A CHARACTERIZATION OF BOUNDED KRULL PRIME RINGS

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In [9] we defined the concept of non commutative Krull prime rings from the point of view of localizations and we mainly investigated the ideal theory in bounded Krull prime rings (cf. [9], [10]).

The purpose of this paper is to prove the following:

**Theorem.** Let $R$ be a prime Goldie ring with two-sided quotient ring $Q$. Then $R$ is a bounded Krull prime ring if and only if it satisfies the following conditions:

1. $R$ is a regular maximal order in $Q$ (in the sense of Asano).
2. $R$ satisfies the maximum condition for integral right and left $\nu$-ideals.
3. $R/P$ is a prime Goldie ring for any minimal prime ideal $P$ of $R$.

As corollary we have

**Corollary.** Let $R$ be a noetherian prime ring. If $R$ is a regular maximal order in $Q$, then it is a bounded Krull prime ring.

In case $R$ is a commutative domain, the theorem is well known and its proof is easy (cf. [11]). We shall prove the theorem by using properties of one-sided $\nu$-ideals and torsion theories.

Throughout this paper let $R$ be a prime Goldie ring, not artinian ring, having identity element 1, and let $Q$ be the two-sided quotient ring of $R$; $Q$ is a simple and artinian ring. We say that $R$ is an order in $Q$. If $R_1$ and $R_2$ are orders in $Q$, then they are called equivalent (in symbol: $R_1 \sim R_2$) if there exist regular elements $a_1, b_1, a_2, b_2$ of $Q$ such that $a_1Rb_1 \subseteq R_2$, $a_2Rb_2 \subseteq R_1$. An order in $Q$ is said to be maximal if it is a maximal element in the set of orders which are equivalent to $R$. A right $R$-submodule $I$ of $Q$ is called a right $R$-ideal provided $I$ contains a regular element of $Q$ and there is a regular element $b$ of $Q$ such that $bI \subseteq R$. $I$ is called integral if $I \subseteq R$. Left $R$-ideals are defined in a similar way. If $I$ is a right (left) $R$-ideal of $Q$, then $O_I(I) = \{x \in Q \mid xI \subseteq I\}$ is an order in $Q$ and is equivalent to $R$. Similarly $O_I(I) = \{x \in Q \mid Ix \subseteq I\}$ is an order in $Q$ and is equivalent to $R$. They are called a left order and a right order of $I$ respectively.
We define the inverse of \( I \) to be \( I^{-1} = \{ q \in \mathbb{Q} \mid Iq \subseteq O(I) \} \). Evidently \( I^{-1} = \{ q \in \mathbb{Q} \mid Iq \subseteq O(I) \} \). Following [2], we define \( I^* = (I^{-1})^{-1} \). If \( I = I^* \), then it is said to be a right (left) \( v \)-ideal. If \( R \) is a maximal order, then \( I^{-1} = I^{-1-1} \) and so \( I^{-1} \) is a left (right) \( v \)-ideal, and the concept of right (left) \( v \)-ideals coincides with one of right (left) \( v \)-ideals defined in [9]. So the mapping: \( I \rightarrow I^* \) of the set of all right (left) \( R \)-ideals into the set of all right (left) \( v \)-ideals is a *-operation in the sense of [9].

**Lemma 1.** Let \( R \) be a maximal order in \( \mathbb{Q} \) and let \( S \) be any order equivalent to \( R \). Then \( S \) is a maximal order if and only if \( S = O(I) \) for some right \( v \)-ideal \( I \) of \( \mathbb{Q} \).

**Proof.** If \( S = O(I) \) for some right \( v \)-ideal \( I \) of \( \mathbb{Q} \), then it is a maximal order by Satz 1.3 of [1]. Conversely assume that \( S \) is a maximal order, then there are regular elements \( c, d \) in \( R \) such that \( cSD \subseteq R \). So \( SD \) is a right \( R \)-ideal and is a left \( S \)-module. Hence \( (SD)^{-1} \) is a left \( R \)-ideal and is a right \( S \)-module. Similarly \( I = (SD)^{-1} \) is a right \( v \)-ideal and is a left \( S \)-module so that \( O(I) \subseteq S \). Hence \( S = O(I) \).

