A THEOREM ON LATTICES OF A COMPLEX
SOLVABLE LIE GROUP

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(Received February 4, 1976)

1. Introduction

A discrete subgroup $\Gamma$ of a Lie group $G$ is called a lattice of $G$ if the homogeneous space $G/\Gamma$ is of finite volume. It is known that any lattice $\Gamma$ of a solvable Lie group $G$ is uniform, i.e., such that $G/\Gamma$ is compact. In this note we shall prove the following theorem.

**Theorem.** Let $G$ be a connected complex solvable Lie group and $\Gamma$ be a lattice of $G$. Suppose that $\Gamma$ is nilpotent. Then $G$ is nilpotent.

It is known that Theorem is not true in general for real solvable Lie group ([1] Chapter 3, Example 4).

2. Proof of Theorem

First we note that our theorem will be valid in general if it is proved for the case where $G$ is simply connected. In fact, let $\tilde{G}$ be the universal covering group with the projection $\pi: \tilde{G} \to G$. Then $\Gamma=\pi^{-1}(\Gamma)$ is a lattice in $\tilde{G}$ and it is nilpotent, since the kernel of $\pi$ is contained in the center of $\Gamma$. Thus $\tilde{G}$ is nilpotent by Theorem for the case where the complex solvable Lie group is simply connected, and so is $G$.

From now on assume that $G$ is simply connected. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $I$ the canonical complex structure. We denote by $\mathfrak{n}$ the maximal nilpotent ideal of $\mathfrak{g}$ regarded as real Lie algebra. Since $\mathfrak{n}$ is given by $\{X \in \mathfrak{g} \mid \text{ad}(X)\text{ is nilpotent}\}$, $\mathfrak{n}$ is invariant by $I$, so that $\mathfrak{n}$ is a complex subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^k$ denote $[\mathfrak{g}, \mathfrak{g}^k]$ where we put $\mathfrak{g}^0=\mathfrak{g}$. Then $\{\mathfrak{g}^k\}$ is a descending sequence of ideals. Put $\mathfrak{g}^\infty=\text{inf} \mathfrak{g}^k$. It is obvious that $\mathfrak{g}^\infty$ equals $\mathfrak{g}^m$ for some $m$ and is a complex subalgebra. We thus have a sequence of ideals:

$$\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}^\infty.$$  

*) Partially supported by Yukawa Foundation.
Let \( g^c \) denote the complexification of \( g \). Then \( g^c = g^+ + g^- \) (direct sum), where \( g^- = \{ X \in g^c | IX = \pm \sqrt{-1}X \} \). By Theorem of Lie, we can take a basis \( \{ X_1, \ldots, X_n \} \) of the complex solvable Lie algebra \( g^+ \) such that

1) \( \{ X_{r+1}, \ldots, X_n \} \) is a basis of \( (g^-)^+ \)
2) \( \{ X_{r+1}, \ldots, X_n \} \) is a basis of \( [g^+, g^-] \)
3) \( \{ X_{s+1}, \ldots, X_n \} \) is a basis of \( n^+ \), where \( n^+ = \{ X \in n^c | IX = \sqrt{-1}X \} \), \( n^c \) being the complex subalgebra spanned by \( n \).
4) the subspaces \( g^+_p \) \( (p=1, \ldots, n) \) spanned by \( \{ X_p, \ldots, X_n \} \) are ideals of \( g^- \).

Put \( Y_j = \frac{1}{2} (X_j + \bar{X}_j) \) for \( j=1, \ldots, n \). Then \( IY_j = \frac{\sqrt{-1}}{2} (X_j - \bar{X}_j) \) and \( \{ Y_1, IY_1, \ldots, Y_n, IY_n \} \) is a basis of \( g \) (over \( \mathbb{C} \)). Moreover, if \( g_{2j-1} \) (resp. \( g_{2j} \)) denotes the real vector space spanned by \( \{ Y_j, IY_j, \ldots, Y_n, IY_n \} \) (resp. \( \{ IY_j, Y_{j+1}, \ldots, Y_n, IY_n \} \)). Then \( g_i \) \( (i=1, \ldots, 2n) \) are subalgebras of \( g \) and \( g_{2i+1} \) is contained in \( g_i \) as an ideal. Since \( G \) is simply connected, it follows that every element \( g \in G \) can be written in one and only one way in the form

\[ g = (\exp t_1 Y_1)(\exp s_1 IY_1) \cdots (\exp t_n Y_n)(\exp s_n IY_n), \]

where \( t_j = t_j(g), s_j = s_j(g) \) \( (j=1, \ldots, n) \) are real numbers (cf. [2]). Since \( [IY_j, Y_j] = 0 \) for \( j=1, \ldots, n \),

\[ g = \exp (t_1 Y_1 + s_1 IY_1) \cdots \exp (t_n Y_n + s_n IY_n). \]

