Yoshimatsu, Y.
Osaka J. Math.
15 (1978), 435-459

H-PROJECTIVE CONNECTIONS AND *H*-PROJECTIVE TRANSFORMATIONS

YASHIRO YOSHIMATSU

(Received May 16, 1977)

Introduction

Let M be an n-dimensional complex manifold. We write J for its natural almost complex structure. Let ∇ be an almost complex affine connection without torsion on M. A curve $c(t)$ in M is called an *H-planner* curve with respect to ∇ if

$$\nabla_{c'}c' = ac'+bJc' \tag{0.1}$$

for certain smooth functions a and b. Two almost complex affine connections ∇ and ∇' without torsion are said to be *H-projectively equivalent* if they have their H-planner curves in common. From the result of T. Otsuki and Y. Tashiro, this is equivalent to existence of a 1-form ρ on M satisfying

$$\nabla_X Y-\nabla'_X Y = \rho(X)Y+\rho(Y)X-\rho(JX)JY-\rho(JY)JX \tag{0.2}$$

for arbitrary vector fields X and Y ([5], [8]). By an *H-projective* transformation of ∇, we mean a biholomorphic transformation $f: M\to M$ such that $f^*\nabla$ and ∇ are H-projectively equivalent. For example, let $\boldsymbol{P}^n(\boldsymbol{C})=L/L_0$ be the n-dimensional complex projective space of lines in $\boldsymbol{C}^{n+1}$ with the usual connection, where

$$\begin{aligned} L &= SL(n+1, \boldsymbol{C}), \\ L_0 &= \left\{\begin{pmatrix} a & u \\ 0 & B\end{pmatrix}\in SL(n+1, \boldsymbol{C}) \,|\, B\in GL(n, \boldsymbol{C})\right\}. \end{aligned} \tag{0.3}$$

Then $L/$(center) is the group of all H-projective transformations.

In the present paper, we shall study H-projective equivalence from the view point of L_0-structure of second order, studied by N. Tanaka and T. Ochiai. In fact, we shall show that H-projective equivalence of ∇ and ∇' is the same as $\boldsymbol{P}^n(\boldsymbol{C})$-equivalence in [6] and [4] (Theorem 1). Therefore, using their results, the family $\{\nabla\}$ of almost complex affine connections without torsion which are H-projectively equivalent to ∇ uniquely determines a Cartan connection ω of type $\boldsymbol{P}^n(\boldsymbol{C})$. This enables us to show that the group of all H-projective transformations of ∇ is a Lie group of finite dimension (Theorem 2). Then we shall prove that a curve $c(t)$ is an H-planner curve with respect to ∇ if and only if the development of $c(t)$ into $\boldsymbol{P}^n(\boldsymbol{C})$ by ω is an H-planner curve in $\boldsymbol{P}^n(\boldsymbol{C})$ (Theorem 3).

An H-planner curve $c(t)$ with respect to ∇ is called an *H-geodesic* of ∇ if $a=0$ and b is a constant in (0.1). An almost complex affine connection ∇ without torsion is said to be *H-complete* if any H-geodesic $c(t)$ of ∇ can be defined for all $t\in\boldsymbol{R}$. When ∇ is the Kaehler connection of a Kaehler metric ds^2, H-completeness of ∇ is equivalent to completeness of ds^2 (Theorem 4). An almost complex affine connection without torsion is said to be *of Kaehler type* if its Ricci tensor is hermitian (i.e., symmetric and J-invariant). In this case we shall show that an H-planner curve $c(t)$ with $a=0$ in (0.1) is an H-geodesic if the development of $c(t)$ is an H-geodesic in $\boldsymbol{P}^n(\boldsymbol{C})$ (Theorem 5). Finally we shall prove

Theorem 6. *Let ∇ and ∇' be H-complete connections of Kaehler type with parallel Ricci tensors S and S' respectively. Suppose that either $S=0$ or S has at least one negative eigenvalue at one point, and that ∇ and ∇' are H-projectively equivalent. Then we have $\nabla=\nabla'$.*

When ∇ and ∇' are the Kaehler connections of complete Kaehler metrics and both S and S' are parallel and negative semi-definite, the above result has been obtained by S. Ishihara and S. Tachibana [1].

Finally we remark that the present paper has been motivated by the paper of N. Tanaka on real projective transformations [7].

I would like to express my gratitude to my thesis advisor, Professor T. Ochiai for his valuable suggestions and encouragement.

NOTATION

Throughout this paper the following standard conventions will be adopted. $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denotes the real (resp. complex) number field. For $z\in\boldsymbol{C}$, Re(z) is the real part of z. We write $\boldsymbol{R}^m$ (resp. $\boldsymbol{C}^m$) for the m-dimensional standard real (resp. complex) vector space. An element of $\boldsymbol{R}^m$ (resp. $\boldsymbol{C}^m$) is considered as a column vector. We denote by $e_1,\cdots,e_m$ the canonical basis of $\boldsymbol{R}^m$ or $\boldsymbol{C}^m$. For $x\in\boldsymbol{R}^m$ or $\boldsymbol{C}^m$, ${}^tx$ denotes the transpose of x. The general linear group acting on $\boldsymbol{R}^m$ (resp. $\boldsymbol{C}^m$) and its Lie algebla are denoted, respectively, by $GL(m,\boldsymbol{R})$ (resp. $GL(m,\boldsymbol{C})$) and $\mathfrak{gl}(m,\boldsymbol{R})$ (resp. $\mathfrak{gl}(m,\boldsymbol{C})$). We write 1_m for the identity $m\times m$ matrix. For an $m\times m$ matrix A, det A denotes the determinant of A.

For a point p of manifold N, $T_p(N)$ is the tangent space to N at p. For a differentiable mapping f, f_* and f^* are the differential and the codifferential of f respectively. For a Lie group G, its Lie algebla is written by the corresponding German letter $\mathfrak{g}$. For a G-principal bundle $Q\rightarrow M$, R_a denotes the right translation by an element a of G acting on Q. For an element A or $\mathfrak{g}$, A^* denotes the fundamental vector field on Q corresponding to A.

1. H-projective equivalence

Let M be an m-dimensional manifold. Let us denote by $j^r(f)$ the r-frame at $p=f(0)$ given by a diffeomorphism f of a neighborhood of the origin 0 of $\boldsymbol{R}^m$ onto an open subset of M. The set $G^r(m)$ of r-frames at $0\in\boldsymbol{R}^m$ is a Lie group with multiplication defined by the composition of jets. The set $F^r(M)$ of r-frames of M is a principal bundle over M with natural projection π^r satisfying $\pi^r(j^r(f))=f(0)$, and with structure group $G^r(m)$. $F^1(M)$ is nothing but the bundle of linear frames.

We have a natural inclusion of $GL(m, \boldsymbol{R})$ into $G^r(m)$, defined by $g\to j^r(g)$ for $g\in GL(m, \boldsymbol{R})$. In particular $GL(m, \boldsymbol{R})$ and $G^1(m)$ are isomorphic by this inclusion. We shall identify $GL(m, \boldsymbol{R})$ with $G^1(m)$ and consider $GL(m, \boldsymbol{R})$ as a subgroup of $G^r(m)$ by this inclusion.

Let f be a diffeomorphism of M onto a manifold N. Then f induces a bundle isomorphism $f^{(r)}$: $F^r(M)\to F^r(N)$ defined by

$$f^{(r)}(j^r(h)) = j^r(f\cdot h) \quad \text{for } j^r(h)\in F^r(M).$$

We have a natural projection ν: $F^2(M)\to F^1(M)$ defined by $\nu(j^2(f))=j^1(f)$ ($j^2(f)\in F^2(M)$). A cross-section s: $F^1(M)\to F^2(M)$ is said to be *admissible* if we have

$$s(xa) = s(x)a \quad \text{for } x\in F^1(M) \text{ and } a\in GL(m,\boldsymbol{R}).$$

The $\boldsymbol{R}^m$ (resp. $\mathfrak{gl}(m,\boldsymbol{R})$)-component of the canonical form Θ on $F^2(M)$ (see [2] for the meaning of terminology) is denoted by Θ_{-1} (resp. Θ_0).