**Lemma 2.** Let \( R, S \) be maximal orders in \( \mathbb{Q} \) such that \( R \sim S \), and let \( \{ I_i \} \), \( I \) be right \( R \)-ideals. Then

1. If \( \cap_i I_i \) is a right \( R \)-ideal, then \( \cap_i I_i^* = (\cap_i I_i)^* \).
2. If \( \sum I_i \) is a right \( R \)-ideal, then \( \sum I_i^* = (\sum I_i)^* \).
3. If \( J \) is a left \( R \) and right \( S \)-ideal, then \( (IJ)^* = (I^*J)^* = (IJ)^* = (I^*J)^* \).
4. If \( I^{-1} \) is a right \( \Lambda \)-ideal and \( (I^*I)^* = R \) and \( (I^*I)^* = T \), where \( T = O(I^*) \).

**Proof.** The proofs of (1) and (2) are similar to ones of the corresponding results for commutative rings (cf. Proposition 26.2 of [4]).

To prove (3) assume that \( IJ \subseteq cS \), where \( c \) is a unit in \( \mathbb{Q} \). Then we have \( (I^*J) \subseteq cS \) and \( (IJ)^* \subseteq cS \), because

\[
(c^{-1}I)J \subseteq S \Rightarrow c^{-1}I \subseteq J^{-1} \Rightarrow c^{-1}IJ \subseteq J^{-1} \subseteq S \Rightarrow (IJ)^* \subseteq cS, \quad \text{and} \quad c^{-1}I \subseteq J^{-1} \Rightarrow c^{-1}IJ \subseteq c^{-1}I \subseteq cS. \]

Hence \( (IJ)^* \) contains \( (IJ)^* \) and \( (I^*J)^* \) by Proposition 4.1 of [9]. The converse inclusions are clear. Therefore we have \( (IJ)^* = (I^*J)^* = (IJ)^* \). From these it is clear that \( (IJ)^* = (I^*J)^* \).

To prove (4), assume that \( I^{-1} \) is a right \( \Lambda \)-ideal and \( c \) is a unit in \( \mathbb{Q} \). Then we have \( c^{-1}I^{-1} \subseteq I^{-1} \) so that \( c^{-1}I = O(I^{-1}) = R \) and thus \( R \subseteq cR \). Hence \( (I^*I)^* \subseteq R \) by Proposition 4.1 of [9]. The converse inclusion is clear. Therefore \( (I^*I)^* = R \). Similarly \( (I^*I)^* = T \).

Let \( R \) be a maximal order in \( \mathbb{Q} \). We denote by \( F^*(R) (F^*(R)) \) the set of right (left) \( v \)-ideals and let \( F^*(R) = F^*(R) \cap F^*(R) \). It is clear that \( F^*(R) \) becomes a lattice by the definition; if \( I, J \in F^*(R) \), then \( I \cup J = (I+J)^* \), and the meet "\( \cap \)" is the set-theoretic intersection. Similarly \( F^*(R) \) and \( F^*(R) \) also become
lattices. For any $I \in F^*_v(R)$ and $L \in F^*_v(R)$, we define the product "o" of $I$ and $L$ by $I \circ L = (IL)^*$. It is clear that $I \circ L \subseteq F^*_v(S) \cap F^*_v(T)$, where $S = O_f(I)$ and $T = O_f(L)$. In particular, the semi-group $F^*_v(R)$ becomes an abelian group (cf. Theorem 4.2 of [2]). For convenience we write $F'_v(R)$ for the sublattice of $F^*_v(R)$ consisting of all integral right $v$-ideals. Similarly we write $F'_{fr}(R)$ and $F'_{fr}(R)$ for the corresponding sublattices of $F^*_v(R)$ and $F^*_v(R)$ respectively.

Let $M$ and $N$ be subsets of $\mathbb{Q}$. Then we use the following notations: $(M: N)_r = \{x \in \mathbb{R} | xN \subseteq M\}$, $(M: N)_t = \{x^R | xN \subseteq M\}$. When $N$ is a single element $q$ of $\mathbb{Q}$, then we denote by $q ~\sim~ M$ the set $(M: N)_r$.

Lemma 3. Let $R$ be a maximal order in $\mathbb{Q}$. Then
(1) If $I \in F^*_v(R)$ and $q \subseteq \mathbb{Q}$ then $q^{-1}I = (I^{-1}q + R)^{-1}$ and so $q^{-1}I \subseteq F'_v(R)$.
(2) If $I \in F^*_v(R)$ and $J$ is a right $R$-ideal, then $(I: J)_r \subseteq F'(R)$ or 0.
(3) If $I \in F^*_v(R)$ and $J \in F^*_v(R)$, then $(I \circ J)^{-1} = J^{-1} \circ I^{-1}$.
(4) If $I, J \in F^*_v(R)$ and $L \in F^*_v(R)$, then $(I \cup J) \circ L = I \circ L \cup J \circ L$.

