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ON GROUPS WITH A STANDARD COMPONENT OF KNOWN TYPE

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1. Introduction and notation

Let G be a finite group containing a standard subgroup of known isomorphism type, centralized by a 4-group. Then it is shown that G is a known group or G is of Conway Type. The proof requires information about the classes of involutions and centralizers in the automorphism groups of the known sporadic groups, and that information is summarized below in tabular form, as it is of independent interest.

The main theorem is a step toward the classification of finite groups of component type. To put the result in the proper setting we include the following definitions and background material.

A group A is *quasisimple* if A is its own commutator group and, modulo its center, A is simple. A component of a group is a subnormal quasisimple subgroup. The *core* of a group is its largest normal subgroup of odd order. *A 2-component* of a group is a subnormal subgroup A such that A is its own commutator group and A is quasisimple modulo its core. G is of *component type* if the centralizer in G of some involution contains a 2-component. This is equivalent to requiring that the centralizer is not 2-constrained.

The following important conjecture of J. G. Thompson seems close to being established:

B-conjecture: Let G be a finite core free group. Then 2-components of centralizers of involutions are quasisimple.

A subgroup K of G is *tightly embedded* in G if K has even order while K intersects its distinct conjugates in subgroups of odd order. A *standard subgroup* of G is a quasisimple subgroup A of G such that $K=C_G(A)$ is tightly embedded in G, $N_G(A)=N_G(K)$, and A commutes with none of its conjugates. It is shown in [1] and [14] that:

Component Theorem. Let G be a finite group of component type satisfying the B-conjecture and contained in the automorphism group of a

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simple group. Then, with known exceptions, G contains a standard subgroup.

Let $\mathcal{K}$ consist of the simple Chevalley groups, of both ordinary and twisted type, the alternating groups, and the 25 known sporadic groups listed below in Table 1. $\mathcal{K}$ contains all the finite simple groups known at the moment. Indeed existence proofs for two of the groups, and uniquencess theorems for still others, do not now exist, and in those cases we include in $\mathcal{K}$ all simple groups satisfying the defining properties of the (potential) group.

Theorem. *Let G be a finite group with $O(G)=1$, A a standard subgroup of G, and $X=\langle A^G\rangle$. Assume $Z(A)\in\mathcal{K}$ and the 2-rank of the centralizer in G of A is at least 2. Then the pair A,X is one of the following:*
(1) *$A=X$.*
(2) *A is an alternating group A_n and X is A_{n+4}.*
(3) *A is $L_2(4)$ and X is the Mathieu group M_{12}.*
(4) *A is $L_2(4)$ and X is the Hall-Janko group HJ.*
(5) *A is $L_3(4)$ and X is the sporadic Suzuki group Sz.*
(6) *A is a covering of $L_3(4)$ and X is Held's group He.*
(7) *A is $Sz(8)$ and X is Rudavalis' group Ru.*
(8) *A is $G_2(4)$ and X is of Conway Type.*

A group X is of Conway Type if X is simple, X possesses a standard subgroup $A\cong G_2(4)$, and there is a subgroup B of order 3 in A such that $E(C_A(B))=L\cong SL_3(4)$ and $\langle L^{C(B)}\rangle/B$ is isomorphic to Sz. Presumably a group of Conway Type is isomorphic to Conway's largest group Co_1.

The case $A/Z(A)\cong L_3(4)$ was done by Cheng Kai Nah [5] and the case $A/Z(A)$ a Bender group was done by Griess, Mason, and Seitz [19]. We appeal to their work rather than duplicating the proof.

Certain information about the involutions in the automorphism group of A is necessary to the proof. If A is a Chevalley group of odd characteristic this information is minimal. The appropriate facts are established in Section 4. If A is a Chevalley group of even characteristic, detailed information is required. This information is contained in [4], which is crucial to the proof. Less detailed information is required if A is a sporadic group. We do however determine the conjugacy classes of involutions in the automorphism group of A and the general nature of the isomorphism type of the centralizer of a representative in each class. These facts are summarized in Table 1. Column 1 gives the simple group G. Column 2 gives the order of the outer automorphism group of G. Columns 3 and 4 give the number of classes of involutions contained in G and in Aut (G)-G, respectively. Column 5 gives the general isomorphism type of the centralizers. By convention the centralizers of the classes in G are listed first. $G_n/G_{n-1}/\cdots$ denotes a group with normal series $1=H_0\lhd H_1\lhd\cdots\lhd H_n=G$ with $H_m/H_{m-1}\cong G_m$. Q^nD^m denotes the central pro-

Table 1

G	$\lvert Out(G)\rvert$	classes in G	classes not in G	centralizers
M_{11}	1	1	0	$GL_2(3)$
M_{12}	2	2	1	$S_3/Q^2 \quad Z_2\times S_5 \quad Z_2\times A_5$
M_{22}	2	1	2	$S_4/E_{16} \quad L_3(2)/E_8 \quad F_{20}/E_{16}$
M_{23}	1	1	0	$L_3(2)/E_{16}$
M_{24}	1	2	0	$L_3(2)/D^3 \quad S_5/E_{64}$
J_1	1	1	0	$Z_2\times A_5$
HJ	2	2	1	$A_5/QD \quad E_4\times A_5 \quad PGL_2(7)$
J_3	2	1	1	$A_5/QD \quad L_2(17)$
Mc	2	1	1	$A_8/Z_2 \quad M_{11}$
Ly	1	1	0	A_{11}/Z_2
HS	2	2	2	$S_5/Q^2{*}Z_4 \quad Z_2\times Aut(A_6) \quad S_5/E_{16} \quad S_8$
He	2	2	1	$L_3(2)/D^3 \quad Z_2/L_3(4)/E_4 \quad S_7/Z_3$
Sz	2	2	2	$\Omega_6^-(2)/Q^3 \quad Z_2/L_3(4)\times E_4 \quad Aut(HJ) \quad Aut(M_{12})$
Ru	1	2	0	$S_5/2^{11} \quad Sz(8)\times E_4$
ON	2	1	1	$Z_2/L_3(4)/Z_4 \quad J_1$
Co_3	1	2	0	$Sp_6(2)/Z_2 \quad Z_2\times M_{12}$
Co_2	1	3	0	$Sp_6(2)/Q^4 \quad A_8/E_{16}\times D^3 \quad Aut(A_6)/E_2 10$
Co_1	1	3	0	$\Omega_8^+(2)/Q^4 \quad Aut(M_{12})/E_{2^{11}} \quad Z_2/G_2(4)\times E_4$
$M(22)$	2	3	3	$U_6(2)/Z_2 \quad Z_2/U_4(2)/Z_2\times Q^4$ 2-constrained $2^{16}\cdot 3^3$ $Aut(\Omega_8^+(2)) \quad Z_2\times Sp_6(2) \quad O_8^-(2)/E_{64}$
$M(23)$	1	3	0	$M(22)/Z_2 \quad Z_2/U_6(2)/E_4 \quad S_3/U_4(2)/E_4\times Q^4$
$M(24)'$	2	2	2	$Z_2/M(22)/Z_2 \quad S_3/\Omega_6^-(3)/Z_3/Q^6 \quad M(23) \quad S_3/U_6(2)/E_4$
F_5	2	2	1	$Z_2/HS/Z_2 \quad A_5$ wreath $Z_2/Q^4 \quad S_{10}$
F_3	1	1	0	A_9/Q^4
F_2	1	4	0	$Z_2/{}^2E_6(2)/Z_2 \quad Z_2/F_4(2)\times E_4 Co_2/D^{11} \quad O_6^+(2)/E_2 16/E_2 9$
F_1	1	2	0	$F_2/Z_2 \quad Co_1/Q^{12}$

duct of n copies of the quaternion group of order 8 and m copies of the dihedral group of order 8, with identified centers. E_n is an elementary abelian group of order n.

Most of the information listed in Table 1 is already known and much appears in the literature. Some is collected in an unpublished table of N. Burgoyne. Proofs and references to proofs of the facts in Table 1 appear within. In many cases more detailed information is included.

In addition to the notation and terminology defined above we also use Bender's notation $F^*(G)$ for the Generalized Fitting subgroup of G. $F^*(G)=E(G)F(G)$, where $E(G)$ is the join of the components of G and $F(G)$ is the Fitting subgroup of G. $L(G)$ is the join of all 2-components of G. $G^{\mathcal{A}}$ denotes the smallest normal subgroup H of G such that G/H is solvable with abelian Sylow 2-groups. Given a permutation representation of G on a set

Ω, G^{Ω} denotes the image of G under this representation.

A quasisimple group A satisfies hypothesis II if whenever a noncyclic elementary abelian 2-group T acts faithfully on A, with T Sylow in a 2-nilpotent tightly embedded subgroup of TA, then $T \leq AC(A)$.

$\Omega_n^{\varepsilon}(2^m)$ is the commutator group of the n-dimensional orthogonal group over $GF(2^m)$ defined by a quadradic form of sign ε.

$I(x)$ is the set of fixed points of a permutation x.

The concept of "admissibility" is defined in Section 2.

2. Preliminary results

In this section we collect a number of lemmas which will be used in the proof of the main theorem.

(2.1) Let K be tightly embedded in G, $R \in Syl_2(K)$ and $\Phi(R)=1$. Assume $F^*(G)$ is simple and $K \ntrianglelefteq G$. Then $0(K)R$ is tightly embedded in G. Further $C_K(r)$ is solvable for each $r \in R^{\#}$.

Proof. As $F^*(G)$ is simple and K is not normal in G, Theorem 4 of [1] implies either K is 2-constrained or $O^{2'}(K)/O(K) \cong L_2(2^n)$. In the former case $O_{2',2}(K)=O(K)R$ is tightly embedded in G. In the latter case $RO(K)$ is $C(r)$-invariant for each $r \in R^{\#}$ and $N(RO(K))$ is transitive on $R^{\#}$, so $RO(K)$ is tightly embedded in G.

The following will be used as an induction tool in the proof of the main theorem:

(2.2) Let K be a solvable tightly embedded subgroup of G and $R \in Syl_2(K)$. Assume L is a quasisimple subgroup of G normal in $N(K)$ and $R \in Syl_2(C(L))$. Then either

(1) L is standard in G, or
(2) $\langle L^G \rangle = L \times L^g$, $R \leq L^g$, and L is a Bender group.

Proof. Let $H=C(L)$. $L \trianglelefteq N(K)$ so $N(K) \leq N(L)$. As $R \in Syl_2(H)$ and K is tightly embedded in G, it follows that H is tightly embedded in G. Also $N(H)=H(N(H)\cap N(R)) \leq HN(K) \leq N(L)$, so $N(H)=N(L)$. Therefore, L is either standard or there exists a conjugate $A=L^g$ of L in H, and we may assume the latter.

Let $T \in Syl_2(L)$. Then $T \leq L \leq C(A)$, so $H \leq C(T) \leq N(C(A)) \leq N(A)$. Hence $A \trianglelefteq H$. Now as R is Sylow in H either $A \leq \Gamma_{1,R}(A) \leq N(K)$, or $R \in Syl_2(A)$ and $N_A(K)$ is strongly embedded in A. In the former case $A=[A, A\cap R] \leq K$, impossible as K is solvable. Hence $R \in Syl_2(A)$, so that $A=O^{2'}(H)$, and A is a Bender group. Moreover $D=L \times A$ satisfies the hypothesis of Theorem 5 in [1], so that theorem implies that $D \trianglelefteq G$.

(2.3) Let K be a solvable tightly embedded subgroup of G. Then $L(N(K))= L(C(t))$ for each involution $t\in K$.

Proof. 2.1 and 2.7 of [1].

Let T be a noncyclic elementary abelian 2-group. A quasisimple group A is *T-admissible* if T acts faithfully on A, T is Sylow in a 2-nilpotent tightly embedded subgroup of TA, and

(2.4) Either $|T|=4$ or $N_T(T^a)\leq C(T^a)$ for each $a\in A$.

(2.5) $O^2(C_A(t))^{\mathcal{A}}\leq C(T)$, each $t\in T^\#$.

Recall $X^{\mathcal{A}}$ is the smallest normal subgroup Y of X such that X/Y is solvable with abelian Sylow 2-groups. A is said to be *admisible* if A is T-admissible for some noncyclic elementary 2-group T.

(2.6) Assume A is T-admissible, $Z(A)\leq C(T)$, and for each $t\in T^\#$, $F^*(C_A(t))/Z(A)$ is a 2-group. Then T is a TI-set in AT.

Proof. $T\in Syl_2(X)$, X a 2-nilpotent tightly embedded subgroup of AT. Let $Y=O(X)$. It suffices to show $[T, Y]=1$. Let $W=C_Y(t)$. Then $W\leq C_A(O_2(C_A(t))$, so as $F^*(C_A(t))/Z(A)$ is a 2-group, $W\leq Z(A)$. Therefore $Y=\langle C_Y(t)\colon t\in T^\#\rangle\leq Z(A)\leq C(T)$.

(2.7) Assume the hypothesis of 2.6 with $A\cap T=1$. Then $[C_A(t), T]=1$ for each $t\in T^\#$.

Proof. $[C_A(t), T]\leq A\cap T=1$ by 2.6.

3. Standard subgroups

Recall that a quasisimple group A is *standard* in G if $K=C_G(A)$ is tightly embedded in G, $N_G(A)=N_G(K)$, and A commutes with none of its conjugates. In this section we operate under the following hypothesis:

Hypothesis 3.1. A is standard in G and A satisfies hypothesis II. $O(G)=1$ and $m(C_G(A))>1$. A is not normal in G.

Set $K=C_G(A)$, $\overline{N(A)}=N_G(A)/K$, and let $R\in Syl_2(K)$. By Theorem 3 in [2]:

(3.2) $\Phi(R)=1$.

(3.3) Let $g\in G-N(A)$ and $T\in Syl_2(K^g\cap N(A))$. Assume $T\neq 1$ and R is T-invariant. Then

(1) Either R has order 4 or $T\in Syl_2(K^g)$ and $[T, R]=1$.
(2) If $T\in Syl_2(K^g)$ then $T\leq AK$
(3) There exists $g\in G-N(A)$ with $T\in Syl_2(K^g)$.

(4) $C_R(T)\cong T$.

Proof. [2].

Given 3.3 we may choose $g\in G-N(A)$ such that a Sylow 2-group T of $K^g\cap N(A)$ is Sylow in K^g, and T centralizes R. Define V to be the weak closure of R in the centralizer of $R^G\cap C_G(RT)$. This notation is maintained thorughout this section.

(3.4) $\langle A^G\rangle=F^*(G)$ is simple.

Proof in [3].

(3.5) (1) V is an elementary abelian 2-group .
(2) $V=R(V\cap A)=T(V\cap A^g)$.

Proof. (1) is immediate from 3.2 and the definition of V. (2) follows from 3.3.3.

(3.6) Assume

(*) For each $R^x\leq V$, $N_{A^x}(V)/C_{A^x}(V)$ has a characteristic cyclic subgroup regular on $(V\cap A^x)^\#$.
Then either
(1) $[N_A(V), V]=T$, $[N_{A^g}(V), V]=R$, and $R^G\cap V=\{R, T\}$.
(2) $[N_A(V), V]=[N_{A^g}(V), V]=V_0$,
$$V-V_0=\bigcup_{Q\in R^G\cap V} Q^\#$$
and $N(V)^{\langle R^G\cap V\rangle}$ is 2-transitive.

Proof. Let $X=\langle O^2(N_A(V), O^2(N_{A^g}(V)\rangle$. Then X in tis action on V satisfies the hypothesis of lemma 3.1 in [1], so that lemma implies $V_0=[N_A(V), V]$ is X-invariant, and either X acts on R or $V-V_0$ is the disjoint union of $q=|V_0|$ conjugates of $R^\#$ under X, with $N_A(V)$ transitive on $R^X-\{R\}$. Notice that in this second case V_0 is the only nontrivial X-invariant subspace of V.

Suppose R is X-invariant. Then R and V_0 are X-invariant subspaces of V, so applying the remarks above to T, T must also be X-invariant. As V_0 is the unique $N_A(V)$-invariant subspace disjoint from R, $T=V_0$. Similarly $R=[V, A^g\cap N(V)]$. Moreover in this case R and T uniquely determine each other in V.

Suppose R is not X-invariant and $R^y=Q\in(R^G\cap V)-R^X$. As $V-V_0$ is the disjoint union of conjugates of $R^\#$ under X, $Q\leq V_0$. Hence applying the argument above to the pair R, Q in place of the pair R, T, we conclude $Q=V_0$ and $R=[V, A^x\cap N(V)]$. Similarly $T=[V, A^x\cap N(V)]$, a contradiction.

It follows that either (1) or (2) holds, and the proof is complete.

(3.7) Assume $W\leq A$ such that

(a) $L=E(C_A(W))$ is quasisimple.
(b) $R\in Syl_2(C(WL)\cap N(R))$.
Then either
(1) L is standard in $C(W)$, or
(2) $\langle L^{C(W)}\rangle=L\times L^c$, $R\leq L^c$, and $L\cong L_2(|R|)$.

Proof. By 2.1, 3.2, and 3.4, $O(K)R$ is tightly embedded in $C(W)$. (b) implies R is Sylow in the centralizer of WL. Hence 2.2 yields the desired result.

(3.8) Assume the hypothesis of 3.7 with $T\leq RL$, $T\cap L=1$, R not normal in $C(WT)$, and if $L\cong L_2(|R|)$ assume $R^G\cap L$ is empty. Then L is standard and nonnormal in $C(W)$.

Proof. Assume L is not standard in $C(W)$. Then by 3.7, $L\cong L_2(|R|)$ and R is contained in a conjugate L^c of L. But then $R^{c^{-1}}\in R^G\cap L$, contrary to hypothesis.

So L is standard. Assume $L\trianglelefteq C(W)$. Then $H=C(LW)\trianglelefteq C(W)$. $T\leq RL$ and $T\cap L=1$, so $RL=TL$ with $R=O_2(TL\cap H)\trianglelefteq C(TW)$, contrary to hypothesis.

Recall that for a group X, X^a is the smallest normal subgroup Y of X such that X/Y is solvable with abelian Sylow 2-groups.

(3.9) Let $t\in T^\#$. Assume the commutator group of Out(A) is of odd order. Then

(1) $(O^2(C_A(t))C_A(t)')^a\leq A^g\leq C(T)$.
(2) $O^2(C_{\bar{A}}(\bar{t}))^a\leq C(\bar{T})$.
(3) A is T-admissible.