Proposition 1 (S. Kobayashi [2]). *For an admissble crosssection s: $F^1(M)\to F^2(M)$, $s^*\Theta_0$ is an affine connection on M without torsion. And this defines a one-to-one correspondence between affine connections on M without torsion and admissible cross-sections.*

Let $u^1,\cdots,u^m$ be a local coordinate system in M, and let $y^1,\cdots,y^m$ be the natural cooordinate system in $\boldsymbol{R}^m$. Each 2-frame u (resp. $a\in G^2(m)$) has a unique polynomial respresentation $u=j^2(f)$ (resp. $a=j^2(f)$) of the form

$$f^i(y) = u^i+\sum u^i_j y^j+\frac{1}{2}\sum u^i_{jk}y^j y^k$$

$$(\text{resp. } f^i(y) = \sum a^i_j y^j+\frac{1}{2}\sum a^i_{jk}y^j y^k),$$

where $u^i_{jk}=u^i_{kj}$ (resp. $a^i_{jk}=a^i_{kj}$), and $f^i(y)$ is the *i-th* coordinate of $f(y)$ with respect to $u^1,\cdots,u^m$ (resp. $y^1,\cdots,y^m$). We shall consider (u^i, u^i_j, u^i_{jk}) (resp. (a^i_j, a^i_{jk})) as a local coordinate system in $F^2(M)$ (resp. a coordinate system in $G^2(m)$). In the same way, a local coordinate system (u^i, u^i_j) in $F^1(M)$ and a coordinate system (a^i_j) in $G^1(m)$ are defined. The action of $G^2(m)$ on $F^2(M)$ is then given by

$$(u^i, u^i_j, u^i_{jk})(a^i_j, a^i_{jk}) = (u^i, \sum u^i_q a^q_j, \sum u^i_q a^q_{jk} + \sum u^i_{l,r} a^l_j a^r_k)\,. \tag{1.1}$$

Let s be the cross-section corresponding by Proposition 1 to an affine connection ∇ without torsion. Then the local expression of s is

$$s(u^i, u^i_j) = (u^i, u^i_j, -\sum u^q_j \Gamma^i_{ql} u^l_k)\,, \tag{1.2}$$

where Γ^i_{ql} are the Christoffel's symbols of ∇ with respect to $u^1,\cdots,u^m$ ([2]).

Let L and L_0 be as in (0.3). We shall consider $L_0/(\text{center})$ as a subgroup of $G^2(n)$ as follows. Let $\pi\colon \boldsymbol{C}^{n+1}-\{0\}\to \boldsymbol{P}^n(\boldsymbol{C})$ be the Hopf fibering. Identifying the subset

$$\left\{\pi\begin{pmatrix}1\\ z\end{pmatrix}\in P^n(\boldsymbol{C})\,|\,z\in\boldsymbol{C}^n\right\}$$

of $\boldsymbol{P}^n(\boldsymbol{C})$ with $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$, $a\in L_0$ can be considered as a local diffeomorphism of $\boldsymbol{R}^{2n}$ leaving the origin 0 of $\boldsymbol{R}^{2n}$ fixed. Here $\boldsymbol{C}^n$ is identified with $\boldsymbol{R}^{2n}$ by the correspondence $(z^1,\cdots,z^n)\in\boldsymbol{C}^n\to(x^1,\cdots,x^n,y^1,\cdots,y^n)\in\boldsymbol{R}^{2n}$, $z^i=x^i+\sqrt{-1}y^i$, $x^i,y^i\in\boldsymbol{R}$, $i=1,\cdots,n$. It can be easily verified that $j^2(a)=\text{id}$ if and only if a is the identity transformation of L/L_0. Hence $L_0/(\text{center})$ can be identified with the group of 2-jets $\{j^2(a)\,|\,a\in L_0\}$. By a straightforward computation we have

Lemma 1.1. *The expression of*

$$a=\begin{pmatrix}1 & t_v\\ 0 & 1_n\end{pmatrix}(\textit{mod center})\in L_0/(\textit{center})$$

as an element of $G^2(n)$ *is given by* (δ^i_j, a^i_{jk}) *with*

$$a^i_{jk}=\delta^i_j\rho_k+\delta^i_k\rho_j-\phi^i_j\rho_s\phi^s_k-\phi^i_k\rho_s\phi^s_j\,, \tag{1.3}$$

where

$$\rho_k=\begin{cases}-v^k & \textit{if } 1\leqq k\leqq n\\ v^k & \textit{if } n+1\leqq k\leqq 2n,\end{cases}\qquad (\phi^i_j)=\begin{pmatrix}0 & -1_n\\ 1_n & 0\end{pmatrix},$$

v^k *being k-th component of* $v\in\boldsymbol{C}^n=\boldsymbol{R}^{2n}$.

Let us denote the Lie algebras of L and L_0 by $\mathfrak{l}$ and $\mathfrak{l}_0$ respectively. Subalgebras $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ of $\mathfrak{l}$ are defined, respectively, as follows:

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix}0 & 0\\ u & 0\end{pmatrix}\in\mathfrak{l}\,|\,u\in\boldsymbol{C}^n\right\}$$

$$\mathfrak{g}_0 = \left\{\begin{pmatrix} z & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{l} \mid A \in \mathfrak{gl}(n, \boldsymbol{C})\right\}$$

$$\mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & {}^t v \\ 0 & 0 \end{pmatrix} \in \mathfrak{l} \mid v \in \boldsymbol{C}^n\right\}.$$

In the following, $\mathfrak{g}_{-1}$ and, $\mathfrak{g}_1$ are identified, respectively, with $\boldsymbol{C}^n$ and its dual space $(\boldsymbol{C}^n)^*$. And $\mathfrak{g}_0$ is identified with $\mathfrak{gl}(n, \boldsymbol{C})$ by the correspondence

$$\begin{pmatrix} z & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{g}_0 \to A - z1_n \in \mathfrak{gl}(n, \boldsymbol{C}).$$

Therefore we can consider $GL(n, \boldsymbol{C})$ as a subgroup of $L_0/(\text{center})$ by the injection

$$B \in GL(n, \boldsymbol{C}) \to \begin{pmatrix} (\det B)^{-1/n+1} & 0 \\ 0 & (\det B)^{-1/n+1} B \end{pmatrix} \text{ (mod center)} \in L_0/(\text{center})$$

Put $L_1 = \exp \mathfrak{g}_1$. Then

$$(1.4) \qquad L_0/(\text{center}) = GL(n, \boldsymbol{C}) \cdot L_1 \quad \text{(semi-direct)}.$$

For the remainder of this section we suppose that M is a complex manifold of complex dimension n. Let ∇ be an almost complex affine connection without torsion on M and let γ be its connection form on the bundle $C(M)$ of complex linear frames. By Proposition 1 there exists an admissible cross-section l: $F^1(M) \to F^2(M)$ corresponding to ∇. Let ι denote the inclusion map $C(M)$ into $F^1(M)$. Then $s = l \cdot \iota$ is an imbedding of $C(M)$ into $F^2(M)$ such that $s^*\Theta_0 = \gamma$ and $s(xa) = s(x)a$ for $x \in C(M)$ and $a \in GL(n, \boldsymbol{C})$. Thus $C(M)$ can be considered as a $GL(n, \boldsymbol{C})$-subbundle of $F^2(M)$. The group extension of $C(M)$ to $L_0/(\text{center})$ with respect to (1.4) will be denoted by $Q(\nabla)$.

