SOME REMARKS ON DEGENERATE CAUCHY PROBLEMS IN GENERAL SPACES

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1. Introduction. We will consider problems of the form

\[ u'' + s(t)u' + Ar(t)u - A^2 a(t)u + b(t)u = f \]
\[ u(0) = u'(0) = 0 \]

where \( A \) is the generator of a locally equicontinuous group \( T(t) \) in a complete separated locally convex space \( E \) (cf. [8; 14]), \( u \in C^\infty(E) \), \( f \in C^\infty(E) \), \( s \), \( r \), \( a \), and \( b \) are continuous real valued functions, while \( a(t) > 0 \) for \( t > 0 \) with \( a(0) = 0 \). This is an extension of the Cauchy problem for Tricomi equations and various general versions of (1.1)-(1.2) have been considered for example in [1; 2; 7; 8; 10; 15; 16; 18; 22; 23; 24]; for an extensive bibliography see [8]. We will adapt a method of Hersh [13] as extended by the author in [4, 5; 6; 8], to solve (1.1)-(1.2) and prove some uniqueness theorems. The behavior of \( \int_{\tau}^{T} (r^2/a) (\xi) d\xi \) as \( \tau \to 0 \) again turns out to play a critical role in uniqueness (as in [7; 8; 23; 24]) and is related to conditions of Krasnov [15] and Protter [18] in their specific contexts. Let us note that a typical case involves \( A^2 = \Delta \) in a suitable space \( E \) (cf. [8]).

2. Following [4; 5; 6; 8; 13] we replace \( A \) by \(-d/dx\) in (1.1) and consider

\[ w'' + s(t)w' - r(t)w - a(t)w_{xx} + b(t)w = 0 \]

where \( w(t) \in \mathcal{G}_x \) (detailed properties are indicated below). Let us Fourier transform (2.1) in the \( x \) variable, writing formally \( \hat{w}(t) = \mathcal{F}w(t) = \int_{-\infty}^{\infty} w(t) \exp ixy \ dx \), to obtain

\[ i\hat{w}'' + s(t)i\hat{w}' + iyr(t)i\hat{w} + a(t)y^2i\hat{w} + b(t)i\hat{w} = 0 \]

It will be convenient to eliminate the \( b(t) \) term as follows. Let \( \hat{\omega}(t) = \hat{\omega}(t) \exp \int_{0}^{t} \gamma(\xi)d\xi \) where \( \gamma(t) \) satisfies the Riccati equation...
(2.3) \[ \gamma' + \gamma + \gamma^2 + b = 0; \quad \gamma(0) = 0 \]

(see below for details). Then \( \theta \) satisfies

(2.4) \[ \theta'' + (2\gamma(t) + s(t))\theta' + (a(t)y^2 + iy\tau(t))\theta = 0 \]

and it will be easier to deal with (2.4). In order to produce a suitable function \( \gamma(t) \) we note that if one sets \( \gamma = \alpha'/\alpha \) then \( \alpha \) satisfies

(2.5) \[ \alpha'' + s(t)\alpha' + b(t)\alpha = 0 \]

(cf. [12]) and we choose \( \alpha \) to be the unique solution of (2.5) satisfying \( \alpha(0) = 1 \) with \( \alpha'(0) = 0 \). Then \( \gamma(0) = 0 \) and the continuous function \( \gamma \) will remain finite on some interval \( 0 \leq t \leq T < t_0 < \infty \) where \( t_0 \) is the first zero of \( \alpha(t) \). It is sufficient for us to solve (1.1) on such an interval since for \( t \geq T \) the equation (1.1) is not degenerate and can be handled by standard techniques (cf [3; 17]).

