Rego, C.J.
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REMARKS ON DIFFERENTIAL OPERATORS ON ALGEBRAIC VARIETIES

C. J. REGO

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We will work over an algebraically closed field of characteristic zero. Let A be a (commutative) k-algebra and $I\subset A\otimes_k A$ the 'diagonal ideal', i.e., the ideal generated by $z\otimes 1-1\otimes z$, $z\in A$. We consider $A\otimes_k A$ as an A-module via

$$a(x\otimes y) = ax\otimes y$$

Recall that the A-module of n-jets (over k), $P^n(A|k)\overset{df}{=\!=} A\otimes_k A/I^{n+1}$ is of finite type when A is a localization of a k-algebra of finite type. Note that there is a splitting

$$P^n(A|k) = A\otimes_k A/I^{n+1} \overset{\curvearrowleft}{\rightarrow} A\otimes_k A/I \approx A$$

so that $P^n(A|k)\approx I/I^{n+1}\oplus A$. We write $P_0^n(A|k)$ for I/I^{n+1}. The k-homomorphism d_n: $A\to P^n(A|k)$ defined by $d_n(x)=\overline{x\otimes 1-1\otimes x}$ is called the 'universal' differential operator of order n on A.

Now suppose A is a local ring obtained from a k-algebra of finite type with residue field k. We wish to pose the question:

$Q(*)$: When does $P_0^n(A|k)$ have a free direct summand for n sufficiently large.

REMARK. For A regular P_0^n is a free A module so we are interested in A non-regular.

To understand the above question recall the definition of a differential operator on A. Say $B=k[X_1, \cdots X_s]_{(X_1,\cdots X_s)}$ and $A=B/\mathfrak{A}$. The differential operators on B of order $\leqslant n$ are the obvious ones, i.e., elements of $\mathrm{Hom}_k(B. B)$

$$D=\sum_{i_1+\cdots+i_s\leqq n} b_{i_1,\cdots i_s}\frac{\partial^{i_1}}{\partial X_1^{i_1}}\cdots\frac{\partial^{i_s}}{\partial X_s^{i_s}}, \quad b_{i_1}\cdots_{i_s}\in B$$

where the $\partial/\partial X_i$'s are the standard derivations of B. They form a B-module denoted by $\mathrm{Diff}^n(B|k)$. An n-th order differential operator on A is a D as above with $D(\mathfrak{A})\subset\mathfrak{A}$. We identify two operators whose difference sends B into $\mathfrak{A}$, to get an A module, $\mathrm{Diff}^n(A|k)$. By the definition we get $\mathrm{Diff}^n\subset\mathrm{Diff}^{n+1}$. We can identify Diff^n with the A-dual of P^n via

$$\varphi\in\mathrm{Hom}_A(P^n, A),\ \varphi \underset{\eta}{\mapsto} \varphi\circ d_n\in\mathrm{Diff}^n(A\,|\,k)\,.$$

In particular, Diff^n is a reflexive A-module-a fact we shall need. Note that $\mathrm{Diff}^1\approx A\oplus\mathrm{Der}_k(A)$ and $P_0^1\approx\Omega^1_{A|k}$, the module of Kaehler differentials. For generalities on Diff^n see [3].

Now our question comes down to asking whether there exists a differential operator D without constant term of some order for which there is an $x\in A$ with $D(x)=1$. The main observation of this note is the

Theorem. *Let A be a reduced, local one dimensional k-algebra with residue field k. Then*

(i): *if $\hat{A}$ the completion of A, is a domain, $Q(*)$ is true for A,*
(ii): *if $\hat{A}$ is not a domain with maximal ideal as conductor then $Q(*)$ is negative for A.*

Notice that we should not ask for D to be a derivation because of the

Lemma (Zariski). *Let D be a derivation of A and $x\in A$ with $D(x)=1$. Then $\hat{A}\approx A_1[[X]]$, i.e., A is analytically a product.*

The composition of two differential operators is a differential operator of higher order. One may ask if Diff^n is generated by the composites of elements of Diff^1. Y. Nakai conjectured that if Diff^n is generated by Diff^1 for all $n\geqslant 0$ then A is regular. It is more or less immediate (and checked below) that if $Q(*)$ has an affirmative answer for a class of rings then the Nakai conjecture is true for that class. We should remark that the Nakai conjecture has been settled in dimension one in [2]: it is also verified below in the dimension one analytically reducible case (which does not follow from the theorem). Finally we observe (Prop. 2) that an affirmative solution of the Nakai conjecture would settle the Zariski-Lipman problem which states that if $\mathrm{Der}_k(A)$ is a free A-module then A is regular.