Theorem 1. *Let ∇_1 and ∇_2 be two almost complex affine connections without torsion. Then ∇_1 and ∇_2 are H-projectively equivalent if and only if $Q(\nabla_1) = Q(\nabla_2)$.*

Proof. Let $z^A = x^A + \sqrt{-1}\, x^{A+n}$, $A = 1, \cdots, n$, be a complex local coordinate system in an open subset U of M. We define the natural almost complex stucture J on M by

$$J(\partial/\partial x^A) = \partial/\partial x^{A+n}, J(\partial/\partial x^{A+n}) = -\partial/\partial x^A, \ A = 1, \cdots, n.$$

It follows from (1.2) that the injections

$$s_1\colon C(M) \to Q(\nabla_1) \quad \text{and} \quad s_2\colon C(M) \to Q(\nabla_2)$$

corresponding respectively to ∇_1 and ∇_2 are expressed as follows:

$$s_1(x^i, x^i_j) = (x^i, x^i_j, -\sum_{m,l=1}^{2n} x^m_j (\Gamma_1)^i_{ml} x^l_k),$$

$$s_2(x^i, x^i_j) = (x^i, x^i_j, -\sum_{m,l=1}^{2n} x^m_j (\Gamma_2)^i_{ml} x^l_k)\,,$$

where $(\Gamma_1)^i_{ml}$ and $(\Gamma_2)^i_{ml}$ are respectively the Christoffel's symbols of ∇_1 and ∇_2 with respect to $x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n}$. Note that $(x^i, \delta^i_j) \in C(M)$.

Assume that $Q(\nabla_1) = Q(\nabla_2)$. Then there exists a C^∞-map $a\colon U \to L_0$ such that

$$s_1(x^i, \delta^i_j) = s_2(x^i, \delta^i_j)a,\ a = (a^i_j, a^i_{jk}) \tag{1.5}$$

By the above formulas for local expression of s_1 and s_2, we see that $a^i_j = \delta^i_j$. This means $a(U) \subset L_1$.
By (1.1) we have

$$-(\Gamma_2)^i_{jk} + a^i_{jk} = -(\Gamma_1)^i_{jk}\,.$$

It follows from Lemma 1.1 that there exist real functions $\rho_1, \cdots, \rho_{2n}$ such that

$$(\Gamma_2)^i_{jk} - (\Gamma_1)^i_{jk} = \delta^i_j \rho_k + \delta^i_k \rho_j - \sum_s \phi^i_j \rho_s \phi^s_k - \sum_s \phi^i_k \rho_s \phi^s_j\,.$$

Let J^i_j be the local expression of J with respect to $x^1, \cdots, x^{2n}$, then $J^i_j = \phi^i_j$. Thus we obtain

$$(\Gamma_2)^i_{jk} - (\Gamma_1)^i_{jk} = \delta^i_j \rho_k + \delta^i_k \rho_j - \sum_s J^i_j \rho_s J^s_k - \sum_s J^i_k \rho_s J^s_j\,. \tag{1.6}$$

This shows that (ρ_i) is a 1-form. Thus ∇_1 and ∇_2 are H-projectively equivalent (cf. the definition in Introduction).

Conversely assume that ∇_1 and ∇_2 are H-projectively equivalent, *i.e.*, ∇_1 and ∇_2 are related by the formula (1.6). Define $a = (\delta^i_j, a^i_{jk}) \in L_1$ by (1.3). Then (1.5) holds. Thus we see $Q(\nabla_1) = Q(\nabla_2)$. q.e.d.

Let ∇ be an almost complex affine connection without torsion and let $s\colon C(M) \to Q(\nabla)$ be the cross-section corresponding to ∇. For a biholomorphic transformation $f\colon M \to M$, define an admissible cross-section $s'\colon C(M) \to F^2(M)$ by $s' = (f^{(2)})^{-1} \cdot s \cdot f^{(1)}$. Since $f^{(2)}$ leaves Θ invariant, s' is the admissible cross-section corresponding to $\nabla' = f^*\nabla$. Thus we have $f^{(2)}(Q(\nabla')) = Q(\nabla)$. Therefore $Q(\nabla) = Q(\nabla')$ if and only if $f^{(2)}(Q(\nabla)) = Q(\nabla)$. Applying Corollary 11–1 in [4] to our case, we obtain

Theorem 2. *Let ∇ be an almost complex affine connection without torsion. Then the group of all H-projective transformations of ∇ is a Lie group of finite dimension.*

2. The development of an H-planner curve with respect to a Cartan connection of type $P^n(C)$

Let M be a manifold of dimension n, G a Lie group, K a closed subgroup of G with $\dim G/K=n$ and Q a principal bundle over M with structure group K. A *G/K-Cartan connection* in the bundle Q is a 1-form ω on Q with values in the Lie algebra $\mathfrak{g}$ of G satisfying the following conditions:

i) $R_h{}^*\omega=Ad(h^{-1})\omega, \quad h\in K$

ii) $\omega(A^*)=A, \qquad A\in\mathfrak{k}$

iii) $\omega(X)\neq 0$ for every nonzero vector X of Q.

A G/K-Cartan connection is said to be a $P^n(C)$-Cartan connection when $G=L/(\text{center})$ and $K=L_0/(\text{center})$, L and L_0 being as in (0.3).

Let P be the group extension of Q to G, i.e., $P=Q\times_K G$. Then a Cartan connection ω in Q can be uniquely extended to a connection form on P, denoted by $\tilde{\omega}$. Let $c(t)$ be a curve in M and let $z(t)\in P$ be a horizontal lift of $c(t)$ with respect to $\tilde{\omega}$ such that $z(0)\in Q$. Then there exists a curve $a(t)\in G$ such that $z(t)a(t)\in Q$. The development $c^*(t)$ of $c(t)$ at $c(0)$ by ω is defined by

$$c^*(t) = z(0)\cdot a(t)0\in Q\times_K G/K\,,$$

where 0 denotes the origin of G/K ([3]). We shall often identify $c^*(t)$ with the curve $a(t)0\in G/K$.

We shall consider the case when $G=L/(\text{center})$ and $K=L/_0(\text{center})$, L and L_0 being as in (0.3). We call a curve $c(t)$ in $\boldsymbol{P}^n(\boldsymbol{C})$ a *projective line* if there exists a 2-dimensional complex subspace W of $\boldsymbol{C}^{n+1}$ such that $c(t)\in\pi(W-(0))$. Let M be an n-dimensional complex manifold with an almost complex affine connection ∇ without torsion. Let us denote by θ the canonical form on $C(M)$ and by γ the connection form on $C(M)$ corresponding to ∇. We see in Section 1 that ∇ gives rise to a K-structure $Q(\nabla)$ of second order, i.e., K-subbundle of $F^2(M)$, and the injection $s\colon C(M)\to Q(\nabla)$. We know that there exists a Cartan connection ω on $Q(\nabla)$ satisfying

(2.1) $$s^*\omega_{-1}=\theta \text{ and } s^*\omega_0=\gamma\,,$$

where ω_{-1} and ω_0 are respectively $\mathfrak{g}_{-1}$-component and $\mathfrak{g}_0$-component of ω.

We shall prove

Proposition 2.1. *Let ∇ be an almost complex affine connection on a complex manifold M and let ω be any Cartan connection on $Q(\nabla)$ satisfying* (2.1). *Then, a curve in M is H-planner if and only if its development with respect to ω is a projective line.*

This follows directly from following Lemmas 2.2 and 2.3.

Lemma 2.1. *Let $c(t)$ be a curve in M and let $x(t)$ be a horizontal lift of $c(t)$ in $C(M)$. Define $v(t)\in\boldsymbol{C}^n$ by*

(2.2) $$c'(t) = x(t)v(t)$$

Then

$$\nabla_{c'} c' = ac' + bJc' \tag{2.3}$$

for certain smooth functions a and b if and only if

$$v(t) = \exp\left(\int_0^t (a(t) + \sqrt{-1}\, b(t))dt\right) v(0)\,. \tag{2.4}$$

Proof. From the difinition of covariant derivative, we obtain

$$\nabla_{c'(t)} c'(t) = x(t) v'(t)\,.$$

By (2.2),

$$a(t)c'(t) + b(t)Jc'(t) = x(t)\,(a(t) + \sqrt{-1}\, b(t)) v(t)\,.$$

Therefore (2.3) holds if and only if

$$v'(t) = (a(t) + \sqrt{-1}\, b(t)) v(t)\,. \tag{2.5}$$

We have (2.4) if and only if (2.5) holds. q.e.d.