Now following [11] we write (2.4) as a system

(2.6) \[ \begin{align*}
\dot{\theta}(t) &= P(y, t)\theta(t); \\
\dot{P}(y, t) &= \begin{bmatrix}
0 & y \\
-ir - ay & -s - 2\gamma
\end{bmatrix}
\end{align*} \]

where \( \theta_1 = y\theta \) and \( \theta_2 = \theta' \). We look for solutions \( \hat{Y} \) and \( \hat{Z} \) of (2.6) satisfying

(2.7) \[ \begin{align*}
\hat{Y}(\tau) &= \begin{bmatrix}
0 \\
Y_t
\end{bmatrix}, \\
\hat{Z}(\tau) &= \begin{bmatrix}
y \\
Z_t
\end{bmatrix}
\end{align*} \]

where \( 0 \leq \tau \leq t \leq T \). The functions \( \hat{Z}(t, \tau, y) \) and \( \hat{Y}(t, \tau, y) \), together with their inverse Fourier transforms, will be called resolvants. It is easily shown following [7; 8; 19] that

(2.8) \[ \hat{Z}_c = (ay^2 + iy\tau)(\tau)\hat{Y} \]

(2.9) \[ \hat{Y}_c = -\hat{Z} + (s + 2\gamma)(\tau)\hat{Y} \]

Now by well known theorems (cf [3; 9; 12]) there exist solutions \( \hat{Y}(t, \tau, y) \) and \( \hat{Z}(t, \tau, y) \) of (2.4) (i.e. (2.6)), satisfying the prescribed initial conditions, which are continuous in \( (t, \tau, y) \) and analytic in \( y \) for \( 0 \leq \tau \leq t \leq T < \infty \) and \( y \in \mathbb{C} \). Moreover by a clever argument in [11] if one writes the solution of (2.6) in the form

(2.10) \[ \hat{\theta}(t, \tau, y) = \hat{Q}(t, \tau, y)\hat{\theta}(\tau, \tau, y) \]
where $Q(\tau, \tau, y)=I$ then $||Q(t, \tau, y)|| \leq |c| \exp \varepsilon |y| (t-\tau)$ where $|| \cdot ||$ denotes the matrix operator norm (so $|q_{ij}| \leq ||Q||$ in particular when $Q=(q_{ij})$). Thus the entries in $Q$ are entire analytic functions of $y$ of exponential type $\leq \varepsilon(t-\tau) \leq \varepsilon T$. This proves

**Lemma 2.1.** The functions $\hat{Y}(t, \tau, y)$ and $\hat{Z}(t, \tau, y)$ are continuous in $(t, \tau, y)$ for $0 \leq \tau \leq t \leq T$ and $y \in \mathbb{C}$ while, for $(t, \tau)$ fixed, $y \hat{Y}$, $y \hat{Z}$, $\hat{Y}_t$, and $\hat{Z}_t$ are entire analytic functions of exponential type $\leq \varepsilon T$.

In order to invoke the Paley-Wiener-Schwartz theorem later (cf. [8; 11; 20]) we examine the growth of $\hat{Y}$, $\hat{Z}$, etc. for real $y$. Thus writing first $\hat{Y}=\varphi + i\psi$ we obtain from (2.4)

\begin{align}
\varphi'' + (2\gamma+s)\varphi' + ay^2\varphi - yr\psi &= 0; \\
\psi'' + (2\gamma+s)\psi' + ay^2\psi + yr\varphi &= 0.
\end{align}

Multiply the first equation in (2.11) by $\varphi'$ and the second by $\psi'$ and add, observing that $\hat{Y} \hat{Y}' = \varphi\varphi' + \psi\psi' + i(\varphi\psi' - \psi\varphi')$ for example so that in particular

\[ d/dt |\hat{Y}|^2 = 2 \text{Re} \hat{Y} \hat{Y}' = 2(\varphi\varphi' + \psi\psi') \text{ while } |yr(\varphi\psi' - \psi\varphi')| = |yr| \text{ Im } \hat{Y} \hat{Y}' \leq \sqrt{2}(y^2r^2) |\hat{Y}^2 + |\hat{Y}'|^2|.
\]

This yields then

\begin{align}
\frac{d}{dt} |\hat{Y}'|^2 + 2(2\gamma+s) |\hat{Y}'|^2 + ay^2 \frac{d}{dt} |\hat{Y}|^2 &\leq \\
(y^2r^2) |\hat{Y}'|^2 + |\hat{Y}'|^2
\end{align}

