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ON THE VALUES OF CHARACTERS OF SEMI-SIMPLE GROUPS OVER FINITE FIELDS

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Introduction. Let G be a connected semi-simple algebraic group defined over the finite field $\boldsymbol{F}_q$ and let F be a Frobenius endomorphism of G. The principal purpose of this work is to give the values of discrete series characters of G^F (the group of F-fixed points of G) at regular unipotent elements. These values are shown to be sums of products of Gauss sums and are in a sense dual to the Jacobi sums discussed by Weil in [10]. They are computed by taking certain Fourier transforms over the first Galois cohomology group $H^1(F, Z)$, where Z is the centre of G.

The group G can be embedded ([2], 1.21) in a reductive group $\bar{G}$ with connected centre and compatible F-structure and the method employed here is to study the characters of G^F via those of $\bar{G}^F$, about which a certain amount is known ([2], §10 and [6]). In particular we give a characterization of the discrete series characters (Theorem 3.7) of $\bar{G}^F$ which have a Whittaker model, and prove some results about the restriction of characters from $\bar{G}^F$ to G^F. It transpires that to compute the character values above, one must decompose the restriction of $R_{\bar{T}}^{\theta}$ to G^F, where $\bar{T}$ is minisotropic and θ is in general position, and this decomposition depends on a certain subgroup $W_T^F(\theta)$ of $H^1(F, Z)$. It is actually over this group that one takes the Fourier transform to perform the computation.

In §3 of the present work the role of $H^1(F, Z)$ in studying restrictions is made explicit and §4 the same theme is exploited in the situation of T^F-orbits of linear characters of U^F. In §5 some character sums are computed in terms of certain invariants of the root systems of the split groups and in §6 these are evaluated by the Fourier transform technique.

The main result for the split groups is given in general form in Theorem 7.4, which incorporates the evaluation of the Fourier transforms, while the individual classical groups are treated explicitly later in §7. Finally, §8 outlines the corresponding procedure for the non-split case, and the computations are carried out for the finite unitary groups.

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NOTATION: For any group B, $\prod^l B$ denotes the direct product $B\times B\times\cdots\times B$ (l times), while for an abelian group A, A^d denotes the subgroup of d^{th} powers in A.

1. Some exact sequences. In this section we recall some (well-known) constructions which shall be used throughout the remainder of this work. Let H be a connected linear algebraic group defined over $\boldsymbol{F}_q$ with Frobenius map F and let Z be an F-fixed finite subgroup of the centre of H.

Lemma 1.1. *We have an exact sequence*

$$1\to Z^F\to H^F\xrightarrow{\varphi}(H/Z)^F\xrightarrow{\delta}Z/(F-1)Z\to 1 \tag{1.1.1}$$

[Here, and elsewhere in this work, H^F denotes the group of F-fixed points of H].

Proof. To define δ, notice that hZ is F-fixed $(h\in H)\Leftrightarrow h^Fh^{-1}\in Z$. Hence $\delta(hZ)=(F-1)h$ is well-defined on $(H/Z)^F$, to within multiplication by an element of $(F-1)Z$. The surjective nature of δ follows from Lang's theorem, which says that F-1 is surjective on H when H is connected. The remainder of the proof is routine.

Note that the sequence (1.1.1) is the well-known sequence in Galois cohomology:

$$1\to H^0(F,Z)\to H^0F,H)\to H^0(F,H/Z)\to H^1(F,Z)\to H^1(F,H)=1$$

which comes from the exact sequence $1\to Z\to H\to H/Z\to 1$. We therefore often write $H^1(F,Z)$ for $Z/(F-1)Z$. We remark also that (as R. Steinberg has pointed out) this sequence provides an easy proof of the equality $|H^F|=|(H/Z)^F|$.

Corollary 1.2. *Let G be as in the introduction-viz. a semi-simple group with (finite) centre Z, and suppose that T is an F-fixed maximal torus of G. Then we have the following exact sequences*:

$$1\to Z^F\to G^F\to(G/Z)^F\to Z/(F-1)Z\to 1 \tag{1.2.1}$$
$$1\to Z^F\to T^F\to(T/Z)^F\to Z/(F-1)Z\to 1 \tag{1.2.2}$$

We now embed G in a connected reductive group $\bar{G}$ which has connected centre as follows (c.f. [2], §1.21): let $Z\to\bar{Z}$ be an embedding of Z in a torus defined over $\boldsymbol{F}_q$ (e.g. one can take $\bar{Z}$ to be any F-fixed maximal torus of G); then take $\bar{G}$ to be the pushout of the diagram

$$\begin{tikzcd} G \arrow[r, dotted] & \bar{G} \\ Z \arrow[u] \arrow[r] & \bar{Z} \arrow[u, dotted] \end{tikzcd}$$

i.e. $\bar{G}=G\times\bar{Z}/\{(z, z^{-1})\,|\,z\in Z\}$. We are thus provided with monomorphisms $G\to\bar{G}$ and $\bar{Z}\to\bar{G}$ which are F-equivariant, and we have

Lemma 1.3. (i) *$\bar{Z}$ is the centre of $\bar{G}$, $Z=\bar{Z}\cap G$, and $Z^F=\bar{Z}^F\cap G^F$,*

(ii) *$\bar{G}/\bar{Z}$ is canonically isomorphic to G/Z and $(G/Z)^F\cong\bar{G}^F/\bar{Z}^F$,*

(iii) *Any F-stable maximal torus T of G is contained in an F-stable maximal torus $\bar{T}$ of $\bar{G}$ such that $\bar{G}=G.\bar{T}$ and $G\cap\bar{T}=T$.*

Proof. (i) is obvious. To see (ii), recall that the isomorphism $G/Z\to\bar{G}/\bar{Z}$ is F-equivariant and hence $(G/Z)^F\cong(\bar{G}/\bar{Z})^F$. Since $\bar{Z}$ is connected, by Lang's theorem $g\bar{Z}$ is F-stable $\Leftrightarrow g\bar{Z}$ contains an element of $\bar{G}^F$. Hence $(\bar{G}/\bar{Z})^F\cong\bar{G}^F/\bar{Z}^F$. For (iii) one can take $\bar{T}=T\times\bar{Z}/\{(z, z^{-1})\,|\,z\in Z\}$.

Corollary 1.4. *We have, in the above notation, that*

$$\bar{G}^F = G^F\bar{T}^F \text{ and } G^F\cap\bar{T}^F = T^F .$$

Proof. Clearly $\bar{G}^F=(G\bar{T})^F\geqslant G^F\bar{T}^F$. Now $G^F\cap\bar{T}^F=T^F$ and so $|G^F\bar{T}^F|=|G^F|\,|\bar{T}^F|\,|T^F|^{-1}$. On the other hand by taking $H=\bar{G}$ and $\bar{T}$ respectively in 1.1.1 we see that $|\bar{G}^F|=|G^F|\,|\bar{Z}^F|$ and $|\bar{T}^F|=|T^F|\,|\bar{Z}^F|$. Thus $|G^F\bar{T}^F|=|G^F|\,|T^F|\,|\bar{Z}^F|\,|T^F|^{-1}=|\bar{Z}^F|$ and the result follows.

Proposition 1.5. *With notation as in Lemma* 1.3, *there exists a canonical surjection δ: $\bar{G}^F\to Z/(F-1)Z$ such that the following sequences are exact*:

$$1\to G^F\bar{Z}^F\to\bar{G}^F\xrightarrow{\delta} Z/(F-1)Z\to 1 \tag{1.5.1}$$

$$1\to T^F\bar{Z}^F\to\bar{T}^F\xrightarrow{\delta} Z/(F-1)Z\to 1 \tag{1.5.2}$$

Proof. From the exact sequence 1.2.1 we have, using Lemma 1.3 (ii)

$$1\to G^F/Z^F\to\bar{G}^F/\bar{Z}^F\to Z/(F-1)Z\to 1 \tag{1.5.3}$$

Now the image of G^F/Z^F in $\bar{G}^F/\bar{Z}^F$ is $G^F\bar{Z}^F/\bar{Z}^F$ and so $\bar{G}^F/G^F\bar{Z}^F\cong Z/(F-1)Z$. The map δ may thus be taken as the composite of the natural map $\bar{G}^F\to\bar{G}^F/\bar{Z}^F$ with the map of 1.5.3 above. The proof of 1.5.2 is the same. For future reference we describe δ explicitly: for $g\in\bar{G}^F\ \exists\ z_g\in\bar{Z}$ such that $gz_g\in G$. Then $\delta(g)=F(z_g)z_g{}^{-1}\cdot(F-1)z$.

We now turn attention to the action of the Weyl group on a torus.

DEFINITION. For an F-stable maximal torus T of G we define $W(T)=N_G(T)/T$. Correspondingly for $\bar{T}$ (an F-stable maximal torus of $\bar{G}$ containing T we define $W(\bar{T})=N_{\bar{G}}(\bar{T})/\bar{T}$.

The map F acts on $W(T)$ and $W(\bar{T})$ and we have corresponding sets of fixed points $W(T)^F=N_{G^F}(T)/T^F$ etc.

Lemma 1.6. *Let T and $\bar{T}$ be as in Lemma* 1.3. *Then there is a canonical* *isomorphim*: $W(T) \to W(\bar{T})$ *which is compatible with* F. *Thus* $W(T)^F \cong = W(\bar{T})^F$.

Proof. For any subgroup L of G which contains Z, write $\bar{L} = L \times \bar{Z} / \{(z, z^{-1}) \mid z \in Z\}$. The observation that $W(\bar{T}) = \overline{N_G(T)} / \bar{T}$ makes the Lemma clear.

Now $W(T)^F (= W(\bar{T})^F)$ acts on $\bar{T}^F$ (and on $\bar{T}$) as a group of automorphisms. For $w \in W(T)^F$ we write $ad\, w$ for the action of w on $\bar{T}^F$ (or $\bar{T}$) and further write $ad\, w: t \to t^w$ where $t^w = wtw^{-1}$ ($t \in \bar{T}$). $W(T)^F$ also acts dually on the complex character group of $\bar{T}^F$. For any character θ of $\bar{T}^F$ we write $\theta^w(t) = \theta(t^w) = (ad\, w)\theta(t)$.

Proposition 1.7. *Let* T *and* $\bar{T}$ *be maximal tori of* G *and* $\bar{G}$ *as above. Then the group* $W(T)^F$ $(= W(\bar{T})^F)$ *acts on* $\bar{T}^F$ *and stabilizes* T^F *and* $T^F \bar{Z}^F$, *Moreover* $W(T)^F$ *acts trivially on the quotient* $\bar{T}^F / T^F$.