Let $c(t)$ be a regular curve in M and let $x(t)$ (resp. $z(t)$ with $z(0) = s(x(0))$) be a horizontal lift of $c(t)$ in $C(M)$ (resp. P) with respect to ∇ (resp. $\tilde{\omega}$). Choose a curve $a(t) \in L$ satisfying

$$z(t)\,[a(t)] = s(x(t)),\ a(0) = 1_{n+1}\,, \tag{2.6}$$

where $[a(t)]$ denotes the image of $a(t)$ by the natural projection $L \to L/(\text{center})$. We may assume that $a(t)$ is smooth since the center of L is discrete. We shall denote the $(A+1)$-th column vector of $a(t)$ by $a_A(t)$ $(0 \leqq A \leqq n)$.

Lemma 2.2. *$a_0(t)$, $a_0'(t)$ and $a_0''(t)$ are linearly dependent for each t if and only if $c(t)$ is H-planner.*

Proof. Differentiating both sides of (2.6), we obtain

$$R_{[a(t)]*} z'(t) + (a(t)^{-1} a'(t))^*{}_{z(t)[a(t)]} = s_*(x'(t))\,.$$

Hence we have

$$a(t)^{-1} a'(t) = \tilde{\omega}(s_* x'(t))\,. \tag{2.7}$$

Let $\tilde{\omega}^A_B$ $(0 \leqq A, B \leqq n)$ denote the $(A+1, B+1)$-component of $\tilde{\omega}(s_*(x'(t)))$. From (2.7) we obtain

$$a_B' = \sum_{A=0}^{n} a_A \tilde{\omega}^A_B \qquad 0 \leqq B \leqq n\,.$$

Hence

$$a_0'' = \sum_{A=0}^{n} a_A \frac{d\tilde{\omega}_0^A}{dt} + \sum_{B=0}^{n} a_B' \tilde{\omega}_0^B$$
$$= \sum_{A=0}^{n} a_A \Big(\frac{d\tilde{\omega}_0^A}{dt} + \sum_{B=0}^{n} \tilde{\omega}_B^A \tilde{\omega}_0^B \Big) .$$

Since $x(t)$ is horizontal with respect to ∇, we have

$$\tilde{\omega}_0^0 = 0 \quad \text{and} \quad \tilde{\omega}_k^j = 0 \qquad 1 \leqq j,\, k \leqq n .$$

Thus we obtain

$$a_0' = \sum_{k=1}^{n} a_k \tilde{\omega}_0^k , \tag{2.8}$$

$$a_0'' = \sum_{k=1}^{n} a_0 \tilde{\omega}_k^0 \tilde{\omega}_0^k + \sum_{k=1}^{n} a_k \frac{d\tilde{\omega}_0^k}{dt} . \tag{2.9}$$

Now suppose that $a_0(t)$, $a_0'(t)$ and $a_0''(t)$ are linearly dependent for each t. Then there exist functions $f(t)$, $g(t)$ and $h(t)$ such that

$$fa_0 + ga_0' + ha_0'' = 0 \tag{2.10}$$

and $$|f| + |g| + |h| \neq 0 .$$

Substituting (2.8) and (2.9) in (2.10), we have

$$(f(t) + h(t) \sum_{k=1}^{n} \tilde{\omega}_k^0 \tilde{\omega}_0^k) a_0 + \sum_{j=1}^{n} \Big(g(t) \tilde{\omega}_0^j + h(t) \frac{d\tilde{\omega}_0^j}{dt} \Big) a_j = 0 .$$

Since $a_0(t), a_1(t), \cdots, a_n(t)$ are linearly independent, this is equivalent to the following:

$$f(t) + h(t) = \sum_{k=1}^{n} \tilde{\omega}_k^0 \tilde{\omega}_0^k = 0 ,$$
$$g(t) \tilde{\omega}_0^j + h(t) \frac{d\tilde{\omega}_0^j}{dt} = 0 \quad \text{for} \quad 1 \leqq j \leqq n .$$

Since $c'(t) \neq 0$, we have $\tilde{\omega}_0^j \neq 0$ for a certain integer j ($1 \leqq i \leqq n$). Hence $h(t) \neq 0$ for each t. Putting

$$F(t) = -g(t)/h(t) ,$$

we obtain

$$F(t) \theta(x'(t)) = d\theta(x'(t))/dt ,$$

which shows that F is a differentiable function. Hence

$$\theta(x'(t)) = \exp\Big(\int_0^t F(t) dt\Big) v_0 , \quad v_0 \in \boldsymbol{C}^n \tag{2.11}$$

i.e.,

$$c'(t) = x(t) \exp \left(\int_0^t F(t)dt\right) v_0 .$$

Therefore if follows from Lemma 2.1 that $c(t)$ is H-planner. Taking the steps backwards, it is now easy to prove the converse.

Lemma 2.3. *$a_0(t)$, $a_0'(t)$ and $a_0''(t)$ are linearly dependent for each t if and only if there exists a 2-dimensional complex subspace W of $\boldsymbol{C}^{n+1}$ in which $a_0(t)$ is contained for every t.*

Proof. First note that a_0 and a_0' are linearly independent for each t. This follows from formula (2.8), because $\tilde{\omega}_0^k \neq 0$ for a certain integer $k(1 \leqq k \leqq n)$ and $a_0(t), a_1(t), \cdots, a_n(t)$ are linearly independent for each t. Let b_A $(0 \leqq A \leqq n)$ be the $(A+1)$-th component of a_0 and define an $(n+1) \times 3$ matrix B by

$$B = \begin{pmatrix} b_0 & b_0' & b_0'' \\ b_1 & b_1' & b_1'' \\ \vdots & \vdots & \vdots \\ b_n & b_n' & b_n'' \end{pmatrix} .$$

We may assume that in an open interval U containing $t=t_0$

$$\det \begin{pmatrix} b_0 & b_0' \\ b_1 & b_1' \end{pmatrix} \neq 0 . \tag{2.12}$$

Now suppose that a_0, a_0' and a_0'' are linearly dependent. Since rank $B=2$, $b_j (j=2, 3, \cdots, n)$ are solutions of the following ordinary linear differential equation of second order:

$$\det \begin{pmatrix} b_0 & b_0' & b_0'' \\ b_1 & b_1' & b_1'' \\ x & x' & x'' \end{pmatrix} = 0 .$$

It follows that there exist constants α_j, β_j $(j=2, \cdots, n)$ such that

$$b_j = \alpha_j b_0 + \beta_j b_1 .$$

Thus we obtain

$$a_0 = b_0 \begin{pmatrix} 1 \\ 0 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} + b_1 \begin{pmatrix} 0 \\ 1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} .$$

This shows that $a_0(t)$ $(t \in U)$ is contained in the 2-dimensional complex subspace W of $\boldsymbol{C}^{n+1}$ spanned by

$$\begin{pmatrix} 1 \\ 0 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

We shall see that such a 2-dimensional subspace is independent of the choice of t_0. In fact, suppose that there exists a 1-dimensional subspace V of $\boldsymbol{C}^{n+1}$ such that $a_0(t)\in V$ for every t in a certain open interval V contained in U. This contradicts (2.12). The proof for the converse is trivial. q.e.d.