Integrating (2.12) now under the assumption that $a \in C^1$ we obtain for $0 < \tau \leq t \leq T$

\begin{align}
|\hat{Y}'|^2 + 2\int_{\tau}^{t} (2\gamma+s) |\hat{Y}'|^2 d\xi + a(t)y^2 |\hat{Y}|^2 &\leq \\
1 + \int_{\tau}^{t} [(a'y^2 + y^2\gamma)] |\hat{Y}|^2 + |\hat{Y}'|^2 d\xi
\end{align}

where $\hat{Y}=\hat{Y}(\xi, \tau, y)$ etc. in the integrations. This type of inequality can be treated by use of Gronwall type lemmas as in [7; 8; 23]. Thus set $P=a'y^2 + y^2\gamma$ and $Q=1-2(2\gamma+s)$ so that $|Q| \leq \varepsilon$ on $[0, T]$ by the continuity of $\gamma$ and $s$. Then add $\varepsilon \int_{\tau}^{t} a'y^2 |\hat{Y}'|^2 d\xi$ to the right side of (2.13), without changing the inequality, and setting $\Xi=|\hat{Y}'|^2 + ay^2 |\hat{Y}|^2$ we have

\[ \Xi \leq 1 + \int_{\tau}^{t} P |\hat{Y}'|^2 d\xi + \varepsilon \int_{\tau}^{t} \Xi d\xi \]

A straightforward application of the Gronwall lemma (cf. [3]) yields

\begin{align}
\Xi &\leq B(t, \tau) + \int_{\tau}^{t} P |\hat{Y}'|^2 E(t, \xi) d\xi
\end{align}
where $E(t, \xi) = \exp (t - \xi)$. Now forget the $|\dot{Y}|^2$ term in $\Xi$ and following a Gronwall type procedure written out in [8] we get immediately from (2.15) for $P \geq 0$

$$\dot{Y}^2 \leq E(t, \tau) \exp \int_{\tau}^{t} \hat{P} d\xi$$

where $\hat{P} = a' + r^2/a$. Integrating the $a'/a$ term and rearranging these results

**Lemma 2.2.** Given $a \in C^1, b, r, s \in C^0, \hat{P} \geq 0,$ and $\dot{Y}$ the solution of (2.4) satisfying $\dot{Y}(\tau, \tau, y) = 0$ with $\dot{Y}(\tau, \tau, y) = 1$ it follows that

$$a(\tau) \dot{Y}(t, \tau, y)^2 \leq E(t, \tau) \exp \int_{\tau}^{t} (r^2/a) d\xi$$

for $y$ real and $0 < \tau \leq t \leq T$.

Let now $F(t, \tau) = \exp \left( -\int_{\tau}^{t} (r^2/a) d\xi \right)$ and $F(\tau) = F(T, \tau)$ so $F(\tau) \leq F(t, \tau)$. Then since $E(t, \tau) \leq \exp \varepsilon T = k$ we have from (2.17) the inequality

$$a(\tau) F(\tau) \dot{Y}(t, \tau, y)^2 \leq k.$$  

Note that $F(\tau)$ may tend to zero as $\tau \to 0$ while $a(\tau) \to 0$ by assumption, but for $\tau > 0$ both $F(\tau)$ and $a(\tau)$ are positive. Similarly, as in [2], we obtain from (2.14)-(2.16)

$$|\dot{Y}(t, \tau, y)|^2 a(\tau) F(\tau) \leq k$$

where $k = \max a(t)$ on $[0, T]$, and going back to (2.4) we have for $Q(\tau) = (a(\tau) F(\tau))^{1/2}$

$$Q(\tau) \dot{Y}(t, \tau, y) \leq 2\gamma(t) + s(t) \left| Q(\tau) \right| \dot{Y} + \left( |yr(t)| + a(t) \gamma \right) Q(\tau) \left| \dot{Y} \right| + k_1 + k_2 |y|$$