Proof. As the kernel of the homomorphism of A to $\bar{A}$ is the center of AT, $O^2(C_{\bar{A}}(\bar{t}))=O^2(C_A(t))/Z(A)$. Hence (1) implies (2). Also (1) and 3.2 and 3.3 imply (3). As the commutator group of Out(A) is of odd order, $O^2(C_A(t))C_A(t)'\leq O^2(N(A^g))=D$ and D/A^g has abelian Sylow 2-subgroups. By 2.1, $K^g\cap C(t)$ is solvable, so (1) follows.

(3.10) Assume $A\cong L_2(4)$ with $R^G\cap A$ empty. Then either

(1) $\langle A^G\rangle\cong HJ$, or
(2) $\langle A^G\rangle\cong M_{12}$ and there exists an involution t fused into R inducing an outer automorphism on A and acting nontrivially on R.

Proof. [3].

(3.11) Assume $\bar{A}$ is a Bender group. Then one of the following holds:

(1) $A\trianglelefteq G$.

(2) $A\simeq L_2(4)$ and $\langle A^G\rangle\simeq M_{12}$, HJ or A_9.
(3) $A\simeq Sz(8)$ and $\langle A^G\rangle\simeq Ru$.

Proof. [19].

4. Chevalley groups of odd characteristic

Hypothesis 4.1. $G=G(q)$ is a Chevalley group with $q=p^e$ odd and $G\neq L_2(q)$ or ${}^2G_2(q)$. Let Δ be a root system, $U\in Syl_p(G)$, H a p-complement in $N_G(U)$, and for $s\in\Delta$ let U_s be the corresponding root subgroup of G and $V_s=\Omega_1(U_s)$. Let r be the root of highest height in Δ, $V=V_r$, $J=\langle V, V_{-r}\rangle$, and $\langle t\rangle=Z(J)$.

(4.2) Assume 4.1. Then

(1) $J\simeq SL_2(q)$ and $t\in H$.
(2) $N_G(J)=XJH$ where $[X, J]=1$ and X is the Levi factor of the parabolic subgroup $P=N_G(V)$.
(3) If G is not isomorphic to $\Omega_n^\varepsilon(q)$ then $N_G(J)=C_G(t)$, so that J is tightly embedded in G.
(4) If $G\simeq\Omega_n^\varepsilon(q)\neq\Omega_8^+(q)$ then $X=X_1J^w$, for some $w\in W$, and $C_G(t)=X_1JJ^wH\langle w\rangle$.
(5) If $G\simeq\Omega_8^+(q)$ there exists a 4-group W_1 in W such that XJ is the central product of four conjugates of J under W_1 and $C_G(t)=XJHW_1$.
(6) The isomorphism class of X and the weak closure of V in the centralizer of t are given in Table 4.2.

Proof. Let G have rank l. Statement (1) is well known. Write

$$G=\bigcup_w UHwU_w^-$$

Table 4.2

$G(q)$	X	$\langle V^G\cap C(t)\rangle$
$L_n(q)$	$SL_{n-2}(q)$	XJ
$PSP_n(q)$	$SP_{n-2}(q)$	XJ
$U_n(q)$	$SU_{n-2}(q)$	XJ
$\Omega_n^\varepsilon(q)$	$SL_2(q)SO_{n-4}^\varepsilon(q)$	XJ, unless $n=7$ or $n=8$, $\varepsilon=-1$, where JJ^w.
$G_2(q)$	$SL_2(q)$	J
${}^3D_4(q)$	$SL_2(q^3)$	J
$F_4(q)$	$SP_6(q)$	XJ
${}^2E_6(q)$	$SU_6(q)$	XJ
$E_6(q)$	$SL_6(q)/Z_{(q-1,3)}$	XJ
$E_7(q)$	$SO_{12}^+(q)$	XJ
$E_8(q)$	$E_7(q)$	XJ

the Bruhat decomposition of G. The representation of elements is unique and $N_G(J)\leq C_G(t)=C_U(t)N_N(t)C_U(t)=\langle C_U(t), C_N(t)\rangle$.

The structure of P is known (eg. [7], [13]). $P=QXH$, where $Q=O_p(P)$ and X is the Levi factor of P. In fact

$$P=\langle B, s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_l\rangle$$

for some i, except for $G=A_l(q)$, where

$$P=\langle B, s_2, \cdots, s_{l-1}\rangle$$

Then $X=\langle U_{\pm\alpha_j}: j\neq i\rangle$ or $\langle U_{\pm\alpha_j}: j\neq 1, l-1\rangle$ if $G=A_l(q)$. If w_0 is the word of greatest length in the generators $s_1, \cdots, s_l$ of W, then $J^{w_0}=J$, $(\Delta^+)^{w_0}=\Delta^-$, and $P^{w_0}=Q^{w_0}XH$.

Now Q is special with $Q'=V$ and hence $(Q^{w_0})'=V^{w_0}=V_{-r}$. Also $O^{p'}(C(V))=QX$, so $X\leq C(J)\leq C(t)$.

Next, t inverts Q/V. This can be checked directly using the structure of P. An easy proof is obtained in most cases using the results of sections 3 and 4 of [7] and sections 9 through 11 of [13] to note that X acts irreducibly on Q/V.

It follows that $C_U(t)=(U\cap X)V$, $C(t)\cap U^{w_0}=(U^{w_0}\cap X)V_{-r}$ and $\langle C_U(t), C_{U^{w_0}}(t)\rangle=\langle C_{U_s}(t): s\in\Delta\rangle=XJ$. Also $C_N(t)$ normalizes $\langle C_{U_s}(t): s\in\Delta\rangle=XJ$.

If X contains no component in J^W then $C_N(t)\leq C(t)\cap N(J)\leq J(N(V)\cap C(t))=JXH$. So in this case $C_G(t)=\langle C_U(t), C_N(t)\rangle=JXH=N(J)$. Moreover this occurs unless $G\cong\Omega_n^\varepsilon(q)$. Here we use the fact that $PSp_4(q)$, $U_4(q)$, and $L_4(q)$ are isomorphic to $\Omega_5(q)$, $\Omega_6^-(q)$ and $\Omega_6^+(q)$, respectively.

In the remaining cases X has the form $X=X_1J^w$, where $X_1\cong\Omega_{n-4}^\varepsilon(q)$. Checking the root system we find w may be chosen to interchange J and J^w and to normalize X_1. Morever with the exception of $\Omega_8^+(q)$, X_1 contains no component in J^w. Hence (4) holds. For $G=\Omega_8^+(q)$, $X_1=J^{w_1}J^{ww_1}$, and we set $W_1=\langle w, w_1\rangle$ to obtain (5). (6) is easy to check.

(4.3) Assume 4.1 and let $\tilde{G}$ be the Universal Chevalley group of type $G(q)$, and $\tilde{J}=\langle\tilde{V}_r, \tilde{V}_{-r}\rangle$ be defined in $\tilde{G}$ in the same way J and V are defined in G. Then $\tilde{J}$ is isomorphic to $SL_2(q)$ and is tightly embedded in $\tilde{G}$.

Proof. Let K be the preimage in $\tilde{G}$ of J under the homomorphism of $\tilde{G}$ onto G. Then $K=O^{p'}(K)\times Z(\tilde{G})$ and $\tilde{J}=O^{p'}(K)\cong J\cong SL_2(q)$. Let $\langle z\rangle=Z(\tilde{J})$. It remains to show $N_{\tilde{G}}(J)=C_{\tilde{G}}(z)$. It suffices to establish this fact in some nontrivial homomorphic image $\bar{G}$ of $\tilde{G}$. For $G\ncong\Omega_n^\varepsilon(q)$, set $\bar{G}=G$ and use 4.2.3. For $G\cong\Omega_n^\varepsilon(q)$ set $\bar{G}=\mathrm{Spin}^\varepsilon(n, q)$, and check the Clifford algebra (eg. [24], 23.4) to obtain the result.

(4.4) Assume 4.1. Let $S\in Syl_2(G)$ and $\sum=\{J^g: J^g\cap S\in Syl_2(J^g)\}$. Then $\langle\sum\rangle$ is the central product of the members of $\sum$.

Proof. It suffices to prove the corresponding result in the Universal Chevalley group $\tilde{G}$ of type $G(q)$. By 4.3, $\tilde{J}$ is tightly embedded in $\tilde{G}$ and has quaternion Sylow 2-groups. Let $\Delta=\{K \cap S: K \in \sum\}$. Then the members of Δ are tightly embedded in S, so by 2.5 in [2], distinct members of Δ commute. Thus for distinct members $\tilde{J}$ and $\tilde{J}^g$ in $\sum$, the involution z^g in $Z(\tilde{J}^g)$ centralizes a Sylow 2-group of $\tilde{J}$. Hence $\tilde{J} \leq C(z^g) \leq N(\tilde{J}^g)$. By symmetry $\tilde{J}^g$ acts on $\tilde{J}$, so $[\tilde{J}, \tilde{J}^g] \leq \tilde{J} \cap J^g = 1$.

(4.5) Assume 4.1. Then $G=\langle O_p(N_G(V), J\rangle$.

Proof. Let $Q=O_P(N(V))$ and $M=\langle Q, J\rangle$. As $N(Q)=N(V)=Q \times H$ with $XH \leq N(J)$ we have $N(Q) \leq N(M)$. Thus $N(M \geq \langle N(Q), J\rangle = G$, so $G=M$.

Hypothesis 4.6. T is a noncyclic elementary abelian 2-group acting on a group G and Sylow in a 2-nilpotent tightly embedded subgroup K of G.

(4.7) Assume 4.6 with $SL_2(q) \cong J \trianglelefteq G$, q odd, and $C_T(J) \neq 1$. Then $[J, T] = 1$.

Proof. As $J \trianglelefteq G$ and $C_T(J) \neq 1$, $J \trianglelefteq N(K)$. Suppose $t \in T - C(J)$. Then $[J, t] \leq K \cap J$, so $[J, t]$ is a 2-nilpotent normal subgroup of J. Hence either $[J, t] \leq Z(J)$ or $q=3$ and $[J, t]=O_2(J)$. In the first case $J=O^2(J) \leq C(t)$. The second case is impossible rs $O_2(J)$ is quaternion while K has abelian Sylow 2-groups.

(4.8) Let U be a 4-group acting on a central product L of groups $L_i \cong SL_2(q)$, q odd, which are permuted by U. Assume U moves L_1. Then $L_1 \leq \Gamma_{1,U}(L)$.

Proof. If $q>3$ this is a corollary to 2.8 in [1]. Morever the same proof works if $q=3$.

Theorem 4.9. *Assume G is quasisimple with $Z(G)$ a 2-group and $\bar{G}= G/Z(G) \cong G(q)$, or $G \cong L_2(3)$ or $SL_2(3)$. Assume T is a 2-group acting faithfully on G and GT satisfies hypothesis 4.6 Then*
(1) $G \cong L_2(q)$, $3 \leq q \leq 9$
(2) $T \leq GC(G)$
(3) *If* $q>5$ *then* $T \leq G$.

Let G be a minimal counter example to Theorem 4.9. We first show
(4.10) $\bar{G} \ncong L_2(q)$ or ${}^2G_2(q)$.

Proof. If $\bar{G} \cong {}^2G_2(q)$ then $|\mathrm{Out}(\bar{G})\colon \bar{G}|$ is odd and $C_{\bar{G}}(\bar{x})$ is maximal in $\bar{G}$ with $\langle \bar{x}\rangle = Z(C(\bar{x}))$ for each involution $\bar{x}$ in $\bar{G}$. Further $Z(G)=1$. So $G= \Gamma_{1,T}(G) \leq N(K)$, a contradiction.

So assume $\bar{G} \cong L_2(q)$. By 3.5 and 3.6 in [1], $T \leq GC(G)$, T is a 4-group, and $q \leq 9$. Moreover if $G \cong L_2(q)$, $q > 5$, then $[C_G(t), T] \neq 1$, so $T \leq G$. Hence we may take $G \cong SL_2(q)$, $q \leq 9$. Now $T^{\#} = \{t_i : 1 \leq i \leq 3\}$ and $t_i = g_i c_i$, g_i and c_i elements of order 4 in G and $C(G)$ respectively with $g_i^2 = c_i^2 = z$ generating $Z(G)$. Then $g_3 c_3 = t_3 = t_1 t_2 = (g_1 g_2)(c_1 c_2)$, so $g_1 g_2 = g_3^{\pm 1}$. Hence $Q = \langle g_1, g_2 \rangle$ is quaternion. Now $z = [t_1, g_2] \in [K, C(t_2)] \leq K$, impossible as T acts faithfully on G.

(4.11) $\bar{G} \cong L_2(q)$ or ${}^2G_2(q)$.

Proof. Assume $\bar{G} \not\cong L_2(q)$ or ${}^2G_2(q)$. Then $\bar{G}$ satisfies 4.1. Take $S \in \mathrm{Syl}_2(G)$ to be T-invariant with $\bar{J} \cap \bar{S} \in \mathrm{Syl}_2(\bar{J})$. Let J_1 be the preimage in G of $\bar{J}$ and set $J = O^2(J_1)$. Then $J_1 = Z(G)J$. We show $[T, J] = 1$. Then $\bar{T}$ centralizes $\bar{V}$, so $\bar{Q} = O_p(N(\bar{V})) = \Gamma_{1,\bar{T}}(\bar{Q}) = \Gamma_{1,T}(Q)C(G)/C(G) \leq N(K)C(G)/C(G)$, as $\bar{Q}$ is of odd order. Hence by 4.5, $G \leq N(K)C(G)$, so as G is quasisimple and $KC(G)$ is solvable, $[G, T] \leq [KC(G), G] = 1$, a contradiction.

So it remains to show $[T, J] = 1$. If T acts on J this follows from 4.7 and 4.10. So assume T does not act on J. We show $\langle J^T \rangle$ is the central product of the groups in J^T and hence by 4.8, $J \leq \Gamma_{1,T}(G) \leq N(K)$. Thus $[J, T] \leq K$. But as $T \not\leq N(J)$, $[J, T]$ is not 2-nilpotent, a contradiction.

Suppose $\bar{G} \not\cong G_2(q)$. Then T acts on $\bar{J}^G$ and we appeal to 4.4. So assume $\bar{G} \cong G_2(q)$. Then $\bar{G}$ has one class of involutions, so we may assume T centralizes the involution z in J. Now $O^2(C_{\bar{G}}(z))$ is the central product of $\bar{J}$ and $\bar{L} \cong SL_2(q)$, so again the result follows. This completes the proof of 4.11, and hence also of Theorem 4.9.

Theorem 4.12. *Assume A is standard and non-normal in G with $O(G) = 1$, $m(C(A)) > 1$, and $A/Z(A) \cong G(q)$, q odd. Then either*
(1) *$A \cong L_2(5)$ and $\langle A^G \rangle \cong HJ$, M_{12} or A_9, or*
(2) *$A \cong L_2(9)$ and $\langle A^G \rangle \cong A_{10}$.*

Proof. If $A/Z(A)$ is isomorphic to $L_2(5) \cong A_5$ or to $L_2(9) \cong A_6$, Then we appeal to the main theorem of [3] to obtain (1) and (2). So assume otherwise.

By 4.9, A satisfies hypothesis II. Hence we may adopt the notation of section 3. A second application of 4.9 implies $Z(A)$ is of odd order, $A/Z(A) \cong L_2(7)$, and $T \leq A$. Now there euists an involution $a \in N_A(T) - C(T)$. As $[a, T] \neq 1$, a induces an outer automorphism on A^g. However $[a, R] = 1$, and by 4.9, $R \leq A^g$, whereas an outer automorphism of $L_2(7)$ centralizes no 4-group in $L_2(7)$. The proof is complete.

5. A fusion lemma

In this section we assume the following hypothesis:

Hypothesis 5.1. $V = R \oplus U \oplus W$ is a finite dimensional vector space over

$GF(2)$ with $m=|R|>2$ and $q=|U|=|W|$. X is a group of automorphisms of V and $A\times B\trianglelefteq N_X(R)$ with A and B cyclic groups such that $[A, U]=0=[B, W]$, A is regular on $W^\#$, and B is regular on $U^\#$, and $[AB,R]=0$. Define

$$\sum = \bigcup_{R^\#} r^X, \quad \Omega = R^X, \quad \Gamma = U+W-(U\cup W).$$

Assume:
(1) For $T\in\Omega-\{R\}$, $R\cap T=0$ and the projection $P(T)$ of T on $U+W$ is contained in $U^\#$, $W^\#$, or Γ.
(2) If $T\leq R+U$ then either $T=U$ and $(R+T)\cap\Omega=\{R, T\}$ or $T\cap U=0$. The same holds with U replaced by W.
(3) There exists $T\in\Omega-\{R\}$ with $P(T)^\#\subseteq\Gamma$.

(5.2) Either

(1) $\sum=V-(U+W)$ and $|\Omega|=q^2$, or
(2) $q=m=4$ and U and W are in Ω.

The proof involves a series of reductions. Assume 5.2 to be false.

(5.3) If $g\in(AB)^\#$, $T\in\Omega$ with $P(T)^\#\subseteq\Gamma$ and t and t^g are in $T^\#$, then $T^\#\subseteq\Gamma$.

Proof. $t^g\in T\cap T^g$ so by 5.1.1, $T=T^g$. Then $T=C_T(g)+[T,g]$. $C_T(g)$ is contained in $R+U$ or $R+W$, say the former, so if $C_T(g)\neq 0$ then $P(T)^\#\nsubseteq\Gamma$. Thus $T=[T,g]\leq[V,AB]=U+W$. So $T^\#=P(T)^\#\subseteq\Gamma$.

(5.4) If $T\cap(U+W)\neq 0$ then $T\leq U+W$.

Proof. By (2) we may take $P(T)^\#\subseteq\Gamma$. Let $t\in T^\#\cap(U+W)$. Then $t\in\Gamma$. Assume $s\in T^\#-\Gamma$. Then $s=r+c$, $r\in R^\#$, $c\in\Gamma$. $r+c+t=s+t\in T^\#$ and hence $c+t\in\Gamma$ by 5.1.1. $(AB)^\Gamma$ is transitive so there exists $g\in(AB)^\#$ with $c+t=c^g$. Then $s^g=s+t$, so by 5.3, $T\leq U+W$.

(5.5) Let $T\in\Omega-\{R\}$ and $k=|T^{AB}|$.

Then one of the following holds:
(1) $T=U$ or $T=W$ and $k=1$.
(2) $T^\#\subseteq(R+U)-U$ or $T^\#\subseteq(R+W)-W$ and $k=q-1$.
(3) $T^\#\subseteq\Gamma$ and $k=(q-1)^2/(m-1)$.
(4) $P(T)^\#\subseteq\Gamma$, $T\cap(U+W)=0$, and $k=(q-1)^2$.

