F$ by equivariancy. So the "only if" part is trivial.

Now we assume that $F$ admits a $k$-field. Then there are $k$ cross sections $s_1, s_2, \ldots, s_k$ of $\tau(F)$ which are linearly independent at each point of $F$. Since $\tau(F)$ is a subvector bundle of $\tau(M)|F$, we may regard $s_1, \ldots, s_k$ as $G$-cross sections of $\tau(M)|F$. As in the proof of Theorem 1.1 we may extend $s_1$ to a nowhere vanishing $G$-cross section $s_1'$ of $\tau(M)$. Choosing a $G$-invariant Riemannian metric on $\tau(M)$, let $E$ be the $G$-invariant subvector bundle of $\tau(M)$ which is orthogonal to the image of $s_1'$. The assumption

$$\dim N(G_x) - \dim G_x \geq k$$

enables us to extend $s_2$ to a nowhere vanishing $G$-cross section $s_2'$ of $E$ by the same method in the proof of Theorem 1.1. By repeating this process we may extend $s_3, \ldots, s_k$ to a $G$-cross sections $s_3', \ldots, s_k'$ of $\tau(M)$ such that $s_1', \ldots, s_k'$ are linearly independent at each point of $M$. So we obtain a $G$-$k$-field on $M$.

5. Concluding lemma

We will conclude this paper by proving the following lemma which was used in the proof of Theorem 1.1. $G$ acts trivially on intervals considered.

**Lemma 5.1.** Let $E \rightarrow X$ be a $G$-vector bundle. Let $A, B$ be $G$-invariant subspaces with $C = A \cap B$ compact. We assume that there is a $G$-invariant subspace $D$ of $C$ such that $C$ is equivariantly deformable to $D$, i.e., there is a $G$-homotopy $F: C \times [0, 1] \rightarrow C$ such that $F(x, 0) = x$ and $F(x, 1) \in D$ for all $x \in C$. We also assume that there is a $G$-invariant neighborhood $U$ of $C$ in $A$ such that there is a $G$-homeomorphism $\varphi: C \times [0, 3] \approx U$ with $\varphi(x, 0) = x$ for all $x \in C$. Let

$s_A: A \rightarrow S(E)|A$

and

$s_B: B \rightarrow S(E)|B$

be $G$-cross sections which agree on $D$. Then there is a $G$-cross section $s: A \cup B \rightarrow S(E)|A \cup B$
which agrees with \( s_A \) on \( A - U \) and \( s_B \) on \( B \).

Proof. By Proposition 2.3 there is a \( G \)-bundle map
\[
F: (S(E) | C) \times [0, 1] \to S(E) | C
\]
which covers \( F \) and is the identity on \( (S(E) | C) \times \{0\} \), and also there is a \( G \)-bundle isomorphism
\[
\varphi: (S(E) | C) \times [0, 3] \approx S(E) | U
\]
which covers \( \varphi \) and is the identity on \( (S(E) | C) \times \{0\} \). We define a \( G \)-bundle map
\[
K: (S(E) | C) \times [0, 2] \to S(E) | C
\]
by
\[
K(v, t) = \begin{cases} F(v, t) & \text{if } 0 \leq t \leq 1 \\ F(v, 2 - t) & \text{if } 1 \leq t \leq 2 \end{cases}
\]
for \( v \in S(E) | C \) and \( t \in [0, 2] \). Then we may define a \( G \)-cross section
\[
s_1: C \times [0, 2] \to (S(E) | C) \times [0, 2]
\]
of the \( G \)-bundle \( (S(E) | C) \times [0, 2] \) by
\[
s_1(x, t) = \begin{cases} K^* s_B(x, t) & \text{if } 0 \leq t \leq 1 \\ K^* s_A(x, t) & \text{if } 1 \leq t \leq 2 \end{cases}.
\]
This is well-defined since \( K^* s_A \) and \( K^* s_B \) agree on \( C \times \{1\} \). \( s_1 \) satisfies the following equations for all \( x \in C \)
\[
s_1(x, 0) = (s_B(x), 0)
\]
and
\[
s_1(x, 2) = (s_A(x), 2).
\]
We define a map
\[
\lambda: [2, 3] \to [0, 3]
\]
by \( \lambda(t) = 3t - 6 \) for \( t \in [2, 3] \). We denote by \( s_2 \) the \( G \)-cross section of \( (S(E) | C) \times [2, 3] \) which is induced from the \( G \)-cross section \( s_A | U \) by the composition
\[
(S(E) | C) \times [2, 3] \xrightarrow{(id, \lambda)} (S(E) | C) \times [0, 3] \xrightarrow{\varphi} S(E) | U.
\]
\( s_2 \) satisfies the following equations for all \( x \in C \)
\[
s_2(x, 2) = (s_A(x), 2)
\]
and
\[ s_2(x, 3) = \varphi^{-1}s_A\varphi(x, 3). \]

Since \( s_1 \) and \( s_2 \) agree on \( C \times \{2\} \), we obtain a \( G \)-cross section

\[ s_2: C \times [0, 3] \to (S(E)|C) \times [0, 3] \]

from \( s_1 \) and \( s_2 \). Then the induced \( G \)-cross section

\[ (\varphi^{-1})^{*}s_2: U \to S(E)|U \]

satisfies the following equations

\[ (\varphi^{-1})^{*}s_2(x) = s_B(x) \quad \text{for all } x \in C \]

\[ (\varphi^{-1})^{*}s_2(x) = s_A(x) \quad \text{for all } x \in \varphi^{-1}(C \times \{3\}). \]

So we obtain a \( G \)-cross section

\[ s: A \cup B \to S(E)|A \cup B \]

defined by

\[ s(x) = \begin{cases} 
s_A(x) & \text{if } x \in A \setminus U \\
(\varphi^{-1})^{*}s_2(x) & \text{if } x \in U \\
s_B(x) & \text{if } x \in B.
\end{cases} \]

References