Proof. Each element of $W(\bar{T})^F$ is represented by an element g of $N_{G^F}(T)$ (Lemma 1.6) and g clearly normalizes T^F and $T^F \bar{Z}^F$. Since $W(\bar{T})$ acts trivially on $\bar{T}/T$ (because $\bar{T} = T\bar{Z}$ and $ad\, g$ acts trivially on $\bar{Z}$), $W(\bar{T})^F$ acts trivially on $(\bar{T}/T)^F$ which may be identified with $\bar{T}^F / T^F$ since T is connected.

EXAMPLES. Suppose G is simply connected and has irreducible root system R. Then $Z \cong P(R)/Q(R)$ where $P(R)$ and $Q(R)$ are the lattices of weights and roots respectively.

(a) $G = SL(n,k)$, $k = \bar{\boldsymbol{F}}_q$ (the algebraic closure of $\boldsymbol{F}_q$). Here $\bar{G} = GL(n, k)$, $\bar{Z} = k^*$, $Z \cong \boldsymbol{Z}/n_{p'}\boldsymbol{Z}$. F acts on $\bar{Z}$ by raising to the q^{th} power. Hence $Z/(F-1)Z = \boldsymbol{Z}/(n/d)\boldsymbol{Z}$ where $d = gcd(n, q-1)$.

(b) $G = \mathrm{Spin}(4n, k)$, k as in (a). Here $\bar{G}$ is obtained by adding similarities to the Spin representation of G. $Z = \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$, $\bar{Z} = k^* \times k^*$ and F acts on $\bar{Z}$ by raising to the q^{th} power. Thus $Z/(F-1)Z = \begin{cases} 1 \ (q \text{ even}) \\ Z/2\boldsymbol{Z} \times Z/2\boldsymbol{Z} \ (q \text{ odd}). \end{cases}$

2. Splitting of characters. We are concerned in this section with the following question: given an irreducible character ρ of $\bar{G}^F$, how does it split on restriction to G^F? We begin with the following general situation: let K be a normal subgroup of the finite group H and let ρ be an irreducible complex character of H; suppose μ is an irreducible component of the restriction $\rho|_K$ and denote by $H(\mu)$ its centralizer in H. For any finite abelian group A write $(A)^\wedge$ for its complex character group. Now suppose that H/K is abelian; then $(H/K)^\wedge$ can be identified with the set of linear characters of H whose restriction to K is trivial. Thus $(H/K)^\wedge$ acts on the set of irreducible characters of H via $\rho \mapsto \alpha\rho = \alpha \otimes \rho$ ($\alpha \in (H/K)^\wedge$). We write $A(\rho)$ for the set $\{\alpha \in (H/K)^\wedge \mid \alpha\rho = \rho\}$. From Clifford theory we have $\rho|_K = e_K(\rho) \sum \mu^h$, where $e_K(\rho)$ is an integer and the sum is over the distinct H-conjugates. of μ. The results summarized in the following lemma are easy consequences of Frobenius reciprocity and Clifford theory and may be found in ([3], Theorem 1).

Lemma 2.1. *Let H, K, $A=H/K$, ρ, μ, $A(\rho)$ and $H(\mu)$ be as above. Then*

(i) *Two irreducible characters ρ_1 and ρ_2 of H have either disjoint or coincident restrictions to K. Moreover $\rho_1|_K=\rho_2|_K$ if and only if $\rho_1=\alpha\rho_2$ for some $\alpha\in(H/K)^\wedge$.*

(ii) $|A(\rho)|=(\rho|_K, \rho|_K)=e_K(\rho)^2|H/H(\mu)|$ *(usual scalar product of group characters).*

If $e_K(\rho)=1$, which is always the case when H/K is cyclic and which also occurs in many other cases we shall be considering, we have additional information:

Proposition 2.1. *With notation as in Lemma 2.1, suppose that $e_K(\rho)=1$. Then $H(\mu)=\bigcap_{\alpha\in A(\rho)} \ker\alpha$.*

Proof. Write $\ker A(\rho)=\bigcap_{\alpha\in A(\rho)}\ker\alpha$. By duality we have $|A(\rho)|=|H/\ker A(\rho)|$. Comparing this with Lemma 2.1 (ii) we see that $|\ker A(\rho)|=|H(\mu)|$ since $e_K(\rho)=1$.

On the other hand since $H(\mu)$ centralizes all the irreducible components of $\rho|_K$, all the irreducible components of $\rho|_{H(\mu)}$ remain irreducible upon restriction to K. Hence $(\rho|_{H(\mu)}, \rho|_{H(\mu)})=(\rho|_K, \rho|_K)$. Again using Lemma 2.1 (ii) we see that $|A(\rho)|=|A'(\rho)|$ where $A'(\rho)=\{\alpha\in(H/H(\mu))^\wedge \mid \alpha\rho=\rho\}$ is a subgroup of $A(\rho)$. Hence each element of $A(\rho)$ is in $A'(\rho)$, i.e. $H(\mu)\le\ker A(\rho)$. Together with the equality of the orders proved above, this implies the proposition.

Corollary 2.3. *With notation as above, if $\rho|_K$ is multiplicity-free, then the irreducible components of $\rho|_K$ are in 1–1 correspondence with the elements of $A(\rho)$ and $A(\rho)$ permutes them regularly.*

Proof. Clearly the components of $\rho|_K$ correspond 1–1 with $H/H(\mu)$. But by 2.2, $H(\mu)=\ker A(\rho)$ and hence $H/H(\mu)=H/\ker A(\rho)\cong A(\rho)$ (by dualtiy). The result follows.

The observation above also shows that $A(\rho)\cong H/H(\mu)$ permutes the components of $\rho|_K$ regularly (i.e. transitively, and only $1\in A(\rho)$ fixes any component).

For the situation which concerns us we have

Proposition 2.4. *Consider the following sequence of normal subgroups of $\bar{G}^F$: $\bar{G}^F\trianglerighteq G^F\bar{Z}^F\trianglerighteq G^F$. We have*

(i) $\bar{G}^F/G^F\cong(\bar{Z}/Z)^F$

(ii) *Any irreducible character ν of $G^F\bar{Z}^F$ remains irreducible on restriction to G^F.*

(iii) *For any irreducible character ρ of $\bar{G}^F$, if μ is an irreducible component of*

$\rho|_{G^F}$ *then the centralizer* $\bar{G}^F(\mu)$ *contains* $G^F\bar{Z}^F$.

(iv) *If, in* (iii) *we have* $\alpha\in(\bar{G}^F/G^F)^\wedge$ *such that* $\alpha\rho=\rho$, *then ker* $\alpha\geq G^F\bar{Z}^F$.

Proof. (i) is clear since $\bar{G}/G\cong\bar{Z}/Z$; (iii) follows from (ii), since given (ii), it follows that $G^F\bar{Z}^F$ centralizes all irreducible components of $\rho|_{G^F}$. To prove (ii), observe that if $\alpha\in(G^F\bar{Z}^F/G^F)^\wedge$ is such that $\alpha\nu=\nu$, then ker $\alpha\supset$ supp ν (where supp $\nu=\{g\in G^F\bar{Z}^F|\nu(g)\neq 0\}$). Since ν is irreducible, the centre of $G^F\bar{Z}^F$ is represented by scalar matrices in any representation corresponding to ν and so $\bar{Z}^F$ is contained in supp ν. Hence each coset of G^F in $G^F\bar{Z}^F$ contains an element of supp ν and thus ker $\alpha\supset$ supp ν implies that $\alpha=1$. By Lemma 2.1 (ii), the restriction $\nu|_{G^F}$ is irreducible. The same proof demonstrates (iv).

Proposition 2.4 shows that when studying the restriction $\rho|_{G^F}$, it is sufficient to consider $\rho|_{G^F\bar{Z}^F}$.

Corollary 2.5. *With notation as above,*

(i) *The group* $A(\rho)=\{\alpha\in(\bar{G}^F/G^F)^\wedge\,|\,\alpha\rho=\rho\}$ *is isomorphic to a subgroup of* $Z/(F-1)Z$.

(ii) *If* $d=|Z/(F-1)Z|$ *then the number of irreducible components of* $\rho|_{G^F}$ *divides d.*

Proof. For (i), observe that by Proposition 2.4 (iv), $A(\rho)$ is a subgroup of $(\bar{G}^F/G^F\bar{Z}^F)^\wedge\cong Z/(F-1)Z$. (ii) follows simply from the observations that restrictions from $G^F\bar{Z}^F$ to G^F remain irreducible, and that $\bar{G}^F/G^F\bar{Z}^F\cong Z/(F-1)Z$ (Proposition 1.5).

Corollary 2.6. *If Z is cyclic then* $\rho|_{G^F}$ *is multiplicity free for any irreducible character* ρ *of* $\bar{G}^F$.

For in this case $Z/(F-1)Z\cong\bar{G}^F/G^F\bar{Z}^F$ is cyclic.

REMARK. If G has irreducible root system R, since Z is isomorphic to a subgroup of $P(R)/Q(R)$ (where $P(R)$ and $Q(R)$ are the weight and root lattices respectively), Lemma 2.6 includes all cases except where $R=D_{2n}$ and the simply connected covering group of G is Spin $(4n, k)$.

Let T be an F-stable maximal torus of G and let $\bar{T}$ be as in Lemma 1.3. The action of $\bar{T}^F$ on restrictions is described in

Lemma 2.7. *Let* ρ *be an irreducible character of* $\bar{G}^F$ *and let* ρ *be an irreducible component of* $|_{G^F}$. *Then* $\bar{T}^F$ *permutes the irreducible components of* $\rho|_{G^F}$ *transitively and the stabilizer* $\bar{T}^F(\mu)$ *of* μ *contains* $T^F\bar{Z}^F$. *If* $\rho|_{G^F}$ *is multiplicity-free then* $\bar{T}^F/\bar{T}^F(\mu)\cong A(\rho)(=\{\alpha\in(\bar{G}^F/G^F)^\wedge\,|\,\alpha\rho=\rho\})\cong$ *a subgroup of* $Z/(F-1)Z$.

Proof. From Corollary 1.4 we have $\bar{G}^F=G^F\bar{T}^F$, whence $\bar{T}^F$ is transitive on the components of $\rho|_{G^F}$. The remaining statements follow from this observation, together with Proposition 2.4 and Corollary 2.5.

3. Restrictions of Gelfand-Graev characters and cuspidal R_T^θ

In this section we show first that Gelfand-Graev characters have multiplicity-free restrictions to G^F and we give more detailed information on the restriction cuspidal Deligne-Lusztig representations R_T^θ.

