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ON NON-SINGULAR QF-3′ RINGS WITH INJECTIVE DIMENSION ≤1

TAKESHI SUMIOKA

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Let R be a ring and E an injective hull of R_R. We call R right QF-3′ if every finitely generated submodule of E is torsionless. The dimension of a right R-module M is defined as the superior of the lengths n of chains $M_0 \subset M_1 \subset \cdots \subset M_n$ of submodules of M such that each factor module M_{i+1}/M_i is not torsion (under the Lambek torsion theory).

We shall characterize a finite dimensional (i.e. the dimensions of R_R and ${}_RR$ are finite) right and left QF-3′ ring using the dimension of modules with some properties. Next, let R be a noetherian ring. Jans [4] proved that every finitely generated torsionless right R-module is reflexive if and only if R has injective dimension ≤ 1 as a left R-module. When R is (not necessarily noetherian) a commutative integral domain, results analogous to above one were proved by Matlis [6], where he further delt with some properties on torsion modules. These properties, in case of noetherian QF-3′ ring, were investigated by Zaks [11] and Sato [8]. In this note, we shall give characterizations of non-singular noetherian (or artinian) QF-3′ ring with injective dimension ≤ 1 using the Lambek torsion theory.

Throughout we assume that if a ring R is said to be noetherian or QF-3′, etc., we mean right and left noetherian or right and left QF-3′, etc.. Moreover we assume that "every R-module" means "every right R-module and every left R-module," and "R has injective dimension ≤ 1" does "R has injective dimension ≤ 1 as right and left R-modules," etc..

In this note, "torsion theory" means the Lambek torsion theory, which is cogenerated by an injective hull of $R_R({}_RR)$. We denote its torsion radical by t. Let R be a ring and M a right R-module. A chain

$$M_1 \subset M_2 \subset M_3 \subset \cdots (\text{resp. } M_1 \supset M_2 \supset M_3 \supset \cdots)$$

of submodules of M is called *t-chain* (*of* M) if M_{i+1}/M_i (resp. M_i/M_{i+1}) is not torsion for each i. A module M is called *finite dimensional* if any ascending t-chain and any descending t-chain of M terminate. A ring R is called *right finite dimensional* if R_R is finite dimensional, (refer Goldman [3] for these definitions and their properties). The *dimension* of M is said to be *equal to or larger than* n and we denote it by $\dim M \geq n$, if M has a t-chain of length n. The *dimension* of M is said to be *equal to* n if $\dim M \geq n$ and $\dim M \ngeq n+1$, and in particular, $\dim M=0$ if M is torsion, and $\dim M=\infty$ if $\dim M \geq n$ for any positive integer n.

The following lemma is immediate from the fact that if $0 \to A \to B \to C \to 0$ is a short exact sequence of right R-modules, B is torsion if and only if A and C are torsion.

Lemma 1. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of right R-modules. Then we have $\dim M = \dim L + \dim N$.*

By Lemma 1, it is easy to see that a right R-module M is finite dimensional if and only if $\dim M=n$ for some integer $n \geq 0$. Let R be a right non-singular ring and M a torsion-free right R-module. Then we should note that this dimension of M is equal to the Goldie dimension of M. A right R-module M is called *finitely imbedded* (more briefly *FI*), if M is imbedded in some finitely generated right R-module. A ring R is called *right QF*-3′ if R satisfies the following equivalent properties:

(1) Any finitely generated submodule of E is torsionless, where E is an injective hull of R_R.

(2) Every FI torsion-free right R-module is torsionless.

(3) The maximal right quotient ring Q of R is a left quotient ring of R, and every finitely generated submodule of an injective hull $E(Q_Q)$ of Q_Q is torsionless.

The equivalence of these was proved by Masaike [5].

If M is a right R-module, we denote $\mathrm{Hom}_R(M, R)$ and $\mathrm{Ext}^1_R(M, R)$ by M^* and M_*, respectively, and these are naturally regarded as left R-modules.

Lemma 2. *Let R be a right QF-3′ ring and M an FI right R-module. Then M is torsion if and only if $M^*=0$.*

Proof. This is immediate from the definition of FI modules.

