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ADMISSIBLE REPRESENTATIONS FOR SPLIT REDUCTIVE GROUPS DEFINED OVER A FUNCTION FIELD*

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1. Introduction

In this paper we shall study the admissibility of representations for a reductive group defined over a global field k of characteristic $p \neq 0$. For the concept of admissible representations in the sense used here and their connection with the Siegel Formula for number fields see [7]. and [11].

We obtain a necessary and sufficient condition for the convergence of the integral $I_\rho(\Phi)$ associated with the reprresentation ρ. This criterion of convergence involves the weights and their multiplicities in ρ and is analogous to Weil's result for number fields [9 p. 20]. We see that although representations need no longer be completely reducible, the admissibility of the triple (G, X, ρ) depends only on the composition factors of ρ.

As a corollary, for G connected and reductive over k, we see that G_A/G_k has finite volume if and only if the centre of G has no k-split torus. In particular this implies that for G to have any non-trivial absolutely admissibly representation over k, G is necessarily semi-simple. Further, we see that for given G only finitely many different composition series can occur as the composition series of admissible representations.

For G a simply connected, simple and k-split group we obtain a list of composition series that can occur in admissible representations. The list includes all the representations that occur for number fields, as well as many new ones, depending on the characteristic of k. For p=characteristic of k sufficiently large we show the list to be complete. However, for p small we cannot conclude that the list is exhaustive. The difficulty is that for small p it is an unsolved problems as to what the weights and multiplicities of a given irreducible representation ρ_λ are when the highest weight λ is given.

We further show how to obtain the admissible representations when G is a simple group.

The author wishes to thank Professor Igusa for suggesting this investigation of the function field case, as an extension of his results for number fields.

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INDEX OF NOTATIONS

$\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$: the integers, the rationals, the real numbers (respectively)
F_q : the finite field with q elements
G_m : the multiplicative group of the universal domain
$\bar{k}$: the algebraic closure of the field k
For an algebraic group G defined over k,
G_0 : identity component of G
G_k : the subgroup of G of points rational over k
G_A : the adelization of G
$R_{K/k}(G)$: the algebraic group defined over k obtained by restricting the field of definition from K to k.
$X(G)=\mathrm{Hom}(G, G_m)$: the module of rational characters of G
$X_k(G)$: the module of k-rational characters of G
$|x|_A$: the idele norm of an element $x \in G_A$.
$\mathcal{S}(X_A)$: the Schwartz-Bruhat space of X_A, when X is a finite dimensional vector space defined over k.

2. Criterion of convergence

Let k be a function field of transcendence degree one over its finite prime field, F_p. Choose q so that F_q is the exact constant field in k. Let $\rho: G \to \mathrm{Aut}(X)$ be a finite dimensional representation of a reductive group G, all defined over k. The triple (G, X, ρ) is called admissible over k if the integral

$$I_\rho(\Phi) = \int_{G_A/G_k} \sum_{\xi \in X_k} \Phi(\rho(g)\xi) \, |dg|_A$$

of the theta series associated to ρ is absolutely convergent for all $\Phi \in \mathcal{S}(X_A)$. We say (G, X, ρ) is absolutely admissible over k if the corresponding integral is absolutely convergent for all finite extensions $K \supset k$. Igusa [7], has introduced this concept for number fields. In this section we derive a necessary and sufficient condition for ρ to be admissible over k, in terms of the weights of ρ. We use Harder's reduction theory [5].

Let G_0 denote the connected component of the identity of G. Then $G_A/(G_0)_A$ is compact since the proof [2 p. 9] works also for function fields. Hence ρ is admissible over k if and only if the restriction of ρ to G_0 is admissible over k. Thus to study admissibility, we may assume that G is connected.

Let P denote a fixed minimal k-parabolic subgroup of G, with T a maximal torus of P, defined over k and $S \subset T$ a maximal k-split torus of G. Then $P = Z(S) \cdot U$, a semi-direct product of the centralizer of S (in G) with the unipotent radical of P. Further, $U = \prod_{\alpha|S<0} P_\alpha$, where the P_α are the 1-parameter subgroups of G associated to the roots of G (with respect to T) and the ordering is the one implicit in our choice of P. Let $\Delta_P/$ be the algebraic module of P, that is, the module of the inner automorphism of P on U. Then

$$\Delta_P^{-1}|_S = \prod_{\alpha|S<0} \alpha = \text{product of the positive roots with multiplicities (with respect to } S).$$

Let $\{\alpha_1, \cdots, \alpha_r\}$ be the simple roots with respect to this ordering and for each i, $1 \leq i \leq r$, let P_i be the maximal proper parabolic subgroup of G containing P, with corresponds to the subset of $\{\alpha_1, \cdots, \alpha_r\}$ with α_i omitted. Write Δ_i for the restriction of the module Δ_{P_i} to P. Then we have $\alpha_i = \sum_j c_{ij}\Delta_j$ for $c_{ij} \in \boldsymbol{Q}$ (where we regard each character in $X(S) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$). Define $|\alpha_i(y)|_A = \prod_j |\Delta_j(y)|_A^{c_{ij}}$ for $y \in P_A$ and note that this makes sense even though $\alpha_i(y)$ need not be defined. For $c>0$, set $P_A(c) = \{y \in P_A \mid |\alpha_i(y)|_A \leq c \text{ for } 1 \leq i \leq r\}$.

The main result of Harder's reduction theory can now be stated as: there exists $c>0$ such that $G_A = FP_A(c)G_k$, for a suitable compact set F whose exact description is unnecessary for our purpose (it involves translates of a given standard open-compact subgroup.)

To obtain our criterion of convergence, we need to refine this result of reduction theory in terms of the maximal k-split torus S.