Proposition 1. *An affirmative solution of $Q(*)$ for a class of rings, $\mathcal{B}$, implies the Nakai conjecture for $\mathcal{B}$.*

Proof. Let $\mathrm{Diff}^1(A)$ generate $\mathrm{Diff}^n(A)$, $n\gg 0$. We need to show that A or equivalently, $\hat{A}$ is regular. Writing $\hat{A}$ as a quotient of $k[[Y_1,\cdots Y_m]]$ the above definition of $\mathrm{Diff}^n(A)$ can be carried over to $\hat{A}$ in an obvious manner once we define the differential operators on $k[[Y_1,\cdots Y_m]]$ to be generated by the $\dfrac{\partial}{\partial Y_i}$'s. Then $\mathrm{Diff}^n(\hat{A}\,|\,k)$ consists of expressions of the type $\sum a_\alpha\dfrac{\partial^{i_1}}{\partial Y_1{}^{i_1}}\cdots\dfrac{\partial^{i_m}}{\partial Y_m{}^{i_m}}$, $a_\alpha\in\hat{A}$, which preserve the ideal defining $\hat{A}$. Hence $\mathrm{Diff}^n(\hat{A}\,|\,k)=\mathrm{Diff}^n(A\,|\,k)\otimes_A\hat{A}$. (When A is complete we must remember that $\mathrm{Diff}^n(A)$ has a different meaning than when A is of geometric type). We assume now that A is complete.

Suppose there is a D in $\mathrm{Der}_k(A)$ and an $x \in A$ with $D(x)=1$ then $A \approx A_1[[X]]$ by Zariski's lemma. We claim that the hypothesis of Nakai's conjecture is satisfied by A_1. Write $A_1 = \dfrac{k[[Y_1, \cdots Y_d]]}{\mathfrak{A}}$ and let $D_i = \sum \alpha_{ij} \dfrac{\partial}{\partial Y_j}$, $\alpha_{ij} \in A_1$ be generators of $\mathrm{Der}_k(A_1)$. Then Der A is generated by $\partial/\partial x$ and the D_i, where $\partial/\partial x \,|\, A_1$ and $D_i(x)$ are zero. Take a differential operator $\mathscr{D}$ on A_1 and represent it

$$\mathscr{D} = \sum a_\alpha \frac{\partial^{i_1}}{\partial Y_1^{i_1}} \cdots \frac{\partial^{i_d}}{\partial Y_d^{i_d}}, \quad a_\alpha \in A_1$$

We may assume it has no constant term. Put $\mathscr{D}(X)=0$ to extend it to an operator $\bar{\mathscr{D}}$ on $A_1[[X]]$. By hypothesis we can express $\bar{\mathscr{D}}$ in terms of the D_i and $\partial/\partial x$ and since $\partial/\partial Y_i$ and $\partial/\partial x$ commute and $\alpha_{ij} \in A_1$ $\left(\text{so } \dfrac{\partial \alpha_{ij}}{\partial x}=0\right)$ $\bar{\mathscr{D}}$ may be written as $\sum b_\alpha Q_\alpha \dfrac{\partial^i}{\partial X^i}$ where $b_\alpha \in A_i[[X]]$ and Q_α is a monomial in the D_i's. Since $\dfrac{\partial^i}{\partial X^i}$ acts as the zero operator on $A_1 \subset A_1[[X]]$ for $i>0$ putting $\bar{\mathscr{D}} = \mathscr{D}_0 + \mathscr{D}_1$, $\mathscr{D}_0 = \sum_{i>0} b_\alpha Q_\alpha \dfrac{\partial^i}{\partial X^i}$, $\mathscr{D}_1 = \sum b_\xi Q_\xi$ we find $\mathscr{D}_1 \,|\, A_1 = \bar{\mathscr{D}}$. Writing $b_\xi = b_\xi^1 + X b_\xi^2$, $b_\xi^1 \in A_1$, $b_\xi^2 \in A_1[[X]]$ notice that $\sum b_\xi^1 Q_\xi$ restricted to A_1 represents $\mathscr{D}$. As Q_ξ is a monomial in the derivations of A_1 we have verified the hypothesis of Nakai's conjecture for A_1.

By induction on dimension of the ring, A_1 is regular and we are through. So let $D(A) \subset \mathfrak{M}_A$, the maximal ideal of A, for every D in $\mathrm{Der}_k(A)$. By hypothesis every $H \in \mathrm{Diff}^n$, $n \gg 0$ is a composite of derivations (plus a constant term) so if H has no constant term, $H(A) \subset \mathfrak{M}_A$. This contradicts the truth of $Q(*)$ and proves the proposition.