Proof. (1) Since $(I^{-1}q + R)^{-1} \subseteq R$, we get $(I^{-1}q + R)^{-1} \subseteq q^{-1}I$. Let $x$ be any element of $(I^{-1}q + R)^{-1}$. Then $(I^{-1}q + R)x \subseteq R$ so that $x \in R$ and $I^{-1}qx \subseteq R$. Let $S = O_f(I)$. Then it is a maximal order equivalent to $R$ by Lemma 1. It is evident that $S\subseteq I$ and that $(I^{-1}q + R)^{-1} \subseteq S$. Thus, by Lemma 2, we have $q \in S(S\subseteq I) \subseteq (II^{-1})^* = (S\subseteq I)^* = (II^{-1}S\subseteq I)^* = I$. Hence $x \in q^{-1}I$ and so $q^{-1}I = (I^{-1}q + R)^{-1}$. It is clear that $q^{-1}I \subseteq F'_v(R)$ by Corollary 4.2 of [9].
(2) If $(I: J)_r \neq 0$, then it is an $L$-ideal of $\mathbb{Q}$ and $(I: J)_r \subseteq I$. So $((I: J)_r)^* \subseteq (I: J)_r \subseteq I$. Hence $((I: J)_r)^* = (I: J)_r$.
(3) It is clear that $O_f(I \circ J) \subseteq O_f(I)$ and so $O_f(I \circ J) = O_f(I)$ by Lemma 1. Since $O_f(I \circ J)$ and so $O_f(I \circ J) = O_f(I)$ by Lemma 1. Hence $O_f(I \circ J)$ and so $O_f(I \circ J) = O_f(I)$ by Lemma 1.
(4) From Lemma 2, we have: $(I \cup J) \circ L = [(I \cup J) \circ L]^* = [(I \circ J) \circ L]^* = (IL + JL)* = [(IL) + (JL)]* = I \circ L \cup J \circ L$.

Let $R$ be a maximal order. We consider the following condition:

(A): $F'_v(R)$ and $F^*_v(R)$ both satisfy the maximum condition.

If $R$ is a maximal order satisfying the condition (A), then $F^*_v(R)$ is a direct product of infinite cyclic groups with prime $v$-ideals as their generators by Theorem 4.2 of [2]. It is evident that an element $P$ in $F^*_v(R)$ is a prime element in the lattice if and only if it is a prime ideal of $R$.

Following [1], $R$ is said to be regular if every integral one-sided $R$-ideal contains a non-zero $R$-ideal.
Lemma 4. Let $R$ be a regular maximal order satisfying the condition (A) and let $P$ be a non-zero prime ideal of $R$. Then $P$ is a minimal prime ideal of $R$ if and only if it is a prime $v$-ideal.

Proof. Assume that $P$ is a minimal prime ideal. Let $c$ be any regular element in $P$. Then since $(cR)^* = cR$ and $R$ is regular, we get $P \supseteq cR \supseteq (P_i^*)^* \circ \cdots \circ (P_1^*)^*$, where $P_i$ is a prime $v$-ideal. Hence $P \supseteq P_i$ for some $i$ and so $P = P_i$. Conversely assume that $P \not\supset P_0 \neq 0$, where $P_0$ is a prime ideal. Then since $P_0^* (P_0^* P_0) = (P_0^* P_0^*) P_0 \subseteq R P_0 = P_0$ and $P_0^* P_0 \subseteq P_0$, we have $P_0^* \subseteq P_0$ and thus $P_0^* = P_0$. It follows that $P_0$ is a maximal element in $\mathcal{F}'(R)$ by [2, p. 11], a contradiction. Hence $P$ is a minimal prime ideal of $R$.

Remark. Let $R$ be a maximal order satisfying the condition (A). Then it is evident from the proof of the lemma that prime $v$-ideals are minimal prime ideals of $R$.