We recall the definition of Gelfand-Graev characters (see, e.g., [2] §10 or [6] §4): Take an F-stable Borel subgroup $\bar{B}$ of $\bar{G}$, and an F-stable maximal torus $\bar{T}$ of $\bar{B}$. Then $\bar{B}=\bar{T}U$ where U is the unipotent radical of $\bar{B}$ (of course $U<G$). Let $U.$ be the subgroup of U generated by the non-fundamental root subgroups. Then $U/U.$ is commutative and isomorphic to $\prod U_\alpha$, where α runs over the fundamental roots. Let I be the set of F-orbits of fundamental roots (c.f. [6], p. 259) and for $i\in I$ let U_i be the product of the U_α with $\alpha\in i$. The U_i are F-stable and $(U/U.)^F=\prod_{i\in I} U_i^F$. A linear character χ of U^F which is trivial on $U^F.$ (this is true for all χ in good characteristic) is called regular if χ defines a non-trivial character of U_i^F for each $i\in I$. All such regular χ are conjugate under the action of $\bar{T}^F$ since $\bar{Z}$ is connected.

DEFINITION. The induced character $\Gamma=\chi^{\bar{G}^F}$ (for any regular linear character χ of U^F) is called the *Gelfand-Graev character* of $\bar{G}^F$.

It is clear that Γ is independent of χ (since all regular χ are conjugate under $\bar{T}^F$) and it is known (see Steinberg [8], Theorem 49) to be multiplicity free.

Theorem 3.1. *Let ρ be an irreducible component of the Gelfand-Graev character Γ of $\bar{G}^F$. Then the restriction $\rho|_{G^F}$ is multiplicity free.*

Proof. By Frobenius reciprocity, $\rho|_{U^F}$ contains each regular linear character of U^F with multiplicity one. Now all the irreducible components of $\rho|_{G^F}$ are conjugate under $\bar{T}^F$, and since $\bar{T}^F$ maps regular linear characters of U^F to regular linear characters, it follows that each component μ of $\rho|_{G^F}$ contains a regular linear character in its restriction to U_F. Since this occurs with multiplicity one in $\rho|_{U^F}$, the component μ occurs with multiplicity one in $\rho|_{G^F}$.

We now turn to the virtual modules $R_{\bar{T}}^\theta$; we abuse notation by also writing $R_{\bar{T}}^\theta$ for the (generalized) character corresponding to the module. Each $R_{\bar{T}}^\theta$ corresponds to an F-stable maximal torus $\bar{T}$ of $\bar{G}$, and a character $\theta\in(\bar{T}^F)^\wedge$ (recall that $(A)^\wedge$ denotes the complex character group of an abelian group A). The restriction of $R_{\bar{T}}^\theta$ to G^F is R_T^θ, where $T=\bar{T}\cap G$ (T is an F-stable maximal torus of G). Now since $\bar{G}^F/G^F\cong\bar{T}^F/T^F$, for any element $\psi\in(\bar{G}^F/G^F)^\wedge$, we may regard ψ as an element of $(\bar{T}^F/T^F)^\wedge$ by restricting ψ to $\bar{T}^F$. With this identification, we have

Lemma 3.2. ([2], 1.27). *Let $\psi\in(\bar{G}^F/G^F)^\wedge$. Then $R_{\bar{T}}^\theta\psi=R_{\bar{T}}^{\theta\psi}$.*

For any F-stable maximal torus $\bar{T}$ of $\bar{G}$, $W(\bar{T})^F$ acts on $\bar{T}^F$ and hence on $(\bar{T}^F)^\wedge$. For $\theta\in(\bar{T}^F)^\wedge$ we denote by $W^F_{\bar{T}}(\theta)$ the group $\{w\in W(\bar{T})^F \mid \theta^w=\theta\}$. For $T=\bar{T}\cap G$ and $\varphi\in(T^F)^\wedge$ we denote by $W^F_T(\varphi)$ the corresponding subgroup of $W(T)^F$ $(=W(\bar{T})^F)$. For $\theta\in(\bar{T}^F)^\wedge$, $W^F_T(\theta)$ denotes the stabilizer of the restriction to T^F of θ. Clearly $W^F_{\bar{T}}(\theta)\leq W^F_T(\theta)$.

Lemma 3.3. ([2], Theorem 6.8.) *Let* $\theta, \theta'\in(\bar{T}^F)^\wedge$. *Then* $(R^\theta_{\bar{T}}, R^{\theta'}_{\bar{T}})= |\{w\in W(\bar{T})^F \mid \theta^w=\theta'\}|$.

Thus in particular we have that $\pm R^\theta_{\bar{T}}$ is irreducible $\Leftrightarrow W^F_{\bar{T}}(\theta)=\{1\}$.

Proposition 3.4. *Let $\bar{T}$ and T be as in the preamble to Lemma* 3.2, *and take* $\theta\in(\bar{T}^F)^\wedge$. *Then for* $w\in W(T)^F$, θ *and* θ^w *have the same restriction to* T^F *if and only if* $\theta^w=\theta\psi$ *for some* $\psi\in(\bar{T}^F/T^F\bar{Z}^F)^\wedge$.

Proof. Clearly $\theta^w=\theta$ on $T^F\Rightarrow\theta^w=\theta\psi$, with $\psi\in(\bar{T}^F/T^F)^\wedge$. But $\theta^w=\theta$ on $\bar{Z}^F$ since $W(T)^F$ acts trivially on $\bar{Z}^F$. Thus $\psi=1$ on $T^F\bar{Z}^F$. The converse is trivial.

Proposition 3.5. *Let* $\bar{T}$, T *be as above and let* $\theta\in(\bar{T}^F)^\wedge$. *There is a monomorphism* φ: $W^F_T(\theta)/W^F_{\bar{T}}(\theta)\to(\bar{T}^F/T^F\bar{Z}^F)^\wedge\cong Z/(F-1)Z$ *such that* $Im(\varphi)=\{\psi\in(\bar{G}^F/G^F)^\wedge \mid R^\theta_{\bar{T}}\psi=R^\theta_{\bar{T}}\}$. (*Here we identify* $(\bar{G}^F/G^F)^\wedge$ *and* $(\bar{T}^F/T^F)^\wedge$ *as in* 3.2.)

Proof. By 3.2 and 3.3, for $\psi\in(\bar{G}^F/G^F)^\wedge$ we have $R^\theta_{\bar{T}}\psi=R^\theta_{\bar{T}}$ if and only if $\theta\psi=\theta^w$, for some $w\in W(\bar{T})^F$. For such w, we must have $w\in W^F_T(\theta)$, since $\theta\psi$ and θ have the same restriction to T^F. On the other hand, $w\in W^F_T(\theta)$ implies that $\theta^w=\theta\psi_w$, with $\ker\psi_w\geq T^F$, i.e. $\psi_w\in(\bar{T}^F/T^F)^\wedge$. Consider the map φ: $W^F_T(\theta)\to(\bar{T}^F/T^F)^\wedge$ given by $w\to\psi_w$: this is a group homomorphism because by 1.7, $W(T)^F$ acts trivially on $(\bar{T}^F/T^F)^\wedge$; thus for $v, w\in W^F_T(\theta)$, $\theta^{vw}=(\theta^v)^w=(\theta\psi_v)^w=\theta\psi_v\psi_w$, and we have $\psi_{vw}=\psi_v\psi_w$. The map φ clearly has kernel precisely $W^F_{\bar{T}}(\theta)$. The proof of (i) is completed by the observations that Im $\varphi=\{\psi\in(\bar{G}^F/G^F)^\wedge \mid \theta\psi=\theta^w$, some $w\}=\{\psi\in(\bar{G}^F/G^F)^\wedge \mid R^\theta_{\bar{T}}\psi=R^\theta_{\bar{T}}\}$ and that by 3.4, $\mathrm{Im}(\varphi)$ is contained in $(\bar{T}^F/T^F\bar{Z}^F)^\wedge$.

Corollary 3.6. *If* θ *above is in general position* (*i.e.* $W^F_{\bar{T}}(\theta)=\{1\}$) *then* $\rho=\pm R^\theta_{\bar{T}}$ *is irreducible, and* $\rho|_{G^F}$ *is multiplicity free. If* $\bar{T}^F_\theta$ *is the stabilizer in* $\bar{T}^F$ *of any of the components of* $\rho|_{G^F}$ *then* $\bar{T}^F/\bar{T}^F_\theta\cong W^F_T(\theta)$. *Thus* $W^F_T(\theta)$ *permutes the components of the restriction regularly.*

Proof. That ρ is irreducible follows from 3.3. Each character $R^\theta_{\bar{T}}$ has a component in common with Γ (the Gelfand-Graev character) ([1], §10) and so θ in general position implies that $\rho=\pm R^\theta_{\bar{T}}$ is a component of Γ. By 3.1, $\rho|_{G^F}$ is multiplicity free. Hence by 2.4 $\bar{T}^F_\theta$ is the intersection of the ker α with $\alpha\in\varphi(W^F_T(\theta))$. Hence $\bar{T}^F/\bar{T}^F_\theta\cong\varphi(W^F_T(\theta))\cong W^F_T(\theta)$; this group permutes the components of the restriction regularly by 2.3.

We next prove a result characterizing the discrete series characters of a group with connected centre which appear as components of the Gelfand-Graev character Γ. This characterization is instrumental in providing a complete picture of the values of discrete series characters at regular unipotent elements when combined with the results of §§3, 4 and 5.

Theorem 3.7. *Suppose the centre of G is connected. If χ is cuspidal, irreducible and a component of Γ, the Gelfand-Graev representation of G^F, then χ is of the form $\chi=\pm R_T^\theta$, where T is an F-stable maximal torus of G and θ is a non-singular character of T^F.*

Proof. By [2 (Lemma 10.6]) for any irreducible component χ of Γ we have $\chi=\sum \frac{(\chi, R_T^\theta)}{(R_T^\theta, R_T^\theta)} R_T^\theta$, the sum being over the G^F-conjugacy classes of pairs (T, θ) where T is an F-stable maximal torus of G and θ is a character of T^F (note that these R_T^θ form an orthogonal system, so that this statement is merely that χ lies in their linear span). Now if χ is cuspidal, then for any T which is contained in a proper parabolic subgroup P we have $(\chi, R_T^\theta)=0$. For if $P=L.U_P$ is a Levi decomposition of P then by [2, 8.2] $R_T^\theta=(R_{T,L}^\theta{}^*)^{G^F}$ where * denotes the lift from L to P, using the projection $U_P\to P\to L$. Thus by the cusp condition for P, $(\chi, R_T^\theta)=0$. Hence if χ is cuspidal, only R_T^θ with T minisotropic can occur with non zero coefficient on the right hand side of the above expression.