Choose a generator $x \in k$ such that extension $k \supset F_q(x)$ is finite separable. For $w|v$, an extension of a valuation v (on $F_q(x)$) to w on k, let e_w denote the ramification index, thus if $q^{\deg \pi_v}$ generates the value group $v(F_h^\times(x)_v)$, then the value group $w(k_w^\times)$ is generated by $q^{\deg \pi_v/e_v}$. Choose w such that $(\deg \pi_v/e_w)$ is the smallest possible, say s_0, as v and w vary over all valuations and fix $z_0 \in k_w^\times$ such that $|z_0|_w = q^{s_0}$. Then embedding $k_w^\times \subset k_A^o$ as usual, we have $k_A^\times = \theta(k_A^\times)^0$, with θ the subgroup of $k_A^\times$ generated by z_0 and $(k_A^\times)^0$ the ideles of norm 1. So $\theta \xrightarrow{\sim} \boldsymbol{Z}$ and notice how this differs from the standard splitting used for number fields. Hence the isomorphism $S \xrightarrow{\sim} (G_m)^s$ of the k-split torus S, gives $S_A = \theta(S)S_A^0$ with $\theta(S) \xrightarrow{\sim} \boldsymbol{Z}^s$ (this isomorphism depending on the original $S \xrightarrow{\sim} (G_m)^s$) and $S_A^0 = \{x \in S_A \mid |\chi(x)|_A = 1 \text{ for all } \chi \in X(S)\}$. Now, since $X_k(P) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to X(S) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, we can choose a basis $\{\chi_1, \cdots, \chi_s\}$ for $X_k(P)$ and defining

$$\begin{aligned} \chi\colon P_A &\to (\boldsymbol{R}_+^\times)^s \\ y &\to (|\chi_1(y)|_A, \cdots, |\chi_s(y)|_A) \end{aligned}$$

we have $\chi(P_A) \supset \chi(S_A)$ with the latter a subgroup of finite index (since for k a function field the $|\chi_i(x)|_A$ are integral powers of q for $x \in S_A$). So choosing coset representatives we have $P_A = \bigcup_{i=1}^t p_i S_A P_A^0$, with t finite and $P_A^0 = \{y \in P_A \mid |\chi(y)|_A = 1 \text{ for all } \chi \in X_k(P)\}$. Thus $P_A = \bigcup_{i=1}^t p_i \theta(S) P_A^0$ and so

$$P_A(c)\subset \bigcup_{i=1}^{t} p_i\theta_{c_i}(S)P_A^0\,,\quad \text{where}$$
$$c_i = c/\min_{1\le j\le r} |\alpha_j(p_i)|_A \quad \text{and}$$
$$\theta_{c_i}(S) = \{\theta\in\theta(S)\,|\,|\alpha_j(\theta)|_A\le c_i 1\le i\le r\}\ .$$

Further $P_A^0=Z(S)_A^0\cdot U_A$ and since $Z(S)$ is k-anisotropic, we have $Z(S)_A^0/Z(S)_k$ compact [5], whence $P_A^0=MP_k$ for M on compact set. Moreover, if $c\ge 1$, $\theta_c(S)/\theta_1(S)$ is finite so enlarging the compact sets if need be, we obtain,

Proposition 1. (Reduction theory for reductive groups over a fuction field) *There exist compact subsets F_1, F_2 of G_A such that*

$$G_A = (F_1\theta_+F_2)G_k,\quad \textit{where}$$
$$\theta_+ = \theta_1(S) = \{\theta\in\theta(S)\,|\,|\alpha_j(\theta)|_A\le 1 \quad \textit{for}\quad 1\le j\le r\}$$
$$\simeq \{(x_1,\cdots,x_s)\in \boldsymbol{Z}^s\,|\,x_i\le 0 \quad \textit{for}\quad 1\le i\le r\}$$

under the isomorphism $S\stackrel{\sim}{\to}(G_m)^s$.

Lemma 2. *For every $c>0$ and every compact subset $F\subset P_A$, the set $\bigcup_{\theta\in\theta c(S)}\theta F\theta^{-1}$ is contained in a compact subset F' of P_A. Moreover, if $F\subset P_A^0$, then F' can be chosen in P_A^0.*

Proof. $P_A=Z(S_A)\cdot U_A$, a semi-direct product and writing F_S=projection of F on $Z(S)_A$, F_u=projection of F on U_A, we have $\theta F_S\theta^{-1}=F_S$, while by the structure theory of reductive groups, $\theta\tau_\alpha(x)\theta^{-1}=\tau_\alpha(\alpha(\theta)x)$ where the τ_α are the morphisms defining the 1-parameter subgroups P_α of G. Hence if $|x|_A$ bounded, so is $|\alpha(\theta)x|_A$ for $\theta\in\theta_c(S)$ and $\alpha>0$, hence $\bigcup_\theta \theta F_u\theta^{-1}\subset a$ compact set.

By the definition of P_A^0, $|\Delta_P|_A$ takes the value 1 there, hence P_A^0 is unimodular. Let $d\theta$, dp_0 denote Haar measures on $\theta(S)$, P_A^0 (respectively), so as usual $|\Delta_P(\theta)|_A^{-1}\,d\theta dp_0$ is a right invariant Haar measure on $\theta(S)P_A^0$. We derive now the analogue of Weil's Lemma 4 [9] for function fields.

Lemma 3. *Let G be a connected reductive group over k, with dg a Haar measure on G_A. There exists a compact subset $C_0\subset G_A$ such that: given any measurable function $F{:}G_A/G_k\to\boldsymbol{C}$ and a measurable function F_0: $\theta_+\to\boldsymbol{C}$ such that*

$$|F(g\theta)|\le F_0(\theta)\quad \textit{for all}\quad g\in C_0,\quad \theta\in\theta_+$$

we have
$$\int_{G_A/G_k}|F(g)|\,dg\le \textit{constant}\int_{\theta_+}F_0(\theta)\,|\Delta_p(\theta)|_A^{-1}d\theta\ .$$

Proof. We have $G_A\supset P_A\supset\theta(S)P_A^0$, with each quotient compact. For a suitable scalar multiple

$$dg\,|_{\theta(S)P_A^0} = |\Delta_P(\theta)|_A^{-1} d\theta dp_0, \quad \text{as each is right invariant.}$$

By Proposition 1, $G_A = F_1\theta_+ F_2 G_k$ with $F_2 \subset P_A^0$, hence

$$\int_{G_A/G_k} |F(g)|\,dg \leq \text{constant} \int_{C_1\theta_+ C_2} |F(g)|\,dg$$

But by lemma 2, $\bigcup_{\theta\in\theta_+} \theta F_2\theta^{-1} \subset F_3 \subset P_A^0$,
for suitable compact F_3, whence $\theta_+ F_2 \subset F_3\theta_+$. Choose $C_0 \supset F_1F_3$ to be any compact subset of G_A. This will be the set whose existence we are asserting. Thus, since $|F(g\theta)| \leq F_0(\theta)$ for $g \in C_0$,, $\theta \in \theta_+$, we see

$$\begin{aligned}\int_{G_A/G_k} |F(g)|\,dg &\leq \text{constant} \int_{C_0\theta_+} |F(g)|\,dg \\ &\leq \text{constant} \int_{\theta_+} F_0(\theta)\,|\Delta_P(\theta)|_A^{-1} d\theta\,.\end{aligned}$$

Let now $\rho: G \to \mathrm{Aut}(X)$ be a representation, defined over k and choose a basis for X so that ρ restricted to S, the chosen maximal k-split torus of G, is diagonal. Then the characters of S that occur are called the weights of ρ and let $P(\rho)$ denote the set of weights of ρ. Recall that G is a connected reductive group defined over k.