Lemma 3. *Let R be a finite dimensional right QF-3′ ring. Then the following statements for a right R-module M are equivalent:*

(1) *M is finite dimensional torsionless.*

(2) *M is FI torsionless.*

(3) *M is FI torsion-free.*

(4) *M is imbedded in a finitely generated free right R-module.*

Proof. We shall only show that (1) implies (4), for (4)⇒(3)⇒(2)⇒(1) are clear. Assume (1). It follows from the definition of "finite dimensional" that there is a finitely generated submodule N of M such that M/N is torsion. Hence

M^* is imbedded in N^*. Since N^* is imbedded in a finitely generated free left R-module, so is M^*, and in particular M^* is also finite dimensional. Therefore, using a similar discussion for M^* it follows that M^{**} is imbedded in a finitely generated free right R-module. Thus (4) is satisfied since the canonical map $M \rightarrow M^{**}$ is monomorphic.

Proposition 1. *Let R be a right QF-3′ ring. Then the following statements are equivalent:*

(1) *R is right finite dimensional.*

(2) *R satisfies the ascending chain conditions on annihilator right ideals and annihilator left ideals.*

Proof. First we note that the condition (2) is clearly equivalent to a condition that R satisfies both the chain conditions on annihilator right ideals. If I is a right ideal of R, I is right annihilator if and only if R/I is torsionless. Hence, (1) implies (2). Assume (2), and let I and I' be right ideals of R such that $I \subset I'$ and I'/I is not torsion. For a subset A of R, we denote by $r(A)$ (resp. $l(A)$) the right (resp. left) annihilator ideal of A. Put $J = rl(I)$ and $J' = rl(I')$, and let K/I be the torsion submodule of R/I. Then, since R/J is torsion-free, J/I contains K/I, i.e., $J \supset K$. By the assumption, a cyclic torsion-free right R-module R/K is torsionless, and hence K is an annihilator right ideal of R. Therefore we have $K = J$ from $I \subset K \subset I$, and so J/I is torsion. Since I'/I is not torsion, so is J'/J, and in particular $J \subsetneqq J'$. This implies by the condition (2) that any ascending t-chain and any descending t-chain of R_R terminate. Thus the condition (1) is satisfied.

REMARKS (i) In Proposition 1, (1) implies (2) without the assumption that R is right QF-3′. Hence, for a semi-prime ring R, R is right finite dimensional if and only if R is right Goldie.

(ii) From right-left symmetry of the condition (2) in Proposition 1, it follows that a left QF-3′ and right finite dimensional ring is also left finite dimensional.

Lemma 4. *Let R be a right QF-3′ ring, M an FI right R-module and N a submodule of M. Then the monomorphism $(M/N)^* \rightarrow M^*$ derived by the canonical epimorphism $M \rightarrow M/N$ is isomorphic if and only if N is torsion.*

Proof. "If" part is trivial, and we shall show "only if" part. Assume that the map $(M/N)^* \rightarrow M^*$ is isomorphic. In case M is torsionless, we easily see $N=0$. Now let M be an FI right R-module and M' the torsion submodule of M. Then, we have a following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' \cap N & \longrightarrow & M' & \longrightarrow & M'+N/N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N/M' \cap N & \to & M/M' & \to & M/M'+N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0\,. & &
\end{array}$$

Moreover, from this we obtain the following commutative diagram with exact rows and isomorphic columns:

$$\begin{array}{ccccccc}
0 & \to & (M/M'+N)^* & \to & (M/M')^* & \to & (N/M' \cap N)^* \\
 & & \wr\| & & \wr\| & & \wr\| \\
0 & \longrightarrow & (M/N)^* & \longrightarrow & M^* & \longrightarrow & N^*\,.
\end{array}$$

Since R is right QF-3$'$, M/M' is torsionless, and so we have $N/M' \cap N=0$ by the above case, and hence $N \subset M'$. Thus N is torsion.