Theorem. *Let A be a reduced, local one dimensional k-algebra of geometric type with residue field k. Then*

(i): *if $\hat{A}$ is a domain $Q(*)$ is affirmative for A,*
(ii): *if $\hat{A}$ is reducible with maximal ideal as conductor then $Q(*)$ is negative for A.*

Proof. Let $\bar{A}$ be the normalization of A, I the diagonal ideal of $A \otimes_k A$ and $\bar{I}$ the diagonal ideal of $\bar{A} \otimes_k \bar{A}$. The picture

$$\begin{array}{ccc} A \otimes_k A / I^n & \xrightarrow{\psi} & A \\ \downarrow & & \downarrow \\ \bar{A} \otimes_k \bar{A} / \bar{I}^n & \xrightarrow{\bar{\psi}} & \bar{A} \end{array} \qquad \psi, \bar{\psi}, \text{ homomorphisms, } \psi \text{ induced by } \bar{\psi}$$

shows that the differential operators of $\bar{A}$ which send A into A are differential operators of A. Note that we do not claim that every differential operator of A is a restriction of one on $\bar{A}$-which is in fact false. (However, by the theorem of Seidenberg every derivation on A comes from one on $\bar{A}$). As in the proof of the proposition, remembering $\mathrm{Diff}^n(\hat{A})=\mathrm{Diff}^n(A)\otimes_A\hat{A}$ we suppose A complete.

(i): Suppose A is a domain so that $\bar{A}\approx k[[Z]]$. We identify A as a subring of $k[[Z]]$. Let $(Z^c)\,\bar{A}$ be the conductor of $\bar{A}$ in A, i.e., $Z^j\in A\ j\geqslant c$. Write D for $\partial/\partial Z$, the generator of $\mathrm{Der}_k(\bar{A})$. We will construct

$$H = a_0D^c+a_1D^{c+1}+\cdots+a_cD^{2c},\ a_i\in\bar{A}$$

satisfying

$$H(Z^c)=1,\quad H(\bar{A})\subset A\,,$$

so, in particular, $H\in\mathrm{Diff}^{2c}(A)$ and satisfies the requirements of the theorem. Put $a_i=h_iZ^i$ where h_i are constants to be chosen presently. For $0\leqslant j\leqslant c$ write

$$D^{c+j}(Z^{c+i}) = \begin{cases} d_{ij}Z^{i-j} & i\geqslant j,\ d_{ij} \text{ integers} \\ 0 & i<j \end{cases}$$

suppose we want

$$H(Z^c)=1,\ H(Z^{c+j})=0,\ 0<j\leqslant c \tag{**}$$

for which it suffices to solve the linear equations

$$\begin{aligned} &d_{00}h_0 = 1 \\ &d_{10}h_0+d_{11}h_1 = 0 \qquad\qquad d_{ij}\in Z \\ &\vdots \\ &d_{c0}h_0+\cdots+d_{cc}h_c = 0\,. \end{aligned}$$

Since these equation are clearly solvable we can choose a_i so that H satisfies (**). Further $H(Z^i)=0,\ i<c$ and for $j>0$, $H(Z^{2c+j})=Z^cb(Z)\in A$ as $Z^c\bar{A}\subset A$. Hence H is our required operator. This proves (i).

(ii): Assuming A complete as before we can write $A=k[[Z_1]]\oplus\cdots\oplus k[[Z_r]]$. Suppose there is an H with $H(\bar{A})\subset A$ and $H(x)=1$, some x in A. Let us derive a contradiction. Note that H operates on the total quotient ring of A which is the same as that of $\bar{A}$. So H is a meromorphic operator on $\bar{A}$ and can be written as a row $(H_1, H_2, \cdots H_r)$ each H_i of the form $\sum_j h_{ij}\dfrac{\partial^j}{\partial Z_i^j}$, $h_{ij}\in k((Z_j))$. Writing $x=(x_1, \cdots x_r)$ we have assumed $H_1(x_i)=1$ for all i. Observe that for $\lambda_i\in k$, $\lambda=(\lambda_1, \cdots\lambda_r)$ and $y=(y_1, \cdots y_r)\in\bar{A}$, we have $H(\lambda y)=\lambda H(y)$. In particular $\lambda H(x)=H(\lambda x)=(\lambda_1, \cdots\lambda_r)$. Now if $\lambda_i\neq\lambda_j$ for some (i, j) then $(\lambda)\notin A$ since A being local its elements have same 'value' at each maximal ideal of $\bar{A}$. Hence $H(\lambda x)\notin A$ for λ not in the diagonal of $k\oplus\cdots\oplus k$. To get a contradiction it suffices to verify that $\lambda x \in A$ and since the maximal ideal is in the conductor we need only choose x a non unit. If x is a unit we can write $x=x_0+x_1$, x_0 a constant and as H has no constant term $H(x_0)=0$ so $H(x_1)=1$. Replace x by x_1 to get the contradiction and complete the proof of the theorem.