Proof. This follows easily from 5.1.1, 5.1.2, and 5.3.

Let α and β be the number of AB orbits of type 5.5.1 and 5.5.2, respectively. By 5.1.2, $\alpha+\beta\leq 2$.

(5.6) There exists $T\in\Omega-\{R\}$ with $T\nleq U+W$.

Proof. If $\Omega-\{R\}\subseteq U+V$ then $|\sum\cap(R+T)|=2(m-1)<|\sum\cap(T+S)|$ for all distinct T and S in $\Omega-\{R\}$, a contradiction.

(5.7) If $\alpha\neq 0$ then $\alpha=\beta=1$.

Proof. Assume $U\in\Omega$ but $\beta=0$. Then by 5.4 and 5.6 there exists $T\in\Omega$ with $T\cap(U+W)=0$ and $P(T)^{\#}\subseteq\Gamma$. Hence U, and possibly W, are the only members S of Ω such that $|\sum\cap(S+R)|=2(q-1)$. Also $R+U=C_V(A)$ and $W=[V,A]$, so W is the unique $C(R)\cap C(U)$-invariant complement to $R+U$ and hence R and W play the same role with respect to U as U and W play to R. Thus $\{R, U\}$ or $\{R, U, W\}$ is a set of imprimitivity for the action of X on Ω. Let Δ be the set of imprimitivity containing T, and S a second member of Δ. $\sum\cap(T+S)=T^{\#}\cup S^{\#}$ so $T+S=(T+S\cap(U+W))\cup((T+S)\cap R+U)\cup(T+S)\cap(R+W)\cup T\cup S$. Hence $m=q=4$. $|\Delta|$ divides $|AB|=9$, so $|\Delta|=3$ and U and W are in Ω. As this is the second case of 5.2, we have a contradiction.

(5.8) $\beta>0$.

Proof. Assume $\beta=0$. By 5.7, $\alpha=0$. By 5.6, there exists $T\in\Omega$ with $T\cap(U+W)=0$ and $P(T)^{\#}\subseteq\Gamma$.

Suppose $\Gamma\subseteq\sum$. Then $(R+S)^{\#}\subseteq\sum$ for all $S\in\Omega-\{R\}$, whereas there exists P and Q in $\Omega\cap(U+W)$ with $(P+Q)^{\#}\nsubseteq\sum$. So by 5.5, $\Omega=\{R\}\cup T^{AB}$. In particular X is 2-transtitve on Ω and by a result of Hering, Kantor, and Seitz, $q-1=r$ is a prime and X^{Ω} is contained in the automorphism group of $L_2(r^2)$. Further $\{A, B\}$ is invariant under $N_X(R)$, so $(r+1)/2\leq 2$ and hence $r=3$ and $q=4$. Let x be an element of order 4 in $N_X(R)^{\Omega}$. Then $x^2=y$ centralizes the 4-group R and fixes exactly two points of Ω. Also y centralizes vectors $u\in U^{\#}$ and $w\in W^{\#}$, and then the coset $R+u+w$. But $R+u+w$ intersects three members of Ω, which must be fixed by y, a contradiction.

(5.9) $\beta=2$, $\sum=V-(U+W)$, and $|\Omega|=q^2$.

Proof. By 5.8, $\beta>0$, so we may assume $\Delta=(R+U)\cap\Omega$ is of order q. Let $R\neq R^g\in\Delta$. Now if Z is an A-invariant subspace of V then either $W\leq Z$ or $Z\leq C_V(A)$. Further A centralizes R^g so A acts nontrivially on U^g or W^g. Hence $W=U^g$ or $W=W^g$. Now $R+U=(R+W)^g$ or $(R+U)^g$, respectively, and as $U=(R+U)-\sum$, $U=W^g$ or $U=U^g$. Thus $\{U, W\}=\{U^g, W^g\}$.

Suppose $W=U^g$. $Y=\langle N_X(R), g\rangle$ acts on $\{U, W\}$. Further for $R^g\neq T\in(R^g+W)\cap\Omega$, $P(T)^{\#}\subseteq\Gamma$ and $T\in R^Y$. Finally $R\cap(U+W)=0$, so $R^y\cap(U+W)=0$ for all $y\in Y$. Therefore $R^Y=\{R\}\cup((R+W))\cap\Omega)\cup\Delta\cup T^{AB}$ is of order q^2 by 5.5. Next $\Omega=R^Y$ or $R^Y\cup S^{AB}$, $S\in\Omega\cap(U+W)$. In the first case 5.2.1

holds. In the second by 5.5, $|\Omega|=q^2+(q-1)/(m-1)$, so $|R^Y|>|\Omega-R^Y|$. But as $N(R)\leq Y$, R^Y is a set of imprimitivity for X on Ω, a contradiction.

Hence $U=U^g$, so Δ^X is a system of imprimitivity for the action of X on Ω. In particular q divides the order n of Ω.

By 5.5, $n=1+\alpha+\beta(q-1)+\gamma(q-1)^2$ where γ is 1, $(m-1)^{-1}$, or $(m-1)^{-1}m$, and $\alpha+\beta\leq 2$. $n\equiv 0 \bmod q$ and $m>2$, so either $\beta=2$, $n=q^2$, and $\sum=V-(U+W)$, or $\alpha=0$, $\beta=1$, $n=q^2$, and $\sum=(V-(R+W))\cup R^\#$, or $\alpha=\beta=1$, $n=2q$, $m=q$, and $\sum=((R+U)-U)\cup((U+W)-U)$.

In the last case $N(R+U)=N(U+W)$, whereas $N(U+W)$ moves W, while we showed above that $N(R+U)$ acts on W. In the second case $R+W=\langle V-\sum\rangle$ and then $R=R+W\cap\Omega$ is X-invariant, a contradiction.

This completes the proof of 5.2.

6. $L_3(2^n)$

Theorem 6.1. *Let A be standard and nonnormal in G with $O(G)=1$, $A/Z(A)\cong L_3(q)$, q even, and $m(C_G(A))>1$. Then either*
(1) $Z(A)=1$ and $G\cong Sz$.
(2) $Z(A)$ is a 4-group and $G\cong He$.

Proof. We prove $q=4$ and appeal to the theorem of Cheng Kai Nah [5]. By 20.1 in [4], A satisfies hypothesis II. Thus we may choose notation as in section 3. Set $Z=V\cap A$.

Assume $q=2$. By 4.9, $A\cong L_3(2)$ and $T=Z$ is a 4-group. Notice $N_A(T)\cong S_4$. Let a be an involution in $N_A(T)$ with $[T,a]\neq 1$. Then a induces an outer automorphism on A^g, so $\langle a\rangle A^g\cong PGL_2(7)$. But this is impossible as a centralizes the 4-group $R\leq A^g$.

Therefore we may take $q>4$. Hence $Z(A)$ has odd order. (*eg*. [9]). Let $t\in T^\#$ and $Z_0=O_2(Z(C_A(t)))$. There exists a nontrivial cyclic subgroup W of order $(q-1)/(q-1,3)$ in $C_A(t)$. Let $P\in Syl_2(C_A(t))$. Then $[P,W]=P$. As the outer automorphism group of A is abelian, $P=[P,W]\leq(AK)^g$ and then $Z_0=\Phi(P)\leq A^g$. As $Z(A)$ has odd order, $T\cap A=1$. Hence 2.7 implies $C_A(t)=C_A(T)$, so $Z=Z_0$. That is Z is a root subgroup of A.

Now $TP\in Syl_2((AK)^g$ and W centralizes the root group Z of A^g, so W induces a group of inner automorphisms on A^g with $E(C_{A^g}(W))\cong E(C_A(W))=L\cong L_2(q)$. In particular R is not normal in $C(WT)$. Also WL is not centralized by any involutory automorphism of A, so by 3.8, L is a nonnormal standard subgroup of $C(W)$. As $q>4$, 3.11 yields a contradiction.

7. Classical groups of even characteristic

In this section A is quasisimple with $A/Z(A)$ isomorphic to $L_n(q)$, $U_n(q)$,

$Sp_n(q)$, or $\Omega_n^\varepsilon(q)$, $n\geq 4$, and q even. Exclude $L_4(2)\cong A_8$ and $Sp_4(2)\cong S_6$. If $A/Z(A)$ is orthogonal take $n\geq 8$.

Theorem 7.1. *Assume A is standard in G with $m(C(A))>1$. Then $A\trianglelefteq G$.*

The proof involves a series of reductions. Let G be a counter example. By 20.1 in [4], A satisfies hypothesis II. Thus we may choose notation so that hypothesis 3.1 is satisfied. By 3.3 we may choose $g\in G-N(A)$ so that $T\in Syl_2(K^g)$. That is the notation of section 3 holds. The results in [4] show $F^*(C_A(a))$ is a 2-group for each 2-element $a\in A-Z(A)$, so by 2.8, T is a TI-set in AT. By 3.9, A is T-admissible. In particular hypothesis 22.1 of [4] is satisfied and we may appeal to 22.2 of [4].

Let $P=RT\cap A$, $t\in T^\#$, and $\{p\}=P\cap tR$. p is one of a canonical set of representatives for the classes of involutions in A denoted by j_l, a_l, b_l, or c_l, where l is a parameter associated with p called its *rank*. Applying 22.2 in [4] we find:

(7.2) $Z(A)$ is of odd order, $P^\#$ is fused in A and either

(I) $P\leq J=O_2(C_A(p)\cap C(p^A\cap C(p)))$, $|T|\leq q$ and one of the following holds:
(1) $J=\alpha(p)$ and $\mathrm{Aut}_A(J)$ is cyclic of order $q-1$ and regular on $J^\#$.
(2) $A=Sp_n(q)$, $b=b_l$, $l>1$, $J=\alpha(a)\alpha(b)$ where a and b are of type a_{l-1} and b_1 respectively, and $\mathrm{Aut}_A(J)\cong Z_{q-1}\times Z_{q-1}$ is regular on $J-(\alpha(a)\cup\alpha(b))$.
(3) $A=Sp_n(q)$, $p=c_l$, $J=\alpha(a)\alpha(b)$ where a and b are type a_l and b_1, respectively, and $\mathrm{Aut}_A(J)$ is as in (2).
(4) $A=\Omega_n^\varepsilon(q)$, $p=c_l$, $J=\beta(p)$ and $\mathrm{Aut}_A(J)$ is cyclic of order $q-1$ and regular on $J^\#$.
or,
(II) $T=P$ has order 4 and either
(5) $A/Z(A)=L_n(2)$, $t=j_2$, and $T\leq\Phi(S)$ where $S\in Syl_2(C_A(t))$.
(6) $A\cong Sp_n(2)$ and $t=c_2$.

$\alpha(p)$ and $\beta(b)$ are certain normal subgroups of $C_A(p)$ isomorphic to the additive group of $GF(q)$. They are discussed in Section 11 of [4].

(7.3) $q>2$.

Proof. Assume $q=2$. Then 7.2.5 or 7.2.6 holds. In particular T is a 4-group contained in A and there is a conjugate a of t under A with $[a, T]=\langle t\rangle$.

As $[a, T]\neq 1$, a induces an outer automorphism on A^g. But if $A\cong Sp_n(2)$ then as $n>4$, the outer automorphism group of A is of odd order. Hence $A/Z(A)\cong L_n(2)$ and the outer automorphism group of A has order 2. Therefore $C_A(T)\leq(KA)^g$, and then by 7.2.5, $T\leq\Phi(C_A(T))\leq A^g$, a contradiction.

The *primary involutions* of A are the transvections (type j_1) of $L_n(q)$ or $U_n(q)$, the transvections (type b_1) of $Sp_n(q)$, or the involutions of type a_2 in $\Omega_n(q)$.

(7.4) $T\cap A=1$. Further we may choose T so that p is a primary involution of A.

Proof. If 7.2.1 or 7.2.4 holds then by 3.6 either $T\cap A=1$ or $T=P=J$. Suppose 7.2.2 or 7.2.3 holds. Recall V is the weak closure of R in the centralizer of $R^G\cap RT$. Then $V=RJ$. Moreover 7.2 and 3.6 imply that hypothesis 5.1 is satisfied and hence 5.2 implies either $T\cap A=1$ and there is a conjugate T_1 of T under $N(V)$ such that $T_1\cap A=1$ and $P_1=RT_1\cap A\leq\alpha(p_1)$, for some primary invoolution p_1 in J, or $\alpha(p_1)\in R^G$ for some primary involution p_1 in J.

Among all G-conjugates T of R in $C(R)$ choose T so that $T=P$ if possible and, subject to this restriction, so that the rank l of p is minimal, and if A is orthogonal, choose p to be primary if possible.

Suppose p is not primary. Then by remarks in the first paragraph, 7.2.2 and 7.2.3 do not hold. Next by 11.3 and 11.6 in [4] there is a conjugate P^a of P such that $|JJ^a\cap J^A|=q-1$ and $JJ^a=\alpha_1\alpha_2$ where the groups α_i are α or β groups of involutions of smaller rank, or if A is orthogonal and p is of type c_2, the α_i are primary.

Assume $T=J$. Then $|TT^a\cap G^A|=q-1\geq 3$, so by 3.6, $|TT^a\cap T^G|=q$. Hence an involution of samller rank, or a primary involution of A, is contained in a conjugate of T, contradicting the choice of T. Thus by choice of T, $R^x\cap A=1$ for all $x\in G$. Hence by 3.6, TT^a contains $|T|(|T|-1)$ involutions in the set $\sum$ of elements fused into $T^\#$ under G.

Now the elements in P are of the form $\alpha(b)=\alpha_1(bu_1)\alpha_2(bu_2)$, for fixed $u_i\in F^\#=GF(q)^\#$, with b ranging over some additive subgroup B of F. Further we may pick a so that $\alpha(b)^a=\alpha_1(bu_1)\alpha_2(bcu_2)$, for some fixed $c\in F$, with $\alpha(d)$ and $\alpha(dc)$ distinct elements of $P^\#$ for some $d\in\theta$. That is a acts on α_2 and centralizes α_1. Thus PP^a contains the $|T|-1$ elements $\alpha_2(b(c+1)u_2)$, $b\in\theta^\#$, and the element $\alpha_1(d(c+1)u_1)$. So as $|TT^a\cap\sum|=|T|(|T|-1)$, some element of $\sum$ projects on one of these elements, again contradicting the choice of T.

Therefore p is primary. Hence if $T=J$, then 11.7 in [4] implies that $T\leq C_A(T)^\infty\leq A^g$, a contraciction. This completes the proof of 7.4.

From now on choose T so that p is a primary involution. By 11.8 and 11.9 in [4] we may choose $W\leq C_A(T)$ with $W\cong L_2(q()$ and a Sylow 2-group of W conjugate under A to J.

Now $A/Z(A)\cong X_n(q)$, $X=L$, U, Sp, or Ω. If $X=L$, U or Sp then by 11.8 in [4], $L=E(C_A(W))$ is isomorphic, modulo its center, to $X_{n-2}(q)$. If $A\not\cong L_4(q)$ or $U_4(q)$ set $Y=W$. If $A\cong L_4(q)$ or $U_4(q)$ let $Y_1=O(C_A(W))$. Then Y_1 is cyclic of order $q-1$ or $q+1$, respectively. In this case set $Y=Y_1W$. Then $L=E(C_A(Y))$ and by 11.8, YL is not centralized by an involutory automorphism of A.

Next suppose $A\cong\Omega_n^\varepsilon(q)$, $n\geq 8$. Then $E(C_A(W))=W_2\times W_0$, where $W_2\cong L_2(q)$ and $W_0\cong\Omega_{n-4}^\varepsilon(q)$, by 11.9 in [4]. In this case set $Y=WW_2$ and $L=W_0$, unless $n=8$. If $n=8$ let $Y=WW_0$ and $L=W_2$. By 11.9 in [4], A admits no involutory automorphism centralizing YL.

In any case it is possible to choose Y so that $J\leq L$. In particular T centralizes Y.

(7.5) L is standard but not normal in $C_G(Y)$.

Proof. As A admits no involutory automorphism centralizing YL, $R\in Syl_2(C(YL)\cap N(R))$. Next $X=C_A(T)^\infty$, so $X\leq C_{A^g}(R)^\infty$. By symmetry, $X=C_{A^g}(R)^\infty$. Hence the isomorphism class of $C_{A^g}(p)^\infty$ is determined and by 11.10 and 11.11 in [4] this implies there is an automorphism γ of A^g such that $X^{g\gamma}=X$.

Let w be an involution in W. Notice $W\leq X$. Then w is a primary involution of A in X, so by 11.14 and 11.15, $w^{g\gamma}$ is a primary involution of A^g, and $\alpha(w^{g\gamma})=\alpha(w)^{g\gamma}$. Now by 11.8 and 11.9 in [4], $C_{A^g}(W)\cong C_A(W)$. In particular R is not normalized by $A^g\cap C(W)$. Hence if $Y=W$ then 3.8 completes the proof.

So assume $Y\neq W$. If $A\cong L_4(q)$ or $U_4(q)$ then Y_1 centralizes W and $\alpha(p)$, so Y_1 induces a group of automorphisms on A^g centralizing $A^g\cap C(W)$. Thus again R is not normalized by $A^g\cap C(Y)$. So assume $A\cong\Omega_n^\varepsilon(q)$. If $n>8$ then we have symmetry between W and W_2, so the embedding of W_2 in $A^g\cap C(W)$ is determined up to an automorphism and again we find R is not normalized by the centralizer of $Y=WW_2$ in A^g. Finally if $n=8$ one can again check that the embedding of $C_Y(W)$ in $A^g\cap C(W)$ is determined up to an automorphism so that R is not normal in $A^g\cap C(Y)$. The proof is complete,

(7.6) Let $B=\langle R^{C(Y)}\rangle$. Then $q=4$, $A\cong L_4(4)$, $U_4(4)$, $Sp_4(4)$, or $\Omega_8^\varepsilon(4)$, and $B\cong HJ$.

Proof. By 7.5, L is standard in B and $L\neq B$. Therefore by 3.10, 3.11, 6.1, and induction on the order of G, $L\cong L_2(4)$ and $B\cong HJ$ or $\mathrm{Aut}(M_{12})$, or $L\cong L_3(4)$ and $B/Z(B)\cong Sz$.

Suppose $L\cong L_2(4)$. Then $A\cong L_4(4)$, $U_4(4)$, $Sp_4(4)$, or $\Omega_8^\varepsilon(4)$, so we may assume $B\cong Aut(M_{12})$. Now by 3.10 there is a conjugate b of t under B inducing an outer automorphism on L with $[R,b]\neq 1$. As $[R,b]\neq 1$, b induces an outer automor automorphism on A. Then $[Y,b]=1$ and $L\langle b\rangle\cong S_5$. But A does not admit such an automorphism.