Lemma 5. *Let R be a right finite dimensional right QF-3$'$ ring. If M is an FI right R-module, we have dim $M_R \leq dim_R\,(M^*)$.*

Proof. In case dim $M=0$, the assertion is trivial. In case dim $M=1$, we have dim $M^* \geq 1$, since M^* is non-zero torsionless by Lemma 2. Assume that the assertion is satisfied for dim $M \leq n-1$, and let the dimension of M be equal to n. Then there is a submodule L of M such that dim $L=1$ and so dim $M/L=n-1$. Then the map $(M/L)^* \to M^*$ is not isomorphic by Lemma 4, and hence we have an exact sequence $0 \to (M/L)^* \to M^* \to K \to 0$ with non-zero torsion-free left R-module K. By inductional assumption, we have dim $(M/L)^* \geq n-1$. Therefore dim $M^* \geq$ dim M.

Let M and N be right R-modules. For a homomorphism f: $M \to N$, we denote its dual map by f^*: $N^* \to M^*$. On the other hand, by φ_M: $M \to M^{**}$ we denote the canonical homorphism of M to M^{**}.

Theorem 1. *Let R be a right finite dimensional ring. Then the following conditions are equivalent:*

(1) *R is QF-3$'$.*

(2) *dim $M=$dim M^* for every FI module M.*

(3) *dim $M=$dim M^* for every FI torsion-free module M.*

Proof. (1) implies (2). Let dim $R_R=n$. Since, by Lemma 5 and Remark (ii), dim $R_R \leq \dim_R (R^*) \leq \dim (R^{**})_R$, we have dim $R_R=$dim ${}_RR$. Therefore we only show that dim $M=$dim M^* for every FI right R-module M, because the same arguments hold for FI left R-modules. If dim $M=0$, this assertion is trivial. Suppose that M is a cyclic right R-module and N is the kernel of an epimorphism $R \to M$. If the dimension of M is n, we have $M_R \simeq R_R$, and so $\dim M = \dim M^*$. If $\dim M = n-1$, and so $\dim N = 1$, then we have $\dim M^* = n-1$ by Lemma 4 and Lemma 5. Suppose $\dim M = n-2$ and $\dim N = 2$. Let L be a submodule N such that $\dim L = 1$. Since R/L is a cyclic right R-module with dimension $n-1$. we have $\dim (R/L)^* = n-1$ as the above case. Therefore, by Lemma 4 and Lemma 5, $\dim M^* = n-2$. Iterating this discussion we have $\dim M = \dim M^*$ for any cyclic right R-module M. Next suppose that M is a finitely generated right R-module. Then, using inductuon on the number of generators of M, the assertion is easily showed by Lemma 5. In the case where M is an FI right R-module, there is a finitely generated submodule N of M such that M/N is torsion, i.e., $\dim M = \dim N$. Then, from the exact sequence $0 \to N \to M \to M/N \to 0$, we have the monomorphism $M^* \to N^*$ and so $\dim M^* \leq \dim N^*$. Since N is finitely generated, $\dim N = \dim N^*$. Thus we have $\dim M = \dim M^*$ by Lemma 5.

(2) implies (3). This is trivial.

(3) implies (1). Since $\dim {}_RR = \dim R_R$, R is left finite dimensional. Let L be a finitely generated torsion-free left R-module, and let $0 \to K \to G \to L \to 0$ be an exact sequence with a finitely generated free left R-module G. Then we have an exact sequence $0 \to L^* \to G^* \to K^* \to L_* \to 0$, which implies $\dim L_* = 0$ by the assumption. Next let M be a finitely generated torsion-free right R-module. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J & \to & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \varphi_F} & & \downarrow{\scriptstyle \varphi_M} & & \\ 0 & \to & J'' & \to & F^{**} & \to & M^{**} & \to & (F^*/J')_* \to 0\,, \end{array}$$

where F is a finitely generated free right R-module and J' is the annihilator of J in F^*. Since F^*/J' is torsionless, $\dim (F^*/J')_* = 0$, so $\dim \operatorname{Im} \varphi_M = \dim M$. This implies that φ_M is monomorphic. Thus R is right QF-3′ and similarly left QF-3′.