(upon using (2.18)-(2.19) and the continuity of $a, r, s, \text{and} \gamma$). Next, setting $\hat{W}(t, \tau, y) = Q(\tau) \dot{Y}(t, \tau, y)$, from Lemma 2.1 and the estimate (2.18) arising from Lemma 2.2 we know that the functions $y \to \hat{W}(t, \tau, y)$ are entire of exponential type $\leq \varepsilon T$ and are bounded uniformly by a constant for $y$ real and $0 \leq \tau \leq t \leq T$. Further we know that the $\hat{W}(t, \tau, \cdot)$ are analytic in the same region (note that the $Q(\tau)$ factor arising from (2.18) is only needed to produce a uniform bound for $y$ real as $\tau \to 0$—the function $\hat{W}(t, \tau, y)$ is continuous in $(t, \tau, y)$ for $0 \leq \tau \leq t \leq T$ and $y \in \mathcal{C}$). Writing $\dot{Y}(t, \tau, y) = \sum_{n=0}^{\infty} a_n(t, \tau) y^n$ we have $y \dot{Y}(t, \tau, y) = \sum_{n=0}^{\infty} a_n(t, \tau) y^{n+1}$

$$= \sum_{n=1}^{\infty} a_{n-1} y^n$$

and by definition one has then $1 = \lim sup k \log k / -\log |a_{n-1}|$ as $k \to \infty$ (cf. [8; 20]). Consequently we can write $\lim sup (n+1)\log(n+1) / -\log |a_n| = 1$ which implies $\lim sup n \log n / -\log |a_n| = 1$ so $\dot{Y}(t, \tau, \cdot)$ is of exponential
type along with \( y \dot{Y}(t, \tau, \cdot) \). Further, since the type of such a function \( g(y) \) is

defined by \( \limsup \log |g(y)| / |y| \) as \( |y| \to \infty \), we see from \( \limsup \log |yg(y)| / |y| = \limsup (\log |y| + \log |g(y)|) / |y| = \limsup \log |g(y)| / |y| \) that the functions
\( \dot{Y}(t, \tau, \cdot) \) are also of exponential type \( \leq cT \) for \( 0 \leq \tau \leq t \leq T \). Now for \( y \) real with \( |y| \leq R_0 \) say \( |T^{(t, r, \tau, y)}| \) is bounded by continuity in \((t, \tau, y)\) and by (2.18)
\[
|T(t, \tau, y)| \leq k^{1/2} |y|
\]
bounded for \( |y| > R_0 \). From the Paley-Wiener-Schwartz theorem it then follows that \( W(t, \tau, \cdot) \in \mathcal{E}'_s \) with supp \( W \) contained in a fixed compact set for \( 0 \leq \tau \leq t \leq T \). Similar conclusions apply to \( W_t \) and \( W_{tt} \) from Lemma 2.1, (2.4), and the estimates (2.19)-(2.20). Reasoning as in [8] one can verify that \( W_t \) and \( W_{tt} \) indeed represent the derivatives of \( W \) in \( \mathcal{E}_s' \) and we can state