Theorem 4 (Criterion for convergence). *The integral $I_\rho(\Phi)$ is absolutely convergent for all $\Phi \in \mathcal{S}(X_A)$ if and only if*

$$C_\rho = \int_{\theta_+} \prod_{\lambda\in P(\rho)} \sup\,(1,\ |\lambda(\theta)|_A^{-m_\lambda})\,|\Delta_P(\theta)|_A^{-1} d\theta < \infty$$

where m_λ = the multiplicity of the weight λ in ρ.

If this condition for convergence is satisfied, then in fact the convergence of $I_\rho(\Phi)$ is uniform on all compact subsets of $\mathcal{S}(X_A)$. In this case I_ρ defines a positive tempered measure on X_A.

Proof. (I) *sufficiency of $C_\rho < \infty$.*

Let C_0 be the compact subset of lemma 3. For given $\Phi \in \mathcal{S}(X_A)$, the family of functions "$x \to \Phi(\rho(g)x)$" ($g \in C_0$, $x \in X_A$), being parametrized by a compact set C_0 forms a compact subset of $\mathcal{S}(X_A)$, hence there exists $\Phi_0 \in \mathcal{S}(X_A)$ such that $|\Phi(\rho(g)x)| \leq \Phi_0(x)$ for all $g \in C_0$. Thus, by lemma 3,

$$\begin{aligned}|I_\rho(\Phi)| &= \left|\int_{G_A/G_k} \sum_{\xi\in X_k} (\Phi(\rho(g)\xi)\,dg\right| \\ &\leq \text{constant} \int_{\theta_+} \sum_{\xi\in X_k} \Phi_0(\rho(\theta)\xi)\,|\Delta_P(\theta)|_A^{-1} d\theta \qquad \cdots\cdots(1)\end{aligned}$$

But $X_k = \bigoplus_{\lambda} X_\lambda$ where $\rho(s)x_\lambda = \lambda(s)x_\lambda$ for $s \in S$, $x_\lambda \in X_\lambda$ with m_λ=dimension of X_λ.

Let $\{a_{\lambda,i}\}$ be a basis for X_λ over k so that then $X_v = X_k \otimes_k k_v$ for every valuation v and use this basis to define X_v^0, the $\mathfrak{o}_v$-rational points of X_v. Then $X_A = X_w \otimes X'$, where w is the valuation singled out by our splitting $S_A = \theta(S)S_A^0$. We may choose $\Phi_0 = \Phi_w \otimes \Phi'$, with each factor locally constant and of compact support, say support $\Phi_w = a^{-1}X_w^0$, support $\Phi' = b^{-1}\prod_{v \neq w} X_v^0$ for suitable a, b. Then $X_w = X_k \otimes_k k_w$ and $X_w \ni \xi = \sum_{\lambda,i} \xi_{\lambda,i} a_{\lambda,i}$, with $\xi_{\lambda,i} \in k_w$. Then $\rho(\theta)\xi = \sum_{\lambda,i} \lambda(\theta)\xi_{\lambda,i}a_{\lambda,i}$.

In particular, if $\xi \in X_k$, the coefficients $\xi_{\lambda,i} \in k$ and for given $\theta \in \theta_+$,

$$0 \neq \Phi_0(\rho(\theta)\xi) = \Phi_w(\rho(\theta)\xi)\Phi'(\xi)$$

if and only if
$$\begin{cases} \rho(\theta)a\xi \in X_w^0 \quad \text{and} \\ \qquad b\xi \in \prod_{\neq wv} X_v^0 \end{cases}$$

But
$$\rho(\theta)a\xi = \sum_{\lambda,i} \lambda(\theta)a\xi_{\lambda,i}a_{\lambda,i}$$
$$b\xi = \sum_{\lambda,i} b\xi_{\lambda,i}a_{\lambda,i}$$

so $\Phi_0(\rho(\theta)\xi) \neq 0$ if and only if for each

$$\lambda,\ i \begin{cases} \lambda(\theta)a\xi_{\lambda,i} \in \mathfrak{o}_w \\ \text{and} \\ \qquad b\xi_{\lambda,i} \in \mathfrak{o}_v \quad \text{all} \quad v \neq w \end{cases} \qquad \cdots\cdots(2)$$

Now, for fixed λ, i $\lambda(\theta)a\xi_{\lambda,i} \in \mathfrak{o}_w$ if and only if

$$ord_w(\lambda(\theta)) + ord_w(\xi_{\lambda,i}) \geq n_w$$

Similarly need $ord_v(\xi_{v\lambda,i}) \geq n_n$ (all $v \neq w$)
where n_w, n_v depend on a, b. Let $\mathfrak{d}_0$ be the divisor

$$\mathfrak{d}_0 = \sum_{\text{all } v} -n_v v \quad \text{and put} \quad \mathfrak{d}_\lambda(\theta) \quad \text{for the divisor}$$
$$\mathfrak{d}_\lambda(\theta) = \mathfrak{d}_0 + ord_w(\lambda(\theta))w.$$

Hence (2) is satisfied if and only if, for each λ, i

$$\xi_{\lambda,i} \in \{\xi \in k \mid div\ \xi + \mathfrak{d}_\lambda(\theta) > 0\} = L(\mathfrak{d}_\lambda(\theta))$$

in the usual notation. By the Riemann-Roch theorem,

$$l(\theta) = \dim_{F_q} L(\mathfrak{d}_\lambda(\theta)) = deg(\mathfrak{d}_\lambda(\theta)) + 1 - g + l(\kappa - \mathfrak{d}_\lambda(\theta))$$

where g=the genus of k and κ is the canonical divisor. Notice that $l(\kappa - \mathfrak{d}_\lambda(\theta)) = 0$

if $deg(\mathfrak{d}_\lambda(\theta))>2g-2$. We have $deg(\mathfrak{d}_\lambda(\theta))=deg\, \mathfrak{d}_0+ord_w(\lambda(\theta))$, so writing $N_0=\max\{2g-1-deg\, \mathfrak{d}_0, 0\}$, we can estimate $l(\theta)$, as follows:

(a) if $ord_w(\lambda(\theta))\geq N_0$, $l(\theta)=\text{constant}+(ord_w(\lambda(\theta)-N_0)$, the constant being $deg\, \mathfrak{d}_0+1-g+N_0$, hence independent of θ, or

(b) if $ord_w<N_0$, we write the divisor

$$\begin{aligned}\kappa-\mathfrak{d}_\lambda(\theta) &= (\kappa-\mathfrak{d}_0-N_0 w)+(N_0-ord_w(\lambda(\theta)))w \\ &= \mathfrak{d}+(N_0-ord_w(\lambda(\theta)))w\,.\end{aligned}$$

Hence $\qquad l(\kappa-\mathfrak{d}_\lambda(\theta))\leq l(\mathfrak{d})+(N_0-ord_w(\lambda(\theta)))$

and so

$$\begin{aligned}l(\theta) &= deg(\mathfrak{d}_\lambda(\theta))+1-g+l(\kappa-\mathfrak{d}_\lambda(\theta)) \\ &\leq deg\, \mathfrak{d}_0+1-g+l(\mathfrak{d})+N_0, \text{ a constant independent of } \theta!\end{aligned}$$

Noting that $|\lambda(\theta)|_A=q^{-s_0 ord_w(\lambda(\theta))}$, we obtain that the number of $\xi_{\lambda,i}$ which satisfy (2)

is
$$\leq\begin{cases} c_\lambda|\lambda(\theta)|_A^{-1} & \text{if } \ |\lambda(\theta)|_A^{-1}\geq q^{N_0}\geq 1 \\ c_\lambda & \text{if } \ |\lambda(\theta)|_A^{-1}<q^{N_0}\end{cases}$$

which can be rewritten to be $\leq c_\lambda \sup(1, |\lambda(\theta)|_A^{-1})$.

Thus cardinality
$$\begin{aligned}&\{\xi\in X_k|\Phi_0(\rho(\theta)\xi)\neq 0\} \\ &\leq\text{constant}\prod_\lambda \sup(1, |\lambda(\theta)|_A^{-m_\lambda})\end{aligned}$$

and we have majorized $I_\rho(\Phi)$ by C_ρ, from (1).

We have further that, since all the functions in any compact subset of $\mathcal{S}(X_A)$ have a common bound by a function also in $\mathcal{S}(X_A)$, the convergence of $I_\rho(\Phi)$ for every Φ, automatically implies the statement of uniform convergence, whence by Lemma 2 [9], I_ρ defines a positive tempered measure on X_A.

(II) *necessity of* $C_\rho<\infty$:

The subset θ_+ is closed in G_A and we have $\theta_+\gamma\cap\theta_+\neq\phi$ for $\gamma\in G_k$ only if $\gamma=1$, since G_k is diagonally embedded, while $\theta_+\subset k_w^\times$. Hence $dg|_{\theta_+}=|\Delta_P(\theta)|_A^{-1}d\theta$ relates the measures. Choosing

$$\begin{aligned}\Phi\geq 0,\ \infty &> \int_{G_A/G_k}\sum_{\xi\in X_k}\Phi(\rho(g)\xi)\,dg \\ &\geq\int_{\theta_+}\sum_{\xi\in X_k}\Phi(\rho(\theta)\xi)|\Delta_P(\theta)|_A^{-1}d\theta\end{aligned}$$

Now, if further $\Phi=\prod\Phi_v$ where each Φ_v is the characteristic function of X_v^0, we see as before that for $\xi\in X_k$, $\Phi(\rho(\theta)\xi)=\Phi_w(\rho(\theta)\xi)\Phi'(\xi)\neq 0$

if and only if for all λ, i
$$\begin{cases} ord_v(\xi_{\lambda,i})\geq 0 & v\neq w \\ ord_w(\lambda(\theta))+ord_w(\xi_{\lambda,i})\geq 0 \end{cases}$$

Put $\qquad L(\lambda(\theta)) = \{\xi \in k \mid div\ \xi + ord_w(\lambda(\theta))\cdot w > 0\}$

with $\quad l(\lambda(\theta)) = \dim_{F_q} L(\lambda(\theta))$, then $\sum_{\xi \in X_k} \Phi(\rho(\theta)\xi) = \prod_\lambda q^{m_\lambda l(\lambda(\theta))}$.

By Riemann-Roch, have

$$l(\lambda(\theta)) = ord_w(\lambda(\theta)) + 1 - g + \dim_{F_q} L(\kappa - ord_w(\lambda(\theta))\cdot w)$$

(i) if $\qquad |\lambda(\theta)|_A^{-1} < 1,\ q^{s_0 m_\lambda l(\lambda(\theta))} \geq 1 = \sup(1, (\lambda(\theta))_A^{-m_\lambda})$

(ii) if $\qquad |\lambda(\theta)|_A^{-1} \geq 1,\ q^{s_0 m_\lambda l(\lambda(\theta))} \geq$

$$q^{s_0(m_\lambda ord_w \lambda(\theta)) + m_\lambda(1-g)} = c_\lambda \sup(1, (\lambda(\theta))_A^{-m_\lambda})$$

Hence $\qquad \sum_{\xi \in X_k} \Phi(\rho(\theta)\xi) \geq \text{constant}. \quad \prod_\lambda \sup(1, (\lambda(\theta))_A^{-m_\lambda})$

i.e. $\qquad \infty > I_\rho(\Phi) \geq \text{constant}. \quad C_\rho$ with a non-zero constant.