So assume $L\cong L_3(4)$. Then $A\cong L_5(4)$. So $C_A(Y)\cong GL_3(4)$. This is impossible since Sz does not admit an automorphism of order 3 inducing an outer automorphism on L.

If $A\cong L_4(4)$ or $U_4(4)$, let $D=Y_1$. If $A\cong\Omega_8^\varepsilon(4)$, let $D=W_0$. Then $B\leq C(Y)$

$\leq C(D)$. Moreover by symmetry between DL and $DW=Y$, $HJ\cong\langle R^{C(DL)}\rangle=\langle W^{C(DL)}\rangle\leq C(D)$. Therefore

(7.7) B is contained in but not normal in $C(D)$.

(7.8) $A\cong Sp_4(4)$.

Proof. Assume A is not $Sp_4(4)$. Let $S\in Syl_2(W)$ and $X=C(D)$. Claim $B\trianglelefteq C_X(U)=C$, for all $1\neq U\leq S$. Set $\bar{C}=C_X(U)/U$. Assume first that $U=S$. As $B\cong HJ$, $R=O_2(K)$ so $O_2(K)S=RS=R^hS=O_2(K^h)S$ for $R^h\leq RS$. Therefore $\bar{R}$ is tightly embedded in $\bar{C}$, so $\bar{L}$ is standard in $\bar{C}$. As $\bar{B}\leq\bar{C}$, 3.10 implies $\bar{B}=\langle\bar{L}^{\bar{C}}\rangle$, so that $B=\langle L^C\rangle\trianglelefteq C$. Next assume U has order 2. Suppose $c\in C$, $r\in R^\#$ and $r^c\in RU-R$. As $N_B(R)\leq C(U)$ is transitive on $R^\#$ we may take $r^c\in rU$, so that $c^2\in C(r)\cap C(r^c)\leq N(R)\cap N(R^c)$. Therefore c acts on $RR^c=RS$ and then $S=[R,c]$. So $c\in N(rU)\cap N(S)\leq N(rU)\cap N(B)$ and hence as $rU\cap B=\{r\}$ we have a contradiction. It follows that $\bar{R}$ is tightly embedded in $\bar{C}$, and as above, $B\trianglelefteq C$.

Now $E(C_C(B))=E(C_C(RL))=1$, so $B=E(C)$. As this holds for each $1\neq U\leq S$, B is standard in C. Now 7.7 and 17.1 (which will be proved independently) yield the result.

(7.9) $A\not\cong Sp_4(q)$.

Proof. Assume $A\cong Sp_4(q)$. By symmetry between L and W, $\langle R^{C(L)}\rangle=E\cong HJ$. Let X be a subgroup of order 5 in L. Then $C_B(X)=X\times H$ where $R\leq H\cong L_2(4)$ (eg. p. 429 in [20]). Also $C_B(X)\cap E\leq C_B(X)\cap C(L)=C_B(L)\cong A_4$, so $C_B(X)\cap E$ is not normal in $C_B(X)$ and hence E is not normal in $C_G(X)$. But $C(XW)\cap N(R)=KX$. so by 3.7 W is standard in $C(X)$. Therefore as $W\leq E\leq C(X)$, 3.10 implies $E=\langle W^{C(X)}\rangle\trianglelefteq C(X)$, a contradiction.

This completes the proof of Theorem 7.1.

8. Exceptional groups of characteristic 2

In this section we assume that A is a quasisimple group with $A/Z(A)$ an exceptional Chevalley group of characteristic 2, or the Tits group ${}^2F_4(2)'$. We exclude $G_2(2)$, as its commutator group $U_3(3)$ was handled in section 4. We prove

Theorem 8.1. *Assume A is standard in G with $O(G)=1$ and $m(C(A))>1$. Then either*

(1) *$A\trianglelefteq G$, or*

(2) *$\langle A^G\rangle$ is of Conway type.*

By [4], A satisfies hypothesis II, so we may choose notation as in section 3. Let $t\in T^\#$. We begin a series of reductions.

Recall the definition of $X^{\mathcal{A}}$ given in section 1.

(8.2) $(C_A(t)'O^2(C_A(t)))^{\mathcal{A}} \leq A^g \leq C(T)$.

Proof. $\mathrm{Out}(\bar{A})'$ has odd order so we may appeal to 3.9.

(8.3) $Z(A)$ has odd order.

Proof. Assume not. By [9], $\bar{A} \cong G_2(4)$, $F_4(2)$, or ${}^2E_6(2)$.

Assume $\bar{A} \cong G_2(4)$. Then A is the covering group of $G_2(4)$ and $\langle z \rangle = Z(A)$ is of order 2. By section 18 of [4], $\bar{A}$ has 2 classes of involutions represented by root involutions $\bar{a}$ and $\bar{b}$ of long and short roots, respectively. By [16], a is an involution while b is of order 4. Hence by 3.5, $\bar{t} \neq \bar{b}$, so we may tkae $\bar{t} = \bar{a}$. By 18.4 in [4], $C_{\bar{A}}(\bar{a}) = \bar{L}\bar{U} = C_{\bar{A}}(\bar{a})^{\infty}$ where $\bar{U} = O_2(C_{\bar{A}}(\bar{a}))$ and $\bar{L} \cong L_2(4)$ contains a conjugate of $\bar{b}$. As b has order 4, $L \cong SL_2(5)$, so $z \in L$. By 8.2, $C_A(a) = C_A(a)^{\infty} \leq A^g$. Then $z \in A^g$ and $C_A(a) = A^g \cap C(z)$. But one checks that $\bar{a}$ has 240 square roots in $\bar{U}$, while there are more than 240 conjugates of b in U squaring to z, a contradiction.

So $\bar{A} \cong F_4(2)$ or ${}^2E_6(2)$. We take $\bar{t}$ to be one of the involutions in 13.1 or 14.1 of [4]. Now $Z(A)$ is the kernel of the homomorphism of A on to $\bar{A}$, so $O^2(C_{\bar{A}}(\bar{t})) = O^2(C_A(t))Z(A)/Z(A)$. Thus by 8.2 $O^2(C_{\bar{A}}(\bar{t}))^a$ centralizes $\bar{T}$. However if $\bar{t} \neq U_\alpha(1)U_\beta(1)$ or $U_r(1)U_s(1)$ in [4], then $\langle \bar{t} \rangle = C_{\bar{A}}(O^2(C_{\bar{A}}(\bar{t}))^a)$. Further if $\bar{t} = U_r(1)U_s(1)$ then $C_{\bar{A}}(O^2(C_{\bar{A}}(\bar{t}))^a) = \langle \bar{t}, \bar{u} \rangle$ where $\bar{u}$ is a root involution. We conclude $\bar{t} = U_\alpha(1)U_\beta(1)$ and $\bar{T}^{\#}$ is fused under $\bar{A}$.

Then $O^2(C_{\bar{A}}(\bar{t})) = O_{2,3}(C_{\bar{A}}(\bar{t}))$ and $Z(C_{\bar{A}}(\bar{t})) = Z(O^2(C_{\bar{A}}(\bar{t})) = \langle \bar{t}, \bar{u} \rangle$. Hence $\bar{T} \cap Z(C_{\bar{A}}(\bar{t})) = \langle \bar{t} \rangle$, so by 3.9, $T \cap A \neq 1$, and we may choose $t \in T \cap A$. By 2.8, T is a TI-set in AT, so $T \cap A$ is a TI-set in A. Let $s \in T - \langle t \rangle$ and set $X = O_{2,3}(C_A(s))$. We have shown $\bar{t} \notin Z(\bar{X})$, so as $t^X \subseteq T \cap A$, $T \cap A$ has order at least 4. If $T \cap A$ has order 4 then as $C_A(t)$ acts on $T \cap A$, $O^2(C_A(t)) \leq C_A(T \cap A)$, so that $\bar{T} \cap \bar{A} \leq Z(C_{\bar{A}}(\bar{t})) = \langle \bar{t} \rangle$. Consequently $|T \cap A| > 4$.

By [16], $|Z(A)| \leq 4$, so it suffices to show $T \cap A \leq Z(A^g)$, that is $T \cap A \leq A^g$. Since t is chosen arbitrarily from $(T \cap A)^{\#}$, it suffices to show $t \in A^g$. From the presentation of the covering group of $\bar{A}$ in [16] we see that $Z(A) \leq O_2(C_A(t))'$. Thus as $O_2(C_{\bar{A}}(\bar{t})) \leq O^2(C_{\bar{A}}(\bar{t}))$, $t \in O^2(C_A(t))$. However $|N(A): AK| \leq 6$, K^g has an abelian Sylow 2-group T, and $C_A(t) \leq N(T)$, so if $t \notin A^g$, then $tA^g \notin O^2(C_A(t))A^g/A^g$, a contradiction. Hence the proof of 8.3 is complete.

Throughout the remainder of this section let p be the projection of t on A and P the projection of T on A. We take p to be in the set Δ of canonical involutions of A defined in the section of [4] corresponding to A. There Δ is linearly ordered. Define the *rank* $r(p)$ of p to be its place in that order. In particular the root involutions have smallest rank. p is said to be *degenerate* if $\bar{A} \cong F_4(q)$ or ${}^2E_6(q)$ and $r(p) = 3$ or $\bar{A} \cong F_4(q)$ and $r(p) = 4$. Let Z be a Sylow 2-group of $Z(C_A(t))$. Inspecting the centralizers given in [4] we find:

(8.4) (1) If p is nondegenerate then $Aut_A(Z)\cong Z_{q-1}$ and is regular on $Z^\#$.

(2) If p is degenerate then $Z=Z_1\times Z_2$ where Z_i is the root group of a root involution, $Z-(Z_1\cup Z_2)\subseteq p^A$, and $Aut_A(Z)\cong Z_{q-1}\times Z_{q-1}$.

(8.5) Assume $\bar{A}\cong G_2(q)$ and $r(p)=2$ or $\bar{A}\cong E_7(q)$ and $r(p)=4$ or 5. Then $\bar{T}\leq\bar{Z}$.

Proof. $C_A(p)=ZC_A(p)^\infty$. By 8.2, $C_A(p)^\infty\leq C(T)$ and as Z is in the center of $C_A(p)$, $Z\leq C(T)$.

(8.6) (1) $\bar{T}\leq\bar{Z}$.

(2) Either $T\cap A=1$ or $\bar{A}\cong E_7(q)$ and $r(p)=4$ or 5. or $\bar{A}\cong G_2(q)$ and $r(p)=2$.

Proof. If $T\cap A=1$ then (1) holds by 2.9, so we may take $p=t$. Moreover by 8.5 we may assume t is not one of the involutions described in 8.6.2. But now inspecting the centralizers in [4] we find $t\in(C_A(t)')^{\mathcal{A}}$, so by 8.2, $t\in A^g$, against 8.3.

(8.7) $q>2$.

Proof. Assume $q=2$. Then by 8.6.1 and 8.4, p is degenerate for each $t\in T^\#$. By 8.4, Z is a 4-group, so by 8.6, $\bar{T}=\bar{Z}$. But then by 8.4, p is a root involution, and hence nondegenerate, for some $t\in T^\#$.

Let V be the weak closure of R in $C(R^G\cap C(RT))$. Let $\sum=\{r^g: r\in R^\#\}$

(8.8) (1) $V=RZ$.

(2) If p is nondegenerate then either
(i) $T=Z$ and $\sum\cap V=R^\#\cup T^\#$ or
(ii) $T\cap A=1$ and $V-Z$ is the disjoint union of q conjugates of R. $\sum\cap Z$ is empty.

(3) If p is degenerate then there exist G-conjugates T_i of R in V such that either
(i) $\bar{T}_i\leq\bar{Z}_i$, $A\cap T=1$, and $\sum\subseteq V-Z$, or
(ii) $T_i=Z_i$.

Proof. (1) follows from 8.4. Moreover V satisfies the hypothesis of 3.6 or 5.1 with $\langle O^2(N_A(V), O^2(N_{A^g}(V)\rangle$ in the role of X, given 8.4. Hence 3.6 and 5.2 imply (2) and (3).

(8.9) Assume $\bar{A}\cong G_2(q)$. Then

(1) If $r(p)=2$ then $Z=T$.
(2) We may choose T so that $r(p)=1$.

Proof. Let $r(p)=2$. Assume first $T\cap A=1$. Let $B=O_2(C_A(p))$. Then

B contains q^2 A-conjugates of Z, so by 8.8, the set Γ of conjugates of R in RB is of order $q^2(q-1)+1$. Moreover $N_A(\Gamma)^\Gamma \cong E_{q^2}GL_2(q)$ is transitive on $\Gamma-\{R\}$ with $O_2(N_A(\Gamma)^\Gamma)$ semiregular on $\Gamma-\{R\}$. Moreover the same holds in A^g, so $N(\Gamma)^\Gamma$ is 2-transitive. But now a result of Shult [29] yields a contradiction.

So by 8.8, $T=Z$. By section 18 in [4], there is a conjugate T^a of T such that TT^a contains q A-conjugates of T and one 2-central root group U. By 8.8 applied to TT^a, $U=A^g \cap TT^a$ and $U \cap \sum$ is empty. Therefore U is a 2-central root group of A^g, so by symmetry between A and A^g, (2) holds.

(8.10) Assume $\bar{A} \not\cong G_2(q)$. Then

(1) $T \cap A=1$ and
(2) We may choose T so that p is in a root subgroup.

Proof. Pick p so that $Z=T$ if possible and, subject to this condition, so that $r(1)$ is minimal.

Assume p is not a root involution. Then by 8.8.3, p is never degenerate. Next by sections 13 through 18 of [4] there is a conjugate Z^a of Z such that ZZ^a contains $(q-1)^2$ conjugates of p and $2(q-1)$ involutions of smaller rank. Hence if $T=Z$ then $|TT^a \cap \sum| \geq (q-1)^2$, so by 8.8.2, $|TT^a \cap \sum|=q(q-1)$. Thus there is a conjugate s of t under G in TT^a with $r(s)<r(t)$, contradicting the choice of p. Therefore by choice of T, $R^x \cap A=1$ for all $R^x \leq N(A)$. So by 8.8, $|TT^a \cap \sum|=|T|(|T|-1)$.

Elements of P have the form $p(b)=U_{\gamma_1}(bu_1)\cdots U_{\gamma_k}(bu_k)$, where U_{γ_i} is a root group, the u_i are fixed elements of $F=GF(q)$, and b varies over some additive subgroup θ of F. We may choose notation so that $ZZ^a=UX$ where $U=U_{\gamma_k}$ and X consists of the elements $X(d)=U_{\gamma_1}(u_1, d)\cdots U_{\gamma_{k-1}}(u_{k-1}d)$, $d \in F$. Moreover we may take $p(b)^a=X(b)U(cb)$ where $p(d)$ and $p(dc)$ are distinct elements of $P^\#$. Thus PP^a contains the $|T|-1$ elements $U((c+1)b)$, $b \in \theta^\#$, and the element $X((c+1)d)$. So as $|TT^a \cap \sum|=|T|(|T|-1)$, some element of $\sum$ projects on one of these elements, again contradicting the choice of p.

So p is a root involution. But now by 8.6, $T \cap A=1$. The proof is complete.

(8.11) Assume $\bar{A} \cong {}^2F_4(q)$. Then we may pick T so that $r(p)=2$.

Proof. Assume not. Then $r(p)=1$. By section 18 in [4] there is a conjugate Z^a of Z such that $|p^A \cap ZZ^a|=2(q-1)$. By 8.10 and 8.8, $|TT^a \cap \sum|=|T|(|T|-1)$. Therefore there exists a conjugate s of t in TT^a with $p(s) \in ZZ^a-p^A$. Hence $r(p(s))=2$.

If $\bar{A} \cong G_2(q)$ pick T so that $r(p)=1$ and if $\bar{A} \cong {}^2F_4(q)$ pick T so that $r(p)=2$. In the remaining cases choose T so that p is a root involution. Let $r \in R^\#$.

(8.12) $C_A(T)^\infty=C_{A^g}(R)^\infty$ and $r(\bar{r}^\gamma)=r(p)$ for some $\gamma \in Aut(\bar{A}^g)$.

Proof. By 8.2, $C_A(T)^\infty \le C_{A^g}(R)^\infty$. By symmetry between R and T we have equality. Now inspecting the centralizers in [4] we find $C_A(T)^\infty$ determining p up to conjugacy in $Aut(\bar{A})$.

We now define a subgroup Q or $C_A(T)^\infty$. If $\bar{A}\cong G_2(q)$ or ${}^2F_4(q)$, or if $\bar{A}\cong {}^3D_4(q)$ and $r(p)=2$, let Q be a Hall subgroup of order $q-1$ in $C_A(T)^\infty$. If $\bar{A}\cong {}^3D_4(q)$ and $r(p)=1$, let Q be a Hall subgroup of order q^2+q+1 in $C_A(T)^\infty$. In all other cases let Z_3 and Z_4 be A-conjugates of Z centralizing T such that $D=\langle Z_3, Z_4\rangle \cong L_2(q)$, and let Q be a Hall subgroup of order $q-1$ in D. Set $L=E(C_A(Q))$. By [4], L is described in Table 8.13:

Table 8.13

$\bar{A}$	$G_2(q)$, $q\neq 4$	$G_2(4)$	${}^3D_4(q)$	${}^3D_4(q)$	${}^2F_4(q)$	$F_4(q)$	${}^2E_6(q)$	${}^2E_6(q)$	$E_6(q)$	$E_7(q)$	$E_8(q)$
$\bar{L}$	$L_2(q)$	$SL_3(4)$	$L_3(q)$	$L_2(q^3)$	$L_2(q)$	$Sp_6(q)$	$U_6(q)$	$\Omega_8^-(q)$	$L_6(q)$	$\Omega_{12}^+(q)$	$E_7(p)$
$r(p)$	1	1	1	2	2	1 or 2	1	2	1	1	1

(8.14) L is standard but not normal in $C(W)$

Proof. By 3.8, 8.9, and 8.10 it suffices to show R is Sylow in $N(R)\cap C(LQ)$, and R is not normalized by $C_{A^g}(Q)$.

Suppose x is a 2-element in $(N(R)\cap C(LQ))-R$. We may assume $x^2\in R$ so $\bar{x}$ induces an involutory automorphism on $\bar{A}$. By [4], $L=O^2(C_{\bar{A}}(\bar{Q}))$, so $\bar{x}$ induces an outer automorphism on $\bar{A}$. Out$({}^2F_4(q))$ is of odd order. If $\bar{A}\cong G_2(q)$ or ${}^3D_4(q)$ then by 19.2 $\bar{x}$ is a field automorphism and $C_L(x)\cong X(q_0)$ where $L\cong X(q)$. Finally if $\bar{A}$ has rank greater than 2 then x acts on $E(C_A(L))=D$ and as $[x, Q]=1$, x centralizes D. But inspecting the possibilities for x given in section 19 of [4], $E(C_A(D\langle x\rangle))<L$.