**Theorem 2.3.** Let the hypotheses of Lemma 2.2 hold with \( Q(\tau) = (a(\tau)F(\tau))^{1/2} \) where \( F(\tau) = \exp(-\int_0^\tau (r^2/|a|) d\xi) \) and set \( \hat{W}(t, \tau, y) = Q(\tau) \dot{Y}(t, \tau, y) \) where \( \dot{Y} \) is the unique solution of (2.4) satisfying \( \dot{Y}(\tau, \tau, y) = 0 \) and \( \dot{Y}_t(\tau, \tau, y) = 1 \). Then \( W = \mathcal{F}^{-1} \hat{W} = W_t, W_{tt} \) and have supports contained in a fixed compact set for \( 0 \leq \tau \leq t \leq T \). Moreover \( (t, \tau) \rightarrow W(t, \tau) \in C_c^0(]0, T[) \) and, for any continuous seminorm \( p \) on \( E \), there is a continuous seminorm \( q \) such that \( p(T(x)e) \leq q(e) \) for \(|x| \leq x_1 \) suitably large and \( e \in E \). The operation \( \langle \cdot, \cdot \rangle \) indicates a pairing between distributions \( S \in \mathcal{E}_s' \) of order \( \leq 2 \) with supp \( S \subset K \) compact and functions \( g \in C^0_c(E) \) on \( R \) (recall here that \( T(x) \) is a group). We define this situation we can think of \( K \subset \bar{K} = \{ x \mid |x| \leq x_0 \} \) and represent \( C^0_c(E) \) on \( \bar{K} \) as \( C^0_c(K) \) (cf. [4; 5; 21]) for details in the present discussion). Then \( S \in C^0_c(K) \) and the pairing \( \langle S, g \rangle \) is well defined with \( S \in \mathcal{E}_s' \) of order \( \leq 2 \) with supp \( S \subset K \) compact and functions \( g \in C^0_c(E) \) on \( K \) (cf. [4; 5; 21]). Given this situation we can think of \( K \subset \bar{K} = \{ x \mid |x| \leq x_0 \} \) and represent \( C^0_c(E) \) on \( \bar{K} \) as \( C^0_c(K) \) (cf. [4; 5; 21]) for details in the present discussion). Then \( S \in C^0_c(K) \) and the pairing \( \langle S, g \rangle \) is well defined with \( S \in \mathcal{E}_s' \) of order \( \leq 2 \) with supp \( S \subset K \) compact and functions \( g \in C^0_c(E) \) on \( K \) (cf. [4; 5; 21]). The map \( \Delta = \Delta \otimes 1 = d^2/|dx^2| \cdot 1: C^0_c(E) \to C^0_c(E) \) is defined by extension from \( C^0_c(\mathbb{R}) \to C^0_c(E) \) and is continuous; it can be transported around under \( \langle \cdot, \cdot \rangle \) in a distribution sense for suitable \( S \) and \( g \) as above (i.e. \( \Delta S, g \rangle = \langle S, \Delta g \rangle \) for \( S \) of order zero, the bracket for \( \langle S, \Delta g \rangle \) being defined in the same way). We remark that in fact \( (S, g) \to \langle S, g \rangle : \mathcal{E}_s \times C^0_c(E) \to E \) is easily seen to be separately continuous for \( S \) restricted as indicated and since \( \mathcal{E}_s \) is barreled \( (S, g) \to \langle S, g \rangle \) will be hypocontinuous on bounded sets in \( C^0_c(E) \). Consider then for \( \tau > 0 \)

\[
(3.1) \quad u(t) = \int_0^t \langle W(t, \xi, \cdot), T(\cdot) h(\xi) \rangle d\xi
\]

We calculate formally in remarking that all the operations are legitimate. First
(3.2) \[ u'(t) = \int W(t, \xi, \cdot), T(\cdot)h(\xi) \, d\xi \]

since \( W(t, t, \cdot) = 0 \) and since \( W'_t(t, t, \cdot) = Q(t) \delta \) there results

(3.3) \[ u''(t) = f(t) + \int W(t, \xi, \cdot), T(\cdot)h(\xi) \, d\xi \]

Now look at our new version of (1.1) and observe that for example

(3.4) \[ Au(t) = \int \langle W(t, \xi, \cdot), AT(\cdot)h(\xi) \rangle \, d\xi \]

\[ = \int \langle W(t, \xi, \cdot), \frac{d}{d\xi} T(\cdot)h(\xi) \rangle \, d\xi \]

\[ = -\int \langle \frac{d}{d\xi} W(t, \xi, \cdot), T(\cdot)h(\xi) \rangle \, d\xi \]

Similarly \( A^2u(t) = \int \langle \Delta W(t, \xi, \cdot), T(\cdot)h(\xi) \rangle \, d\xi \) where \( \Delta = d^2/dx^2 \). Putting \( u(t) \), defined by (3.1), in the modified equation (1.1) we obtain

(3.5) \[ u'' + s(t)u' + Ar(t)u - A^2a(t)u = f(t) \]

\[ + \int \langle W_{tt} + s(t)W_t - r(t) \frac{d}{dx} W - a(t)\Delta W, Th \rangle \, d\xi \]

and the integral term vanishes because \( W_t \), along with \( Y \), satisfies the correspondingly modified equation (2.1). There is no trouble now in passing to the limit \( \tau = 0 \) under our hypotheses and, using \( \gamma \) to transform back to the original equation (1.1), we have proved

**Theorem 3.1.** Let \( a(t) > 0 \) for \( t > 0 \) with \( a(0) = 0 \) and \( a \in C^1 \); let \( b, \gamma, \) and \( \tau \) belong to \( C^0, \hat{P} \geq 0 \) and choose \( T \) as in (2.3)–(2.4); let \( Q \) and \( F \) be defined as in Theorem 2.3 and assume \( h(\cdot) = f(\cdot)Q(\cdot) \in C^0(E) \) on \([0, T]\) with \( Ah(\cdot) \) and \( A^2h(\cdot) \in C^0(E) \) on \([0, T] \), where \( A \) generates a locally equicontinuous group \( T(x) \) in \( E \).