Corollary 5. *Let G be a connected reductive group over k. Then G_A/G_k has finite volume if and only if the centre of G has no non-trivial k-split torus.*

Proof. Consider the trivial representation of G on $X = \{0\}$.
Then $I(\Phi) = \Phi(0)\ \mathrm{vol}(G_A/G_k) < \infty$ if and only if

$$\int_{\theta_+} |\Delta_P(\theta)|_A^{-1} d\theta < \infty .$$

Now $\Delta_P(\theta) = \prod_{\alpha | S < 0} \alpha(\theta) = \prod_{i=1}^r \alpha_i(\theta)^{n_i}$, with $n_i > 0$, integers. Further, $(\bigcap_{i=1}^r \mathrm{Ker}\ \alpha_i)_0 = (S \cap Z(G))_0$, is the maximal k-split torus of $Z(G)$, whence by the duality between tori and their character group, $\{\alpha_1, \cdots, \alpha_r\}$ generate a subgroup of finite index in $X_k(S) = X(S)$ if and only if $Z(G)$ has no (non-trivial) k-split torus. We had $\theta_+ = \{(x_1, \cdots, x_s) \in \mathbf{Z}^s \mid x_i \leq 0$ for $1 \leq i \leq r$, thus

$\int_{\theta_+} |\Delta_P(\theta)_A^{-1} d\theta = \sum_{i,j} \prod_{i=j}^r (q^{s_0})^{n_i}$ where the summation is over $n_i \leq 0$ for $1 \leq i \leq r$ and $\mathbf{Z}^{s-r}$. Thus, volume $(G_A/G_k) < \infty$ if and only if $s = r$, that is, $Z(G)$ has no non-trivial k-split torus.

REMARKS (1) This the analogue for function fields of Borel's result [2, p. 21].

(2) A reductive group G that has an absolutely admissible representation over k, is necessarily semi-simple since it always has the trivial orbit $\{0\}$, whence $vol(G_A/G_k)$ is finite for every finite algebraic extension $K \supset k$, so by the previous corollary, must have zero radical.

The criterion for convergence allows us to make the following simplifications in the study of admissible representations, exactly as is the case for number fields.

Suppose $G^* \to G$ is an isogeny defined over k, write ρ^* for the composition of this isogeny with the representation ρ of G, then since ρ and ρ^* have the same weights, our criterion implies that ρ^* is admissible over k if and only if ρ is admissible over k. Further, if $\rho=\begin{pmatrix}\rho_1 & * \\ 0 & \rho_2\end{pmatrix}$, then ρ admissible over k implies that ρ_1, ρ_2 are both admissible over k. A partial converse is also true, namely if either ρ_1 or ρ_2 is the trivial representation of G repeated a finite number of times (where by trivial representation we mean that every element of G is mapped to 1), then admissibility of either ρ_1 or ρ_2 over k, implies admissibility for ρ, since the factor $\prod_{\lambda} \sup(1, |\lambda(\theta)|_A^{-w\lambda})$ is the same. Moreover, if ρ is a representation of G over k and $K\supset k$ is a finite algebraic extension, then ρ admissible over K implies that ρ is admissible over k since both the integrand and the domain of integration become smaller for k. Conversely, if G is k-split, then clearly an admissible representation over k is absolutely admissible.

While an arbitrary representation ρ need not be completely reducible, our criterion of convergence says that admissibility depends only on the composition factors of ρ.

Thus, to classify the absolutely admissible representations of G, we are led to considering a k-split form of G, which if it is to have any admissibles at all must be semi-simple. Further, we may take a simply connected covering of G. The objective of this paper is to list the composition series that can occur in admissible representations of semi-simple Chevalley groups.

3. Admissibility for a Chevalley group

From here on, G will stand for a connected semi-simple, simply connected Chevalley group over k. The finite dimensional irreducible representations for G are determined by their highest weight. Since we are endeavoring to list the composition series of admissible representations, for economy of expression we may suppose that the representations occuring are all completely reducible.

As before let $\{\alpha_1, \cdots, \alpha_r\}$ denote the simple roots of G with respect to our choice of minimal parabolic subgroups (now a Borel subgroup, since G is k-split) and choice of maximal k-split torus S (now a maximal tours of G). Let further Λ_i, ρ_i denote the corresponding fundamental weights and representations of G. Thus the mapping

$$x \to (\Lambda_1(x), \cdots, \Lambda_r(x)) \quad \text{gives the } k\text{-isomorphism} \quad S \to G_m^r .$$

Let $a_{ij}=\dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ be the entries in the Cartan matrix of G, where $(\ ,\ )$ is any positive, non-degenerate scalar product in $X(S)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, invariant under the Weyl group. For (b_{ij}) the inverse matrix, one has $b_{ij}\geq 0$. This is important for us, so lacking any convenient references to cite, we give a proof in the following:

Lemma 6. *$b_{ij} \geq 0$ for all i, j; for (b_{ij}) the inverse Cartan matrix for G.*

Proof. $a_{ij} \leq 0$ if $i \neq j$, while $a_{ii}=2$. For fixed i, let

$$J = J(i) = \{j \in \{1, 2, \cdots, r\} \mid b_{ij} \leq 0\} \quad \text{while}$$
$$I = I(i) = \{j \in \{1, 2, \cdots, r\} \mid b_{ij} > 0\} .$$

From $\Lambda_i = \sum_j b_{ij}\alpha_j$, we obtain

$$\sum_{j\in J} b_{ij}(\alpha_j, \alpha_k) = (\Lambda_i, \alpha_k) - \sum_{j\in I} b_{ij}(\alpha_j, \alpha_k) .$$

Thus if $k \in J$, we have $\sum_{j\in J} b_{ij}(\alpha_j, \alpha_k) \geq 0$, or multiplying by b_{ik} and summing over J, we obtain

$$(\sum_{j\in J} b_{ij}\alpha_j, \sum_{k\in J} b_{ik}\alpha_k) \leq 0, \quad \text{whence} \quad \sum_{j\in J} b_{ij}\alpha_j = 0 .$$

But since the α_i are linearly independent we have $b_{ij}=0$ for all $j \in J$, as required.

Since any admissible representation can always be extended by any (finite) number of trivial representations and still be admissible, we shall suppose always that the representations under discussion are free from the trivial representation.

Proposition 7. *For given G, the number of different composition series that can arise as the composition series of some admissible representation is finite.*

REMARK. This is precisely the same as for number fields [7], with the proofs identical. For emphasis we departed from the earlier convention of considering only completely reducible representations. We give a proof only to indicate how our choice of section in $k_A^\times = \theta(k_A^\times)^0$ affects the number field argument and to justify later appeals to the number field situation, where we shall omit the details.