It remains to show R is not normalized by $A^g\cap C(Q)$. By 8.12, $Q\le Y=C_{A^g}(R)^\infty$ and $r(\bar{r}^\gamma)=r(p)$ for some $\gamma\in Aut(\bar{A}^g)$. Then Q is contained in a Levi factor X of Y. $r(\bar{r}^\gamma)=r(p)$, so X centralizes a subgroup $D_0=\langle Z, Z_0\rangle\cong L_2(q)$. Thus $D_0\le A^g\cap C(Q)$ and D_0 does not normalize $R\le TZ$. The proof is complete.

(8.15) $\bar{A}\cong G_2(4)$ and $\langle L^{C(W)}\rangle=B$ is isomorphic to Sz, modulo its core.

Proof. Let $B=\langle L^{C(W)}\rangle$. By 8.14, 8.12, 6.1, and induction on the order of G, $L\cong SL_3(4)$ and $B/Z(B)\cong Sz$. Hence $\bar{A}\cong G_2(4)$ or ${}^3D_4(4)$. In the latter case let Y be a Sylow 3-group of $C_A(T)$ centralizing W. Then $Y/Y\cap W$ induces a diagonal automorphism on L. However $B/Z(B)\cong Sz$ does not admit an automorphism of order 3 centralizing R and inducing a diagonal automorhpism on L.

(8.16) G is of Conway type.

Proof. 8.15 and 8.9.

This completes the proof of Theorem 8.1.

9. The Mathieu groups M_n

In this section G is the Mathieu group M_{24} acting 5-transtitively on a set $\Omega=\{1, 2, \cdots, 24\}$. The following facts can be found in section 4 of [32]:

(9.1) G has one class z^G of involutions fixing 8 points and one class t^G of fixed point free involutions. z is 2-central.

(9.2) Elements of order 3 centralized by t are fixed point free.

G_{123} is isomorphic to $L_3(4)$ and acts on $\Omega_3=\Omega-\{1, 2, 3\}$ as on the points of the projective plane $PG(2, 4)$ over $GF(4)$. Choose $z\in G_{123}$ and let $\Delta=I(z)\cap\Omega_3$. Then Δ is the axis of z in $PG(2, 4)$. Let E be the subgroup of G_{123} generated by all elations with axis Δ. Then $E\cong E_{16}$ is the pointwise stabilizer in G_{123}, and hence also in G, of $I(z)$. So $E\trianglelefteq C_G(z)$. Next G_{123} has one class of involutions and hence is transitine on $z^G\cap G_{123}$. So $C(z)^{I(z)}$ is 3-transitive on its 8 points and hence isomorphic to the holomorph of E_8. The stabilizer of a cycle of z is a compliment for E in $C(z)$. Moreover $E=C(e)_{I(e)}$ for each $e\in E^\sharp$, so $N(E)$ is transitive on $E^\sharp$ and by an order argument $N(E)/E\cong GL_4(2)$. Summarizing:

(9.3) $G_{I(z)}=E\cong E_{16}$ and $N(E)/E\cong GL_4(2)$. $C_G(z)$ is the split extension of E by the holomorph of E_8.

The following facts can be found in lemmas 2.17, 5.4, and 5.5 of [22]:

(9.4) $C(t)=RX$ where $E_{64}\cong R\trianglelefteq C(t)$, $X\cong S_5$, and $R=[R, X]$. There is *a* 4-group $U\trianglelefteq C(t)$. $\langle t\rangle=Z(C(t))$.

Now the set Γ of orbits of U on Ω is of order 6. If x is an element of order 5 then $I(x)$ is of order 4, so $I(x)$ is one of these orbits and X is transitive on Γ. Thus

(9.5) UX is transitive on the 12 cycles of t.

(9.6) $G_1\cong M_{23}$ has one class z^{G_1} of involutions. $G_1\cap C(z)$ is the split extension of $E\cong E_{16}$ by $L_3(2)$.

As G has one class z^G of point fixing involutions and $C_G(z)^{I(z)}$ is transitive the first remark follows. As $C(z)_1{}^{I(z)}$ is a complement for $O_2(C(z)^{I(z)})$ in $C(z)^{I(z)}$ the second remark follows.

Next let $G_{12}=L\cong M_{22}$ and set $A=G(\{1, 2\})$. By [9],

(9.7) $A=Aut(M_{22})$.

As G^Ω is 5-transitive we may choose $t\in A$ and $u\in z^G\cap(A-L)$. As E is regular on $\Omega-F(z)$, $C_G(z)$ is transitive on the cycles of z. By 9.5, $C_G(t)$ is transitive on the cycles of t. Therefore:

(9.8) t^A and u^A are the classes of involutions in $A-L$ under L. z^L is the unique class of involutions in L.

The stabilizer of 1 and 2 in $C_G(z)^{I(z)}$ is the stabilizer in $L_3(2)$ of 2 and is isomophic to S_4. Hence

(9.9) $C_L(z)$ is the spit extension of $E\cong E_{16}$ by S_4 and hence is isomorphic to Z_2 wreath S_4.

As the stabilizer of a cycle of z is a complement for E in $C_G(z)$ we get:

(9.10) $C_L(u)$ is the holomorph of E_8.

Recall $C_G(t)$ acts as $PGL_2(5)$ on the 6 orbits $I(x)^{C(t)}$ of U, where x is an element of order 5. Then R is in the kernel of this action and the pointwise stabilizer of a cycle c of t in $I(x)$ is $[x, R]X$ where X is of index 2 in $C(\langle x, t\rangle)$. Thus

(9.11) $C_L(t)$ is the split externsion of $V\cong E_{16}$ by the holomorph of a cyclic subgroup $\langle x\rangle$ of order 5 with $V=[V, x]$.

As a final remark notice that by 9.3

(9.12) $N_A(E)/E\simeq S_5$.

Witt shows in Satz 9 of [33] that there is a subgroup K of G isomorphic to M_{12} acting on Ω with two nonequivalent orbits Γ and Γ' interchanged by an involution b of G acting on K. By [9] $|Aut(M_{12})\colon M_{12}|=2$. Therefore

(9.13) $Aut(M_{12})=K\langle b\rangle=B$.

Choose $1\in\Gamma$ and $2=1^b$. Then as Witt remarks, K_2 acts 3-transitively on Γ with $K_{12}\cong L_2(11)$. Then $\langle b\rangle K_{12}\cong Aut(L_2(11)=PGL_2(11)$. As K_2 is transitive on Γ and there is one class of involutions in $PGL_2(11)-L_2(11)$ it follows that

(9.14) There is one class b^K of involutions in $B-K$.

By Wong [34].

(9.15) K has one class z^K of involutions fixing 4 points and one class t^K of fixed point free involutions. $C_K(z)$ is the split extention of $Q_8{*}Q_8$ by S_3. $C_K(t)\cong Z_2\times S_5$.

As $K_{12}\langle b\rangle\cong PGL_2(11)$ b centralizes an element x of order 5. By 9.15 we may take $[x, t]=1$, and xt is self centralizing in K. Hence $\langle t\rangle$ is Sylow in $C_K(x)$ and and we may take $[b, t]=1$. Then as $[b, x]=1$, b centralizes $E(C_K(t))=J$, and by L-balance, $J\leq L(C(b))$. But b interchanges Γ and Γ' so b is fixed point free on Ω. Therefore by 9.4, $J=E(C_K(b))=E(X)$ and $U=\langle t, b\rangle$. As $\langle t\rangle=$

$Z(C(t))$, $[X, U]\neq 1$, so

(9.16) We may choose b so that $C_K(b)=\langle t\rangle\times E(C_K(t))$.

(9.17) Let H be quasisimple with $\bar{H}=H/Z(H)$ a Mathieu group. Then

(1) If $H\cong M_{12}$ then H satisfies hypothesis II.

(2) If $H/Z(H)\cong M_{12}$, T is a 2-group acting faithfully on H with $m(T)>1$ and $T\in Syl_2(Q)$, where Q is tightly embedded in HT, then $H\cong M_{12}$ and $T=\langle t, b\rangle$ $=C_{HT}(E(C_H(T))\cong E_4$, for some non 2-central involution t of H.

Proof. Assume the hypothesis of (2). By Theorem 4 in [1] we may assume Q is 2-constrained. So if $\bar{b}$ or $\bar{t}$ is in $\bar{T}$, then as $\langle\bar{b}, \bar{t}\rangle=O_{2',2}(C(\bar{s}))$ for each $\bar{s}$ in $\langle b, t\rangle^{\#}$, $\bar{Q}=\langle\bar{b}, \bar{t}\rangle$. Suppose $Z(H)\neq 1$. Then $Z(H)=\langle\pi\rangle$ is of order 2, $t^2=\pi$ and $[b, t]=\pi$. Let $s\in T$ with $\bar{s}=\bar{t}$. Then $t\in C(s)\leq N(T)$, so $\pi=[t, b]$ $\in T$, impossible as T acts faithfully on H. Hence $Z(H)=1$. Now there exists $h\in C(t)\leq N(T)$ with $[h, B]=t$, so $t\in T$.

So we may assume $\bar{T}^{\#}\subseteq\bar{z}^H$. But $\bar{H}=\langle O^2(C_{\bar{H}}(\bar{z}), O^2(C_{\bar{H}}(\bar{z}^g))\rangle$ for any conjugate $\bar{z}^g$ of $\bar{z}$ with $\bar{z}^{-g}\in O_2(C(\bar{z}))-\langle\bar{z}\rangle$. Hence $H\leq\Gamma_{1,T}(H)\leq N(Q)$, and then $[H, O_{2',2}(Q)]=1$, a contradiction.

So we may assume $\bar{H}\ncong M_{12}$, and it remains to show H satisfies hypothesis II. By [9], M_{11}, M_{23}, and M_{24} have trivial outer automorphism groups, so we may take $\bar{H}\cong M_{22}$. Assume T is a noncyclic elementary abelian 2-group acting faithfully on H and Sylow in a 2-nilpotent tightly embedded subgroup Q of HT, with $T\nleq HC(H)$. As $|Aut(H): H|=2$, $T=T_0\langle b\rangle$ where $T_0=T\cap HC(H)$. By a Frattini argument $C_H(s)=O(C_H(s)(C_H(s)\cap N(T))$, each $s\in T^{\#}$. As $O(C_H(s))$ $=1$, $C_H(s)\leq N(T)$.

By 9.8, 9.10, and 9.11, $\bar{T}=\langle\bar{b}\rangle O_2(C_{\bar{H}}(\bar{b}))\cong E_{16}$ or E_{32}, for $\bar{b}$ fused to u or t, respectively. Without loss we take $\bar{z}\in\bar{T}_0$. E is the unique abelian subgroup of rank 4 in $O_2(C_{\bar{H}}(\bar{z}))$ and by 9.12, E is self centralizing in $Aut(M_{22})$. So we may take $\bar{b}=\bar{u}$. Now $C_{\bar{H}}(\bar{u})$ is transitive on $\bar{u}(\bar{T}\cap\bar{H})^{\#}$, so $N_{\bar{H}}(\bar{T})$ is 2-transitive on $\bar{u}(\bar{T}\cap\bar{H})$. So $|\bar{H}|_2\geq|\bar{T}\cap\bar{H}|\cdot|C_{\bar{H}}(\bar{u})|_2=2^9>2^7=|\bar{H}|_2$, a contradiction.

(9.18) Let H be quasisimple with $H/Z(H)\cong M_{11}$, M_{22}, M_{23}, or M_{24}, and assume H is T-admissible. Then $H\cong M_{24}$, T is a 4-group, and $T\leq C_H(T)^{\infty}$.

Proof. Set $\bar{H}=HT/C_{HT}(H)$. $\bar{T}$ centralizes $O^2(C_{\bar{H}}(\bar{t}))$ for each $t\in\bar{T}^{\#}$. It follows that $\bar{H}=M_{11}$, M_{22}, or $\bar{H}=M_{24}$ and $\bar{T}$ is the group U defined in 9.4 Assume the latter. M_{24} has a trivial multiplier (*eg* [9]) so $Z(H)=1$ and $C_{\bar{H}}(\bar{t})=\overline{C_H(t)}$. Now $\bar{U}=\bar{T}=\nleq Z(C_{\bar{H}}(\bar{t}))$, so by 2.9, $H\cap T\neq 1$. Hence as $\bar{T}^{\#}$ is fused, $T\leq H$. Then $T=U\leq C_H(T)^{\infty}$ by 9.4.

So take $\bar{H}=M_{11}$ or M_{22}. By 2.8, $\bar{T}\leq O_2(\overline{C_H(t)})$. Further $O^2(C_{\bar{H}}(\bar{t}))\leq\overline{C_H(t)}$. But if $\bar{H}=M_{11}$ then $O_2(C_{\bar{H}}(\bar{t}))=O_2(O^2(C_{\bar{H}}(\bar{t}))$ is of 2-rank 1, a contradiction.

So $\bar{H} \cong M_{22}$. Let $\bar{X}=\langle \bar{t}^H \cap O^2(C_{\bar{H}}(\bar{t})\rangle$. Then $\langle \bar{t}\rangle = C_{\bar{H}}(\bar{X})$, so as X is T-admissible, T is a 4-group. However $O^2(C_{\bar{H}}(\bar{t}))$ normalizes no 4-group.

10. The Hall-Janko group *HJ*

Let $G=HJ$ and $A=Aut(G)$.

(10.1) (1) G has one class z^G of 2-central involutions and one class r^G of non-2-central involutions.

(2) $C_G(z)$ is the split extension of $Q=O_2(C_G(z))\cong Q_8*D_8$ by A_5. $Q=\langle z^G \cap C(z)\rangle$
(3) $C_G(r)=R\times L$ where $R\cong E_4$, $L\cong A_5$, and $r^G\cap L$ is empty.
(4) Let $S\in Syl_2(G)$ and P the weak closure of R in S. Then P is isomorphic to a Sylow 2-group of $L_3(4)$ and $S=P\langle b\rangle$ where b is a conjugate of z inducing the graph-field automorphism on Q.
(5) $|A: G|=2$ and there is one class a^G of involutions in $A-G$. $C_G(a)\cong PGL_2(7)$.

Proof. (1) and (2) are well known. See [3] for (3) and (4), where it is also shown that $|A: G|=2$ and there is an involution $a\in C_A(r)-B$ with $\langle a\rangle L\cong S_5$ and $[R, a]\neq 1$. We may assume a acts on S. Then a induces the field automorphism on P and we may take $[a, b]=1$. All involutions in aS are fused under S to a or ab. Further $Z(P)\langle b\rangle = C_S(ab)\cong D_8$ and $C_S(a)=\langle b, r\rangle\cong D_{16}$, so $C_S(a)$ is Sylow in $C_G(a)$, and $C(a)$ is transitive on $z^G\cap C(a)$. Hence by the classification of groups with dihedral Sylow 2-groups, $C_G(a)\cong PGL_2(q)$, some odd q. As $C(a)\cap C(r)\cong D_{12}$, we conclude $q=7$. Moreover letting $\langle z\rangle = Z(C_S(a))$, ab is fused to az in $C_G(a)$, so as all involutions in aS are fused under S to ab or az, there is one class of involutions in $A-G$.

(10.2) Assume H is quasisimple with $\bar{H}=H/Z(H)\cong HJ$. Then H satisfies hypothesis II and if H is T-admissible then T projects on a conjugate of the group R in 10.1.3.

Proof. By 2.3, and 10.1, H satisfies hypothesis II. Further $\langle z\rangle = C_G(C_G(z))$ $=C_G(C_G(z)^\infty)$, so each involution in T projects into a conjugate of R. Now by 2.3, T projects onto a conjugate of R.

11. The Janko group J_3

In this section $G\cong J_3$. By [25]:

(11.1) G has one class z^G of involutions. $C_G(z)$ is the extension of $Q=O_2(C(z))$ $\cong Q_8*D_8$ by A_5. $C(z)=C(z)^\infty$.
(11.2) Let $G\trianglelefteq B\le Aut(G)$ with $|B: G|=p$ prime. Then either

(1) $p=2$ and $C_B(z)/Q \cong S_5$, or
(2) $C_B(z)=Z(C_B(z))C_G(z)$.

Proof. By 11.1 and a Frattini argument $B=GC_B(z)$. As $Aut(Q)$ is the extension of E_{16} by S_5, either (1) holds or $C_B(z)=C_B(Q)C_G(z)$. In the latter case as $|C_B(Q): \langle z \rangle|=p$ and $C(z)=C(z)^\infty$, (2) holds.

(11.3) A Sylow 17-group X of G is of order 17. $N_G(X)/X \cong Z_8$ and $X=C_G(X)$.

Proof. Lemma 5.6 in [25].

(11.4) Let $A=Aut(G)$. Then $|A: G|=2$ and $C_A(z)/Q \cong S_5$.

Proof. By [23], $|A: G| \geq 2$. We show $|A: G| \leq 2$ and $\langle z \rangle = Z(C_A(z))$, and then apply 11.2. First if $|A: G|>2$ then by 11.2 we may choose $B \leq A$ with $|B: G|=p$ prime and $C_B(z)=Z(C_B(z))C_G(z)$. Hence it suffices to assume B exists and then exhibit a contradiction.

Let $X \in Syl_{17}(G)$ and Y a complement for X in $N_G(X)$. We may take $z \in Y$. $Aut(X) \cong Z_{16}$, so by a Frattini argument Y is contained in an abelian complement W to X in $N_B(X)$. As $Y=C_G(Y)$, $W=C_B(Y)=YC_B(z)$ and hence $C_B(z)=\langle z \rangle \times \langle b \rangle$ where $\langle b \rangle = Z(N_B(X))$. But now $G=\langle C_G(z), X \rangle \leq C(b)$, a contradiction.

(11.5) There is one class of involutions in $A-G$ with representative a. $C_G(a) \cong L_2(17)$.

Proof. Let $X \in Syl_{17}(G)$, $Y \in Syl_2(N_G(X))$, and $Y \leq Y_1 \in Syl_2(N_A(X))$. We may assume $z \in Y$. Suppose Y_1 is cyclic. The image of Y_1 in $C_A(z)/Q \cong O_4^-(2)$ is cyclic of order 4 and hence acts without fixed points on the nonsingular vectors of $Q/\langle z \rangle$. On the other hand Y_1 centralizes $\Phi(Y) \leq Q$, and $\Phi(Y)$ is of order 4, so that $\Phi(Y)/\langle z \rangle$ is a nonsingular point of $Q/\langle z \rangle$. Hence $Y_1 = Y \times \langle a \rangle$, where a is an involution centralizing X.