Then, after modification by a factor \( \exp \int_0^\tau \gamma(\xi) \, d\xi \), \( u(t) \) given by (3.1) with \( \tau = 0 \) is a solution of (1.1)–(1.2) on \([0, T]\).

4. We go now to questions of uniqueness and will have to determine some properties of the other resolvent \( \tilde{Z}(t, \tau, y) \). First we duplicate our procedure (2.11)–(2.12) in order to estimate \( |\tilde{Z}| \) and \( |\tilde{Z}_t| \) for \( y \) real. This yields

(4.1) \[ \frac{d}{dt} |\tilde{Z}'|^2 + 2s(t) |\tilde{Z}'|^2 + a(t) y^2 \frac{d}{dt} |\tilde{Z}'|^2 \]

\[ \leq y^2 r^2(t) |\tilde{Z}'|^2 + |\tilde{Z}'|^2 \]

(4.2) \[ |\tilde{Z}'|^2 + 2 \int_s(\xi) |\tilde{Z}'|^2 d\xi + a(t) y^2 |\tilde{Z}'|^2 \]
We will develop now a uniqueness procedure based on [6; 8] which uses the following formal calculations, valid for $\tau > 0$. Define first

\begin{align}
R(t, \xi) &= \langle Z(t, \xi, \cdot), T(\cdot)u(\xi) \rangle; \\
S(t, \xi) &= \langle Y(t, \xi, \cdot), T(\cdot)u'(\xi) \rangle
\end{align}

where $u$ is any solution of our modified equation (1.1) (i.e. $s(t)$ is replaced by $s(t) = s(t) + 2\gamma(t)$ and $b(t) = 0$) with $f = 0$. For $\tau > 0$, $Y$, $Z$, $Y'$, and $Z'$ belong to $\mathcal{C}_x$ with supports contained in a fixed compact set so (4.9) makes sense, as do the following computations (cf. (2.8)-(2.9)), but we will mercifully omit detailed examination of each step. Thus

\begin{align}
R_t &= \langle Z_t, Tu' \rangle + \langle Z, Tu' \rangle = \langle Z, Tu' \rangle \\
&\quad - \langle a(\xi)\Delta Y, Tu' \rangle - \langle \tau(\xi) \frac{d}{dx} Y, Tu \rangle = \langle Z, Tu' \rangle \\
&\quad + \langle Y, \tau(\xi)ATu \rangle - \langle Y, a(\xi)Au' \rangle
\end{align}

\begin{align}
S_t &= \langle Y_t, Tu' \rangle + \langle Y, Tu'' \rangle = \langle Y, Tu'' \rangle \\
&\quad - \langle Z, Tu' \rangle + \langle s(\xi)Y, Tu' \rangle = \langle Y, Tu'' \rangle \\
&\quad + \langle Y, s(\xi)Tu' \rangle - \langle Z, Tu' \rangle.
\end{align}

Letting $\varphi(t, \xi) = R(t, \xi) + S(t, \xi)$ we have from (4.10)-(4.11)

\begin{align}
\varphi_t &= \langle Y, T(u'' + \dot{s}u' + rAu - aAu) \rangle = 0.
\end{align}

Consequently $\varphi(t, t) = \varphi(t, \tau)$ which implies that

\begin{align}
u(t) &= \langle Z(t, \tau, \cdot), T(\cdot)u(\tau) \rangle \\
&\quad + \langle Y(t, \tau, \cdot), T(\cdot)u'(\tau) \rangle = \langle F^{1/2}(\tau)Z(t, \tau, \cdot), T(\cdot)F^{-1/2}(\tau)u(\tau) \rangle \\
&\quad + \langle Q(\tau)Y(t, \tau, \cdot), T(\cdot)Q^{-1}(\tau)u'(\tau) \rangle
\end{align}

Now let $\tau \to 0$ and if $F^{-1/2}(\tau)u(\tau)$ and $Q^{-1/2}(\tau)u'(\tau) \to 0$ as $\tau \to 0$ we have $u(t) \equiv 0$. Hence, referring back to the original equation (1.1) via $\gamma$ as before we have proved

**Theorem 4.3.** Let $u$ satisfy (1.1) (modified) under the stipulations that $F^{-1/2}(\tau)u(\tau) \to 0$ and $Q^{-1/2}u'(\tau) \to 0$ as $\tau \to 0$. Assume the hypotheses of Lemma 2.2. Then $u$ is unique.