Let $I$ be any right ideal of $R$. Then we denote by $\sqrt{I}$ the set $\cup \{(s^{-1} I : R), |s \in I, s \in R\}$. Following [3], if $\sqrt{I}$ is an ideal of $R$, then we say that $I$ is primal and that $\sqrt{I}$ is the adjoint ideal of it. A right ideal $I$ of $R$ is called primary if $JA \subseteq I$ and $J \nsubseteq I$ implies that $A^* \subseteq I$ for some positive integer $n$, where $J$ is a right ideal of $R$ and $A$ is an ideal of $R$. We shall apply these concepts for integral right $v$-ideals.

Lemma 5. Let $R$ be a maximal order satisfying the condition (A) and let $I$ be a meet-irreducible element in $\mathcal{F}'(R)$. Then $I$ is primal, and $\sqrt{I}$ is a minimal prime ideal of $R$ or 0, and $\sqrt{I} = (x^{-1} I : R)$, for some $x \in I$.

Proof. If $\sqrt{I} = 0$, then the assertion is evident. Assume that $\sqrt{I} \neq 0$. By Lemma 3, $(s^{-1} I : R)$ is a $v$-ideal or 0. Hence the set $S = \{(s^{-1} I : R), |s \in I, s \in R\}$ has a maximal element. Assume that $(s^{-1} I : R)$, and $(t^{-1} I : R)$, are maximal elements in $S$. Then $(s R + I) (s^{-1} I : R) \subseteq I$ implies that $(s R + I)^*(s^{-1} I : R) \subseteq I$ by Lemma 2 and so $(s^{-1} I : R) \subseteq (I : (s R + I)^*)$. The converse inclusion is clear. Thus we have $(s^{-1} I : R) = (I : (s R + I)^*)$. Similarly $(t^{-1} I : R) = (I : (t R + I)^*)$. Since $I$ is irreducible in $\mathcal{F}'(R)$, we have $I \subseteq (s R + I)^* \cap (t R + I)^* = I$. Let $x$ be any element in $J$ but not in $I$. Then it follows that $(x^{-1} I : R) \subseteq (s^{-1} I : R)$, $(t^{-1} I : R)$, so that $\sqrt{I} = (x^{-1} I : R) = (s^{-1} I : R)$, which is a $v$-ideal. Hence $I$ is primal. If $A B \subseteq \sqrt{I}$ and $A \nsubseteq \sqrt{I}$, where $A$ and $B$ are ideals of $R$, then $x A B \subseteq I$ and $x A \nsubseteq I$. Let $y$ be any element in $x A$ but not in $I$. Then $y B \subseteq I$ and so $B \subseteq (y^{-1} I : R) \subseteq \sqrt{I}$. Thus $\sqrt{I}$ is a prime ideal of $R$. It follows that $\sqrt{I}$ is minimal from the remark to Lemma 4.

A right ideal of $R$ is said to be bounded if it contains a non-zero ideal of $R$. 
**Lemma 6.** Let $R$ be a maximal order satisfying the condition (A) and let $I$ be an irreducible element in $F'_*(R)$. If $I$ is bounded, then it is primary and $(\sqrt{I})^n \subseteq I$ for some positive integer $n$.

Proof. Since $I \in F'_*(R)$ and is bounded, $(I:R)$ is non-zero and is a $v$-ideal. Write $(I:R)=((P_1^*)^* \cdots (P_k^*)^*)$, where $P_i$ are prime $v$-ideals. For any $i (1 \leq i \leq k)$, we let $B_i=(P_i^*)^* \cdots (P_{i-1}^*)^* o (P_{i+1}^*)^* \cdots (P_k^*)^*$. Then $B_i \nsubseteq I$ and $B_i (I:R) \subseteq I$, because $F^*_s(R)$ is an abelian group. Thus $P_i^* \subseteq \sqrt{I}$ and so $P_i^* = \sqrt{I}(1 \leq i \leq k)$ by Lemma 5. Therefore $(\sqrt{I})^n \subseteq I$. It is evident that $I$ is primary.

If $A$ is an ideal of $R$, then we denote by $C(A)$ those elements of $R$ which are regular mod $(A)$.

**Lemma 7.** Let $R$ be a maximal order satisfying the condition (A). Let $P$ be a prime $v$-ideal. Then

1. $C(P) = C((P^n)^*)$ for every positive integer $n$.
2. $C(P) \subseteq C(0)$.