Now by 11.4, a induces a transvection in $O_4^-(2)$ on $Q/\langle z \rangle = O_2(C_G(z))/\langle z \rangle$, and all involutions in $C_A(z)-G$ are fused under $C(z)$ into aQ. There is an element x of order 3 in $C(z)$ with $[x, a] \leq Q$. Let $\langle c \rangle / \langle z \rangle = [a, Q/\langle z \rangle]$ and $P/\langle z \rangle = C_{Q/\langle z \rangle}(a)$. $P=\langle c \rangle * [P, x]$ with $[P, x]$ quaternion. We show $C_Q(a) \cong D_8$, so $[P, x, a] \neq 1$ and hence $[x, a] \neq 1$. This implies $C(a) \cap C(z)$ is a 2-group.

Let $S \in Syl_2(C_G(z))$ with $C_S(a) \in Syl_2(C_G(a))$. Then S is isomorphic to a Sylow 2-group of $L_3(4)$ extended by a graph-field automorphism and a induces an outer automorphism on S, so $S\langle a \rangle$ is isomorphic to a Sylow 2-group of $Aut(L_3(4))$ (*eg.* 3.3 in [25]). Hence as $Y \leq C_S(a)$, $C_S(a) \cong D_{16}$ and then $C_Q(a) \cong D_8$.

Therefore $C_S(a)=C(a) \cap C(z) \cong D_{16}$. Moreover $\langle Q, x \rangle$ is transitive on the involutions in aQ, so $C(z)$ is transitive on the involutions in $C_A(z)-G$, and hence there is one class of involutions in $A-G$. Further $C(a)$ is transitive on $z^G \cap C(a)$, so appealing to the classification of groups with dihedral Sylow 2-groups, $C_G(a) \cong L_2(17)$.

(11.6) Let X be quasisimple with $X/Z(X) \cong J_3$. Then

(1) X satisfies hypothesis II.
(2) X is not admissible.

Proof. (1) follows from 2.3, 11.1, and 11.5 $\langle z\rangle = C_G(C_G(z)) = C_G(C_G(z)^{\infty})$, so X is not admissible.

12. The Higman-Sims group HS

Let $G=HS$ and $A=Aut(G)$. A is the automorphism group of a strongly regular graph $\mathcal{S}$ on a set Ω of 100 vertices. Let ∞ be a distinguished vertex of Ω, $\Delta=\Delta(\infty)$ the set of 22 vertices joind to ∞ in $\mathcal{S}$, and $\Gamma=\Gamma(\infty)$ the set of 77 remaining vertices. A_∞ is the automorphism group of M_{22} and G_∞ is M_{22}. A_∞ acts 3-transitively on Δ and the vertices in Γ can be regarded as the fixed point sets of involutions in G_∞ on Δ. The members of Δ are called points and the members of Γ *blocks*. For $\alpha \in \Delta$, $\Delta(\alpha)$ is the set of blocks containing α. (Recall a block is a set of fixed points of an involution). Two blocks are adjacent in $\mathcal{S}$ if they are disjoint.

By 9.8, $\mathcal{S}_\infty$ has one class z^{G_∞} of involutions. Let $B=I(z) \in \Gamma$ and $\alpha \in B$. Let $H=A_\infty$. By [26]

(12.1) $C_G(z)$ is the extension of $T=O_2(C(z)) \cong Q_8^* Q_8^* Z_5$ by S_5 with $C(z)/T$ acting as the stabilizer of a nonsingular point on the orthogonal space $T/\langle z\rangle$.

Again by 9.8, $H-G_\infty$ has two classes u^H and t^H of involutions with u fixing 8 points of Δ and t acting without fixed points on Δ. $C=G_\infty \cap C(t)$ is the split extension of E_{16} by the holomorph of Z_5. Choose $S \in Syl_2(C(t))$ with $z \in Z(S)$. Then as $[t, z]=1$, t acts on $I(z)=B$. Let K be the stabilizer in H of B. By 9.12, $K^B \cong S_6$. $K-G$ has 3 classes of involutions t^K, s^K and r^K where t and r act fixed point freely on B and s fixes 4 points of B. r centralizes an element of order 3 in K, while as remarked above, $C_H(t)$ has order prime to 3. Hence

(12.2) $t^K = t^H \cap K$, $s^K \cup r^K = u^H \cap K$.

Let N be the number of fixed blocks of t. Counting the set of pairs (D, t^h), where D is a block fixed by t^h, we have $N|t^H| = 77|t^K|$. We conclude $N=5$.

Now $C=G_\infty \cap C(t)=XY$ where $Y=O_2(C)$ and $X=N_C(X_5)$ where X_5 is a Sylow 5-group of C. As $t^K=t^H \cap K$, C is transitive on $\Gamma \cap I(t)$ and then X_5 is regular on these blocks. Thus $Y=C_A(t)_{I(t)}$. $I(u)$ has order at least 9 so

$u^G \neq t^G$ and hence $t^G \cap H = t^G$. So $C_G(t)$ is transitive on $I(z)$ and hence $C_G(t)^{I(t)} \cong PGL_2(5)$.
Summarizing:

(12.3) $C_G(t)$ is the extension of $Y = O_2(C_G(t)) \cong E_{16}$ by $PGL_2(5)$ and $t^G \cap H = t^H$. $C_G(t)$ acts irreducibly on Y.

By [15].

(12.4) G has two classes z^G and v^G of involutions. There are two classes t^G and u^G of involutions in $A - G$. $|C_G(u)| = 8!$.

As $u^G \cap H = u^H$, u fixes $|C_A(u)\colon C_H(u)| = 8!/2^6 \cdot 3 \cdot 7 = 30$ vertices. Hence u fixes 21 blocks. By 12.2 $C_H(u)$ has 2 orbits Γ_1 and Γ_2 on these fixed blocks, where Γ_1 consists of those blocks in which u fixes 4 points and Γ_2 the blocks upon which u acts without fixed points. As 3 points determine a block and u fixeds 8 points of Δ, Γ_1 has order $8 \cdot 7 \cdot 6/4 \cdot 3 \cdot 2 = 14$. Hence Γ_2 has order 7. Each cycle of u on Δ is contained in 2 blocks of Γ_1 and 3 blocks of Γ_2. Hence easy counting arguments show the graph $\mathcal{A}$ induced by $\mathcal{A}$ on $I(u)$ is bipartite so that $C_G(u)$ is 2-transtitve on one of the sets I_1 in the partition. Hence $\mathcal{A}$ is the incidence graph of the projective geometry $PG(3, 2)$, and $C_G(u)$ is the automorphism group of that geometry together with a polarity. Hence

(12.5) $C_G(u) \cong S_8$.

Let $w \in C_G(u)$ correspond to a transvection. Then $C_G(w) \cap C(u) \cong Z_2 \times S_6$, and $W = E(C_G(u) \cap C(u)) \leq L(C(w))$. It follows from 12.1 and 12.4 that we may take $v = w$. By [26],

(12.6) $C_G(v) \cong Z_2 \times Aut(A_6)$.

Thus $W = L(C(v))$ and

(12.7) $C_A(v) = Z_2/(E_4 \times S_6)$.

(12.8) Let X be quasisimple with $X/Z(X) \cong HS$. Then

(1) X satisfies hypothesis II.
(2) X is not admissible.

Proof. Let T be an elementary 2-group acting faithfully on X and Sylow in a 2-nilpotent tightly embedded subgroup of XT. Assume first $a \in T - XC(X)$. By 2.3, a does not induce u on X, so a induces t. As Out(X) has order 2, some $b \in T^{\#}$ induces an inner automorphism on X. By 2.3, b induces an automorphism in z^G. As $O^2(C_G(t))$ acts irreducibly on $O_2(C_G(t))$, $T \cap XC(X) = T_0$ projects on $O_2(C_G(t))$. Now $C_G(b)$ acts on T_0 and since we may choose b to project on z, $O^2(C_G(z))$ normalizes the projection of T_0. But $T_0 \cong O_2(C_G(t)) \cong E_{16}$ while

$m(O_2(O^2(C_G(z)))=3$, a contradiction.

So $T\leq XC(X)$ and (1) is established. Also as $\langle x\rangle=C_G(C_G(x)^\infty)$ for each involution $x\in G$, X is not admissible, establishing (2).

13. The Fischer groups

Let $G=M(24)$, the largest of Fischer's three groups generated by 3-transpositions. The following facts are in [11]:

(13.1) (1) G is generated by a class $d^G=D$ of 3-transpositions.

(2) $|G\colon E(G)|=2$, $E(G)$ is simple, and $G=Aut(E(G))$.
(3) $C_{E(G)}(d)=H\cong M(23)$ is simple and $H=Aut(H)$.
(4) Let $d\in S\in \mathrm{Syl}_2(G)$ and $L=\langle S\cap D\rangle$.
Then L is abelian of order 2^{12} and $N_G(L)$ is the non-split extension of M_{24} acting 5-transtitively on $S\cap D$.
(5) Let a, b, c, and d be distinct members of $S\cap D$. The all involutions in L are ufsed under $N(L)$ to d, $t=da$, dab, or $dabc$.
(6) Let $K=C_H(t)$. Then K is quasisimple and $K/\langle t\rangle\cong M(22)$ is simple. $Aut_G(K)=Aut(K)$ and $|Aut(M(22))\colon M(22)|=2$.
(7) $C_K(bt)$ is isomorphic to the covering group of $U_6(2)$.
(8) $M(2n)$ contains a unique class of 3-transpositions for each $n=2, 3, 4$.

We record four elementary facts about groups generated by 3-transpositions:

(13.2) Let a, b, and c be distinct commuting members of a set E of 3-transpositions. Then

(1) $C_E(ab)=C_E(a)\cap C_E(b)$.
(2) $C_E(abc)=C_E(a)\cap C_E(b)\cap C_E(c)$.
(3) If $\langle E\rangle$ is transitive on E then $C_E(a)=C_E(b)$ exactly when $a\in b^{O_2(\langle E\rangle)}$.
(4) If x is an involutory automorphism of $\langle E\rangle$ and $[a, a^x]\neq 1$, then x centralizes a member of E.

Proof. (1), (2), and (4) are easy. See [11], 2.1.3, for (3).

(13.3) Let E be the set of 3-transpositions in $M\cong M(22)$. Then every involutory automorphism of M centralizes a member of E.

Proof. Let x be an involutory automorphism of M and assume $C_E(x)$ is empty. By 13.2.4, $[u, u^x]=1$ for each $u\in E$, so x centralizes the involution $s=uu^x$. By 13.1.7, $C_M(u)$ is a covering of $U_6(2)$ over u. Moreover u^x is a transvection in $C_M(u)/\langle u\rangle$, so $J=\langle C_E(u)\cap C_E(u^x)\rangle=AB$ where $A=\langle A\cap E\rangle\cong U_4(2)$ and $B=O_2(J)$. Further by [11], 16.1.10, A^J is the unique class of E-subgroups complementing B in J.

x centralizes s, so by 13.2.1, x acts on J. If x acts nontrivially on J/B then by 12.2 and the proof of 19.8 in [4], $[v, v^x] \notin B$ for some $v \in J \cap E$, against 13.2.4. So x centralizes J/B. Now by uniqueness of A^J and a Frattini agrument, $x=yb$ for some $b \in B$ and $y \in N(A)$. Then y centralizes A. Moreover A acts irreducibly on $B/Z(J)$, so y centralizes $B/Z(J)$ and then B. So $[y, J]=1$. Now A acts in its natural representation on $B/Z(J)$, so every member of $B/Z(J)$ is centralized by some $v \in A \cap E$. Further by 13.2.1, $Z(J)v \cap E = \{v\}$, so every member of B is centralized by some member of $A \cap E$. In particular $x=yb$ centralizes a member of $A \cap E$.

(13.4) (1) Every involution in G centralizes a member of D.

(2) Every involution in H fixes a 3-transposition of H.

Proof. $t=da$ is a 3-transposition of H in the center of a Sylow 2-group of H, so (2) is immediate. Let x be an involution in G and suppose $C_D(x)$ is empty. By 13.2.4 we may assume $d^x=a$, so that x centralizes t. By 13.2.1, $K\langle d\rangle = C_D(t)$ is x-invariant, so x acts on $K=E(K\langle d\rangle)$. By 13.3 we may assume x acts on $\langle bd, t\rangle$. Now x acts on $\langle b, d, t\rangle \cap D = \{a, b, d\}$ by 13.1.4. So x centralizes b.

(13.5) (1) d, t, dab, and $dabc$ are representatives for the conjugacy classes of involutions in G.

(2) Let u, v, and w be distinct commuting 3-transpositions in $M \cong M(23)$ or $M(22)$. Then u, uv, and uvw are representatives for the conjugacy classes of involutions in M.

Proof. L is weakly closed in S and L is abelian, so $N(L)$ controls fusion in L. By 13.1.5, any involution in L is fused to d, da, dab, or $dabc$, while by 13.1.4 none of these involutions is fused in $N(L)$. So to prove (1) it suffices to show each involution in G is the product of commuting 3-transpositions. By 13.4, each involution in G is conjugate to an involution in $C_G(d)=\langle d\rangle H$. So as t^H is the set of 3-transpositions in H, (1) is reduced to (2).

Next if $M \cong M(23)$ or $M(22)$, E is the set of 3-transpositions of M, and $T \in Syl_2(M)$, then $N(T \cap E)^{T \cap E}$ is M_{23} or M_{22} and all involutions in $\langle T \cap E\rangle$ are fused to exactly one of u, uv, or uvw in $N(T)$, where u, v, and w are distinct members of $T \cap E$. Hence we may repeat the argument above, and reduce (2) to showing that any involution in $C_M(v)$ is the product of commuting 3-transpositions, where $M \cong M(22)$ and v is a 3-transposition of M. However by 13.1.7, $C_M(v)/\langle v\rangle \cong U_6(2)$, so as every involution in $U_6(2)$ is the product of 1, 2, or 3 transvections, the proof is complete.

(13.6) (1) $C_G(d)=\langle d\rangle \times H \cong Z_2 \times M(23)$.

(2) Let $e\in D-C(\langle a, d\rangle)$ and $S=\langle d, a, e\rangle$. Then $S\cong S_4$ and $C_G(t)=\langle s\rangle(\langle d\rangle\times K)$ where $s\in t^S$ induces an outer automorphism on K and $K/\langle t\rangle\cong M(22)$.
(3) $C_G(dab)=\langle dab\rangle\times J$ where $E(J)$ is the covering group of $U_6(2)$ and $J/Z(J)\cong Aut(U_6(2))$.
(4) Let $z=dabc$ and $X=\langle C_D(z)\rangle$. Then $O_2(X)\cong Q^6$, $X/O_2(X)$ is the perfect central extension of Z_3 by $O_6^-(3)$, $C_G(z)/X\cong S_3$, and $C_G(X)=\langle z\rangle$.

(5) $C_G(S)=\langle C_D(S)\rangle\cong Aut(\Omega_8^+(2))$.

Proof. (1) follows from 13.1.3. Pick e and S as in (2). As $(ad)^2=(ae)^3=(de)^3=1$, $S\cong S_4$, so there is a conjugate s of t under S with $[d, s]=t$. Now (2) follows from 13.1.6 and 13.2.1. (5) follows from 18.3.12 and 18.3.14 in [11].

By 13.1.7. and 13.2.2, $B=\langle C_D(dab)\rangle=\langle dab\rangle\times E(B)$ where $E(B)$ is the covering group of $U_6(2)$. By 13.1.4, $C_G(dab)$ induces S_3 on $\{d, a, b\}$ and hence (3) follows.

Finally let $z=dabc$, $X=\langle C_D(z)\rangle$ and $Y=\langle C_D(z)\cap C(d)\rangle$. Then $Y=\langle d\rangle\times O^2(Y)$where $\Phi(O_2(Y))=\langle z\rangle$, $Z(O^2(Y))=\langle da, db, dc.\rangle$, and $O_2(Y)\cong E_8\times Q^4$, with $Y/O_2(Y)\cong U_4(2)$. Also $\{d, a, b, c\}=Z(Y)\cap D$. Moreover by 13.5.2, $d, a, da, ab, dab, abc,$ and $dabc$ are representatives for the $C(d)$ classes of involutions in $C(d)$. So by 13.5.1, $z^G\cap C(d)=z^{C(d)}$. Hence X is transitive on $X\cap D$. Now by 13.2.3, $\{d, a, b, c\}=d^{O_2(X)}$. So $\langle da, db, dc\rangle\leq O_2(X)$. As $O_2(O^2(Y))$ is generated by conjugates of $\langle da, db, dc\rangle$, $O_2(O^2(Y))\leq O_2(X)$. $O_2(Y)=O_2(C(d)\cap C(z))$ and $|O_2(X): O_2(X)\cap C(d)|=|d^{O_2(X)}|=4$. As $O_2(O^2(Y))\cong E_4\times Q^4$ we conclude $O_2(X)\cong Q^6$.

Now $YO_2(X)/O_2(X)\cong Z_2\times U_4(2)$. Also $\langle C_D(e)\cap C_D(d)\rangle=W\cong 3O_8^+(3)$. So if v is a conjugate of z in W, $\langle C(v)\cap D\cap W\rangle$ is solvable. Hence by the main theorem of [11], $X/O_2(X)$ is isomorphic to $O_6^-(3)$ modulo its center. By [35], $O_6^-(3)\not\leq O_{12}^+(2)$, so $Z(X/O_2(X))\cong Z_3$. By a Frattini argument $C(z)=X(C(z)\cap C(d))$, so as $C(z)\cap C(d)$ induces S_3 on $\{a, b, c\}$, the proof of (4) is complete.

(13.7) Let E be the class of 3-transpositions in $H\cong M(23)$ and $u, v,$ and w distinct commuting members of E. Then

(1) $C_H(u)$ is quasisimple with $C_H(u)/\langle u\rangle\cong M(22)$.
(2) $C_H(uv)=\langle s\rangle J$ where J is the covering group of $U_6(2)$ and s is a conjugate of uv inducing an outer automorphism on J.
(3) $C_H(uvw)$ is 2-constrained with $C_H(uvw)/(C_H(u)\cap C_H(v)\cap C_H(w))\cong S_3$.

Proof. This follows from 13.1 and 13.2.