**Remark 4.4.** The condition $\dot{P} \geq 0$ has been discussed in [7; 8; 23; 24].

In general the requirements of Theorem 4.3 regarding the growth of $u(\tau)$ and $u'(\tau)$ as $\tau \to 0$ are too strong (cf. [7]) although the solution $u$ of (1.1) given by (3.1) could be made to satisfy them by imposing further hypotheses on $f$. It is therefore of some interest to consider the case when $F(\tau) \to 0$ as $\tau \to 0$ and the relation of this to certain conditions of Krasnov [15] and Protter [18] has been
\begin{align*}
& \leq a(\tau)y^2 + \int_\tau^t \left( |(a'y^2 + y^2r^2)| \dot{Z}|^2 + |\dot{Z}'|^2 \right) d\xi.
\end{align*}

Setting \( P = a'y^2 + y^2r^2 \) as before and \( Q = 1 - 2\xi \) with \(|\dot{Q}| \leq \tilde{c} \) on \([0, T]\), we write
\begin{align*}
\Xi &= |\dot{Z}'|^2 + ay^2|\dot{Z}|^2 \\
\text{and add } \tilde{c} \int_\tau^t ay^2|\dot{Z}|^2 d\xi
\end{align*}
to the right side of (4.2) to obtain
\begin{equation}
\Xi \leq a(\tau)y^2 + \int_\tau^t P|\dot{Z}|^2 d\xi + \tilde{c} \int_\tau^t \Xi d\xi
\end{equation}
Consequently as in (2.15) there results
\begin{equation}
\Xi \leq a(\tau)y^2 E(t, \tau) + \int_\tau^t P|\dot{Z}|^2 d\xi + \tilde{c} \int_\tau^t \Xi d\xi
\end{equation}
and as in (2.16) we obtain
\begin{equation}
a(t)y^2|\dot{Z}|^2 \leq a(\tau)y^2 E(t, \tau) \exp \int_\tau^t \dot{P} d\xi
\end{equation}
which yields

**Lemma 4.1.** Given the hypothesis of Lemma 2.2 on a, b, r, s, \( \dot{P} \), with \( \hat{Z}(t, \tau, y) \) the unique solution of (2.4) satisfying \( \hat{Z}(\tau, \tau, y) = 1 \) and \( \dot{Z}(\tau, \tau, y) = 0 \) it follows that for \( y \) real and \( 0 \leq \tau \leq t \leq T \)
\begin{equation}
|\hat{Z}(t, \tau, y)|^2 \leq E(t, \tau) \exp \int_\tau^t (r^2/\alpha) d\xi
\end{equation}
which can be written as \( F(\tau)|\hat{Z}(t, \tau, y)|^2 \leq E(t, \tau) \).

Similarly, as in (2.19)-(2.20), we could estimate \(|\dot{Z}|\) and \(|\ddot{Z}|\) but this will not be needed here. Instead we want estimates on \( \ddot{Y} \) and \( \dddot{Z} \) which will follow from (2.8)-(2.9). Thus, from (2.8) one obtains, using (2.18),
\begin{equation}
|Q(\tau)\dddot{Z}| \leq \tilde{k} + \hat{k}_1 |y|
\end{equation}
while, using (2.18) and (4.6), we get from (2.9)
\begin{equation}
|yQ(\tau)\dddot{Y}| \leq \tilde{k}_2 + \hat{k}_3 |y|
\end{equation}
From their expressions (2.8)-(2.9) (and reasoning about \( \hat{Z} \) from Lemma 2.1 as was done for \( \hat{Y} \) before Theorem 2.3) we know that \( \hat{Y} \) and \( \hat{Z} \) are entire functions in \( y \) of exponential type \( \leq \epsilon T \). The estimates (4.7)-(4.8) and an argument as in Theorem 2.3 then proves (cf. Lemma 4.1)