On the other hand by [2, §10.7] the irreducible components of Γ are parametrized by the geometric conjugacy classes x of pairs (T, θ), and we have

$$\chi = \rho_x = \sum \frac{(\pm 1)}{(R_T^\theta, R_T^\theta)} R_T^\theta$$

where the sum is precisely over the G^F-conjugacy classes of pairs (T, θ) which lie in the geometric conjugacy class x. Further, each geometric conjugacy class contains a "maximally split" pair (T, θ) [2, 5.25] and by [2, 5.27] if (T, θ) is maximally split and T is minisotropic then θ is non-singular. Hence if $\chi=\rho_x$ is cuspidal, the geometric conjugacy class x contains a (maximally split) pair (T, θ) such that T is minisotropic and θ is non-singular. We show finally that for such a pair (T, θ) the geometric conjugacy class coincides with the G^F-conjugacy class, which will complete the proof.

From [2, 5.21. 5,5.24] there is a bijection $(T, \theta)\to(T', \theta')$ between G^F-conjugacy classes of pairs (T, θ) where T is an F-stable maximal torus of G and θ is a character of T^F and G^{*F}-conjugacy classes of pairs (T', θ') where T' is an F-stable maximal torus of the dual group G^* of G and θ' is an element of T' such that (T_1, θ_1) and (T_2, θ_2) are geometrically conjugate if and only if θ_1' and θ_2' are G^{*F}-conjugate. Now if in (T, θ) we have that θ is non-singular, then in the corresponding G^{*F}-class of (T', θ') we have θ' is regular, and is therefore contained in a unique maximal torus of G^*, viz. T'. Hence for such (T, θ) if (T_1, θ_1) is in its geometric conjugacy class then $\theta_1'=ad\, g(\theta')$ for some $g\in G^{*F}$ and by regularity, we have $T_1'=ad\, g(T')$, whence (T', θ') and $(T_1, '\theta_1')$ are G^{*F}-conjugate and so (T, θ) and (T_1, θ_1) are G^F-conjugate.

Consequently we have $\chi=\dfrac{\pm 1}{(R_T^\theta, R_T^\theta)} R_T^\theta$ where T is minisotropic and θ is non-singular, and the result follows since for such (T, θ), $(R_T^\theta, R_T^\theta)=1$.

EXAMPLE. The discrete series characters $J^{\langle\theta\rangle}$ of $GL(n, q)$ (see [4]) are of the form in 3.6 above. They correspond to a character $\theta\in(F_{q^n}^*)^\wedge$. Here $W(\bar{T})^F$ is cyclic of order n, and acts via $\theta\to\theta^{q^i}$. Hence θ is in general position if θ has period n under the Frobenius map. If $G^F=SL(n, q)$, we have $T^F=(\bar{T})^{q-1}$; thus for θ in general position, if $\theta\,|_{T^F}$ has (reduced) period m under Frobenius, then $W_T^F(\theta)\simeq \boldsymbol{Z}/(n/m)\boldsymbol{Z}$, and n/m divides $d=(n, q-1)$. This case has been treated in [4].

4. Character values and orbits of regular linear characters

The remainder of this work is directed towards the computation of the value of an irreducible cuspidal character of G^F on a regular unipotent element. Since the regular unipotent elements of G^F are not all conjugate, we shall actually compute the set of values on the various classes of regular unipotents. We assume henceforth that the *characteristic p of $\boldsymbol{F}_q$ is good for G.* With this assumption, it was proved in [5] that

Theorem 4.1. *If U is an F-stable maximal unipotent subgroup of G, λ is an irreducible complex character of U^F of degree greater than* 1, *and u is a regular unipotent element of G contained in U, then* $\lambda(u)=0$.

Thus for any irreducible character μ of G^F, to compute $\mu(u)$ it is sufficient to compute $\sum m_\chi \chi(u)$ over the linear characters χ of U which are constituents of $\mu\,|_{U^F}$ (here the multiplicity of χ in $\mu\,|_{U^F}$ is m_χ). For the case where the centre is connected we have the following statements (which were proved in [6] for the adjoint case - i.e. for trival centre, but the general case has the same proof):

Theorem 4.2. *Let ρ be a cuspidal irreducible character of $\bar{G}^F$. Then $\rho\,|_{U^F}$ either contains no linear character, or has linear content precisely the sum of all the regular linear characters, each occuring with multiplicity one. The two situations correspond to $(\rho, \Gamma)=0$ or $(\rho, \Gamma)=1$, where Γ is the Gelfand-Graev character.*

The connection between this and cuspidal characters of G^F is established by

Proposition 4.3. *μ is an irreducible cuspidal character of $G^F \Leftrightarrow$ each compo-*

nent of the induced character $\mu^{\bar{G}^F}$ is cuspidal $\Leftrightarrow \mu$ is a component of $\rho|_{G^F}$ for an irreducible cuspidal character ρ of $\bar{G}^F$.

Proof. For a proper character ρ of $\bar{G}^F$, ρ cupsidal means $(\rho, 1_{V^F}^{\bar{G}^F})=0$ for each unipotent radical V of an F-stable parabolic subgroup of $\bar{G}$. Now μ (on G^F) is cuspidal $\Leftrightarrow (\mu, 1_{V^F}^{G^F})=0$ for the same set of unipotent radicals V (since these are contained in G). Further $(\mu^{\bar{G}^F}, 1_{V^F}^{\bar{G}^F})=\sum(\mu, 1_{gV^Fg^{-1}}^{G^F})_{G^F\cap gG^Fg^{-1}}$ by the Mackey formula (where the sum is over a set of $\bar{G}^F/G^F$ coset representatives). But gVg^{-1} is a unipotent radical if V is; hence if μ is a cusp form, so is $\mu^{\bar{G}^F}$. The converse is simple. The second equivalence follows by Frobenius reciprocity.

Now 4.1 and 4.2 enable one to compute (see [5] §4) $\rho(u)$ for ρ an irreducible cuspidal character of $\bar{G}^F$ and u regular unipotent (note that in good characteristic, the regular unipotent elements form a single conjugacy class if the centre is connected): referring to the notation in the preamble to 3.1, we have $\rho(u)=0$ (if $(\rho, \Gamma)=0$) or $\rho(u)=\sum_{\chi_i\in(U_i^F)^\wedge}(\prod_{i\in I}\chi_i(1))=(-1)^{|I|}$ if $(\rho, \Gamma)=1$. Thus $\rho(u)=(-1)^{|I|}(\rho, \Gamma)$. In view of 4.3, to evaluate $\mu(u)$ for μ an irreducible cuspidal character of G^F and u regular unipotent we can assume that μ is a component of $\rho|_{G^F}$ where ρ is an irreducible cuspidal character of $\bar{G}^F$. If $(\rho, \Gamma)=0$, then $\mu|_{U^F}$ contains on linear character of U^F and so $\mu(u)=0$. Thus we can restrict ourselves to the case where $(\rho, \Gamma)=1$. We first record

Lemma 4.4. *The regular unipotent class of $\bar{G}^F$ splits into $d=|Z/(F-1)Z|$ classes in G^F. These are permuted regularly by $\bar{T}^F/T^F\bar{Z}^F\cong Z/(F-1)Z$ (where T, $\bar{T}$ are as in* 1.3).

Proof. Let u be a regular unipotent element in U^F. Then the centralizer $G_u=Z.U_u$ and its connected component is $G_u^0=U_u$. Thus $G_u/G_u^0=Z$ and so the class of u splits into $|H^1(F,Z)|=|Z/(F-1)Z|=d$ classes in G^F by [7], §3.4. These d classes are clearly conjugate under $\bar{T}^F$ since $\bar{G}^F=G^F\bar{T}^F$, and each is stable under $T^F\bar{Z}^F$. Hence the second statement. Henceforth let $\bar{B}=\bar{T}U$ be an F-stable Borel subgroup of $\bar{G}$, $\bar{T}$ an F-stable maximal torus of $\bar{B}$ and U the unipotent radical of $\bar{B}$.

Proposition 4.5. *The group T^F has d orbits on the set of regular linear characters of U^F. These are permuted regularly by $\bar{T}^F/T^F\bar{Z}^F$.*

Proof. Consider the exact sequence 1.2.2:
$1\to Z^F\to T^F\to(T/Z)^F\to Z/(F-1)Z\to 1$. We see that T^F/Z^F is a subgroup of $(T/Z)^F$ of index d. But (T/Z) is an F-stable maximal torus of the adjoint group of G. Hence by Theorem C' of [5] $(T/Z)^F$ acts regularly on the regular linear characters of U^F. Thus T^F/Z^F, and hence T^F, has d orbits. The second statement follows from the isomorphism $(T/Z)^F\cong\bar{T}^F/\bar{Z}^F$.

From 4.4 and 4.5 we have labellings of the regular unipotent classes c_z and orbits Ω_z of regular linear characters ($z\in\bar{T}^F/T^F\bar{Z}^F\cong Z/(F-1)Z$) such that for $z'\in Z/(F-1)Z$, $(c_z)^{z'}=ad\ z'(c_z)=c_{zz'}$ and $(\Omega_z)^{z'}=ad\ z'(\Omega_z)=\Omega_{zz'}$.

Now for a cuspidal character ρ of $\bar{G}^F$ such that $(\rho, \Gamma)=1$, if μ is a component of $\rho|_{G^F}$ then the linear characters in $\mu|_{U^F}$ from a union of T^F-orbits of regular linear characters of U^F; in fact we have

Proposition 4.6. *Let $\bar{T}^F(\mu)$ be the stabilizer of μ in $\bar{T}^F$ (c.f. 2.7). Then the linear content of $\mu|_{U^F}$ is precisely a $\bar{T}^F(\mu)$-orbit of regular linear characters of U^F.*

Proof. $\bar{T}^F(\mu)$ contains $T^F\bar{Z}^F$ and since $(\rho, \Gamma)=1$, by 3.1, 2.3 and 2.7, $\bar{T}^F/\bar{T}^F(\mu)$ permutes the $\bar{T}^F$ conjugates of μ regularly. By 4.5, $\bar{T}^F/\bar{T}^F(\mu)$ also permutes the $\bar{T}^F(\mu)$-orbits of regular linear characters regularly. The result follows.

The isotropy group $\bar{T}^F(\mu)$ corresponds to a unique subgroup $X(\mu)$ of $H^1(F, Z)$, viz. $X(\mu)=\bar{T}^F(\mu)/\bar{T}^F\bar{Z}^F$ and a $T^F(\mu)$-orbit of regular linear characters of U^F is a union of the Ω_z, taken over a coset of $X(\mu)$. Thus the set of these orbits is permuted regularly by $Y(\mu)=H^1(F, Z)/X(\mu)$, as is the set of $\bar{G}^F$-conjugates of μ. We therefore label the former $\Omega_y=\bigcup_{z\in y}\Omega_z$ and latter as μ_y in such a way that Ω_y is the set of (regular) linear characters in the restriction $\mu_y|_{U^F}$.