Corollary 1. *Let R be a finite dimensional QF-3′ ring. Then we have:*

(1) *If M is a finite dimensional right R-module, M^* is reflexive.*

(2) *Let M be a finitely generated right R-module, and let f: $L \to M$ be a monomorphism. Then M_* and Coker f^* are torsion.*

Proof. (1) This follows from a fact that the composition map of natural maps φ_{M^*}: $M^* \to M^{***}$ and $(\varphi_M)^*$: $M^{***} \to M^*$ is identity, and Theorem 1. (2) The assertion for M_* is immediate from the equivalence of (1) and (2) in Theorem 1. Hence $(M/L)_*$ is also torsion, and so the left R-module Coker f^* imbedded in $(M/L)_*$ is torsion.

Proposition 2. *Let R be a finite dimensional QF-3' ring. Then, for a finite dimensional torsion-free right R-module M, $\dim M = \dim M^*$ if and only if M is torsionless.*

Proof. Suppose $\dim M = \dim M^*$ and $\dim M = n$. We shall show by induction on n that the canonical map $M \to M^{**}$ is monomorphic. The result holds for $n=1$, since $M^* \neq 0$. Assume the result for positive integer $\leq n-1$. There exists a submodule N of M such that $\dim N = 1$ and M/N is a torsion-free right R-module with $\dim M/N = n-1$. Then an exact sequence $0 \to N \to M \to M/N \to 0$ derives the exact sequence $0 \to (M/N)^* \to M^* \to L \to 0$, where L is the image of the derived map $M^* \to N^*$. Now, let A be a finite dimensional right R-module. Then there is a finitely generated submodule B of A such that A/B is torsion, and so $\dim A^* \leq \dim B^*$. Hence we have $\dim A^* \leq \dim A$, since $\dim B = \dim B^*$ by Theorem 1. Therefore in particular, $\dim N^* \leq 1$ and $\dim (M/N)^* \leq n-1$. On the other hand, $\dim M^* = n$. Consequently $\dim N^* = 1$ and $\dim (M/N)^* = n-1$, and so N and M/N are torsionless by inductional assumption. Moreover $\dim L = 1$, so the dual map $\iota^*: N^{**} \to L^*$ of the inclusion $\iota: L \to N^*$ is monomorphic. Consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\varphi_M"] & M/N \arrow[r] \arrow[d, "\varphi_{M/N}"] & 0 \\
0 \arrow[r] & L^* \arrow[r] & M^{**} \arrow[r] & (M/N)^{**} &
\end{tikzcd}$$

with exact rows, where $N \to L^*$ is the composition of the maps $\varphi_N: N \to N^{**}$ and $\iota^*: N^{**} \to L^*$. Then the middle map is monomorphic, since so are both the out sides. Thus M is torsionless. The converse is followed from Lemma 3 and Theorem 1.

The following lemma is a slight extension of Matlis [6, Lemma, p. 19], and this extension is essentially necessery in the proof of Theorem 3.

Lemma 6. *Let R be a ring, and let $0 \to N \to M \to L \to 0$ be an exact sequence of right R-modules. Suppose that N is reflexive, M is torsion-free and L, L_* and L_{**} are simple torsion. Then M is reflexive.*

Proof. By the sequence

$$0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0 \qquad \cdots\cdots\cdots\cdots(\mathrm{A}),$$

an exact sequence $0 \to M^* \xrightarrow{f^*} N^* \xrightarrow{\delta} L_*$ is obtained. Suppose that f^* is isomorphic. Then its dual map f^{**} is also isomorphic. Since, in a commutative diagram

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "\varphi_N"] & M \arrow[d, "\varphi_M"] \\
N^{**} \arrow[r, "f^{**}"] & M^{**},
\end{tikzcd}$$