EXAMPLE 2.1. $S=SU(n+1,\boldsymbol{C})/(\text{center})$ acts transitively on $P^n(\boldsymbol{C})$ in a natural manner. Let H be the isotropy subgroup of S at

$$0=\pi\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\in \boldsymbol{P}^n(\boldsymbol{C}).$$

Since each $f\in S$ is a transformation of $P^n(\boldsymbol{C})$ and a neighborhood of 0 in $P^n(\boldsymbol{C})$ is identified with a neighborhood of 0 in $\boldsymbol{R}^{2n}$ in a natural way, the 1-jet $j_0^1(f)$ can be considered as a 1-frame of $P^n(\boldsymbol{C})$ at $f(0)$. The set of all 1-frames thus obtained defines an H-subbundle of the bundle $C(P^n(\boldsymbol{C}))$ of complex linear frames, which may be identified with the bundle S over $P^n(\boldsymbol{C})$. L and L_0 being as in (0.3), let G and K denote $L/(\text{center})$ and $L_0/(\text{center})$ respectively. Then the set of all 2-frames $\{j_0^2(f)|\in G\}$ defines a K-subbundle of $F^2(P^n(\boldsymbol{C}))$, and this can be identified with the bundle G over $P^n(\boldsymbol{C})$. The Maurer-Cartan form ω of G is a G/K-Cartan connection in G. Define an injection s: $C(P^n(\boldsymbol{C}))\to F^2(P^n(\boldsymbol{C}))$ by $s(xa)=\iota(x)a$ for $x\in S$ and $a\in GL(n, \boldsymbol{C})$, ι being the inclusion map of S into G. Then the bundle G is the group extension of $C(P^n(\boldsymbol{C}))$ by s to the group K. The 1-form $s^*\omega|\mathfrak{g}_0$ on $C(P^n(\boldsymbol{C}))$, restriction of values of $s^*\omega$ to the Lie algebra $\mathfrak{g}_0$ of $GL(n,\boldsymbol{C})$, corresponds to the Kaehler connection ∇ on the symmetric space $P^n(\boldsymbol{C})=S/H$. Thus ω is a Cartan connection corresponding to ∇ and, in fact, ω is the normal Cartan connection (see section 4 for the meaning of terminology) [4]. ω can be uniquely extended to a connection form $\tilde{\omega}$ on the bundle $G\times_K G$ over $P^n(\boldsymbol{C})$. A horizontal lift of a curve $c(t)=a(t)0\in P^n(\boldsymbol{C})$ ($a(t)\in G$) with respect to $\tilde{\omega}$ is $z(t)=a(t)\cdot a(t)^{-1}a(0)\in G\times_K G$. In fact, noting that $R_{a(0)^{-1}a(t)}z(t)$ belongs to the subbundle G, we have by the definition of $\tilde{\omega}$

$$\begin{aligned}\tilde{\omega}(z'(t)) &= \tilde{\omega}(R_{a(t)^{-1}a(0)*}R_{a(0)^{-1}a(t)*}(z'(t)))\\ &= Ad(a(0)^{-1}a(t))\tilde{\omega}(R_{a(0)^{-1}a(t)*}z'(t)))\\ &= Ad(a(0)^{-1}a(t))\,(\omega(a'(t))+Ad(a(t)^{-1}a(0))\,(a(0)^{-1}a(t)\,(a(t)^{-1}a(0))'))\\ &= Ad(a(0)^{-1}a(t))\,(a(t)^{-1}a'(t)+(a(t)^{-1})'a(t))=0\,.\end{aligned}$$

Here we may assume $a(t)$ is locally differentiable, since $z(t)$ is independent of the choice of $a(t)\in G$. Thus $c^*(t)=a(0)^{-1}a(t)0\in P^n(\boldsymbol{C})$ is the development of $c(t)$ with respect to ω.

Applying Proposition 2.1 to the case when $M=P^n(\boldsymbol{C})$, we obtain

Corollary 2.1. *A curve in $P^n(\boldsymbol{C})$ is H-planner if and only if it is a projective line.*

By Proposition 2.1 and Corollary 2.1 we have

Theorem 3. *The assumptions and notation being as in Proposition* 2.1, *a curve in M is H-planner if aud only if its development with respect to ω is H-planner.*

3. *H*-completeness

We have defined an H-geodesic and H-completeness in Introduction. In this section we shall prove the following:

Theorem 4. *Let M be a connected Kaehler manifold with a Kaehler metric g and let ∇ be the Kaehler connection of g. Then H-completeness of ∇ is equivalent to completeness of g.*

Proof. Completeness of g follows from H-completeness of ∇ since a geodesic of g is clearly an H-geodesic of ∇. Assume that g is complete. Let $c(t)$ $0\leqq t<L$ be an H-geodesic, i.e.,

$$\nabla_{c'}c' = bJc' \qquad b\text{: constant.} \tag{3.1}$$

We shall show that this H-geodesic can be extended beyond L. Let $x(t)$ be a horizontal lift of $c(t)$ in the unitary frame bundle with respect to g. We can choose such a horizontal lift because ∇ is the Kaeler connection of g. Then $c'(t)=x(t)v(t)$, where $v(t)=\exp(\sqrt{-1}\,bt)v(0)$ by Lemma 2.1. Let $\{t_k\}$ be an infinite sequence such that $t_k\to L$ $(k\to\infty)$. Then

$$\begin{aligned} d(c(t_k), c(t_l)) &\leq \left|\int_{t_k}^{t_l} g(c'(t), c'(t)dt\right| \\ &= |t_k-t_l|\,|v(0)|\,, \end{aligned}$$

where d denotes the distance function defined by g and $|v(0)|$ denotes the usual norm of $v(0)$ in $\boldsymbol{C}^n$. This shows that $\{c(t_k)\}$ is a Canchy sequence in M with respect to d and hence converges to a point, say p. The limit point is independent of the choice of a sequence $\{t_k\}$ converging to L. Let $x^1, x^2, \cdots, x^{2n}$ be a local coordinate system in a relatively compact coordinate neighborhood U of p. The local expression of (3.1) in U is

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = bJ^i_j\frac{dx^j}{dt}\,. \tag{3.2}$$

The exists a positive number δ such that $\{c(s)\mid L-\delta\leqq s<L\}\subset U$. Since the length of c' is constant, $\{dx^j/dt(s)\mid L-\delta<s<L\}$ are bounded. It follows from (3.2) that $\{|d^2x^j/dt^2(s)|\mid L-\delta<s<L\}$ are also bounded, and less than a constant N. Let $\{s_k\}$ be an infinite sequence such that $s_k\to L$ $(k\to\infty)$. Then

$$\left|\frac{dx^j}{dt}(s_m)-\frac{dx^j}{dt}(s_l)\right|=\left|\int_{s_l}^{s_m}\frac{d^2x^j}{dt^2}dt\right|\leqq N|s_m-s_l|\ .$$

This shows that $\{dx^j/dt(s_k)\}$ is a Cauchy sequence in $\boldsymbol{R}$, hence converges to a real number. The limit is independent of the choice of a sequence $\{s_k\}$ converging to L. Since $c(t)$ and dx^i/dt converge when $t\to L$, the solution of (3.2) can be extended beyond L. This completes the proof of Theorem 3.

4. A connection of Kaehler type

In this section we shall prove a certain property of a connection of Kaehler type defined in Introduction. The result will be used to prove Theorem 5 and Theorem 6 in the following sections.