Let c_z be a regular unipotent class of $G^F(z\in H^1(F, Z))$.

Lemma 4.7. *We have, with notation as above*

$$\mu_y(c_z) = \mu_{z^{-1}y}(c_1)$$

where $H^1(F, Z)$ acts on $Y(\mu)=H^1(F, Z)/X(\mu)$ in the obvious way.

This is clear. It shows that the computation of the various character values can be reduced to the evaluation of the set of conjugates of μ at a fixed regular unipotent element. One has one such sum for each $y\in Y(\mu)$ and explicitly, for a fixed regular unipotent u

$$\mu_y(u) = \sum_{\chi\in\Omega_y}\chi(u) = \sum_{z\in y}\sum_{\chi\in\Omega_z}\chi(u) \tag{4.7.1}$$

where y is regarded as a coset of $X(\mu)$ in $H^1(F, Z)$. The latter formula shows that to compute $\mu(u')$ for μ a component of $\rho|_{G^F}$ and u' regular unipotent it suffices to compute

(a) The group $X(\mu)=\bar{T}^F(\mu)/T^F\bar{Z}^F\leq H^1(F, Z)$.

(b) The sums $\sum_{\chi\in\Omega_z}\chi(u)=\sigma_z$, for u a fixed regular unipotent element of G^F ($z\in H^1(F, Z)$).

For the characters which concern us, the group $X(\mu)$ has already been determined in terms of the Weyl group in 3.5 and 3.6, given the characterization of cuspidal characters which are components of Gelfand-Graev furnished by Theorem 3.7.

We now show that the computation of the sums (b) can be reduced to the case where G is simply connected. Let $\pi: \tilde{G} \to G$ be the simply connected covering of G and let $\tilde{T} = \pi^{-1}(T)$. Then $\tilde{T}$ is an F-stable maximal torus of $\tilde{G}$ and $\tilde{T}^F$ acts on the regular linear characters of U^F ($\pi^{-1}(U^F) \cong U^F$). Writing $\tilde{Z} = \ker \pi$, we have from 1.1.1:

$$1 \to \tilde{Z}^F \to \tilde{T}^F \to (\tilde{T}/\tilde{Z})^F = T^F \to H^1(F, \tilde{Z}) \to 1 .$$

Hence $\tilde{T}^F/\tilde{Z}^F$ is a subgroup of T^F of index $|H^1(F, \tilde{Z})|$, and contains Z^F since $\tilde{T}$ contains $Z(\tilde{G})$. Thus a T^F-orbit of regular linear characters is a union of $|H^1(F, \tilde{Z})|$ $\tilde{T}^F$-orbits; moreover the $\tilde{T}^F$-orbits are permuted regularly by $H^1(F, Z(\tilde{G}))$ (apply 4.5 to $\tilde{G}$), of which $H^1(F, \tilde{Z})$ is a subgroup. For $s \in H^1(F, Z(\tilde{G}))$,write σ_s for the sum corresponding to (b) (consider $\tilde{T}^F$-orbits); then for $z \in H^1(F, Z) = H^1(F, Z(\tilde{G}))/H^1(F, \tilde{Z})$ we have $\sigma_z = \sum_{s \in z} \sigma_s$ (z regarded as a coset of $H^1(F, \tilde{Z})$). More explicitly, we have the following chain of subgroups: $(\tilde{T}/Z(\tilde{G}))^F \geq T^F/Z^F \geq \tilde{T}^F/Z(\tilde{G})^F$ and the successive quotients are $H^1(F, Z)$ and $H^1(F, \tilde{Z})$. Now $(\tilde{T}/Z(\tilde{G}))^F$ acts transitively on the regular linear characters and the subgroups have orbits as described above.

To compute the sums σ_z ($z \in H^1(F, Z)$) it is therefore sufficient to compute the σ_s ($s \in H^1(F, Z(\tilde{G}))$ and add these over $H^1(F, \tilde{Z})$-cosets; thus it suffices to consider the case $G = \tilde{G}$, i.e. we may assume G is simply connected.

We summarize some of the results of this section in

Theorem 4.8. *Let G be a semi-simple group and let u be a regular unipotent element of U^F (notation as above). Suppose μ is an irreducible cuspidal character of G^F which is a component of $\rho|_{G^F}$ where ρ is an irreducible (cuspidal) character of $\tilde{G}^F$ such that $\rho(u) \neq 0$. Then μ defines a subgroup $X(\mu)$ of $H^1(F, Z(\tilde{G}))$ ($\tilde{G}$ being the universal cover of G) and the values of the $|H^1(F, Z)|$ components of $\rho|_{G^F}$ at u are given by the sums $\sum_{s \in zX(\mu)} \sigma_s$ corresponding to cosets of $X(\mu)$ in $H^1(F, Z(\tilde{G}))$ and $\sigma_s = \sum_{\chi \in \Omega_s} \chi(u)$, Ω_s being a T^F-orbit of regular linear characters of U^F.*

Fixing u has the advantage that the ρ_z may be regarded as being function values of a function $\sigma: H^1(F, Z) \to \boldsymbol{C}$. These values will later be computed by taking a Fourier transform of the function σ.

5. Some character sums-the split case. The remainder of this work is addressed to the problem of evaluating the sums

$$\sum_{\chi \in \Omega_z} \chi(u) = \sigma_z$$

where $z \in H^1(F, Z)$, u is a fixed regular unipotent element of U^F and the sum is over the regular linear characters of U^F in the T^F-orbit Ω_z. The result 4.8 shows that one may confine attention to the simply connected case. In this section we make the additional assumption that the torus T is $\boldsymbol{F}_q$-split, i.e. all of the characters of T are defined over $\boldsymbol{F}_q$. [Notation: $B=TU$ is an F-stable Borel subgroup as in the previous sections.] We denote by k the algebraic closure $\bar{\boldsymbol{F}}_q$ of $\boldsymbol{F}_q$. In the split case T^F is just the set of $\boldsymbol{F}_q$-rational points of T.

Now suppose that G is simply connected and that T is $\boldsymbol{F}_q$-split. Let $\alpha_1, \cdots, \alpha_l$ be the fundamental roots of G with respect to T and let $\omega_1, \cdots, \omega_l$ be the fundamental weights. Consider the maps

$$\begin{aligned} \omega\colon\ & T \to \textstyle\prod^l k^*;\ \omega(t) = (\omega_1(t), \cdots, \omega_l(t)) \\ \alpha\colon\ & T \to \textstyle\prod^l k^*;\ \alpha(t) = (\alpha_1(t), \cdots, \alpha_l(t))\,. \end{aligned}$$

Suppose $\alpha_i=\sum_{j=1}^{l} a_{ij}\omega_j\ (a_{ij}\in\boldsymbol{Z})$. Define
$\tilde{\alpha}\colon \prod^l \boldsymbol{F}_q^* \to \prod^l \boldsymbol{F}_q^*$ by $\tilde{\alpha}(t_1, \cdots, t_l)=(s_1, \cdots, s_l)$ where $s_i=\prod_{j=1}^{l} t_j^{a_{ij}}$.

We then have:

Proposition 5.1. (i) *ω is an $\boldsymbol{F}_q$-rational isomorphism*: $T\to\prod^l k^*$.
(ii) *$\omega|_{T^F}$ is an isomorphism*: $T^F\to\prod^l\boldsymbol{F}_q^*$.
(iii) *The following diagram commutes*:

$$\begin{tikzcd} T^F \arrow[r, "\omega"] \arrow[d, "\alpha"'] & \prod^l \boldsymbol{F}_q^* \arrow[dl, "\tilde{\alpha}"'] \\ \prod^l \boldsymbol{F}_q^* & \end{tikzcd}$$

Proof. (i) follows because T is $\boldsymbol{F}_q$-split and the ω_i form a basis of the character group $X(T)$.

(ii) follows since ω is $\boldsymbol{F}_q$-rational; hence ω maps the $\boldsymbol{F}_q$-rational points of T onto the $\boldsymbol{F}_q$-rational points of $\prod^l k^*$, viz. onto $\prod^l \boldsymbol{F}_q^*$.

(iii) is the definition of the map $\tilde{\alpha}$: we have

$$s_i = \prod_{j=1}^{l} t_j^{a_{ij}} = \sum_{j=1}^{l} a_{ij}\omega_j(t) = \alpha_i(t)\,.$$

Henceforth "Im(α)" will refer to $\alpha(T^F)$.

The relevance of this to evaluating the sums σ_z is as follows: a linear character χ of U^F is given by a sequence $\chi=(\chi_1, \cdots, \chi_l)$ where $\chi_i\in(U^F_{\alpha_i})^\wedge=(\boldsymbol{F}_q^+)^\wedge$, and χ is regular if and only if χ_i is non-trivial for each i ([6], §3, recall that we are assuming good characteristic). Since $\boldsymbol{F}_q$ is self dual, we may obtain all the characters in $(\boldsymbol{F}_q^+)^\wedge$ as follows: fix a non-trivial $\lambda\in(\boldsymbol{F}_q^+)^\wedge$ and then any $\lambda'\in(\boldsymbol{F}_q^+)^\wedge$ is given by $\lambda'=\lambda^a\ (a\in\boldsymbol{F}_q)$ where $\lambda^a(x)=\lambda(ax)$. Now we have seen (4.7) that we may choose the regular unipotent element u arbitrarily. Thus we take

$$u = \prod_{i=1}^{l} x_{\alpha_i}(1).$$

Hence for $\chi=(\chi_1, \cdots, \chi_l)$ we have

$$\chi(u) = \chi_1(1)\chi_2(1) \cdots \chi_l(1).$$

Moreover for $t \in \bar{T}^F$,

(5.2) $$\chi^t(u) = \chi_1(\alpha_1(t))\chi_2(\alpha_2(t)) \cdots \chi_l(\alpha_l(t))$$

Thus we have

(5.3) $$(\chi_1, \cdots, \chi_l)^t = (\chi_1^{\alpha_1(t)} \cdots \chi_l^{\alpha_l(t)}).$$

Now choose a set $\{z\}$ of representatives for $T^F\bar{Z}^F$ in $\bar{T}^F$ (recall $\bar{T}^F/T^F/\bar{Z}^F \cong H^1(F, Z)=Z/(F-1)Z$). Then for any fixed regular linear χ, the set $\{\chi^z \mid z \in H^1(F, Z)\}$ is a set of representatives for the T^F-orbits of regular linear characters. Thus by (5.2)

(5.4) $$\begin{cases} \sigma_z = \sum_{t \in T^F} \chi^z(\alpha_1(t), \cdots, \alpha_l(t)) \\ \quad = \sum_{(s_1, \cdots, s_l) \in Im(\alpha)} \chi^z(s_1, \cdots, s_l) \\ \quad = \sum_{(s_1, \cdots, s_l) \in Im(\alpha)} \chi_1(s_1) \cdots \chi_l^z(s_l). \end{cases}$$

Proposition 5.5. *We have $\prod^l \boldsymbol{F}_q^*/Im\ \alpha \cong H^1(F, Z)=Z/(F-1)Z$.*

Proof. We have an exact sequence $1 \to Z \to T^\alpha \to \prod^l k^* \to 1$, since $Im\ \alpha$ is closed and has dimension l. Hence as in 1.1.1 we have the following exact sequence:

$$1 \to Z^F \to T^F \xrightarrow{\alpha} (\textstyle\prod^l k^*)^F = \prod^l \boldsymbol{F}_q^* \to H^1(F, Z) \to 1.$$

Hence $\prod^l \boldsymbol{F}_q^*/Im\ \alpha \cong H^1(F, Z)$ and the result follows.