φ_N and f^{**} are isomorphic, the exact sequence (A) splits. This is a contradiction because M is torsion-free and L is torsion. Thus f^* is not isomorphic, that is $\delta \neq 0$, which implies that δ is epimorphic, since L_* is simple. Thus we have an exact sequence $0 \to N^{**} \xrightarrow{f^{**}} M^{**} \xrightarrow{\delta'} L_{**}$. δ' is epimorphism, since f^{**} is not isomorphic. Therefore we have a map $\eta: L \to L_{**}$ and the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\varphi_N"] & M \arrow[r, "g"] \arrow[d, "\varphi_M"] & L \arrow[r] \arrow[d, "\eta"] & 0 \\ 0 \arrow[r] & N^{**} \arrow[r, "f^{**}"] & M^{**} \arrow[r, "\delta'"] & L_{**} \arrow[r] & 0 . \end{tikzcd}$$

Assume that φ_M is not epimorphic. Put $M_0 = \operatorname{Im} \varphi_M$. Since φ_N is isomorphic, $\operatorname{Im} f^{**} \subset M_0$. Therefore, since L_{**} is simple, $M_0 = \operatorname{Im} f^{**}$, which implies (A) splits. This is a contradiction. Thus φ_M is an epimorphism. Therefore η is isomorphic, and consequently φ_M is isomorphic.

Let R be a ring, and let M be a finitely generated torsion right R-module such that M_* is torsion. Consider an exact sequence $0 \to L \to P \to M \to 0$ with finitely generated projective right R-module P. Then we have an exact sequence $0 \to P^* \to L^* \to M_* \to 0$ derived from the above sequence. Therefore, since M^* is torsion, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\varphi_L"] & P \arrow[r] \arrow[d, "\varphi_P"] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L^{**} \arrow[r] & P^{**} \arrow[r] & M_{**} & \end{tikzcd}$$

with exact rows, where $M \to M_{**}$ is the map induced by the left side square:

$$\begin{tikzcd} L \arrow[r] \arrow[d] & P \arrow[d] \\ L^{**} \arrow[r] & P^{**} . \end{tikzcd}$$

We denote this map by μ_M: $M \to M_{**}$. Clearly μ_M does not depend on selection of the finitely generated projective right R-module P.

Theorem 2. *Let R be a noetherian non-singular QF-3′ ring with the maximal quotient ring Q. Then the following statements are equivalent:*

(1) *For every cyclic torsion R-module M, μ_M is isomorphic.*

(2) *Any finitely generated submodule of Q is reflexive.*

(3) *R has injective dimension ≤ 1.*

Proof. (3) implies (1) and (2). These follows from Sato [8] and Jans [4]. (2) implies (3). Let I be an essential right ideal of R. Since any essential right ideal is reflexive and I^* is a noetherian left R-module, R/I has a composition series. Hence any cyclic torsion right R-module, and consequently, any finitely generated torsion right R-module has a composition series. Next, we have an isomorphism $\mathrm{Hom}_R(R, Q) \simeq \mathrm{Hom}_R(I, Q)$, since R/I is torsion and Q is injective. On the other hand, there is a monomorphism $\mathrm{Hom}_R(I, R) \to \mathrm{Hom}_R(I, Q)$. Thus I^* is imbedded in ${}_RQ$, and so every submodule of I^* is reflexive by the assumption. Therefore, since we have an exact sequence $0 \to R^* \to I^* \to (R/I)_* \to 0$, we easily see that if R/I is simple torsion, so is $(R/I)_*$. Similarly, if S is simple torsion as a left R-module, so is S_* as a right R-module. Now let M be an essential submodule of a finitely generated free right R-module F. Then there is an essential submodule N of M such that N is isomorphic to a finite direct sum of right ideals of R, and so N is reflexive. Since F/N is finitely generated torsion, F/N and consequently M/N are artinian. Therefore, from Lemma 6, it follows by induction on the length of composition series of M/N that M is reflexive. Thus, by Jans [4], we have inj. dim. ${}_RR \leq 1$. Similarly, we can show inj. dim. $R_R \leq 1$.