(13.8) Let $M=M(22)$, $A=Aut(M)$, E the class of 3-transpositions in M, and $u, v,$ and w distinct commuting members of E. Then

(1) $C_M(u)$ is quasisimple with $C_M(u)/\langle u\rangle \cong U_6(2)$. $C_A(u)=\langle s\rangle C_M(u)$ where s is an involution inducing an outer automorphism on $C_M(u)$.
(2) $C_A(uv)$ and $C_A(uvw)$ is 2-constrained with $C_A(uv)/(C_A(u)\cap C_A(v))\cong Z_2$ and $|C_A(uvw)|=2^{16}\cdot 3^3$.
(3) There are 3 classes of involutions in $A-M$ with representatives s, su, and suv.
(4) $C_M(s)\cong Aut(\Omega_8^+(2))$.
(5) $C_M(su)=C_M(s)\cap C_M(u)\cong Z_2\times Sp_6(2)$.
(6) $C_M(suv)$ is the extension of E_{64} by $O_6^-(2)$ acting in its natural representation with uv corresponding to a nonsingular point.

Proof. By 13.1.6 $N_G(K)/C_G(K)$ is isomorphic to A and $|A: M|=2$. Let $e\in D-C_G(\langle a, d\rangle)$ and s_1 a conjugate of ad under $\langle a, d, e\rangle$ with $[d, s_1]=ad$. Let s be the image of s_1 in A. By 13.6.2, $A=M\langle s\rangle$.

By 13.6.5 and 13.2.1 we have $\langle C_D(\langle a, d, s_1\rangle)\rangle=\langle C_D(\langle a, d, e\rangle)=C_G(\langle a, d, e\rangle)\rangle \cong Aut(\Omega_8^+(2))$. This yields (4), and shows we may choose b and c to be contained in and fused under $C_G(\langle a, d, s_1\rangle)$. Now E is the image of $C_D(ad)$ in M, so we may take u and v to be the image of b and c, respectively. Then (1) follows from 13.6.3 and (2) follows from (1), 13.2, and an easy calculation.

Next as s induces an outer automorphism of $C_M(u)$, 19.8 in [4] implies s, su, sv, and svu are representatives for the $C_A(u)$ classes of involutions in $C_A(u)-M$. Notice that s_1 and s_1bc are involutions while s_1db and s_1dc are elements of order 4, so s and suv are not fused in A to su or sv. In addition $C_M(u)\cap C(s)\cong Z_2\times Sp_6(2)$ and $C_M(u)\cap C(sv)=C_M(u)\cap C(s)\cap C(v)$. By 13.3 every involution in $A-M$ is fused to one of s, su, sv, or suv.

Recall that s_1b is fused to s_1c in $C(\langle a, d, s_1\rangle)$, so su is fused to sv in A. By (4), $C_M(s)$ is transitive on $C_E(s)$, so $C_M(v)$ is transitive on $s^M\cap C(v)$. Thus s is not fused to suv in A. Hence (3) is established.

su is the image of efb where $s_1=ef$, $e, f\in D$. By 13.6.3, $C_G(efb)$ acts on $\{e, f, b\}$, so $C_G(efb)\cap C(ad)$ acts on $\{e, f, b\}\cap C(ad)=\{b\}$. Therefore $C_M(su)=C_M(s)\cap C_M(u)\cong Z_2\times Sp_6(2)$, proving (5).

Next $C(u)$ is transitive on $(suv)^M\cap C(u)$ and hence $C(suv)$ is transitive on $C_E(suv)$. Let $Y=\langle E\cap C(suv)\cap C(u)\rangle$ and $X=\langle C_E(suv)\rangle$. $Y=\langle u\rangle\times Y_1$ where Y_1 is the centralizer of a transvection in $Sp_6(2)$. Also $\{u, v\}=Z(Y)\cap E$ so by 13.2.4, $uv\in O_2(X)\cong E_{64}$. $YO_2(X)/O_2(X)\cong Z_2\times Sp_4(2)$, so by the main theorem of [11], $X/O_2(X)\cong O_6^\varepsilon(2)$ acts in its natural representation on $O_2(X)$ with $\langle uv\rangle=C_X(Y)$ a nonsingular point. By a Frattini argument, $C_M(suv)=X(C(u)\cap C(suv))=X$.

X is the image of $\langle C(ad)\cap C(efbc)\cap D\rangle$ so ad induces an automorphism of $W=\langle C_D(efbc)\rangle$ such that $\langle C(ad)\cap DO_2(W)/O_2(W)\rangle$ has an $O_6^\varepsilon(2)$ composition factor. But by 13.6.4, $W/O_2(W)\cong O_6^-(3)$, so $O_6^\varepsilon(2)$ is of characteristic 3. Hence $X/O_2(X)\cong O_6^-(2)\cong O_5^-(3)$. This completes (6).

(13.9) Let X be quasisimple with $X/Z(X)$ isomorphic to $M(22)$, $M(23)$, or $M(24)'$. Then

(1) X satisfies hypothesis II.
(2) X is not admissible.

Proof. By 13.1.3, $M(23)$ has a trivial outer automorphism group. By 13.1.2, $G=Aut(M(24)')$, and then 13.5.1, 13.6, and 2.3 imply $M(24)'$ satisfy hypothesis II. Suppose $X/Z(X)\cong M(22)$ and T is an elementary abelian 2-group acting on X and Sylow in a tightly embedded 2-nilpotent subgroup of TX with $T\not\leq XC(X)$. By 13.8 and 2.3, some $t\in T^{\#}$ induces suv on $X/Z(X)$. By 2.8, $C_X(t)$ *acts* on T so as $C_{X/Z(X)}(t)=O^2(C_{X/Z(X)}(t))$ *acts* irreducibly on $O_2(C_{X/Z(X)}(t))$, some $r\in T^{\#}$ projects on uv. But now tr induces s on $X/Z(X)$, against 2.3. The proof of (1) is complete.

By 13.6, $C_{E(G)}(C_G(x)^a)=\langle x\rangle$ for each involution x in $E(G)$, so $M(24)'$ is not admissible.

Suppose $\bar{X}\cong X/Z(X)\cong M(23)$. We adopt the notation of 13.7, setting $\bar{X}=H$. By 2.3 and 13.7, $\bar{T}\cap\bar{E}$ is empty. Next $C_H(J)=\langle u, v\rangle$, so as $\bar{T}\cap E$ is empty, 2.3 implies $\bar{T}\cap(uv)^H$ is empty. Hence by 13.5, $\bar{T}^{\#}\subseteq(uvw)^H$. Finally by 13.7, $C_H(C_H(uvw)^{\mathcal{A}})\equiv\langle u, v, w\rangle$, so as $(uvw)^H\cap\langle u, v, w\rangle=\{uvw\}$, we have a contradiction.

Finally assume $\bar{X}\cong M(22)$. Let $\bar{u}, \bar{v}$, and $\bar{w}$ be commuting 3-transpositions in $\bar{X}$. By 13.5 each involution in $\bar{X}$ is fused to $\bar{u}$, $\bar{u}\bar{v}$, or $\bar{u}\bar{v}\bar{w}$, By 13.8 and 2.3, $\bar{u}\notin\bar{T}$. $\langle\bar{u},\bar{v}\rangle=C_{\bar{X}}(C_{\bar{X}}(\bar{u}\bar{v})^{\mathcal{A}})$, so as $\bar{u}\notin\bar{T}$, $\bar{u}\bar{v}\notin\bar{T}$. Thus each involution in $\bar{T}$ is fused to $\bar{u}\bar{v}\bar{w}$, and we may take $\bar{t}=\bar{u}\bar{v}\bar{w}$, $t\in T^{\#}$. Let $\bar{J}=C_{\bar{X}}(\bar{u})$. Then $\bar{J}/\langle\bar{u}\rangle\cong U_6(2)$ and by 13.6.3, J is quasisimple. By 21.7 and 10.6 in [4], $C_J(t^J\cap C(t))=\langle uvw, Z(J)\rangle$. By symmetry among u, v, and w, $C_X(t^X\cap C(t))\leq\langle uvw, Z(X)\rangle$, so by 2.4, $T\cong E_4$. By 2.6, $C_X(t)\leq N(T)$, while by 21.7 and 10.6 in [4], $\langle u, u, Z(J)\rangle(J)/Z(J)$ is the only 4-group in $J/Z(J)$ normalized by $C_J(t)/Z(J)$. Hence $\bar{T}\leq\langle\bar{u},\bar{v},\bar{w}\rangle$. But $\bar{t}^{\bar{X}}\cap\langle\bar{u},\bar{v},\bar{w}\rangle=\{\bar{t}\}$, a contradiction.

14. Conway's second group Co_2

Let $G=Co_2$. We record some facts about G found in [30]:

(14.1) (1) G has 3 classes of involutions with representatives z, a, and $z\pi$.

(2) $C_G(z)$ is the split extension of $E=O_2(C(z))\cong Q^4$ by $S\cong Sp_6(2)$ with $E=[E, S]$.
(3) Set $I=C_G(a)$. Then $O_2(I)=W_0\times D_0$ where $W_0\cong E_{16}$ and $D_0\cong D^3$ are I-invariant, and $I/O_2(I)\cong A_8$ acts as $L_4(2)$ on W_0 and as $\Omega_6^+(2)$ on $D_0/\langle a\rangle$.
(4) Set $M=C_G(z\pi)$, $X=C_M(z)$, and $J=O_2(M)$. Then $|M:X|=2$, $J=D\times D^u$ where $D\cong E_{32}$ and $u\in M-X$, $X/J\cong S_6$, $M/J\cong Aut(A_6)$, and X/J acts in its natural representation on the permutation modules D and D^u.

(5) Let γ and β be G-conjugates of z such that γ is a transvection in S and $\alpha=\gamma\beta$ is of order 3. Then $N_G(\langle\alpha\rangle)=\langle\alpha, \gamma\rangle\times C$ where $C\cong Aut(U_4(2))$.
(6) $G_1=\langle E, N_G(\langle\alpha\rangle)\rangle=\langle\gamma\rangle G_0$ where $G_0\cong U_6(2)$ and γ induces a graph automorphism on G_0.
(7) G acts as a rank 3 group on the set Ω of cosets of G_1 in G. G is a normal subgroup of index 1 or 3 in the automorphism group of the rank 3 graph $\mathcal{S}$ of this representation.

Proof. Specific references in [30] are as follows: (1), 2.8; (2), 1.1, 2.1, and 2.11; (3), 2.12; (4), 2.13 and the discussion on pages 101 and 102; (5), 4.1; (6), 4.3; (7), section 5.

(14.2) $Aut(G)=Aut(\mathcal{S})=G$.

Proof. Let $A=Aut(G)$. By 14.1.7 it suffices to show $A=GN_A(G_1)$, and $N_A(G_1)=G_1$. By a Frattini argument $A=GC_A(z)$. A second Frattini argument implies $C_A(z)=C_G(z)C_A(\langle z, \alpha\rangle)$. By 14.1.5, $N_A(\langle\alpha\rangle)=B\times N_G(\langle\alpha\rangle)$ where $B=C_A(N_G(\langle\alpha\rangle)$. In particular $B\leq C(z)\leq N(E)$, so by 14.1.6, B acts on G_1. Then by 14.1.7, $A=Aut(\mathcal{S})$ and B has order 1 or 3.

Assume B has order 3. We show B centralizes $C_G(z)$. Then B centralizes $G_1=\langle E, N_G(\langle\alpha\rangle)\rangle$. By 14.1.7, G_1 is maximal in G, so B centralizes $G=\langle G_1, C_G(z)\rangle$, a contradiction. B acts on $C_G(z)=ES$ centralizing $\langle\alpha, \gamma\rangle\times C_S(\alpha)\cong S_3=Sp_4(2)$, so $[B, S]\leq E$. Assume B does not centralize $C_G(z)$. As S acts irreducibly on $E/\langle z\rangle$, $C_E(B)=\langle z\rangle$. $\overline{BC_G(z)}=BC_G(z)/E$ acts on $E/\langle z\rangle=V$, preserving a quadratic form of sign $+$, so $\overline{BC(z)}\cong Z_3\times Sp_6(2)\leq O_8^+(2)$. Let $g\in\bar{S}$ have order 7. $[V, g]$ is of dimension 3 or 6 and as $\bar{B}$ acts without fixed points on $[V, g]$, it must be the latter. Then $C_V(g)$ is nondegenerate of dimension 2, and as $\bar{B}$ acts without fixed points on $C_V(g)$, $C_V(g)$ is of sign$-$. Hence $[V, g]$ is of sign$-$. But the order of $O_6^-(2)$ is not divisible by 7. The proof is complete.

(14.3) Let A be quasisimple with $A/Z(A)\cong Co_2$. Then

(1) A satisfies hypothesis II.
(2) A is not admissible.

Proof. (1) follows from 14.2. By 14.1, $\langle x\rangle=C_G(C_G(x)^{\mathcal{A}})$ if $x=z$ or a, and $\langle z, \pi\rangle=C_G(C_G(\pi z)^{\mathcal{A}})$. This yields (2).

15. Subgroups of Fischer's Monster

We adopt the notation of R. Griess in discussion subgroups of Fischer's Monster. That is F_n denotes the simple composition factor of the centralizer of a certain element of order n in the Monster. The orders are as follows:

F_5 $2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$
F_3 $2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13\cdot 19\cdot 31$
F_2 $2^{41}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$
F_1 $2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$

(15.1) Let $G=F_5$ and $A=Aut(G)$. Then

(1) $|A: G|=2$.
(2) G has 2 classes of involutions with representatives z and t.
(3) $E(C_G(t)$ is quasisimple with $E(C_G(t))/\langle t\rangle\cong HS$ and $C_G(t)/\langle t\rangle\cong Aut(HS)$.
(4) $C_G(z)$ is the extension of Q^4 by A_5 wreath Z_2.
(5) There is one class of involutions in $A-$G with representative a.
(6) $C_G(a)\cong S_{10}$.

Proof. [21].

(15.2) Let $G=F_3$. Then

(1) $G=Aut(G)$.
(2) G has one class of involutions with representative z.
(3) $C_G(z)$ is the extension of an extraspecial group of order 2^9 by A_9.

Proof. [31].

(15.3) Let $G=F_2$. Then

(1) G is generated by a class $D=d^G$ of 3,4-transpositions.
(2) $C_G(d)=\langle e\rangle H$ where $H=E(C(d))$, $H/\langle d\rangle\cong {}^2E_6(2)$, and e induces a graph automorphism of H.
(3) $D=d\cup b^H\cup e^H\cup a^H\cup v^H$ where ad and vd have order 3 and 4, respectively, and $\langle b, d\rangle/\langle d\rangle$ is a root involution of $H/\langle d\rangle$.
(4) G has 4 classes of involutions with representatives d, $z=db$, $f=de$, and $\theta=dbb'$, where $b'\in b^H$.
(5) $C_G(z)$ is the extension of D^{11} by Co_2.
(6) $C_G(f)=\langle s\rangle(\langle f, d\rangle\times E(C_H(e)))$ where $F=E(C_H(e))\cong F_4(2)$, $[s, d]=f$, and s induces an outer automorphism on F.
(7) $C_G(\theta)$ is the extension of $O_2(C_G(\theta))$ by $O_8^+(2)$ and $\langle\theta\rangle=C_G(O^2(C_G(\theta)))$.

Proof. [12].

(15.4) Let $G=F_2$. Then $G=\langle H, E(C_G(b))\rangle$.

Proof. There exist conjugates b_1 and b_2 of b under H such that bb_1 and bb_2 have order 3 and 4 respectively. Moreover as D is a set of 3,4-transpositions,

e does not centralize $H=\langle b^H\rangle$, and e is not conjugate to b in H, there exists $b_3 \in b^H$ such that eb_3 is of order 4. It follows with 15.3.3, that $D\subseteq\langle H, E(C_G(b))\rangle$, completing the proof.

(15.5) F_2 is its own automorphism group.

Proof. Let $G=F_2$ and $A=Aut(G)$. By a Frattini argument, $A=GC_A(d)$. Moreover $C_G(d)/\langle d\rangle$ is its own automorphism group, so $C_A(d)=XC_A(d)$, where $X=C_A(H)$. Let $Y=O^2(C_G(d)\cap C_G(b))$. Then $C_A(Y)=X\langle b\rangle=C_A(E(C_G(b))\langle d\rangle$, so $|X: C_X(E(C_G(b)))|=2$. Finally by 15.4, $C_X(E(C_G(b)))=1$, so $X=\langle d\rangle$. The proof is complete.

(15.6) Let $G=F_1$. Then

(1) G has two classes of involutions with representatives t and z.
(2) $C_G(t)=H$ is quasisimple with $H/\langle t\rangle\cong F_2$.
(3) $C_G(z)$ is the extension of Q^{12} by Co_1.
(4) If X is a group with involutions t and z with centralizer as in (2) and (3), then $|X|=|G|$.

Proof. [18].

(15.7) Let $G=F_1$ and $s\langle t\rangle/\langle t\rangle$ a 3,4-transposition in $H/\langle t\rangle$. Set $K=E_H(N(\langle s,t\rangle))$. Then K is the covering group of ${}^2E_6(2)$ and $\langle s,t\rangle^{\#}$ is fused in $N(K)$.

Proof. $K/Z(K)\cong {}^2E_6(2)$ and, replacing s by st if necessary, $Z(K)=\langle s\rangle$ or $\langle s,t\rangle$. As ${}^2E_6(2)$ has E_4 as a multiplier, s is an involution. By L-balance $K\leq L(C(s))$, so by 15.6 s is fused to t. Now by symmetry between s and t, $Z(K)=\langle s,t\rangle$. Next by 15.3 there exists $h\in H$ inducing an outer automorphism on K, so $[s,h]=t$. By symmetry between s and t, $N(K)$ induces S_3 on $\langle t,s\rangle^{\#}$.

(15.8) Let $G=F_1$ and r a conjugate of s under H contained in $C_K(\langle t,s\rangle)-\langle t,s\rangle$. Set $T=\langle t,r,s\rangle$. Then T contains a unique conjugate of z, which we take to be z, and $N_G(T)$ is transitive on $T-\langle z\rangle$.

Proof. $\langle rs,t\rangle/\langle t\rangle$ is the center of Sylow 2-group of H, so as $\langle z\rangle$ is the center of a Sylow 2-group of G, rs or rst is fused to z. By 15.7, st is fused to t, so we may take $z=rs$. Now $N_G(K)=K(N(K)\cap N(T))$ by a Frattini argument, and by 15.7, $N(K)\cap N(T)$ has orbits of length 1, 3, and 3 on $T^{\#}$. The orbits of length 3 are fused in $N_H(T)$, completing the proof.