Proof. (1) We shall prove by the induction on $n (>1)$. Assume that $C(P) = C((P^n)^*)$. If $cx \in (P^n)^*$, where $c \in C(P)$ and $x \in R$, then $cx(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$ by Lemma 2. Since $cx \in (P^n)^*$, we get $x \in (P^n)^*$ and so $x(P^{-1})^{n-1} \subseteq P$. Hence $x(P^{-1})^n \subseteq P$. Then we have $(xR + P^n)(P^{-1})^{n-1}P^{n-1} \subseteq P^*$ so that $x \in (P^n)^*$ by Lemma 2. Conversely suppose that $cx \in P$, $c \in C((P^n)^*)$, $x \in R$. Then $cxP^{n-1} \subseteq (P^n)^*$ and so $xP^{n-1} \subseteq (P^n)^*$. Since $(xP + P^n)P^{n-1}(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$, we get $x \in P$ by Lemma 2. Therefore $C(P) = C((P^n)^*)$.

(2) If $0 \neq \cap_n (P^n)^*$, then it is a $v$-ideal by Lemma 2. Write $\cap_n (P^n)^* = (P_1^*)^* \cdots (P_k^*)^*$, where $P_i$ are prime $v$-ideals. This is a contradiction, because $F^*_s(R)$ is an abelian group and $P, P_i$ are minimal prime ideals of $R$. Hence $0 = \cap_n (P^n)^*$. Therefore (2) follows from (1).

If $P$ is a prime ideal of a ring $S$, then the family $T_P = \{ I : \text{right ideal } |^s-I \cap C(P) \neq \phi \text{ for any } s \in S \}$ is a right additive topology (cf. Ex. 4 of [12, p. 18]). The following lemma is due to Lambek and Michler if $S$ is right noetherian. However, only trivial modifications to their proof are needed to establish the more general result.

**Lemma 8.** Let $P$ be a prime ideal of $S$ and let $\overline{S}=S/P$ be a right prime Goldie ring. Then the torsion theory determined by the $S$-injective hull $E(S)$ of $\overline{S}$ coincides with one determined by the right additive topology $T_P$, that is, a right ideal $I$ of $S$ is an element in $T_P$ if and only if $\text{Hom}_S(S/I, E(\overline{S})) = 0$ (Corollary 3.10 of [8]).

**Lemma 9.** Let $R$ be a maximal order satisfying the condition (A) and let $P$
be a prime \(v\)-ideal such that \(\bar{R} = R/P\) is a prime Goldie ring. If \(I\) is any element in \(F'(R)\) such that \(R \cong I \supseteq P\), then \(I \cap C(P) = \emptyset\).

Proof. It is enough to prove the lemma when \(I\) is a maximal element \(F'(R)\). Since \(I^{-1} \supseteq R\), \(P \circ I^{-1} \cap R \supseteq P\). If \(P \circ I^{-1} \cap R = P\), then \(P^{-1} = (P \circ I^{-1})^{-1} \cup *R\), because the mapping: \(J \to J^{-1}\) is an inverse lattice isomorphism between \(F^p(R)\) and \(F^p(R)\). By Lemma 3, \(P^{-1} = I \circ P^{-1} \cup \ast R\). On the other hand \(P \subseteq I\) implies that \(R \subseteq I \circ P^{-1}\). Hence \(P^{-1} = I \circ P^{-1}\) and so \(R = I\), a contradiction. Thus we have \(P \circ I^{-1} \cap R \not\supseteq P\). Let \(a\) be any element in \(P \circ I^{-1} \cap R\) but not in \(P\). Then \(aP \subseteq (P \circ I^{-1})I \subseteq P \ast I^{-1} \circ I = P\) so that \(I \subseteq a^{-1}P \subseteq R\). Since \(a^{-1}P\) is a right \(v\)-ideal by Lemma 3, we get \(I = a^{-1}P\). Then \(\text{Hom}(R/I, E(\bar{R})) \neq 0\), because \(R/I = R/a^{-1}P \cong (aR + P)/P \subseteq \bar{R}\). Now assume that \(I \cap C(P) \neq \emptyset\) and let \(c\) be any element in \(I \cap C(P)\). Then \(cR + P \in T\) by Lemma 3.1 of [6]. Hence \(I \in T\) and thus \(\text{Hom}(R/I, E(\bar{R})) = 0\) by Lemma 8. This is a contradiction and so \(I \cap C(P) = \emptyset\).