(1) implies (3). Let I be an essential right ideal of R. Then, we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "\varphi_I"] & R \arrow[r] \arrow[d, "\varphi_R"] & R/I \arrow[r] \arrow[d, "\mu_{R/I}"] & 0 & \\ 0 \arrow[r] & I^{**} \arrow[r] & R^{**} \arrow[r] & (R/I)_{**} \arrow[r] & (I^*)_* \arrow[r] & 0 . \end{tikzcd}$$

Since $\mu_{R/I}$ is isomorphic, I is reflexive and $(I^*)_*=0$. Therefore, it is easy to see that S_* is simple torsion for every simple torsion module S. Thus the assertion is showed as the proof of (2)$\Rightarrow$(3).

REMARKS (i) From the above commutative diagram, we obtain the following exact sequence:

$$0 \to I \to I^{**} \to R/I \to (R/I)_{**} \to \mathrm{Ext}^1_R(I^*, R) \to 0 .$$

If R is an integral domain, this may be identified with the exact sequence of Matlis [6, Theorem 1.2].

(ii) Let R be a noetherian integral domain with quotient field Q. If A is a finitely generated submodule of Q, A is isomorphic to an ideal of R. Therefore, the equivalence of (2) and (3) in Theorem 2 is an extension of that of (2) and (4) in Matlis [6, Theorem 3.8].

Proposition 3. *Let R be a non-singular right finite dimensional ring. Suppose that for every finitely generated torsion R-module A, A_* is torsion. Then R is QF-3'.*

Proof. From the assumption, it is easily showed that $\dim {}_RR = \dim R_R$, and dim $I^*=1$ for every uniform right ideal I of R, and in particular R is left finite dimensional. Now, let L be a finitely generated torsion-free right R-module. Since R_R is non-singular, there is a finite number of uniform right ideals whose direct sum is isomorphic to an essential submodule of L. Consequently, we have $\dim L^*=\dim L$ from the assumption. Next, let N be an FI torsion-free right R-module. Then N is clearly imbedded in a finitely generated torsion-free right R-module M. Since M is finite dimensional, $N\oplus K$ is essential in M with some finitely generated right R-module K. From the above case, we have $\dim M^*=\dim M$ and $\dim K^*=\dim K$, and therefore $\dim N^*=\dim N$. Similar arguments also hold for FI torsion-free left R-modules. Consequently, R is QF-3′ by Theorem 1.

Lemma 7. *Let R be a noetherian ring. Then the following conditions are equivalent:*

(1) *If M is a finitely generated torsion module, then so is M_*, and M_{**} is isomorphic to M.*

(2) *If M is a finitely generated torsion module, then so is M_*, and μ_M is isomorphic.*

Proof. It is trivial that (2) implies (1). Assume (1). Then it is clear that if S is a simple torsion module, S_* is non-zero. First we show that if S is simple torsion as a right R-module, so is S_* as a left R-module. Let N be a maximal submodule of S_*, and $0\to N\to S_*\to S_*/N\to 0$ the natural exact sequence. Then we have the derived exact sequence $0\to(S_*/N)_*\to S_{**}$, since N is torsion. By the assumption, S_{**} is isomorphic to S, and so S_{**} is simple. This implies $(S_*/N)_*\simeq S_{**}\simeq S$. Therefore $S_*\simeq(S_*N)_{**}\simeq S_*/N$, and so S_* is simple. If an R-module M has a composition series, we denote its length by $l(M_*)$. Let M be a torsion R-module with $l(M)=n$. Then, by induction on n, we can easily show $l(M_*)=n$. Now, let M be a finitely generated torsion right R-module. Then there are a finitely generated free right R-module F and its submodule K such that $F/K\simeq M$. In order to show that M is artinian, let

$$F\supset K_1\supset K_2\supset\cdots\supset K \qquad \cdots\cdots\cdots\cdots\text{(A)}$$

by a chain of submodules of F such that $l(F/K_i)=i$ for each i. Since F/K is a finitely generated torsion right R-module, we may assume that we have a chain

$$F^*\subset K_1^*\subset K_2^*\subset\cdots\subset K^* \qquad \cdots\cdots\cdots\cdots\text{(B)}$$