We can therefore label the cosets of $Im\ \alpha$ in $\prod^l \boldsymbol{F}_q^*$ as $C_z\ (z \in H^1(F, Z))$. Comparing with (5.4) we get

(5.6) $$\sigma_z = \sum_{(s_1, \cdots, s_l) \in C_z} \chi_1(s_1) \cdots \chi_l(s_l)$$

Here $\chi=(\chi_1, \cdots, \chi_l)$ is an arbitrary regular linear character of $(U/U.)^F$, and thus we may take $\chi=(\chi, \cdots, \chi)$ (by abusing notation) where χ (on the right hand side) is a non-trivial element of $(\boldsymbol{F}_q^+)^\wedge$.

We then have

(5.7) $$\begin{aligned} \sigma_z &= \sum_{(s_1, \cdots, r_l) \in C_z} \chi(s_1)\chi(s_2) \cdots \chi(s_l) \\ &= \sum_{(s_1, \cdots, s_l) \in C_z} \chi(s_1 + \cdots + s_l). \end{aligned}$$

Note that changing χ or the labelling C_z has the effect of changing the labelling σ_z by a translation by an element of $H^1(F, Z)$.

6. Fourier transforms and products of Gaussian sums. In this section we show that the sums σ_z can be computed by taking Fourier transforms $\sum_z \psi(z)\sigma_z$ (where $\psi \in H^1(F, Z)^\wedge$) and evaluating these as products of certain Gaussian sums. We proceed now to compute certain of these sums and in the next section show how to apply the computations to the simply connected classical groups.

For this section, we introduce the following notation: g is a generator of F_q^*; we assume $d\,|\,(q-1)$ and denote by C_i the coset $g^i F_q^{*d}$ of F_q^{*d} in F_q^*; note that $F_q^*/F_q^{*d} \cong Z/dZ$. We fix (as in §5) a non-trivial character $\chi \in (F_q^+)^\wedge$ and write $\Delta_i = \sum_{s\in C_i} \chi(s)$.

Let $\psi \in (Z/dZ)^\wedge$. Then $\mathcal{G}(\psi) = \sum_{i\in Z/dZ} \psi(i)\Delta_i = \sum_{s\in F_q^*} \psi(s)\chi(s)$ where ψ is regarded as a character of F_q^* with kernel containing F_q^*. Thus $\mathcal{G}(\psi)$ is a Gauss sum over F_q. We shall evaluate the sums $\sigma_i = \sum_{\pi s_j^{n_j} \in C_i} \chi(s_1)\chi(s_2)\cdots\chi(s_l)$ in terms of the Gauss sums $\mathcal{G}(\psi)$ for a fixed integer sequence $n_1, \cdots, n_l$.

Theorem 6.1. *With notation as above, let $\sigma_i = \sum \chi(s_1)\chi(s_2)\cdots\chi(s_l)$, the sum being over $(s_1, \cdots, s_l) \in \prod^l F_q^*$ such that $\prod_j s_j^{n_j} \in C_i$. For any $\psi \in (Z/dZ)^\wedge$ write $\sigma(\psi) = \sum_{i\in Z/dZ} \psi(i)\sigma_i$ (this is the Fourier transform of the function σ). If one of the integers n_j is ± 1 mod d then*

$$\sigma(\psi) = \mathcal{G}(\psi^{n_1})\,\mathcal{G}(\psi^{n_2})\cdots\mathcal{G}(\psi^{n_l})$$

where $\Delta(\varphi) = \sum_{i\in Z/dZ} \varphi(i)\Delta_i = \sum_{s\in F_q^*} \varphi(s)\chi(s)$.

Proof. We have
$$\begin{aligned}\sigma_i &= \sum_{k_1\in Z/dZ} \sum_{s_1\in C_{k_1}} \chi(s_1) \sum_{\substack{\prod s_j^{n_j}\in C_{i-n_1k_1}\\ j\geq 2}} \chi(s_2)\cdots\chi(s_l)\\ &= \sum_{k_1\in Z/dZ} \Delta_{k_1} \sum_{\prod s_j^{n}\in C_{i-n_1k_1}} \chi(s_2)\cdots\chi(s_l)\\ &= \cdots \text{ (repeating the above procedure)}\\ &= \sum_{k_1,\cdots,k_{l-1}\in Z/dZ} \Delta_{k_1}\Delta_{k_2}\cdots\Delta_{k_{l-1}} \sum_{s_l^{n_l}\in C_{i-n_1k_1-n_2k_2\cdots n_{l-1}k_{l-1}}} \chi(s_l)\,.\end{aligned}$$

Now by assumption one of the n_j is ± 1 mod d; we may without loss take $n_l \equiv \pm 1$ mod d. To fix ideas take $n_l \equiv 1$ mod d (the proof for $n_l \equiv -1$ mod d is the same). Then we have

$$\sigma_i = \sum_{k_1,\cdots,k_{l-1}\in Z/dZ} \Delta_{k_1}, \Delta_{k_2}\cdots\Delta_{k_{l-1}}\Delta_{i-n_1k_1-n_2k_2-\cdots n_{l-1}k_{l-1}}\,.$$

Hence

$$\sigma(\psi) = \sum \psi(i)\sigma_i \sum_{k_1,\cdots k_{l-1}} \mathfrak{s}_{k_1} \cdots \mathfrak{s}_{k_{l-1}} \sum_{i \in Z/dZ} \psi(i) \mathfrak{s}_{i-n_1k_1-\cdots-n_{l-1}k_{l-1}} .$$

But $\psi(i) = \psi(i-n_1k_1-\cdots-n_{l-1}k_{l-1})\psi^{n_1}(k_1)\psi^{n_2}(k_2)\cdots\psi^{n_{l-1}}(k_{l-1})$.

Hence
$$\begin{aligned}\sigma(\psi) &= \sum_{k_1,\cdots,k_{l-1}} \psi^{n_1}(k_1)\mathfrak{s}_{k_1}\cdots\psi^{n_{l-1}}(k_{l-1})\mathfrak{s}_{k_{l-1}} \sum_i \psi(i-n_1l_1-\cdots-n_{l-1}k_{l-1}) \\ &= \mathcal{G}(\psi^{n_1})\,\mathcal{G}(\psi^{n_2})\cdots\mathcal{G}(\psi^{n_{l-1}})\,\mathcal{G}(\psi^{n_l}) \quad \text{since } n_l \equiv 1 \bmod d.\end{aligned}$$

Corollary 6.2. *We have (with notation as in* 6.1)

$$\sigma_i = \frac{1}{d} \sum_{\psi \in (Z/dZ)^\wedge} \bar{\psi}(i)\mathcal{G}(\psi^{n_1}) \cdots \mathcal{G}(\psi^{n_l}) .$$

This is obtained by simply inverting the formula $\sigma(\psi) = \sum_{i \in Z/dZ} \psi(i)\sigma_i$ and using 6.1.

A useful result for evaluating these sums is the following

Lemma 6.3. *We have*

$$\mathcal{G}(\psi)\,\mathcal{G}(\psi^{-1}) = \psi(-1)\cdot q .$$

This is well-known and simple to verify.

EXAMPLE 6.4. Because of their relevance for some of the groups considered in the next section, we give some explicit computations for the case $d=2$. Here we have two cosets C_0 and C_1 of F_q^{*2} with corresponding sums $\mathfrak{s}_0$ and $\mathfrak{s}_1$. There are two characters, 1 and $\mathcal{E}$, of $Z/2Z$ and we have

$$\mathcal{G}(1) = \mathfrak{s}_0 + \mathfrak{s}_1 = -1$$
$$\mathcal{G}(\mathcal{E}) = \mathfrak{s}_0 - \mathfrak{s}_1 .$$

Now $\mathfrak{s}_0$ and $\mathfrak{s}_1$ may easily be computed by evaluating the sum $\sum_{a \pm 0} \sum_{s \in F_q^{*2}} \chi(a(1+s)) = \sum_{\chi \in (F_q^*)^\wedge - \{1\}} \sum_{s \in F_q^{*2}} \chi(a(1+s))$ in two different ways. One obtains $\mathfrak{s}_0 - \mathfrak{s}_1 = \pm\sqrt{(-1)^{(q-1)/2}q}$, the sign being determined by the original choice of χ. We assume χ is chosen so that $\mathfrak{s}_0 - \mathfrak{s}_1 = \sqrt{(-1)^{(q-1)/2}q}$.

Suppose now that, of the integers n_j in 6.1, e are even and e' are odd (of course $e+e'=l$). Then $\sigma(1)=(-1)^l$ $\sigma(\mathcal{E})=(-1)^e$. $(\mathcal{G}(\mathcal{E}))^{e'}=(-1)^{e+e'}$. $((-1)^{(q-1)/2}q)^{e'/2}$.

From (6.2) we therefore have

$$\text{(6.4)} \quad \begin{aligned}\sigma_0 &= \frac{1}{2}((-1)^l + (-1)^l(\sqrt{(-1)^{(q-1)/2}q})^{e'}) = \frac{(-1)^l}{2}(1 + (\sqrt{(-1)^{(q-1)/2}q})^{e'}) \\ \text{and } \sigma_1 &= \frac{1}{2}((-1)^l - (-1)^l(\sqrt{(-1)^{(q-1)/2}q})^{e'}) = \frac{(-1)^l}{2}(1 - (\sqrt{(-1)^{(q-1)/2}q})^{e'})\end{aligned}$$

7. The classical groups (split case). We give explicit computations of the sums σ_z for the split simply connected classical groups - i.e. the simply connected groups (in good characteristic) with irreducible root system of type A,B,C or D. The crux of the computation is to identify the subgroup $Im\ \tilde{\alpha}$ of $\prod^l \boldsymbol{F}_q^*$ (see (5.1) and (5.7)) in a form amenable to the computations of § 6.