For convenience, we write \(M(p)\) for the family of minimal prime ideals of \(R\). If \(R\) is a regular maximal order satisfying the condition \((A)\), then we know from Lemma 4 that a prime ideal \(P\) is an element in \(M(p)\) if and only if it is a prime element in \(F'(R)\).

**Lemma 10.** Let \(R\) be a regular maximal order satisfying the condition \((A)\), \(P \in M(p)\) and let \(I \in F'(R)\). If \(\bar{R} = R/P\) is a prime Goldie ring, then \(I \cup \ast P = R\) if and only if \(I\) contains an ideal \(B\) such that \(B \not\subseteq P\).

Proof. Assume that \(I \supseteq B\), where \(B\) is an ideal not contained in \(P\). Then \(I \supseteq B^*\) and \(B^* \cup \ast P = R\), because \(P\) is a maximal element in \(F'(R)\) (cf. [2, p. 11]). Therefore \(I \cup \ast P = R\). Conversely assume that the family \(S = \{I \in F'(R) \mid I \cup \ast P = R\}\) is not empty and let \(I\) be a maximal element in \(S\). If \(I\) is irreducible in \(F'(R)\), then there exists \(P'\) in \(M(p)\) such that \(I \supseteq P'^n\) by Lemmas 5 and 6. Since \(I \in S\), we have \(P' = P\). If \(n = 1\), then \(R = I \cup \ast P = I\), a contradiction. We may assume that \(I \supseteq P^{n-1}\) and \(n > 1\). Then \((P^{n-1})^* = (I \cup \ast P) \circ (P^{n-1})^* = I \circ (P^{n-1})^* \cup \ast (P^n)^* \subseteq I^* = I\) by Lemmas 2 and 3. This is a contradiction. If \(I\) is reducible, then \(I = I_1 \cap I_2\), where \(I_i \in F'(R)\) and \(I \supseteq I_i\) (\(i = 1, 2\)). There are non zero ideals \(B_i(\not\subseteq P)\) such that \(I_i \supseteq B_i\). Thus \(I\) contains the ideal \(B_1B_2\) not contained in \(P\), a contradiction. Hence \(S = \emptyset\). This implies that if \(I \cup \ast P = R\), then \(I\) contains an ideal not contained in \(P\).

Let \(P\) be a prime ideal of a ring \(S\). If \(S\) satisfies the Ore condition with respect to \(C(P)\), then we denote by \(S_P\) the quotient ring with respect to \(C(P)\).

**Lemma 11.** Let \(R\) be a regular maximal order satisfying the condition \((A)\) and let \(P\) be an element in \(M(p)\) such that \(\bar{R} = R/P\) is a prime Goldie ring. Then
(1) $R$ satisfies the Ore condition with respect to $C(P)$.

(2) $R_p = \lim_{\rightarrow} B^{-1}$, where $B$ ranges over all non zero ideals not contained in $P$.

(3) $R_p$ is a noetherian, local and Asano order.

Proof. (1) It is clear that $T = \lim_{\rightarrow} B^{-1}(B(P): \text{ideal})$ is an overring of $R$. Let $c$ be any element in $C(P)$. Then $c$ is regular by Lemma 7 and so $cR^F_r(R)$. Since $(cR \cup P) \cap C(P) \not= \emptyset$, we have $cR \cup P = R$ by Lemma 9 and so $cR$ contains an ideal not contained in $P$ by Lemma 10. Hence $c^{-1} \in T$. So for any $r \in R$, $c \in C(P)$, there exists an ideal $B$ such that $c^{-1} rB \subseteq R$. It is evident that $B \cap C(P) \not= \emptyset$. Let $d$ be any element in $B \cap C(P)$. Then we have $c^{-1} d - s$ for some $s \in R$, that is, $rd = sc$. This implies that $R$ satisfies the right Ore condition with respect to $C(P)$. The other Ore condition is shown to hold by a symmetric proof.

(2) is evident from (1).