Let ∇ be an almost complex affine connection without torsion on a complex manifold M of complex dimension n. And let Q and $s\colon C(M)\to Q$ be the corresponding L_0/(center)-structure and the injection. We know that there exists a $\boldsymbol{P}^n(\boldsymbol{C})$-Cartan connection ω satisfying (2.1) for any almost complex affine connection without torsion which is H-projectively equivalent to ∇ ([4]). Define a subspace H_q of the tangent space $T_q(Q)$ at $q\in Q$ by

$$H_q=\{X\in T_q(Q)\mid\omega_0(X)=0,\ \omega_1(X)=0\}\ .$$

Then $\omega_{-1}\colon H_q\to\mathfrak{g}_{-1}$ is a linear isomorphism. Put

$$\Omega=d\omega+[\omega,\omega]/2\ .$$

Decompose Ω into $\Omega=\Omega_{-1}\oplus\Omega_0\oplus\Omega_1$, Ω_{-1}, Ω_0 and Ω_1 being $\mathfrak{g}_{-1}$-, $\mathfrak{g}_0$- and $\mathfrak{g}_1$-components of Ω respectively. Let $\{v_i\}_{i=1,2,\cdots,2n}$ be a real basis of $\mathfrak{g}_{-1}$ and let $\{z^i\}$ be its dual basis in $\mathfrak{g}_1$ with respect to the Killing-Cartan form B of $\mathfrak{L}$ which is non-singular on $\mathfrak{g}_{-1}\times\mathfrak{g}_1$. Choose $X_i\in H_q$ such that $\omega_{-1}(X_i)=v_i$. We shall call ω a *$\boldsymbol{P}^n(\boldsymbol{C})$-nomal Cartan connection* if Ω_0 satisfies

$$\sum z^i\Omega_0(X_i,Y)=0\qquad\text{at each point } q\in Q.$$

If $n\geqq 2$, there exists uniquely a $\boldsymbol{P}^n(\boldsymbol{C})$-normal Cartan connection ([4]).

For the $\boldsymbol{P}^n(\boldsymbol{C})$-normal Cartan connection, define $E_x\colon\mathfrak{g}_{-1}\to\mathfrak{g}_1$ $(x\in C(M))$ by

$$E_x(\theta(Y))=s^*\omega_1(Y)\qquad Y\in T_x(C(M))\,. \tag{4.1}$$

E_x is well-defined. In fact, if $\theta_x(Y)=0$, there exists $A\in\mathfrak{gl}\ (n,\boldsymbol{C})$ such that $Y=(A^*)_x$. Hence

$$(s^*\omega_1)(Y) = \omega_1(s_*(A^*)_x) = \omega_1((A^*)_{s(x)}) = 0\,.$$

Let us denote by $C^{p,q}$ $(-1\leqq p\leqq 3)$ the set of all $\mathfrak{g}_{p-1}$-valued q-skew-symmetric multilinear form on $\mathfrak{g}_{-1}$, where $\mathfrak{g}_{-2}=\{0\}$ and $\mathfrak{g}_2=\{0\}$. Define d: $C^{p,q}\to C^{p-1,q+1}$ by

$$dc(y_1,\cdots,y_{q+1}) = \sum_{i=1}^{q+1}(-1)^{i+1}[y^i, C(y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_{q+1})]\,,$$

$y_1,\cdots,y_{q+1}\in\mathfrak{g}_{-1}$. And define d^*: $C^{p,q}\to C^{p+1,q-1}$ by

$$(d^*c)(y_1,\cdots,y_{q-1}) = \sum_{i=1}^{2n}[z^i, c(v_i,y_1,\cdots,y_{q-1})]\,,$$

$y_1,\cdots,y_{q-1}\in\mathfrak{g}_{-1}$, where $\{v_i\}$ denotes a basis of $\mathfrak{g}_{-1}$ and $\{z^i\}$ denotes the dual basis of $\{v_i\}$ in $\mathfrak{g}_1$ with respect to the Killing-Cartan form B of $\mathfrak{L}$.

We shall denote by S the Ricci tensor field of ∇. Define S_x: $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\boldsymbol{R}$ and T_x: $\mathfrak{g}_{-1}\to\mathfrak{g}_1$ for $x\in C(M)$ by

$$S_x(u,v) = S(xu,xv) \text{ and}$$

$$B(T_x(u),v) = S_x(u,v) \tag{4.2}$$

respectively. Then

$$T_x = -d^*dE_x \text{ ([4])}\,. \tag{4.3}$$

For $z\in\mathfrak{g}_1$ and $v\in\mathfrak{g}_{-1}$ we shall denote by $\langle z, v\rangle$ the real part of zv.

Lemma 4.1. *Let ∇ be a connection of Kaehler type on an n-dimensional complex manifold $(n\geqq 2)$. Then*

$$\langle E_x(u), v\rangle = -S(xu,xv)/2(n+1)$$

or equivalently

$$E_x(u)v = -\{S(xu,xv)-\sqrt{-1}\,S(xu,Jxv)\}/2(n+1)\,.$$

In particular, $E_x(v)v$ is real valued.

Proof. We write E for E_x for simplicity. From the definition of the Killing-Cartan form of $\mathfrak{L}$, we obtain

$$B(X,Y)/4(n+1) = \text{Re (\textit{the trace of} } XY), \tag{4.4}$$

for X, $Y\in\mathfrak{L}$. Hence we consider $\mathfrak{L}$ as a real Lie algebra. Since $\{{}^te_i/4(n+1), -\sqrt{-1}\,{}^te_i/4(n+1)\}_{i=1,2,\cdots,n}$ is the dual basis of $\mathfrak{g}_1$ corresponding to a real basis $\{e_i, \sqrt{-1}\,e_i\}_{i=1,\cdots,n}$ of $\mathfrak{g}_{-1}$ with respect to B, we have

$$\begin{aligned} &d^*dE(v) \\ &= \sum_{i=1}^{n}\frac{1}{4(n+1)}[{}^te_i, dE(e_i,v)] + \sum_{i=1}^{n}\frac{1}{4(n+1)}[-\sqrt{-1}\,{}^te_i, dE(\sqrt{-1}\,e_i, v)] \end{aligned} \tag{4.5}$$

$$= \sum_{i=1}^{n} \frac{1}{4(n+1)} \{[{}^t e_i, [e_i, E(v)] - [v, E(e_i)]] + [-\sqrt{-1}\, {}^t e_i, [\sqrt{-1}\, e_i, E(v)] - [v, E(\sqrt{-1}\, e_i)]]\} .$$

On the other hand, for $v \in \mathfrak{g}_{-1}$, $z \in \mathfrak{g}_1$ and $A \in \mathfrak{g}_0$,

$$[v, z] = vz + (zv)1_n ,$$
$$[z, A] = zA .$$

Applying these formulas to (4.5), we obtain

$$d^*dE(v) = \frac{1}{4(n+1)} \sum_{i=1}^{n} \{2E(v) + 2{}^t e_i E(v) e_i - ({}^t e_i v E(e_i) + {}^t e_i E(e_i) v) + (\sqrt{-1}\, {}^t e_i v E(\sqrt{-1}\, e_i) + \sqrt{-1}\, {}^t e_i E(\sqrt{-1}\, e_i) v\} . \tag{4.6}$$

By virtue of (4.2), (4.3), (4.4) and (4.6),

$$-S_x(u, v) = 2(n+1)\langle E(u), v\rangle - \sum_{i=1}^{n} \langle {}^t e_i u E(e_i) + {}^t e_i E(e_i) u, v\rangle + \sum_{i=1}^{n} \langle \sqrt{-1}\, {}^t e_i u E(\sqrt{-1}\, e_i) + \sqrt{-1}\, {}^t e_i E(\sqrt{-1}\, e_i) u, v\rangle . \tag{4.7}$$

Since S_x is symmetric, we have by (4.7)

$$\langle E(u), v\rangle = \langle E(v), u\rangle \quad \text{for any} \quad u, v \in \mathfrak{g}_{-1} . \tag{4.8}$$

Put $u = e_j$ and $v = e_k$ in (4.7). Then we obtain

$$-S_x(e_j, e_k) = (2n+1)\langle E(e_j), e_k\rangle - \langle E\langle(e_k), e_j\rangle + \langle \sqrt{-1}\, E(\sqrt{-1}\, e_j), e_k\rangle + \langle \sqrt{-1}\, E(\sqrt{-1}\, e_k), e_j\rangle .$$

Thus, by (4.8)

$$-S_x(e_j, e_k) = 2n\langle E(e_j), e_k\rangle + 2\langle \sqrt{-1}\, E(\sqrt{-1}\, e_j), e_k\rangle . \tag{4.9}$$