Lemma 7.1. *Let L be a lattice in $\boldsymbol{R}^l$ and let M be a sublattice such that L/M is a finite cyclic group of order n. Suppose $\gamma_1, \cdots, \gamma_l$ is a basis of M. Then there exists a $\boldsymbol{Z}$-linear combination $\gamma=\sum n_i\gamma_i$ such that $\gamma\in nL$ and gcd $(n_1, \cdots, n_l)=1$.*

Proof. By the elementary divisor theorem, there is a basis $\beta_1, \cdots, \beta_l$ of L such that $n\beta_1, \beta_2, \cdots, \beta_l$ is a basis of M. We take $\gamma=n\beta_1$. This is clearly a $\boldsymbol{Z}$-linear combination of the γ_i and $gcd\,(n_1, \cdots, n_l)=1$ since γ is a member of a basis of M.

Now take $L=P(R)$, $M=G(R)$, the lattices of weights and roots of G respectively. Then since G is simply connected, $L/M\cong Z$. Thus in case Z is cyclic, (7.1) applies. Suppose $\{\alpha_1, \cdots, \alpha_l\}$ are the fundamental roots (a basis of $M=Q(R)$) and let $\gamma=\sum n_i\alpha_i$ be as in Lemma 7.1.

Proposition 7.2. *Suppose $Z\cong\boldsymbol{Z}/n\boldsymbol{Z}$ and let $d=(n, q-1)$. With notation as in* §5, *we have $Im\ \alpha=\{(s_1, \cdots, s_l)\,|\,\prod s_j{}^{n_j}\in \boldsymbol{F}^{*d}\}$.*

Proof. Consider the homomorphism

$$\nu\colon \textstyle\prod^l \boldsymbol{F}_q^*\rightarrow \boldsymbol{F}_q^* \text{ given by } \nu(s_1, \cdots, s_l)=s_1{}^{n_1}\cdots s_l{}^{n_l}\,.$$

Since g.c.d. $(n_1, \cdots, n_l)=1$, this is a surjection. Suppose $(s_1, \cdots, s_l)=(\alpha_1(t), \cdots, \alpha_l(t))\in Im\ \alpha(t\in T^F)$. Then $\nu(s_1, \cdots, s_l)=(\sum n_i\alpha_i)(t)=(n\omega)(t)$ for some $\omega\in P(R)$. Thus $\nu(s_1, \cdots, s_l)=\omega(t)^n\in \boldsymbol{F}_q^*$. Hence $\nu(Im\ \alpha)\leqslant \boldsymbol{F}_q^{*d})$. But the index of $Im\ \alpha$ in $\prod^l \boldsymbol{F}_q^*$ is d (see (5.5); $(q-1)Z=dZ$) and since ν is surjective we must have $Im\ \alpha=\nu^{-1}(\boldsymbol{F}_q^*)$.

Corollary 7.3. *The cosets C_i ($i\in\boldsymbol{Z}/d\boldsymbol{Z}$, see 5.5) of $Im\ \alpha$ in $\prod^l \boldsymbol{F}_q^*$ are $C_i=\{(s_1, \cdots, s_l)\,|\,\prod s_j{}^{n_j}\in g^i\boldsymbol{F}_q^{*d}\}$ where g is a generator of $\boldsymbol{F}_q^*$.*

This result, together with (6.2) reduces the computation (in the case where Z is cyclic) to finding the vector γ of (7.1). This leaves only the case of Spin $(4n)$ (which has root system of type D_{2n}), and we treat this separately.

We can now summarize our result for Z cyclic as follows:

Theorem 7.4. *Suppose G is a simply connected group with cyclic centre $Z\cong\boldsymbol{Z}/n\boldsymbol{Z}$, and assume that G is $\boldsymbol{F}_q$-split, so that $H^1(F, Z)=\boldsymbol{Z}/d\boldsymbol{Z}$, where $d=(n, q-1)$. Let μ be an irreducible discrete series character of G^F, and let u be a regular unipotent element of G^F. Let $\alpha_1, \cdots, \alpha_l$ be the fundamental roots of G and* *suppose* $\gamma=\sum_{i=1} n_i\alpha_i$ *s an element of* $Q(R)$ *as described in* (7.1) *and assume* $n_i\equiv\pm 1$ *mod* d *for some* j. *Write* (*for* $i\in \boldsymbol{Z}/d\boldsymbol{Z}$)

$$\sigma_i=\frac{1}{d}\sum_{\psi\in(\boldsymbol{Z}/d\boldsymbol{Z})^\wedge}\overline{\psi(i)}\,\mathcal{G}(\psi^{n_1})\,\mathcal{G}(\psi^{n_2})\cdots\mathcal{G}(\psi^{n_l})\,,$$

where for $\varphi\in(\boldsymbol{Z}/d\boldsymbol{Z})^\wedge$, $\mathcal{G}(\varphi)$ *is the Gaussian sum described in the preamble to* (6.1). *Then there is a subgroup* $X(\mu)$ *of* $\boldsymbol{Z}/d\boldsymbol{Z}$ *such that*

$$\mu(u)=\frac{|X(\mu)|(\Gamma,\mu)_{G^F}}{d}\sum_{i\in jX(\mu)}\sigma_i\,.$$

Here $jX(\mu)$ *is a coset of* $X(\mu)$ *and* Γ *is the Gelfand-Graev character of* $\bar{G}^F$.

We take $X(\mu)=\bar{T}^F(\mu)/T^F\bar{Z}^F$. $X(\mu)$ acts on the T^F-orbits of regular linear characters and hence on the σ_i. If $(\Gamma,\mu)_{G^F}=0$ then $\mu|_{U^F}$ contains no linear character and so $\mu(u)=0$. Otherwise the value on u of the sum of a T^F-orbit of regular linear characters is σ_i and $\mu(u)$ is a sum of these over an $X(\mu)$-orbit (4.8). The formula follows by observing that $(\Gamma,\mu)_{G^F}=(\Gamma,\mu^{\bar{G}^F})_{\bar{G}^F}=\dfrac{d}{|X(\mu)|}$ if $(\Gamma,\mu)_{G^F}\neq 0$.

REMARK. Notice that for any subgroup X of $\boldsymbol{Z}/d\boldsymbol{Z}$, say $X=\boldsymbol{Z}/e\boldsymbol{Z}$ where $e|d$, one can consider the X-orbits of σ_i, and we have $\sum_{i\in jX}\sigma_i=\frac{1}{f}\sum_{\psi\in\boldsymbol{Z}/f\boldsymbol{Z})^\wedge}\overline{\varphi(j)}\,\mathcal{G}(\varphi^{n_1})\cdots\mathcal{G}(\varphi^{n_l})$, where $e=\frac{d}{f}\left(\text{for } j\in\left(\frac{\boldsymbol{Z}/d\boldsymbol{Z}}{X}\right)\right)$ by (6.1). This may also be verified directly.

Corollary 7.5. $\mu(u)$ *is either* 0 *or if* $[\bar{T}^F:\bar{T}^F(\mu)]=f$ $\mu(u)=\frac{1}{f}\sum_{\psi\in(\boldsymbol{Z}/f\boldsymbol{Z})^\wedge}\overline{\psi(i)}\,\mathcal{G}(\psi^{n_1})\,\mathcal{G}(\psi^{n_2})\cdots\mathcal{G}(\psi^{n_l})$ *for some* $i\in\boldsymbol{Z}/f\boldsymbol{Z}$.

This follows directly from (7.4) and the remark above. We now give the n_i explicitly for the various classical groups with cyclic centre; a reference for the computations with the roots is Bourbaki: [1], Ch. VI, §4.

Type A_l: $G=SL(l+1)Z$ and thus $d=(l+1,q-1)$. The vector γ may be taken as $\gamma=\alpha_1+2\alpha_2+3\alpha_3+\cdots+l\alpha_l$. An easy computation show that $\gamma\in(l+1)P(R)$. The discrete series characters of SL are all of the type discussed in the example after 3.6. Thus the integer of (7.5) is $(l+1)/m$ (where m is the reduced period of θ) and we have, for any component μ of $J^{\langle\theta\rangle}|_{SL}$ and regular unipotent u (writing $n=l+1$),

$$\mu(u)=\frac{m}{n}\sum_{\psi\in(\boldsymbol{Z}/(^m/_n)\boldsymbol{Z})^\wedge}\overline{\psi(i)}\,\mathcal{G}(\psi)\,\mathcal{G}(\psi^2)\cdots\mathcal{G}(\psi^l)\left(i\in\boldsymbol{Z}/\left(\frac{n}{m}\right)\boldsymbol{Z}\right).$$

In particular if $n/m=2$ we get the formulae (6.4) with $e'=\left[\dfrac{l+1}{2}\right]$. For $l=1$, this is the well-known formula for $SL(2, q)$.

Using Lemma 6.3, together with fact that the powers i in ψ^i may be taken modulo $l+1$, the product $\mathcal{G}(\psi)\mathcal{G}(\psi^2)\cdots\mathcal{G}(\psi^l)$ may be simplified as follows: suppose the order of ψ is $r(|l+1)$; then

$$\mathcal{G}(\psi)\mathcal{G}(\psi^2)\cdots\mathcal{G}(\psi^{l+1}) = [\mathcal{G}(\psi)\cdots\mathcal{G}(\psi^r)]^{(l+1)/r} \quad \text{and}$$
$$\mathcal{G}(\psi)\mathcal{G}(\psi^2)\cdots\mathcal{G}(\psi^r) = \begin{cases} -\psi^{(r2-1)/8}(-1)q^{(r-1)/2} \text{ if } r \text{ is odd} \\ -\psi^{(r2-2r)/8}(-1)\,(-1)^{(q-1)/4}q^{(r-1)/2} \text{ if } r \text{ is even.} \end{cases}$$

Type B_l: $G=\mathrm{Spin}\,(2l+1, k)$, $G^F=\mathrm{Spin}\,(2l+1, q)$ (characteristic $\neq 2$). Here $Z=\boldsymbol{Z}/2\boldsymbol{Z}=H^1(F, Z)$. We take $\gamma=\alpha_1+2\alpha_2+\cdots+l\alpha_l\in 2P(R)$. $(=2\omega_l)$. Thus we have the same values as for A_l when $d=2$. We have $\mu(u)=0$ or $(-1)^l$ or σ_0 or σ_1 where σ_i is given by (6.4) with $e'=\left[\frac{l+1}{2}\right]$.

Type C_l: $G=Sp(2l, k)$, $G^F=Sp(2l, q)$ (characteristic $\neq 2$). Here again $Z=\boldsymbol{Z}/2\boldsymbol{Z}=H^1(F, Z)$. We take $\gamma=\alpha_l$. Thus $\mu(u)=0$ or $(-1)^l$ or σ_0 or σ_1 where σ_i are again given by (6.4), this time with $e'=1$. Thus Sp behaves as $SL(2)$.