(3) We let $P' = PR_p$. Then clearly $P' = P_p P'$ and $P = P' \cap R$. So we may assume that $\tilde{R} = R/P \subseteq \tilde{R}_p = R_p / P'$ as rings. By (1), $\tilde{R}_p$ is the quotient ring of $\tilde{R}$. Since $\tilde{R}$ is a prime Goldie ring, $\tilde{R}_p$ is the simple artinian ring. Hence $P'$ is a maximal ideal of $R_p$. Let $V'$ be any maximal right ideal of $R_p$. Suppose that $V' \not\subseteq P'$. Then $V' + P' = R_p$. Write $1 = v + pc^{-1}$, where $v \in V'$, $p \in P$ and $c \in C(P)$. Then $c = vc + p$ and so $vc = c - p \in C(P) \cap V'$. This implies that $V' = R_p$, a contradiction and so $V' \subseteq P'$. Hence $P'$ is the Jacobson radical of $R_p$. The ideal $P^{-1} P$ properly contains $P$ so that $C(P) \cap P^{-1} P \not= \emptyset$. It follows that $P^{-1} PR_p = R_p$. Similarly $R_p PP^{-1} = R_p$. Hence $P'$ is an invertible ideal of $R_p$. Therefore $R_p / P^m$ is an artinian ring for any $n$, because $\tilde{R}_p$ is an artinian ring. Let $I'$ be any essential right ideal of $R_p$. It is clear that $I' = (I' \cap R) R_p$. Let $c$ be any regular element of $I' \cap R$. Then, since $cR \subseteq F'(R)$ and $R$ is regular, $cR$ contains a non zero $v$-ideal $(P'^m)^* \circ (P'^{m*})^* \circ \cdots \circ (P'^{*m*})^*$, where $P_i \in M(p)$. So we get $I' \supseteq R_p P^m = P''$. Therefore essential right ideals of $R_p$ satisfies the maximum condition. Since $R_p$ is finite dimensional in the sense of Goldie, $R_p$ is right noetherian. Similarly $R_p$ is left noetherian. Hence $R_p$ is a noetherian, local and Asano order by Proposition 1.3 of [7].

After all these preparations we now prove the following theorem which is the purpose of this paper:

**Theorem.** A prime Goldie ring $R$ is a bounded Krull prime ring if and only if it satisfies the following conditions:

(1) $R$ is a regular maximal order,

(2) $R$ satisfies the maximum condition for integral right and left $v$-ideals,

(3) $R/P$ is a prime Goldie ring for any $P \in M(p)$.

Proof. Assume that $R = \cap_i R_i (i \in I)$ is a bounded Krull prime ring, where $R_i$ is a noetherian, local and Asano order with unique maximal ideal $P_i$. (1) is
clear from Corollary 1.4 and Lemma 1.6 of [10]. Let \( I \) be any right (left) \( R \)-ideal. Then \( I^* = \cap I_R = \cap R_I \) by Proposition 1.10 of [10]. Since \( R_t \) is noetherian, (2) follows from the condition (K3) in the definition of Krull rings. Let \( P_i = P_I \cap R \). It follows that \( \{ P_i | i \in I \} = M(p) \) by Proposition 1.7 of [10] so that (3) is evident from Proposition 1.1 of [9].

It remains to prove that the conditions (1), (2) and (3) are sufficient. Let \( P \) be any element in \( M(p) \). Then \( R \) satisfies the Ore condition with respect to \( C(P) \) and \( R_P \) is a noetherian, local and Asano order by Lemma 11. Hence \( R_P \) is an essential overring of \( R \). It is clear that \( R \subseteq T = \cap R_P \), where \( P \in M(p) \). To prove the converse inclusion let \( x \) be any element of \( T \). Then there is an ideal \( B_P (\subseteq P) \) such that \( xB_P \subseteq R \) by Lemma 11. Let \( B \) be the sum of all ideals \( B_P \). If \( B^* \) is different from \( R \), then \( B^* \) is contained in some \( P \) in \( M(p) \). But \( B^* \subseteq P \) so that \( B^* = R \). Hence we have \( x \in (xR + R) \subseteq (xR + R)^* \) \( B^* = (xB + B)^* \) \( \subseteq R \). Thus we get \( R = \cap R_P \). Let \( c \) be any regular element in \( R \). Then \( cR \) contains a \( \psi \)-ideal \( (P_i^*) \circ \cdots \circ (P_k^*) \), where \( P_i \in M(p) \). It follows that \( cR_P = R_P \) for every \( P \in M(p) \) different to \( P (1 \leq i \leq k) \) by Lemma 11. Hence \( R \) is a bounded Krull prime ring. This completes the proof of the theorem.

**Corollary.** Let \( R \) be a regular, noetherian and prime ring. If \( R \) is a maximal order, then it is a bounded Krull prime ring.

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**References**