Proof. Set $|\Lambda_i(s)|_A = (q^{s_0})^{a_i(x)}$ for $x \in S_A$. Then the identification $S^\sim \to G_m^r$ given by $x \to (\Lambda_1(x), \cdots, \Lambda_r(x))$ gives

$$\begin{array}{ccc} \theta(s) & \simeq & Z^r \\ \cup & & \cup \\ \theta & \to & (a_1(\theta), \cdots, a_r(\theta)) . \end{array}$$

Since
$$\alpha_i = \sum_j a_{ij}\Lambda_j, \ \Lambda_i = \sum_j b_{ij}\alpha_j, \quad \text{we have}$$
$$\theta_+ = \{(a_1, \cdots, a_r) \in Z^r \mid \sum_{j=1}^r a_{ij}a_j \leq 0 \quad \text{for} \quad 1 \leq i \leq r\}$$
$$= \{(b_1, \cdots, b_r) \in Z^r \mid b_i \leq 0 \quad \text{for} \quad 1 \leq i \leq r\} ,$$

under the change of coordinates $b_i = \sum_j a_{ij}a_j$. Moreover, since

$$\delta = \sum_i \Lambda_i = \tfrac{1}{2}\sum_{\alpha>0}\alpha, \quad \text{we have}$$
$$|\Delta_P(\theta)|_A^{-1} = \prod_{\alpha>0}|\alpha(\theta)|_A = \prod_{i=1}^r |\Lambda_i(\theta)|_A^2 = q^{2s_0\sum_j b_{ij}b_j} \quad \text{for} \quad \theta\in\theta_+ .$$

Thus

$$C_\rho = \text{constant} \sum_{\substack{(b_1,\cdots,b_r)\in \boldsymbol{Z}^r\\ b_i\le 0}} \prod_\lambda \sup(1, |\lambda(\theta)|_A^{-m_\lambda})\cdot q^{2s_0\sum_j b_{ij}b_j}$$

where the constant comes from the change of variables (since $a_{ij}\in \boldsymbol{Q}$).

Now, for any irreducible representation in ρ of highest weight λ, with the multiplicity of λ being m_λ in ρ, this irreducible factor occurs at most m_λ in ρ as a composition factor. Writing

$$\lambda = \sum_{i=1}^r e_i\Lambda_i, \quad \text{with} \quad e_i\ge 0, \quad \text{we have}$$
$$\prod_{\mu\in P(\rho)} \sup(1, |\mu(\theta)|_A^{-m_\mu}) \ge \sup(1, |\lambda(\theta)|_A^{-m_\lambda})$$
$$\ge |\lambda(\theta)|_A^{-m_\lambda} = q^{-m_\lambda s_0 \sum_{i,j} e_i b_{ij} b_j}$$

for $\theta\in\theta_+$. Hence

$$\infty > C_\rho \ge \text{constant.} \prod_{j=1}^r \sum_{\substack{(b_1,\cdots,b_r)\in \boldsymbol{Z}^r\\ b_i\le 0}} (q^{s_0})^{(\sum_i (2-m_\lambda e_i)b_{ij})b_j}$$

so

$$m_\lambda \sum_{i=1}^r e_i b_{ij} < 2\sum_{i=1}^r b_{ij}, \quad \text{since} \quad b_{ij}\ge 0 .$$

Hence we obtain the bound:

$$\text{(B)} \qquad m_\lambda e_i b_{ij} \le m_\lambda \sum_{i=1}^r e_i b_{ij} < 2\sum_{i=1}^r b_{ij}$$

for each j, from which our finiteness claim follows.

The existence of the estimate (B) has another important implication. It allows the classification of admissibles to be reduced to the case when G is a simple group. For number fields Igusa could see this directly, since the list of admissibles was shown to be exhaustive, whereas in characteristic $p\neq 0$, we can only do this for p large. Nevertheless, the estimate (B) suffices in all but one case.

Proposition 8. *Let G be a simple group. Then for every admissible, irreducible representation ρ of G we have that: (the degree of ρ)$>$(the multiplicity of ρ in any admissible representation of G), except perhaps for the case when G is of type A_r and $\rho=\rho_1$.*

The proof consists of a case by case examination of the simple groups. For $\rho=\rho_\lambda$, irreducible representation with highest weight λ, one has (degree ρ_λ)$\ge |W\cdot\lambda|$, the cardinality of the orbit of λ under the Weyl group.

Let W_λ denote the stabilizer of λ in W, then $|W\cdot\lambda|-|W: W_\lambda|$ and Chevalley's theorem [10, volume I, p. 14] gives the generators of W_λ, in terms of λ. For example, if G is of type B_n and $\lambda=\sum_{i=1}^{n} e_i\Lambda_i$, with $e_i\neq 0$ for a given $i>1$, we have

$$\begin{aligned} &W_\lambda\subset W(A_{i-1})\times W(B_{n-i}) && \text{if} \quad i\neq n \quad \text{or} \\ &W_\lambda\subset W(A_{n-1}) && \text{if} \quad i=n\,. \end{aligned}$$

Here $W(A_i)$ denotes the Weyl group of the group of type A_i, etc. Since $\Lambda_i=\sum_j b_{ij}\alpha_j$, we have $b_{ij}=\dfrac{2(\Lambda_i, \Lambda_k)}{(\alpha_k, \alpha_k)}$ so (B) can be reinterpreted:

$$m_\lambda e_i b_{ij}=m_\lambda e_i\frac{2(\Lambda_i, \Lambda_j)}{(\alpha_j, \alpha_j)}<\frac{4((\delta, \Lambda_j)}{(\alpha_j, \alpha_j)}$$

whence $m_\lambda<\dfrac{(2\delta, \Lambda_j)}{(\Lambda_i, \Lambda_j)}$ for each j, if $e_i\neq 0$

Bourbaki [4] lists Λ_i, W and for the exceptional groups (b_{ij}), whence $m_\lambda<$ degree ρ_λ for all G, except G of type A_r with $\lambda=\Lambda_1$. For this case, we obtain only $m_\lambda<2r$ while degree $\rho_1\geq r+1$. In the appendix we tabulate the bounds obtained in this manner. They are the numbers $N_i(\Phi)$.

From our classification of irreducible admissible representations we shall see that in fact, Proposition 8 does not have any exceptions.

If characteristic $k=p$, for any representation ρ defined over k, we have also ρ^{Fr}, where one replaces the coefficients that occur in the matrix representing $\rho(g)$ by their *p-th* power [3], hence $P(\rho^{Fr})=pP(\rho)$ is the relationship between the weights. Moreover, in $C_{\rho^{Fr}}$, the product $\prod_{\mu\in P(\rho^{Fr})}\sup(1, |\mu(\theta)|_A^{-m_\mu})$ is precisely $\prod_{\lambda\in P(\rho)}\sup(1, |\lambda(\theta)|_A^{-pm_\lambda})$, whence $C_{\rho^{Fr}}\geq C_\rho$.