Type D_l, l odd: $G=\mathrm{Spin}\ (2l, k)$ $G^F=\mathrm{Spin}^+(2l, q)$ (characteristic $\neq 2$). Here $Z=\boldsymbol{Z}/4\boldsymbol{Z}$ and we may take $\gamma=2\alpha_1+4\alpha_2+6\alpha_3+\cdots+2(l-2)\alpha_{l-2}+(2(l-1)+1)\alpha_{l-1}+(2l+1)\alpha_l$. There are 4 possible sums $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ which may be computed easily using (6.2) by replacing γ by $\gamma'=2\alpha_1+2\alpha_3+\cdots+2\alpha_{l-2}+\alpha_{l-1}+3\alpha_l$. Let ψ be a generator of $(\boldsymbol{Z}/4\boldsymbol{Z})^\wedge$. Then

$$\begin{aligned}\sigma_i &= (-1)^l+(\psi(i)+\overline{\psi(i)})(-1)^{(l-3)/2}\mathcal{G}(\psi^2))^{(l-3)/2}\mathcal{G}(\psi)\mathcal{G}(\psi^3)+\psi^2(i)(-1)^{l-2}\mathcal{G}(\psi^2) \\ &= (-1)^l+(-1)^l\mathcal{G}(\psi^2)+\delta_{i,0}\cdot 2(-1)^{(l-3)/}(\mathcal{G}(\psi^2))^{(l-3/2}\ell(\psi)\mathcal{G}(\psi^3)\end{aligned}$$

where $\delta_{i,0}$ is the Kronecker δ.

Using Lemma 6.3 we see that $\mathcal{G}(\psi)\mathcal{G}(\psi^3)=\psi(-1)q$. Further, $\mathcal{G}(\psi^2)$ is the $\mathcal{G}(\varepsilon)$ of (6.4), so that

$\mathcal{G}(\psi^2) = -\sqrt{(-1)^{p-1/2}q}$. Hence
$\sigma_i = (-1)^l(1+(-1)^{q-1/2}q)+2\delta_{i,0}(-1)^{(lq+q+7l-25)/4}q^{(l+1)/4}(q\equiv 1 \pmod 4))$.

To complete the computations, we now treat the case of D_l (l even), where $G=\mathrm{Spin}\ (2l, k)$ and $4|2l$. Here $Z=\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}\simeq H^1(F, Z)$, (since characteristic $\neq 2$), so that (7.1) does not apply directly. However we do have

$$\alpha_{l-1}+\alpha_l\in 2P(R)$$
$$\alpha_1+2\alpha_2+\cdots+(l-1)\alpha_{l-1}\in 2P(R)\,.$$

Hence in analogy with (6.2) we can consider the homomorphism ν: $\prod^l \boldsymbol{F}_q^{*l}\to \boldsymbol{F}_q^*\times\boldsymbol{F}_q^*$ given by

$$\nu(s_1, \cdots, s_l) = (s_{l-1}s_l, s_1s_2^2\cdots s_{l-1}^{l-1}) = (\nu_1(s_1, \cdots, s_l), \nu_2(s_1, \cdots, s_l))\,.$$

Again as in (7.2), one shows that $Im\ \alpha=\{(s_1,\cdots,s_l)\,|\,\nu(s_1,\cdots,s_l)\in \boldsymbol{F}_q^{*2}\times \boldsymbol{F}_q^{*2}\}$. Now clearly $(\boldsymbol{F}_q^*\times \boldsymbol{F}_q^*)/\boldsymbol{F}_q^{*2}\times \boldsymbol{F}_q^{*2}\cong \boldsymbol{Z}/2\boldsymbol{Z}\times \boldsymbol{Z}/2\boldsymbol{Z}$ and the cosets of $\boldsymbol{F}_q^{*2}\times \boldsymbol{F}_q^{*2}$ can be written $c_{ij}(i,j\in\{0,1\})$ where $c_{ij}=g^i\boldsymbol{F}_q^{*2}\times g^j\boldsymbol{F}_q^{*2}$ (g being a generator of $\boldsymbol{F}_q^*$). Thus the cosets C_{ij} of $Im\ \alpha$ are given by

$$C_{ij}=\{(s_1,\cdots,s_l)\,|\,\nu(s_1,\cdots,s_l)\in C_{ij}\}\ .$$

The sums which we have to evaluate are therefore

$$\sigma_{ij}=\sum_{(s_1\cdots s_l\in C_{ij}}\chi(s_1)\cdots\chi(s_l)\qquad (i,j\in\{0,1\})\,.$$

Now
$$\sigma_{i0}+\sigma_{i1}=\sum_{s_{l-1}s_l\in C_i}\chi(s_1)\cdots\chi(s_l)=\sigma_{i-}\quad\text{and}$$

$$\sigma_{0j}+\sigma_{1j}=\sum_{s_1s_2^2\cdots s_{l-1}^{l-1}\in c_j}\chi(s_1)\cdots\chi(s_l)=\sigma_{-j}$$

can be computed by (6.2).

Moreover
$$\begin{aligned}\sigma_{00}+\sigma_{11}&=\sum_{s_{l-1}s_l\equiv s_1s_3\cdots s_{l-1}\bmod \boldsymbol{F}_q^{*2}}\chi(s_1+\cdots+s_l)\\&=\sum_{s_1s_3s\cdots{}_{l-3}s_l\equiv 1\bmod \boldsymbol{F}_q^{*2}}\chi(s_1+\cdots+s_l)\\&=\sigma_{-0}\\&=\sigma_{10}+\sigma_{00}\,.\end{aligned}$$

Hence $\sigma_{10}=\sigma_{11}$.

Putting this information together with (6.3) we obtain (where $\mathscr{G}(\varepsilon)=\Delta_0-\Delta_1=-\sqrt{(-1)^{p-1/2}q}$ as in (6.3))

$$\sigma_{00}=\frac{(-1)^l}{4}(1+\mathscr{G}(\varepsilon)^2)+\frac{(-1)^{l/2}}{2}\mathscr{G}(\varepsilon)^{l/2}$$

$$\sigma_{01}=\frac{(-1)^l}{4}(1+\mathscr{G}(\varepsilon)^2)-\frac{(-1)^{l/2}}{2}\mathscr{G}(\varepsilon)^{l/2}$$

$$\sigma_{10}=\frac{(-1)^l}{4}(1-\mathscr{G}(\varepsilon)^2)$$

$$\sigma_{11}=\frac{(-1)^l}{4}(1-\mathscr{G}(\varepsilon)^2)\,.$$

This completes the computation for the split classical groups.

8. The non-split case. The methods of the previous three sections can be applied equally well to the non-split groups. In this section we briefly indicate how this is carried out and give results for the unitary groups. Throughout this section we suppose that G is simple and simply connected, and has a set of fundamental roots $\alpha_1,\cdots,\alpha_l$ and corresponding fundamental weights $\omega_1,\cdots,\omega_l$. In the non-split case, it can be shown (c.f. [9]) that there is a permutation τ and power q of the characteristic such that, if F^* is the dual of $F: T\to T$,

$$F^*\alpha_j = q\alpha_{\tau j} \tag{8.1}$$

Define $\alpha: T\to\prod^l k^*$ by $\alpha(t)=(\alpha_1(t), \cdots, \alpha_l(t))$ as in the split case. Since α has kernel Z and $F(Z)\subset Z$, the following diagram defines F': $\prod^l k^*\to\prod^l k^*$

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & \prod^l k^* \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ T & \xrightarrow{\alpha} & \prod^l k^* \end{array}$$

Moreover by 8.1, $(\prod^l k^*)^{F'}\cong \boldsymbol{F}_{l_1}^*\times\cdots\times \boldsymbol{F}_{l_1}^*$ where $l_1, \cdots, l_r$ are the sizes of the orbits of τ. By the argument of Lemma 1.1, we have an exact sequence

$$1 \to Z^F \to T^F \xrightarrow{\alpha} (\textstyle\prod^l k^*)^{F'} \to H^1(F, Z) \to 1 \tag{8.1.1}$$

and again we are faced with the problem of identifying $Im(\alpha)$ in (8.1.1). We show how this is done in the case of the finite unitary groups. The method in general is similar.

EXAMPLE 8.2. The unitary groups $SU(2n+1, q^2)$: Here $(\prod^l k^*)^{F'}\cong \boldsymbol{F}_{q^2}^*\times \boldsymbol{F}_{q^2}^*\times\cdots\times \boldsymbol{F}_{q^2}^*$ (n times) (since $G=SL$ and each l_i is 2), Consider the map γ: $(\prod^l k^*)^{F'}\to \boldsymbol{F}_{q^2}^*$ given by

$$\gamma(a_1, a_2, \cdots, a_n) = a_1^{1-q}a_2^{2(1-q)} \cdots a_n^{n(1-q)}\,.$$

Then $Im(\gamma)=K$ (the kernel of the norm: $\boldsymbol{F}_{q^2}\to \boldsymbol{F}_q$) and for $t\in Im\ \alpha$, $\gamma(t)=(2n+1)\omega(t)\in(\boldsymbol{F}_{q^2}^*)^{2n+1}$.

Thus for such t, $\gamma(t)\in(\boldsymbol{F}_{q^2}^*)^{2n+1}\cap K$. A simple computation shows that this latter group is K^{2n+1}, which has index $(q+1, 2n+1)$ in K. Now $H^1(F, Z)=Z/(F-1)Z=Z/(q+1)Z$ where Z is cyclic, of order $2n+1$. Thus $|H^1(F, Z)|=(q+1, 2n+1)$ and we have identified $Im\ \alpha$ in analogous fashion to the split case as: $Im\ \alpha=\{(a_1, \cdots, a_n)\,|\,\gamma(a_1, \cdots, a_n)\in K^{2n+1}\}$. The character values can now be computed using the results of §6.

EXAMPLE 8.3. The unitary groups $SU(2n, q^2)$: Here

$$(\textstyle\prod^l k^*)^{F'}\cong\prod^{(n-1)}\boldsymbol{F}_{q^2}^*\times \boldsymbol{F}_q^*.$$

Using the method above, it is straightforward to obtain:

$$\begin{aligned} Im\ \alpha = \{&(a_1, \cdots, a_{n-1}, a_n)\,|\,a_1, \cdots, a_{n-1}\in \boldsymbol{F}_{q^2}^*, a_n\in \boldsymbol{F}_q^* \text{ and} \\ &a_1a_2^2\cdots a_{n-1}^{n-1}\in \boldsymbol{F}_{q^2}^{*d} \text{ where } d=(2n,\ q+1)\}\,. \end{aligned}$$

Once again, the computations may now be carried out using §6.

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