Thus we get a biholomorphic map \( \Phi: G \to \mathbb{C}^n \) defined by

\[ \Phi(g) = (t_1(g) + \sqrt{-1}s_1(g), \ldots, t_n(g) + \sqrt{-1}s_n(g)). \]

Let \( \{ 2C_{ij}^k \} \) be the structure constants of the Lie algebra \( g^+ \) with respect to the basis \( \{ X_1, \ldots, X_n \} \). Then we may regard \( \{ C_{ij}^k \} \) as the structure constants of the complex Lie algebra \( g \) with respect to the basis \( \{ Y_1, \ldots, Y_n \} \).

Note that, for \( i=s+1, \ldots, n \),

\[
\text{ad}(X_i) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & r-s & l-r & n-l \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}^t
\]

where \( A_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).
and, for \( i=1, \ldots, s \),

\[
ad(X_i) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & * & A_i & 0 \\
* & * & * & B_i
\end{pmatrix}
\]

where

\[
A_i = \begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}
\quad \text{and} \quad
B_i = \begin{pmatrix}
2C_{i+1}^{l+1} & \cdots & 0 \\
* & 2C_{i}^{l}
\end{pmatrix}.
\]

In the following, decomposition of a matrix in sixteen blocks is always taken in sizes as indicated above.

We note that \( (C_{j_1}, \ldots, C_{j_s}) \neq (0, \ldots, 0) \) for any \( j = l+1, \ldots, n \), by the definition of \( g^n \).

Since \( \text{Ad}(g) = (\exp^{\sum_{j=l}^{n} \text{ad}(Y_j)}) \cdots (\exp \text{ad}(Y_1)) \),

\[
(1) \quad \text{Ad}(g)(Y_1, \ldots, Y_n) = (Y_1, \ldots, Y_n)
\]

where

\[
B_3 = \begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}, \quad
B_7 = \begin{pmatrix}
\exp (\sum_{j=1}^{s} C_{j+1}^{l+j} z_j(g)) & \cdots & 0 \\
* & \exp (\sum_{j=1}^{s} C_{j}^{l} z_j(g))
\end{pmatrix}.
\]

Consider \( g \) as a real Lie algebra and let \( l(g) \) denote the number of eigenvalues different from 1 of \( \text{Ad}(g) \): \( g \rightarrow g \) for \( g \in G \). Define rank \( G = \sup_{g \in G} l(g) \). An element \( g \in G \) is called regular if \( l(g) = \text{rank } G \). Then it is easy to see that \( g \in G \) is regular if and only if \( \exp (\sum_{j=1}^{s} C_{j+k}^{l+j} z_j(g)) \neq 1 \) for all \( k = l+1, \ldots, n \).

**Lemma 1.** Let \( \Gamma \) be a lattice of a simply connected complex solvable Lie group \( G \). Then \( \Gamma \) contains a regular element of \( G \).

Proof. If we denote by \( N \) the connected maximal normal nilpotent Lie group of \( G \), \( N \cap \Gamma \) is a lattice of \( N \) by a theorem of Mostow ([3], [4]). Let \( \pi : G \rightarrow G/N \) be the projection. Then \( \pi(\Gamma) \) is a lattice of \( G/N \) and \( (G/N)/\pi(\Gamma) \) is a complex torus. By the definition of \( \Phi : G \rightarrow C^s \), it is obvious that \( G/N \) is biholomorphic to \( C^s \) by \( G/N \ni \pi(g) \mapsto (z_1(g), \ldots, z_s(g)) \in C^s \). We identify \( G/N \) with \( R^{2s} \) by
\[ \pi(g) = (\text{Re } z_1(g), \text{Im } z_1(g), \cdots, \text{Re } z_s(g), \text{Im } z_s(g)). \]