(15.9) Let $G=F_1$. Then $C_G(z)=\langle C_G(\langle z,t\rangle), C_G(\langle z,s\rangle)\rangle$.

Proof. Let $Q=O_2(C(z))$ and $X=C(\langle z,t\rangle)$. By 15.3, $X/\langle t\rangle$ is the extension of an extraspecial group of order 2^{23} by Co_2. So by 15.6, $t\in Q$ and XQ/Q

is maximal in $C_G(z)/Q$. By 15.8, $s\in t^{C(z)}-\langle t, z\rangle$, so $Q=\langle C_Q(t), C_Q(s)\rangle$ and as XQ/Q is maximal in $C(z)/Q$, the result follows.

(15.10) Let $G=F_1$. Then $G=\langle C_G(z), C_G(t)\rangle$.

Proof. 15.6.4.

(15.11) F_1 is its own automorphism group.

Proof. Let $G=F_1$ and $A=Aut(G)$. By a Frattini argmuent $A=GC_A(t)$. By 15.5 $C_A(t)=C_G(t)X$, where $X=C_A(C_G(t))$. X centralizes $C(t)\cap C(s)$, so $|X: C_X(C_G(s))|=2$. By 15.9, $Y=C_X(C_G(s))$ centralizes $C_G(z)$, and then by 15.10, $Y=1$. The proof is complete.

(15.12) Let A be quasisimple with $A/Z(A)\cong F_n$, $n=1, 2, 3$, or 5. Then

(1) A satisfies hypothesis II.
(2) A is not admissible.

Proof. If $n\neq 5$ then the outer automorphism group of A is trivial by 15.2, 15.5, and 15.11. If $n=5$, 15.1 and 2.3 imply (1). The results in this section show $m(C(C_A(u)^{\mathcal{A}}))<m(C(C_A(v)^{\mathcal{A}}))$ for each pair of involutions u and v in $A/Z(A)$ with $u\in C(C_A(v)^{\mathcal{A}})-\langle v\rangle$. Thus (2) holds.

16. The remaining sporadic groups

(16.1) Let G be the small Janko group J_1. Then

(1) $G=Aut(G)$.
(2) G has one class of involutions with representative z.
(3) $C_G(z)\cong Z_2\times A_5$.

(16.2) Let G be Conway's small group Co_3, let M be McLaughlin's group Mc, and let $A=Aut(Mc)$. Then

(1) $G=Aut(G)$.
(2) $|A: M|=2$.
(3) G has two classes of involutions with representatives z and t.
(4) $C_G(z)$ is the covering group of $Sp_6(2)$.
(5) $C_G(t)\cong Z_2\times M_{12}$.
(6) M has one class of involutions with representative z.
(7) $C_M(z)$ is quasisimple with $C_M(z)/\langle z\rangle\cong A_8$.
(8) There is one class of involutions in $A-M$ with representative t.
(9) $C_M(t)\cong M_{11}$.

Proof. See [9] for (1) and (2) By [10], $A\leq G$ and G acts 2-transitively on the set Ω of cosets of A in G. Moreover (3) holds where we may choose $z\in M$ and $t\in A-M$, with the fixed point sets $I(z)$ and $I(t)$ of z and t on Ω of order 36 and 12, respectively.

Let ∞ be the point of Ω fixed by A and 0 a second point. Set $H=G_{\infty 0}$. By [10], p. 64, H is transitive on conjugates of z and t in H, so $C(u)^{I(u)}$ is 2-transitive, $u=z$ or t. This yields (6) and (8).

(4) and (5) follow from [9]. In particular $C(z)^{I(z)}=Sp_6(2)$ and $C(t)^{I(t)}=M_{12}$. Hence as A is the stabilizer of ∞, $C_A(z)$ acts as $O_6^+(2)\cong S_8$ on $I(z)-\{\infty\}$ and $C_A(t)$ acts as M_{11} on $I(t)-\{\infty\}$. The proof is complete.

(16.3) Let G be Lyon's group Ly. Then

(1) $G=Aut(G)$.
(2) G has one class of involutions with representative z.
(3) $C_G(z)$ is the covering group of A_{11}.

(16.4) Let G be Held's group He and $A=Aut(G)$. Then

(1) G has two classes of involutions with representatives z and r.
(2) $C_G(z)$ is the centralizer of a 2-central involution in the holomorph of E_{16}.
(3) There is a standard subgroup L of G with $r\in R=Z(L)\cong E_4$ and $L/R\cong L_3(4)$. $N_G(L)=L\langle d,f\rangle$ where d and f induce diagonal and field automorphisms on L/R, respectively.
(4) $|A:G|=2$.
(5) There is one class of involutions in $A-G$ with representative a.
(6) $Z(E(C_G(a)))\cong Z_3$ and $C_G(a)/Z(E(C_G(a)))\cong S_7$.

Proof. (1)–(3) are well known and are contained in, or can easily be derived from [22].

Let $T=R^g$ be a distinct conjugate of R contained in L, and set $S=C_L(RT)$. Then S is a Sylow 2-group of L and of L^g. Now $|Aut(L): Aut_G(L)|\leq 2$ with S/R self centralizing in $Aut(L)$. So $C_A(S)=TC_A(L)$. Let $X=C_A(L)$. Then X acts on L^g and centralizes S, so $X=RC_X(L^g)$. By [5], He is the unique group generated by a nonnormal standard subgroup isomorphic to L, so $G=\langle L, L^g\rangle$. Hence $X=R$. As $|Aut(L): Aut_G(L)|\leq 2$, a Frattini argument shows $|A:G|\leq 2$. The existence of an outer automorphism is known and establishes (4). Moreover we have shown $RT=C_A(S)$, and there exists $\sigma\in A-L$ inducing a graph automorphism on L.

Let P be a Sylow 3-group of $N(S)\cap N_A(L)$. Then $N_A(P)\cap N(L)=PD$, where $D\cong E_4$. So we may take $D=\langle f, \sigma\rangle$. In particular σ and $a=\sigma f$ are involutions.

We may assume $z\in C(a)$. Then a induces an outer automorphism on $C_G(z)/O_2(C_G(z))=C\cong L_2(7)$, so all involutions in aC are fused to $aO_2(C(z))$. a

inverts an element c of order 7 in C and c acts without fixed points on $O_2(C(z))/\langle z\rangle$ so by 2.1 in [4], each involution in $C_A(z)-G$ is conjugate to a or az in $C(z)$. Finally we may choose $z\in Rt$, where $\langle t\rangle=[T, a]$. This proves (5).

$N(L)\cap C(\sigma)=\langle r\rangle\times Y$ where $\langle r\rangle=[R, \sigma]$ and $Y=\langle T^{C(\sigma)\cap L}\rangle\cong S_5$. By (5), $\sigma^g=a$, some $g\in G$, so $Y^g\leq X=\langle R^{C(a)}\rangle$.

Let $Q=\langle f, s\rangle$ be a-invariant. $C_Q(a)=\langle R, t, f\rangle\cong Z_2\times D_8$, where $\langle t\rangle=[T, a]$. In particular R is weakly closed in $C_Q(a)$. As $C_X(r^g)\cong S_5$ and R is weakly closed in a Sylow 2-group $C_Q(a)$ of $C_G(a)\cap N(R)$, Theorem 3 in [2] implies $\langle r^g, X\rangle/Z(X)\cong S_7$. As $C(R\langle a\rangle)=O^{2'}(C(R\langle a\rangle))$, $C_Q(a)\leq\langle r^g, X\rangle$, and a Sylow 3-group of $C(R\langle a\rangle)$ is of order 27, (6) follows.

(16.5) Let G be the sporadic Suzuki group Sz and let $A=Aut(G)$. Then

(1) G has two classes of involutions with representatives z and r.
(2) $C_G(z)$ is the extension of Q^3 by $\Omega_6^-(2)$.
(3) There is a standard subgroup L of G with $r\in R=O_2(C_G(L))\cong E_4$ and $L\cong L_3(4)$. $N_G(L)=RL\langle y, e\rangle$ where $\langle R, y\rangle=C_G(L)\cong A_4$, $[R, e]\neq 1$, and e induces a graph-field automorphism on L.
(4) $|A:G|=2$.
(5) There are two classes of involutions in $A-G$ with representatives σ and σr.
(6) $C_G(\sigma)\cong Aut(HJ)$.
(7) $C_G(\sigma r)\cong Aut(M_{12})$.

Proof. (1)–(3) are well known (eg. [28]).

Let $T=R^g$ be a distinct conjugate of R contained in $C_G(R)=RL$ and set $S=C_G(RT)$. Then $S\in Syl_2(RL)$ and by symmetry $S\in Syl_2(TL^g)$. Moreover $Z(S\cap L)$ is the centralizer in $Aut(L)$ of $S\cap L$. So $C_A(S)=TX$, where $X=C_A(LR)$. Then X acts on L^g and centralizes S, so $X=RC_X(TL^g)$. By [5], Sz is the unique group generated by a nonnormal standard subgroup $L\cong L_3(4)$ with $m(C(L))>1$, so $G=\langle RL, TL^g\rangle$. Hence $X=R$

Without loss choose $z\in Z(S)$ and set $H=C_G(z)$. Then $C_A(H)\leq C_A(S)=TR$, so $C_A(H)=\langle z\rangle$. Hence as $Aut(H)\cong O_6^-(2)/E_{64}$, by a Frattini argument, $|A:G|\leq 2$, with $C_A(z)\cong O_6^-(2)/Q_3$ in case of equality. An outer automorphism of G is realized in Co_1, so (4) holds.

Next $\Delta=R^G\cap RT$ is of order 4 with $N_G(\Delta)\cong S_4$. In particular if x induces an involutory automorphism on RT centralizing R then as x centralizes a member of $RT\cap L$, x centralizes a hyperplane of RT and then fixes each member of Δ. Thus $[x, RT]=1$. Similarly $[x, RT]=1$ if $[x, T]=1$, so if $[x, L]=1$ then $x\in C_A(S)=RT$. Thus $C_G(L)=C_A(L)$, so there exists $\sigma\in A$ inducing a graph automorphism on L. σ centralizes T, so $[\sigma, RT]=1$. Thus $N(L)\cap C(\sigma)\cong N(L^g)\cap C(\sigma)\cong A_4\times A_5$. Now $N_L(\sigma)$ is standard and nonnormal in $C_G(\sigma)$, so by 3.10, $K=\langle R^{C(\sigma)}\rangle\cong HJ$. e induces an outer automorphism on K, so $C_G(\sigma)\cong$

$Aut(HJ)$. Similarly $N(L) \cap C(\sigma r) \cong Z_2(E_4 \times A_5)$, so by 3.10, $C_G(\sigma r) \cong Aut(M_{12})$.

Let $J = C_A(z)$ and $Q = O_2(H)$. $J/Q \cong \Omega_6^-(2)$. Define the rank of an involution in J/Q to be the dimension of its commutator space on $Q/\langle z \rangle$. There are two classes of involutions in $J/Q - H/Q$ with rank 1 and 3 respectively. As $C(\sigma) \cap C(z) \cong S_5/Q_8 * D_8$, has σ rank 1. Moreover σ is fused to σz in $C(\sigma r)$, so all involutions in $\sigma C_Q(\sigma)$ are fused. Hence all involutions in J of rank 1 are fused to σ. Hence σr has rank 3. Thus all involutions in $\sigma r Q/\langle z \rangle$ are fused to $\langle \sigma r, z \rangle/\langle z \rangle$, and hence all involutions of rank 3 are fused to σr. This completes the proof of (5), and then of lemma 16.5.

(16.6) Let G be the Rudvalis group Ru. Then

(1) $G = Aut(G)$.
(2) G has two classes of involutions with representatives z and r.
(3) $C_G(z)$ is the extension of a group of order 2^{11} and class 3 by S_5. $\langle z \rangle = C_G(C_G(z)^\infty)$.
(4) $C_G(r) \cong E_4 \times S_z(8)$.

Proof. See [6], page 547 for (1). (2)-(4) are well known; see for example [8], page 53.

(16.7) Let G be a group of O'Nan Type, and set $A = Aut(G)$. Then

(1) $|A:G| \leq 2$.
(2) G has one class of involutions with representative z.
(3) $Z(E(C_G(z))) \cong Z_4$, $E(C_G(z))/Z(E(C_G(z))) \cong L_3(4)$, and $C_G(z) = E(C_G(z))\langle t \rangle$, where t is an involution inducing an outer automorphism on $E(C(z))$.
(4) If $A \neq G$ there is a unique class of involutions $a^G \subseteq A - G$. Further $C_G(a) \cong J_1$.

Proof. [27].

(16.8) Let G be Conway's large group Co_1. Then

(1) $G = Aut(G)$.
(2) G has 3 classes of involutions with representatives z, t, and r.
(3) $C_G(z)$ is the extension of Q^4 by $\Omega_8^+(2)$.
(4) $C_G(t)$ is the extension of $E_{2^{11}}$ by $Aut(M_{12})$.
(5) $C_G(r) = \langle s \rangle (R \times L)$ where $R \times L \cong E_4 \times G_2(4)$, and s is a conjugate of r with $[R, s] \neq 1$ and inducing an outer automorphism on L.

Proof. [28].

(16.9) Let A be quasisimple with $A/Z(A)$ isomorphic to J_1, Mc, Co_3, Ly, He, Sz, Ru, Co_1, or of O'Nan Type. Then

(1) A satisfies hypothesis II.
(2) Assume A is T admissible. Then $AC_{AT}(A)/C_{AT}(A)=\bar{A}\simeq He$, Sz, Ru, or Co_1, and $\bar{T}=O_2(C_{\bar{A}}(\bar{L}))$ where L is standard in $\bar{A}$.

Proof. We have shown $Out(A)=1$ unless $\bar{A}=A/Z(A)$ is Mc, He, Sz, or ON, in which case $\langle\bar{a}\rangle$ is Sylow in $C_{Aut(\bar{A})}(E(C(\bar{a})))$. So 2.3 implies (1).

Assume A is T-admissible. Then $\bar{T}$ centralizes $O^2(C_A(\bar{t}))^{\mathcal{A}}$ for each $t\in T^\#$. Inspecting the possible centralizers we get (2).

17. Proof of the Main Theorem

Theroem 17.1. *Assume A is standard in G with $A/Z(A)$ a sporadic group in K, and $m(C_G(A))>1$. Then $A\trianglelefteq G$.*

The proof involves several reductions.

(17.2) $A/Z(A)\not\simeq M_{12}$.

Proof. Assume $A/Z(A)\simeq M_{12}$. By Theorem 3 in [2] there is a conjugate $K^g\neq K=C_G(A)$ such that a Sylow 2-group T of $K^g\cap N(A)$ is of 2-rank at least 2. By 9.17.2, $Z(A)=1$ and $T=\langle t, b\rangle$ where $t\in A$ and b induces an outer-automorphism on A. $K\leq C(t)$, so $T\in Syl_2(K^g)$ and T centralizes a Sylow 2-group R of, K, by Theorem 2 in [1]. As the outer automorphism group is of order 2 we conclude $R\leq Z(O^2(N(R)))$. But $T\in R^G$ and there exists an involution $a\in A$ with $[a, T]\neq 1$.

(17.3) A satisfies hypothesis II.

Proof. 9.17, 10.2, 11.6, 12.8, 13.9, 14.3, 15.12, and 16.9.

With 17.3 we may adopt the notation of section 3. In particular $K=C_G(A)$, $R\in Syl_2(K)$, and $T\in Syl_2(K^g)$ with R T-invariant. By 3.9, A is T-admissible. Hence by 9.18, 11.6, 12.8, 13.9, 14.3, 15.12, and 16.9:

(17.4) $A/Z(A)\simeq M_{24}$, HJ, He, Sz, Ru, or Co_1, and T is a 4-group with its projection on $A/Z(A)$ uniquely determined up to conjugacy.

(17.5) $Z(A)$ is of odd order.

Proof. The multiplier of M_{24} and He is trivial; the 2 part of the multiplier of Ru HJ, Sz, and Co_1 is of order 2. (eg. [9], [17]). In the latter 4 cases the involutions in $A/Z(A)$, upon which the elements in $T^\#$ project, lift to elements of order 4 in a cover of $A/Z(A)$ over Z_2. This contradicts 3.5. For Co_1 this fact appears in [28], p. 15. The coverings of Sz and HJ are contained in the covering $\cdot O$ of Co_1 with the appropriate 4-groups identified, so the remark follows for Sz and HJ. For Ru we rely on a personal communication from D. Wales.

(17.6) If $A/Z(A)\cong M_{24}$, He, Sz, or Co_1, then $T\leq A$.

Proof. If $A/Z(A)\cong M_{24}$, He, Sz, or Co_1, then for $t\in T^{\sharp}$, $C_A(t)\nleq C_A(T)$, so by 2.9, $A\cap T\neq 1$. Now by 3.6, $T\leq A$.

(17.7) $A/Z(A)\ncong M_{24}$ or He

Proof. Assume $A/Z(A)\cong M_{24}$ or He. By 17.6, $T\leq A$. By 9.18 and 16.4, $T\leq C_A(T)^{\infty}$. But $N_G(T)/A^g$ is solvable, so $T\leq C(A^g)\cap A^g\leq Z(A^g)$, against 17.5.

(17.8) $T\cap A=1$.

Proof. By 17.4 and 17.7, $A/Z(A)\cong HJ$, Sz, Ru, or Co_1. By 3.6, either $T\leq A$ or $T\cap A=1$. Assume $T\leq A$. There exists $a\in A$ such that $|T^A\cap TT^a|=4$. Hence by 3.6, $T^a\cap A^g=1$. But then $T^{ag^{-1}}\cap A=1$., impossible by 3.6, since $T^{ag^{-1}}$ projects on an A-conjugate of T.

We now derive a contradiction, completing the proof of Theorem 17.1. By 17.4, 17.6, and 17.8, $A/Z(A)\cong HJ$ or Ru and $T\cap A=1$. The group V generated by a maximal set Δ of commuting conjugate of R containing R and T is of order 64 or 128 respectively. As $T\cap A=1$, Δ is of order 13 or 25, respectively and $Q^{\Delta}=O_2(N_A(V))^{\Delta}$ is elementary of order 4 or 8, respectively, and acts semiregularly on Δ. Moreover $X^{\Delta}=\langle N_A(V), N_{A^g}(V)\rangle^{\Delta}$ is 2-transitive on Δ, so by a result of Shult [29], $|\Delta|-1$ is a power of 2, a contradiction.

We have established the Main Theorem except in the case where $A/Z(A)$ is an alternating group. Here we appeal to the main theorem of [3]. Thus the proof of the Main Theorem is complete.

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