Further, if ρ is irreducible with highest weight λ, ρ^{Fr} is again irreducible, with highest weight $p\lambda$. This leads to admissible representations which did not occur for number fields and these occur or do not occur depending on p. However, they can only occur if p is small.

To fix the ideas involved, let $\mathfrak{g}_C$ be the Lie algebra over $\boldsymbol{C}$ corresponding to G. Fix a Chevalley basis and denote by $\mathfrak{g}_Z$ the $\boldsymbol{Z}$-span of this basis. Then, we have $\mathfrak{g}=\mathfrak{g}_Z\otimes\bar{k}$ for the Lie algebra of G. Let V_λ be an irreducible of $\mathfrak{g}_C$-module with highest weight λ and let $v_0\in V$ be a maximal vector. For $\mathfrak{u}_C$ the universal enveloping algebra of $\mathfrak{g}_C$, let $\mathfrak{u}_Z$ be a $\boldsymbol{Z}$-form of $\mathfrak{u}_C$ obtained from the Chevalley basis. Then $\mathfrak{u}_Z v_0$ is an admissible lattice, containing v_0 and stable under $\mathfrak{u}_Z$. Tensoring $\mathfrak{u}_Z v_0$ with $\bar{k}$ yields a restricted $\mathfrak{g}$-module $\bar{V}_\lambda$, which is then also a module for the simply connected Chevalley group G and has $v_0\otimes 1\in\bar{V}_\lambda$ as a maximal vector with weight λ. We say a weight $\lambda=\sum_{i=1}^{r} e_i\Lambda_i$

is *restricted* if $0 \leq e_i < p = chr(k)$. Let $\mathfrak{M}$ be the collection of restricted weights for G, with M_λ a restricted irreducible G-module with highest weight λ, for each $\lambda \in \mathfrak{M}$. Then $\{M_\lambda | \lambda \in \mathfrak{M}\}$ exhaust the isomorphism classes of restricted irreducible G-modules and all irreducible G-modules are obtained by Steinberg's method [3] as tensor products from these. Further, the M_λ are homomorphic but not necessarily isomorphic images of $\bar{V}_\lambda$.

Since in characteristic zero, the weights of the irreducible representation with highest weight λ are precisely the weights that occur in $\bar{V}_\lambda$, we see that in characteristic p, the weights in M_λ are a subset of those in characteristic zero, for the same highest weight. Hence any representation, described in terms of the highest weights and their multiplicities which is admissible in characteristic zero, will still be admissible in the function field case.

Moreover, if p is sufficiently large in relation to λ, we have $\bar{V}_\lambda$ irreducible, as given by the following result of Ballard [1].

Proposition 9. *Let G be a simple Chevalley group, over k with root system Φ. Let $\lambda = \sum_{i=1}^{r} e_i \Lambda_i$ be a dominant weight and write $\delta = \sum_i \Lambda_i$. Also, let β_0 be the highest short root of Φ (where if $\beta = \sum_{i=1}^{r} a_i \alpha_i$ is a root, height $(\beta) = \sum_i a_i$). Then, if*

(i) *$p >$ the Coxeter number of Φ and*

(ii) $\dfrac{2(\lambda, \beta_0)}{(\beta_0, \beta_0)} < p - \dfrac{2(\delta, \beta_0)}{(\beta_0, \beta_0)}$

$\bar{V}_\lambda$ is irreducible and $\bar{V}_\lambda \cong M_\lambda$.

REMARKS. (a) Condition (ii) implies that λ is a restricted weight for G, since for every simple G, $\beta_0 = \sum_i a_i \alpha_i$ with $a_i \geq 1$, as can be seen in the table given by Humphreys [6, p. 66].

(b) For G of type A_r and $\lambda = \Lambda_1$, one also has $\bar{V}_\lambda \xrightarrow{\sim} M$ for every p, since the Weyl group operates transitively on the weights of $\bar{V}_\lambda$. Hence $m\rho_1$ is admissible (for k a function field) if and only if $m\rho_1$ is admissible (for k a number field). Hence, by Igusa [7, p. 72] $m\rho_1$ admissible if and only if $m \leq r$, and Proposition 8 is valid for every simple group G and every irreducible admissible representation.

This reduces the problem of classifying the absolutely admissible representations of a given connected and simply connected semi-simple group G, over k to the case when G is absolutely simple. To see this recall that G, defined over k contains only a finite number of connected simple normal subgroups and the Galois group of $\bar{k}/k$ operates on the set of these subgroups, dividing the set into orbits. If $G_1, G_2, \cdots$ are a complete set of representatives of these orbits with $K_1, K_2, \cdots$ the smallest field of definition of $G_1, G_2, \cdots$ then

$G=R_{K_1/k}(G_1)\times R_{K_2/k}(G_2)\times\cdots$. [8, p. 6]. The $R_{K_i/k}(G)$ are the k-simple factors of G. Further, as for number fields we have

Lemma 10. *Let G_1, G_2, $\cdots$ denote not necessarily distinct Chevalley groups over k and let G be their product. Then every irreducible admissible non-trivial representation ρ of G is a product of an irreducible admissible non-trivial representation of G_i for a uniquely determined index i and the canonical projection $G\to G_i$.*

Proof. by Proposition 8 and [7, p. 92].

The discussion on p. 93 [7] is relevant for function fields also, so for G absolutely simple over k, if G_0 is a Chevalley group over k which is $\bar{k}$-isomorphic to G, then G is a k-form of G_0 and the problem of determining which representations of G are absolutely admissible over k, becomes one of determining which ones are $\bar{k}$-equivalent to an admissible representation of G_0. We shall now obtain a list of admissible representations for a simple, simply connected Chevalley group.

4. Admissible representations for simple Chevalley groups

Let G now denote a simple, simply connected Chevalley group over k, with root system Φ. Then Φ is irreducible and as before we let $\{\alpha_1, \cdots, \alpha_r\}$ be a set of simple roots, with $\{\Lambda_1, \cdots, \Lambda_r\}$ the fundamental dominant weights. For ρ an irreducible representation of G with highest weight λ, we write $\rho=\rho_\lambda$ and even ρ_i to stand for ρ_{Λ_i}. Also, let $m\rho$ denote the representation with composition factors all equal to ρ and repeated m times.

Set
$$N_i(\Phi)=\min_{1\leq j\leq r}\left\{\frac{2(\delta, \Lambda_j)}{(\Lambda_i, \Lambda_j)}\right\},$$

where we omit any terms that have $(\Lambda_i, \Lambda_j)=0$. These are listed in the appendix for each group G.