Analogously, we have

$$-S_x(\sqrt{-1}\, e_j, \sqrt{-1}\, e_k) = 2n\langle \sqrt{-1}\, E(\sqrt{-1}\, e_j), e_k\rangle + 2\langle E(e_j), e_k\rangle , \tag{4.10}$$

$$-S_x(e_j, \sqrt{-1}\, e_k) = 2n\langle E(e_j), \sqrt{-1}\, e_k\rangle - 2\langle E(\sqrt{-1}\, e_j), e_k\rangle , \tag{4.11}$$

$$-S_x(\sqrt{-1}\, e_j, e_k) = 2n\langle E(\sqrt{-1}\, e_j), e_k\rangle - 2\langle E(e_j), \sqrt{-1}\, e_k\rangle . \tag{4.12}$$

Since $S(e_j, e_k) = S(\sqrt{-1}\, e_j, \sqrt{-1}\, e_k)$, (4.9) and (4.10) give

$$2(n-1)\langle \sqrt{-1}\, E(\sqrt{-1}\, e_j), e_k\rangle = 2(n-1)\langle E(e_j), e_k\rangle .$$

Since $n \geqq 2$ by assumption, we have

$$\langle E(e_j), e_k\rangle = \langle \sqrt{-1}\, E(\sqrt{-1}\, e_j), e_k\rangle . \tag{4.13}$$

In a similar fashion, (4.11) and (4.12) give

(4.14) $$\langle E(e_j), \sqrt{-1}\, e_k\rangle = \langle \sqrt{-1}\, E(\sqrt{-1}\, e_j), \sqrt{-1}\, e_k\rangle .$$

By virtue of (4.13) and (4.14),

$$E(e_j) = \sqrt{-1}\, E(\sqrt{-1}\, e_j) .$$

Applying this to (4.7), we obtain

$$-S_x(u, v) = 2(n+1)\langle E(u), v\rangle .$$

The second formula in Lemma 4.1 is now easy to show, because the imaginary part of $E(u)v$ is $-\langle E(u), \sqrt{-1}\, v\rangle$. This completes the proof of Lemma 4.1.

5. The development of an H-geodesic with respect to the $P^n(C)$-normal Cartan connection

Let ∇ be a connection of Kaehler type on a complex manifold M. Let us denote by $\{\nabla\}$ the family of almost complex affine connections without torsion which are H-projectively equivalent to ∇. We see in Section 4 that $\{\nabla\}$ determines uniquely a $P^n(C)$-normal Cartan connection. We shall prove

Proposition 5.1. *Assume that the development of a curve $c(t)$ with respect to the normal Cartan connection is contained in $\pi(W-\{0\})$ for a 2-dimensional real subspace W of C^{n+1}. Then, under a certain change of parameter, $c(t)$ is an H-geodesic.*

Proof. By Theorem 3 $c(t)$ is an H-planner curve. Hence $c(t)$ satisfies $\nabla_{c'}c'=ac'+bJc'$ for cetrin real functions a and b. Define a curve $\tilde{c}$ by

(5.0) $$\tilde{c}(T) = c(t), \quad T = \int_0^t \exp\left(\int_0^t a(t)dt\right)dt .$$

Then we have

$$\nabla_{\tilde{c}'}\tilde{c}' = \tilde{b}J\tilde{c}' , \quad \tilde{b}\text{: a real function.}$$

Since $\tilde{c}(t)$ satisfies the assumption of Proposition 5.1, we may assume $\nabla_{c'}c'=bJc'$. Let $x(t)$ be a horizontal lift in $C(M)$. Then by Lemma 2.1,

$$c'(t) = x(t)\,(\exp \sqrt{-1} \int_0^t bdt)v ,$$

v being a certain vector in C^n. This is equivalent to

$$d\theta(x'(t))/dt = \sqrt{-1}\, b\theta(x'(t)) .$$

Here θ denotes the canonical form on $C(M)$. The notation being as in Lemma 2.2, put

$$f(t) = -\sum_{k=1}^{n} \tilde{\omega}_k^0 \tilde{\omega}_0^k .$$

By the definition of $E_{x(t)}: \mathfrak{g}_{-1} \to \mathfrak{g}_1$ given in (4.1), we see

$$f(t) = -E_{x(t)}(\theta(x'(t)))\theta(x'(t)) . \tag{5.1}$$

It follows from Lemma 4.1 that $f(t)$ is a real-valued function. Let $a(t)$ be as in (2.6). Then by (2.8) and (2.9) in Lemma 2.2, we have

$$a_0'' - \sqrt{-1}\, b a_0' + f a_0 = 0 . \tag{5.2}$$

Let c_1 and c_2 be the solutions of

$$c'' - \sqrt{-1}\, b c' + f c = 0 \tag{5.3}$$

with initial values, respectively,

$$\begin{cases} c_1(0) = 1 \\ c_1'(0) = 0 \end{cases} \qquad \begin{cases} c_2(0) = 0 \\ c_2'(0) = 1 . \end{cases}$$

Then

$$a_0 = \begin{pmatrix} c_1 \\ c_2 v \end{pmatrix} .$$

Let W be a 2-dimensional real subspace of $\boldsymbol{C}^{n+1}$ such that

$$\pi\begin{pmatrix} c_1 \\ c_2 v \end{pmatrix} \subset \pi(W - \{0\}) .$$

Since $c_1(0) = 1$ and $c_2(0) = 0$,

$$\pi\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \pi(W - \{0\}) .$$

So there exists a constant $s \in \boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ such that

$$s\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W - \{0\} , \; i,e., \; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in s^{-1}W - \{0\} .$$

Therefore we may assume

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W - \{0\} .$$

Lemma 5.1. *There exists a differentiable function h such that*

$$h\begin{pmatrix} c_1 \\ c_2 v \end{pmatrix} \in W - \{0\} .$$

in an open interval U in which $c_2 \neq 0$.

Proof of Lemma 5.1. Let

$$e_1=\begin{pmatrix}1\\0\end{pmatrix},\ \tilde{\alpha}=\begin{pmatrix}\alpha^0\\ \alpha\end{pmatrix}\qquad (\alpha^0\in \boldsymbol{C},\ \alpha\in \boldsymbol{C}^n)$$

be a basis of W. Putting

$$d(t)=\begin{pmatrix}c_1(t)\\ c_2(t)v\end{pmatrix},$$

we have $d=z(u_1e_1+u_2\tilde{\alpha})$ for certain real valued functions u_1 and u_2, and a complex valued non-zero function z. $u_2\neq 0$ follows from the assumption $c_2\neq 0$. We only have to put $h=1/zu_2$ to complete the proof.

By Lemma 5.1 we see that $h(t_0)d(t_0)$ and e_1 for $t_0\in U$ is a basis of W. So

$$h\begin{pmatrix}c_1\\ c_2v\end{pmatrix}=A\begin{pmatrix}1\\0\end{pmatrix}+Bh(t_0)\begin{pmatrix}c_1(t_0)\\ c_2(t_0)v\end{pmatrix}$$

for certain real-valued functions A and B. Hence

$$c_1/c_2=A/Bh(t_0)c_2(t_0)+c_1(t_0)/c_2(t_0)\,.$$

Put

$$D=c_1/c_2,\quad G=A/B\ \text{ and }\ K=1/h(t_0)c_2(t_0)\,. \tag{5.4}$$

Then

$$D'=G'K \tag{5.5}$$

Lemma 5.2. *Let D be as in* (5.4) *and let U be an open interval in which $c_2(t)\neq 0$. Then*

$$D'=\frac{D'(t_0)(c_2(t_0))^2}{(c_2(t))^2}\exp\left(\sqrt{-1}\int_{t_0}^{t} b\,dt\right)\quad t_0\in U\,. \tag{5.6}$$

Proof of Lemma 5.2. Since c_1 is a solution of (5.3), i.e., $c_1''-\sqrt{-1}\,bc_1'+fc_1=0$, substituting $c_1=Dc_2$ in this equation, we have $D''c_2+(2c_2'-\sqrt{-1}\,bc_2)D'=0$. Hence

$$D''+(2c_2'/c_2-\sqrt{-1}\,b)D'=0$$

Solving this equation on D', we obtain (5.6). This completes the proof of Lemma 5.2.