**Theorem 4.2.** Under the hypothesis of Theorem 2.3, \( F^{1/2}(\tau)Z = F^{1/2}(\tau)\mathcal{F}^{-1}Z \), \( Q(\tau)Z \) (and \( Q(\tau)Z \)), and \( Q(\tau)Y \) belong to \( \mathcal{E}_s ' \) with supports contained in a fixed compact set for \( 0 \leq \tau \leq t \leq T \). The derivatives in \( \tau \) can be taken in \( \mathcal{E}_s ' \) for \( \tau > 0 \) and \( (t, \tau) \rightarrow F^{1/2}Z \) or \( QZ \), \( QZ \), and \( Q(\tau)Y \) are continuous with values in \( \mathcal{E}_s ' \).
discussed in [7; 8]. In this event the requirements of Theorem 4.3 on \( u \) are only that \( u(0)=0 \) and \( a^{-1/2}(\tau)u'(\tau)\to 0 \) as \( \tau \to 0 \). To examine the feasibility of this let \( u \) satisfy the modified equation (1.1) with \( f=0 \), \( u(0)=0 \), and \( u'(0)=0 \). Multiply this equation by \( \exp \int_0^t \delta(\xi) d\xi \) and integrate to obtain (cf. [7; 8])

\[
(4.14) \quad u'(t) = -\int_0^t [Ar(\xi)u - A^2 a(\xi)u] e^{-\int_0^\tau \delta(\eta) d\eta} d\xi .
\]

Let \( p \) be any continuous seminorm in \( E \) so that, since \( \exp(-\int_0^t \delta(\eta) d\eta) \leq M \) on \([0, T]\),

\[
(4.15) \quad p(u'(t)) \leq \int_0^t [r(\xi)p(Au) + a(\xi)p(A^2u)] M d\xi .
\]

Now \( \int_0^t r(\xi)d\xi = \int_0^t a^{1/2}(r/a^{1/2}) d\xi \leq \left( \int_0^t a(\xi)d\xi \right)^{1/2} (\int_0^t (r/a)d\xi)^{1/2} \), whereas \( \int_0^t a(\xi)d\xi = \left( \int_0^t a(\xi)d\xi \right)^{1/2}/2 \). Since \( p(Au) \) and \( p(A^2u) \) will be bounded for a solution \( u \in C^2(E) \) on \([0, T]\) we have for \( \int_0^t (r/a)d\xi \) bounded

\[
(4.16) \quad p(a^{-1/2}(t)u'(t)) \leq a^{-1/2}(t)p(u'(t)) \leq M a^{-1/2}(t) \left( \int_0^t a d\xi \right)^{1/2} + M a^{-1/2}(t) \int_0^t a d\xi \]

\[
\leq M a^{-1/2}(t) \left( \int_0^t a d\xi \right)^{1/2}
\]

Hence \( a^{-1/2}(t)u'(t) \to 0 \) if \( a^{-1/2}(t) \left( \int_0^t a d\xi \right)^{1/2} \to 0 \). This condition is examined in [7; 8; 23; 24] and since oscillations in \( a(t) \) are permitted by the stipulation \( \hat{p}_t \geq 0 \) (or \( a' \geq -r^2 \)) it is not automatically satisfied. However if \( a \) is monotone increasing near \( t=0 \) it is obviously valid since \( \left( \int_0^t a d\xi \right)^{1/2} \leq a(t)^{1/2} t^{1/2} \). Thus it makes sense to state the result (after modification) as

**Theorem 4.5.** Assume the hypothesis of Lemma 2.2 and suppose \( F(\tau) \geq 0 \) on \([0, T]\) with \( a^{-1/2}(t) \left( \int_0^t a(\xi)d\xi \right)^{1/2} \to 0 \) as \( t \to 0 \). Then \( a^{-1/2}(\tau)u'(\tau) \to 0 \) as \( \tau \to 0 \) and if \( u \) satisfies (1.1)–(1.2) with \( f=0 \) it follows that \( u(t) \equiv 0 \) on \([0, T]\).

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References