Finally, put $N(\Phi)=2\sum(N_i(\Phi)+1)\dfrac{(\Lambda_i, \beta_0)}{(\beta_0, \beta_0)}$. An easy check shows that $N(\Phi)$ is greater than the Coxeter number of Φ in every case.

For each type G we list admissible representations. The proof that they are admissible follows from Igusa's work [7], together with the remarks preceding Propositions 7 and 9. For $p>N(\Phi)$, the list is exhaustive. This is the content of Proposition 9. For $p\leq N(\Phi)$ we cannot prove the list to be exhaustive. The result that $P(\rho_{\lambda+\lambda'})=P(_{\lambda\lambda'})+P(\rho_{\lambda'})$, ignoring multiplicities, which was so powerful for number fields, is false if $p\leq N(\Phi)$ (for example $P(\rho^{Fr})=pP(\rho)$). A description of the weights and their multiplicities for ρ_λ when $\lambda\in\mathcal{M}$ is an unsolved problem of modular representation theory. Our results have reduced the question of admissibility in characteristic p, when $p\leq N(\Phi)$, to this unsolved question.

The following theorems now follow from the earlier remarks. Note that $\rho^{\mathrm{Fr}}=\rho_{p\lambda}$.

Theorem 10-A$_r$ $(r\geq 1)$. *Admissible representations are* $a\rho_1+b\rho_r(a+b\leq r)$, $\rho_1+\rho_2$, $\rho_1+\rho_{r-1}$, $\rho_2+\rho_r$, $\rho_{r-1}+\rho_r$, ρ_2, ρ_{r-1}, $a\rho_{p^m\Lambda_1}+b\rho_{p^n\Lambda}$, $(ap^m+bp^n<r)$. *These are the only ones if* $p>\dfrac{3}{2}r(r+1)$.

Theorem 10-B$_r$ $(r\geq 2)$. $a\rho_1$ $(a\leq r-1)$, $a\rho_{pm\Lambda_1}$ $(ap^m\leq r-1)$ *are admissible representations in general, also* $\rho_1+\rho_r$, $2\rho_r$, ρ_1 *for* $r=2, 3$; $2\rho_1+\rho_r$, $\rho_1+\rho_r$, ρ_r *for* $r=4$; ρ_5 *for* $r=5$. *Further if ch.* $k=2$, *we have in addition* $\rho_{2\Lambda_r}$ $(r=2, 3)$, $\rho_{2\Lambda_1}+\rho_4$ $(r=4)$. *If ch* $k>\dfrac{1}{2}r(3r+1)$, *these are the only admissible representations.*

Theorem 10-C$_r$ $(r\geq 3)$. $a\rho_1$ $(a\leq r)$, $\rho_1+\rho_2$, ρ_2, $a\rho_{p^m\Lambda_1}$ $(ap^m\leq r)$ *in general and* ρ_3 *for* $r=3$ *are admissible representations. These are the only ones if* $p\geq 3r^2-3r-5$.

Theorem 10-D$_r$ $(r\geq 4)$. *The following are admissible representations* ρ_1 $(a\leq r-2)$, $a\rho_{p^m\Lambda_1}$ $(ap^m\leq r-2)$ *in genral and also* $\rho_1+\rho_3+\rho_4$,

$$\rho_1+\rho_3,\ \rho_1+\rho_4,\ \rho_3+\rho_4,\ 2\rho_3,\ \rho_3,\ 2\rho_4,\ \rho_4 \quad \textit{for} \quad r=4;$$
$$2\rho_1+\rho_2,\ \rho_1+\rho_4,\ \rho_4,\ 2\rho_1+\rho_5,\ \rho_1+\rho_5,\ \rho_5 \quad \textit{for} \quad r=5;$$

$\rho_1+\rho_5$, ρ_5, $\rho_1+\rho_6$, ρ_6 *for* $r=6$. *Further, if* $k=2$, *also have* $\rho_{2\Lambda_3}$, $\rho_{2\Lambda_4}$ *for* $r=4$, $\rho_{\Lambda_1}+\rho_4$, $\rho_{2\Lambda_1}+\rho_5$ *for* $r=5$. *When* $p>3r^2-9r+2$ *these are all the only admissible representations.*

Theorem 10-E$_6$. $\rho_1+\rho_5$, $2\rho_1$, ρ_1, $2\rho_5$, ρ_5 *are admissible in general aud also* $\rho_{2\Lambda_1}$, $\rho_{2\Lambda_5}$ *for chr* $k=2$. *When* $p>121$ *these the only admissible representations.*

Theorem 10-E$_7$. ρ_1 *is an admissible represensation and it is the only one if* $p>217$.

Theorem 10-E$_8$. *There are no admissible representations when* $p>464$.

Theorem 10-F$_4$. $2\rho_1$, ρ_1 *in general and also* $\rho_{2\Lambda_1}$ *if chr* $k=2$, *are admissible representations. If* $p>94$ *these are the only ones.*

Theorem 10-G$_2$. $2\rho_1$, ρ_1 *in general and also* $\rho_{2\Lambda_1}$ *if chr* $k=2$ *are admissible. If* $p>24$ *these are the only admissible representations.*

In the above list, recall our convention of treating representations as being completely reducible. Hence what we have given are in fact the composition factors of admissible representations. In a subsequent work we shall address the question of completely determining the admissible representations.

5. Appendix

Condition (B): $m_\lambda e_i < N_i(\Phi)$, where $\lambda = \sum_{i=1}^{r} e_i \Lambda_i$ is the highest weight of an irreducible admissible representation ρ_λ.

Type of G	$N_i(\Phi)$	$N(\Phi)$
A_r	$2r+1-i,\ 1 \le i \le r$	$\frac{3}{2}r(r+1)$
B_r	$2r-1-i,\ 1 \le i \le r-1$	$\frac{1}{2}r(3r+1)$
	$2r \quad i=r$	
C_r	$2r+1-i$	$3r^2-3r-5$
D_r	$2r-1-i,\ 1 \le i \le r-2$	$3r^2-9r+2$
	$2r-2,\ i=r-1,\ r$	
E_6	12, 12, 10, 9, 8, 12	121
E_7	17. 14, 11, 8, 10, 13, 17	217
E_8	23, 17, 13, 10, 11, 14, 19, 29	464
F_4	10, 5, 7, 11	94
G_2	5, 3	24

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