of submodules of K^*. Then we have $K^*/F^*\simeq(F/K)_*$ and $K_i^*/F^*\simeq(F/K_i)_*$, and so K^*/F^* is also finitely generated torsion and $l(K_i^*/F^*)=i$. On the other hand K^* is noetherian, which shows that the chain (B) and consequently (A) terminate. Now, we show that the canonical inclusion $\varphi_K\colon K\to K^{**}$ is an isomorphism. Since K^*/F^* is torsion and $F \simeq F^{**}$, we may assume that $K \subset K^{**} \subset F$. Then we have $F^* \subset K^{***} \subset K^*$. Therefore $(\varphi_K)^*\colon K^{***} \to K^*$ is isomorphic, since $(\varphi_K)^*$ is always epimorphic. Which implies $l(F/K) = l(F/K^{**})$, and so φ_K is isomorphic. We have, however, $l(M) = l(M^{**})$. Thus μ_M is isomorphic.

The following Corollary is immediate from Proposition 3, Lemma 7 and Sato [8]. (It seems that an isomorphism $M \simeq \mathrm{Ext}^1_R(\mathrm{Ext}^1_R(M, R), R)$ in Sato [8, Theorem 2.3] means μ_M.)

Corollary 2. *Let R be a noetherian non-singular ring. Then the following statements are equivalent:*

(1) *If M is a finitely generated torsion module, so is M_*, and $M_{**} \simeq M$.*
(2) *R is a QF-3′ ring with injective dimension ≤ 1.*

A ring R is called *right QF*-3 if R has a minimal faithful right R-module. The following lemma is a slight extension of Rutter [7, Corollary 3].

Lemma 8. *Let R be a right perfect ring satisfying the ascending chain condition on annihilator right ideals. If R is right QF-3′, then R is QF-3.*

Proof. By Faith [2] R is semi-primary, and it follows from the proof of Proposition 1 that R is right finite dimensional. Let M be a finitely generated submodule of an injective hull $E(R_R)$ of R_R. Then $\bigcap_{f \in M^*} \mathrm{Ker}\, f = 0$ since M is torsionless. For every subset A of M^*, $M/\bigcap_{f \in A} \mathrm{Ker}\, f$ is torsionless. But M is finite dimensional, which implies that there exist $f_1, \cdots, f_n$ in M^* such that $\bigcap_{i=1}^{n} \mathrm{Ker}\, f_i = 0$, and so M is imbedded in a free right R-module. Therefore $E(R_R)$ is projective by Rutter [7], and R is right QF-3. Thus, by Colby-Rutter [1], R is QF-3.

Corollary 3. *Let R be an artinian ring. Then the following statements are equivalent:*

(1) *R is a non-singular QF-3′ ring with injective dimension ≤ 1.*
(2) *R is hereditary QF-3.*

Proof. Assume (1). By Lemma 8 (or Rutter [7]), R is QF-3. Since R is non-singular and artinian, R has the semi-simple maximal quotient ring. Therefore, by Sumioka [9], R is hereditary. The converse is clear since any QF-3 ring is QF-3′ (see Tachikawa [10], p. 47).

Theorem 3. *Let R be a non-singular artinian ring. Then the following conditions are equivalent:*

(1) *If S is a simple torsion module, then so is S_*.*

(2) *R is hereditary QF*-3.

Proof. It follows from Sato [8] that (2) implies (1). Assume (1). Let M be a finitely generated torsion-free right R-module with $l(M)=n$, where $l(M)$ explesses the length of a composition series of M. By induction on n, we shall show that M is reflexive. Suppose $l(M)=1$. Then, since M is a non-singular simple module, M is projective and in particular reflexive. Let $l(M)=n$, and assume that the result holds for $n-1$. Let N be a maximal submodule of M, and consider the exact sequence $0\rightarrow N\rightarrow M\rightarrow M/N\rightarrow 0$. By inductional assumption, N is reflexive. If M/N is simple torsion-free, then M/N is projective, and so the above sequence splits, and consequently M is reflexive. If M/N is simple torsion, then the result holds, by Lemma 6. Thus every finitely generated torsion-free right R module is reflexive. The similar statement on left R-module is also true. Therefore R is hereditary QF-3 by Jans [4] and Corollary 3.

OSAKA CITY UNIVERSITY

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