REMARK. The statement (i) is much stronger than what is required to prove the Nakai conjecture in the one dimensional case. It can be verified as follows. Put $A \subset \bar{A} = \oplus k[[Z_i]]$ and recall that by Seidenberg's result the derivations of A extend to $\bar{A}$, so they are of the form $\left(\psi_i(Z_i)\frac{\partial}{\partial Z_i}\right)$. Clearly all the $\psi_i(Z_i)$'s are not units in $k[[Z_i]]$—for otherwise $D=\left(\psi_i(Z_i)\frac{\partial}{\partial Z_i}\right)$ would send $(\lambda_i Z_i)$, $\lambda_i \in k$ to $(\lambda_i \psi_i)$ which is not in A for suitable λ_i. We may assume that there is one i for which all derivations of A have i-th entry $\phi(Z_i)\frac{\partial}{\partial Z_i}$, $\phi(Z_i)$ a non-unit in $k[[Z_i]]$—say $i=1$. We will now restrict our attention to the first coordinate so let $\bar{A}=k[[Z]]$, i.e., $r=1$. Write $D_i = W_i(Z)\frac{\partial}{\partial Z}$ for the derivations of A, $W_i(Z)$ of order $m_i \geqq 1$. By hypothesis the operators of A are composites of the D_i's at least for large order. Notice that

$$\left(Z^m\frac{\partial}{\partial Z}\right)\left(Z^n\frac{\partial}{\partial Z}\right) = Z^{m+n}\frac{\partial^2}{\partial Z^2} + nZ^{m+n-1}\frac{\partial}{\partial Z}$$

implies that in a monomial $\prod D_i^{s_i} = \sum_\alpha b_\alpha \frac{\partial^\alpha}{\partial Z^\alpha}$ the order of b_α is not less than α. From this we find that $Z^c\frac{\partial^{c+1}}{\partial Z^{c+1}}$ is not a combination of D_i's for any $c>0$. But for $c \gg 0$ eg. so that $Z^c\bar{A} \subset A$ we note that $Z^c\frac{\partial^{c+1}}{\partial Z^{c+1}}(\bar{A}) \subset A$ which contradicts the hypothesis and proves the conjecture. See [2] for another proof.

Hopefully $Q(*)$ has an affirmative answer in the case when A is normal thus solving the Nakai and Zariski-Lipman problems because of the

Proposition 2. *The truth of the Nakai conjecture implies that if A has $Der_k(A)$ a free module then A is regular.*

Proof. Let A be as above. By [4] we know A is normal so we may suppose $\dim A \geqslant 2$ and A is nonsingular in codimension one. The composities of the derivations $D_1, D_2, \cdots D_m$, $m=\dim A$, $\{D_i\}$ a free basis of $\mathrm{Der}_k(A)$ generate (together with constants) submodules $Diff^n \subset \mathrm{Diff}^n$. It suffices to verify equality for $n \gg 0$. Put $X=\mathrm{Spec}\, A$, $U \subset X$ the set of regular points of X and $Y=X-U$. The monomials in the D_i's form a free basis of Diff^n at points in U since the higher order operators on a regular local ring are generated by derivations and

$\mathit{Diff}^n \subset \mathrm{Diff}^n$. In particular $\mathit{Diff}^n \mid U = \mathrm{Diff}^n \mid U$. Let R be a relation among the monomials in the D_i's. Since at points in U the monomials are independent the coefficients of R are zero on U. Since A is a domain an element vanishing on an open subset of Spec A is zero. Hence R has all coefficients zero.

Hence Diff^n is a free extension of $\mathrm{Diff}^n \mid U$. As X is of depth$\geqslant 2$ at Y and Diff^n (being reflexive) has Y-depth$\geqslant 2$ the local cohomology groups $H_Y^0(\mathrm{Diff}^n) = H_Y^1(\mathrm{Diff}^n) = 0$ [see [1]]. The exact sequence of local cohomology says that $\Gamma(U, \mathrm{Diff}^n \mid U) = \Gamma(X, \mathrm{Diff}^n)$. For the same reason $\Gamma(X, \mathit{Diff}^n) = \Gamma(U, \mathit{Diff}^n \mid U) = \Gamma(U, \mathrm{Diff}^n \mid U) = \mathit{Diff}^n$, as Diff^n is of Y depth$\geqslant 2$. Hence $\mathit{Diff}^n = \mathrm{Diff}^n$ and we are done.

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