Consider the real subspaces \( H_k \) of codimension 1 defined by
\[ H_k = \{(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^2 | \sum_{j=1}^{s} (\text{Re } (C^j_k)x_j - \text{Im } (C^j_k)y_j) = 0 \} \]
for \( k = l+1, \cdots, n \). Since \( \pi(\Gamma) \) is a lattice of \( \mathbb{R}^s \), there are infinitely many different real subspaces of codimension 1 which are generated by \( 2s-1 \) lattice points of \( \pi(\Gamma) \). Hence, there exists a point \( \gamma \in \Gamma \) such that \( \pi(\gamma) \notin H_k \) for \( k = l+1, \cdots, n \). Then \( |\exp(\sum_{j=1}^{s} C^j_k(z_j(\gamma)))| \neq 1 \) for all \( k = l+1, \cdots, n \) and \( \gamma \in \Gamma \) is a regular element of \( G \).

**Lemma 2.** (Mostow) Let \( G \) be a simply connected solvable Lie group and \( \Gamma \) a uniform subgroup of \( G \) containing a regular element. Let \( G^\omega \) denote the connected Lie subgroup of \( G \) corresponding to \( \mathfrak{g}^\omega \). Then \( G^\omega \cap \Gamma \) is uniform in \( G^\omega \).

**Proof.** See [3] Lemma 5.

Proof of Theorem. Suppose that \( G \) is not nilpotent. Then \( G^\omega \neq \{e\} \). Since \( G^\omega \) is a simply connected nilpotent Lie group, \( G^\omega \cap \Gamma \neq \{e\} \) by Lemma 2. Since \( \Gamma \) is nilpotent, \( G^\omega \cap \Gamma \) contains a non-trivial element of the center \( C \) of \( \Gamma \). Choose an element \( \gamma \neq e \) of \( G^\omega \cap \Gamma \cap C \). We can write \( \gamma \) uniquely as
\[ \gamma = (\exp z_{l+1}Y_{l+1}) \cdots (\exp z_{n}Y_{n}) \]
where \( (z_{l+1}, \cdots, z_{n}) \in \mathfrak{C}^{n-l} \).

Note that \( ad(Y_j) \) is represented by the basis \( \{Y_1, \cdots, Y_n\} \) as follows:
\[ ad(Y_j) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_j & B_j & C_j & D_j \end{bmatrix} \quad \text{for } j = l+1, \cdots, n \]
where
\[ A_j = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ C_{j_1}^{l_1} & \cdots & C_{j_s}^{l_s} \end{bmatrix} < j-l, \quad B_j = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \end{bmatrix} < j-l, \quad C_j = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \end{bmatrix} < j-l, \quad D_j = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \end{bmatrix} < j-l. \]
Fix a $j = l + 1, \ldots, n$ and put $\delta_j = (\exp z_1 Y_1) \cdots (\exp z_n Y_n)$. Then $Ad(\delta_j) = (\exp z_1 ad(Y_1)) \cdots (\exp z_n ad(Y_n))$ is written as follows:

$$Ad(\delta_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ P_j & Q_j & R_j & S_j \end{pmatrix}$$

where

$$P_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad C_j \quad \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix}$$

$$Q_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad S_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We claim that if $\gamma_0 \delta_j = \delta_j \gamma_0$ for a regular element $\gamma_0 \in \Gamma$, then $z_j = 0$. Put

$$Ad(\gamma_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix}.$$
Consider the \((j-l, k)\)-component of both hands of (3), by (1) we get
\[
\exp \left( \sum_{i=1}^{s} C_{i}^{j}, x_{0}^{j} \right) C_{j}^{j} z_{j} = C_{j}^{j} z_{j}
\]
for \(k=1, \ldots, s\). Since \(\gamma_{0}\) is a regular element of \(G\), \(\exp \left( \sum_{i=1}^{s} C_{i}^{j}, x_{0}^{j} \right) \neq 1\) and \(C_{j}^{j} z_{j} = 0\) for \(k=1, \ldots, s\). Thus \(z_{j} = 0\), since \((C_{j}^{j}, \ldots, C_{j}^{j}) = (0, \ldots, 0)\).

Now, starting with \(j=l+1\), we get \(z_{j} = 0\) successively for all \(j=l+1, \ldots, n\). This contradicts our assumption \(\gamma \neq e\). Hence, \(G\) is nilpotent, and this proves our Theorem.

**Remark.** The special case of our Theorem has been proved in a stronger form in the section 2 of [5].

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**References**