By (5.5) and (5.6) we have

$$\frac{D'(t_0)(c_2(t_0))^2}{(c_2)^2}\exp\left(\sqrt{-1}\int_{t_0}^{t} b\,dt\right)=G'K\,.$$

Put $K/D'(t_0)(c_2(t_0))^2=l\exp(\sqrt{-1}\,\psi)$, $c_2=r_2\exp(\sqrt{-1}\,\theta_2)$, where l, ψ, r_2 and θ_2 are real functions. Then

$$\exp\{\sqrt{-1}\,(-2\theta_2+\int_{t_0}^{t} b dt-\psi)\} = G'l(r_2)^2 .$$

Since G', l and r_2 are continuous real functions, we have

$$-2\theta_2+\int_{t_0}^{t} b dt-\psi = 0 \pmod{\pi} . \tag{5.7}$$

Differentiating (5.7), we obtain

$$\theta_2' = b/2 . \tag{5.8}$$

Let

$$c_2 = r_2 \exp(\sqrt{-1}\,\theta_2) \tag{5.9}$$

be the expression by polar coordinates. Since c_2 is a solution of (5.3), i.e., $c_2''-\sqrt{-1}\,bc_2'+fc_2=0$, putting (5.9) in this equation, we have

$$\exp(\sqrt{-1}\,\theta_2)\,\{(r_2''-r_2(\theta_2')^2+br_2\theta_2'+fr_2)+\sqrt{-1}\,(2r_2'\theta_2'+r_2\theta_2''-br_2')\} = 0 .$$

Hence

$$2r_2'\theta_2'+r_2\theta_2''-br_2' = 0 . \tag{5.10}$$

Substituting (5.8) in (5.10), we obtain $r_2b_2'=0$. Since $r_2 \neq 0$, we have $b'=0$. This holds in an open interval in which $c_2 \neq 0$. However, since c_2 is a solution of an ordinary linear differential equation of second order, the zero points of c_2 are discrete. Thus b is constant, namely $c(t)$ is an H-geodesic. This completes the proof of Proposition 5.1.

Proposition 5.2. *Let ∇ be a connection of Kaehler type whose Ricci tensor is parallel, and let $c(t)$ be an H-geodesic with respect to ∇ under a certain change of parameter. Then there exists a 2-dimensional real subspace W of C^{n+1} such that the development of $c(t)$ with respect to the normal Cartan connection is contained in $\pi(W-\{0\})$.*

Proof. We may assume that $c(t)$ is an H-geodesic, since existence of such a 2-dimensional real subspace W of C^{n+1} as above is independent of the choice of a parameter. Let $x(t)$ be a horizontal lift in $C(M)$. Then, by Lemma 2.1,

$$c'(t) = x(t)\exp(\sqrt{-1}\,bt)v , \qquad v\in C^n$$

Since $c(t)$ is an H-geodesic, b is a real constant. The notation being as in the proof of Proposition 5.1, we have

$$a_0''-\sqrt{-1}\,ba_0'+fa_0 = 0 .$$

Lemma 4.1 shows that f is a real constant, because the Ricci tensor of ∇ is parallel. We shall denote this constant by $-k$. Let c_1 and c_2 be the solutions of

$$c''-\sqrt{-1}\, bc'-kc=0$$

with initial values, respectively,

$$\begin{cases} c_1(0)=1 \\ c_1'(0)=0 \end{cases} \qquad \begin{cases} c_2(0)=0 \\ c_2'(0)=1\,. \end{cases}$$

Then

$$a_0=\begin{pmatrix} c_1 \\ c_2 v \end{pmatrix}.$$

We only have to prove existence of a 2-dimensional real subspace W of C^{n+1} satisfying $\pi(a_0(t))\subset\pi(W-\{0\})$. Since b and k are real constants, the solutions c_1 and c_2 can be obtained explicitly as follows:

i) If $D=-b^2+4k\neq 0$, then

$$c_1=\frac{1}{2\sqrt{D}}\exp(\sqrt{-1}\, bt/2)\{(-\sqrt{-1}\, b+\sqrt{D})\exp(\sqrt{D}\, t/2)$$
$$+(\sqrt{-1}\, b+\sqrt{D})\exp(-\sqrt{D}\, t/2)\}\,,$$
$$c_2=\frac{1}{\sqrt{D}}\exp(\sqrt{-1}\, bt/2)\{\exp(\sqrt{D}\, t/2)-\exp(-\sqrt{D}\, t/2)\}\,.$$

ii) If $-b^2+4jk=0$ and $k\neq 0$, then

$$c_1=(-\sqrt{-1}\, bt/2)\exp(\sqrt{-1}\, bt/2)+\exp(\sqrt{-1}\, bt/2)\,,$$
$$c_2=t\exp(\sqrt{-1}\, bt/2)\,.$$

iii) If $b=0$ and $k=0$, then

$$c_1=1, \qquad c_2=t\,.$$

Thus we can choose a real basis $\{\alpha, \beta\}$ of W as follows:

i) If $D>0$, then

$$\alpha=\begin{pmatrix} \dfrac{-\sqrt{-1}\, b+D}{2} \\ v \end{pmatrix} \qquad \beta=\begin{pmatrix} \dfrac{\sqrt{-1}\, b+D}{2} \\ -v \end{pmatrix},$$

because

$$\pi(a_0(t))=\pi\left(\exp\left(\frac{\sqrt{D}}{2}t\right)\alpha+\exp\left(\frac{-\sqrt{D}}{2}t\right)\beta\right).$$

i)′ If $D<0$, then

$$\alpha=\begin{pmatrix} \sqrt{D} \\ 0 \end{pmatrix} \qquad \beta=\begin{pmatrix} -\sqrt{-1}\, b \\ 2v \end{pmatrix},$$

because

$$\pi(a_0(t)) = \pi\left(\cos\left(\frac{\sqrt{-D}}{2}t\right)\alpha+\sqrt{-1}\sin\left(\frac{\sqrt{-D}}{2}t\right)\beta\right).$$

ii) If $D=0$ and $k\neq 0$, then

$$\alpha=\begin{pmatrix}\frac{-\sqrt{-1}}{2}b\\ v\end{pmatrix} \qquad \beta=\begin{pmatrix}1\\0\end{pmatrix},$$

because

$$\pi(a_0(t)) = \pi(t\alpha+\beta).$$

iii) If $b=0$ and $k=0$, then

$$\alpha=\begin{pmatrix}1\\0\end{pmatrix} \qquad \beta=\begin{pmatrix}0\\v\end{pmatrix}.$$

q.e.d.

From Propositions 5.1 and 5.2 follows

Corollary 5.1. *Let ∇ be a connection of Kaehler type whose Ricci tensor is parallel. Then a curve $c(t)$ is an H-geodesic with respect to ∇ under a certain change of parameter if and only if there exists a 2-dimensional real subspace W of $\boldsymbol{C}^{n+1}$ such that the development of $c(t)$ with respect to the normal Cartan connection is contained in $\pi(W-\{0\})$.*

We have detailed the development of a curve in $P^n(\boldsymbol{C})$ in Example 2.1. Applying Corollary 5.1 to $M=P^n(\boldsymbol{C})$, we obtain

Corollary 5.2. *A curve $c(t)$ in $P^n(\boldsymbol{C})$ is an H-geodesic under a certain change of parameter if and only if there exists a 2-dimensional real subspace W of $\boldsymbol{C}^{n+1}$ such that $c(t)$ is contained in $\pi(W-\{0\})$.*

By Proposition 5.1 and Corollary 5.2 we have

Theorem 5. *Let ∇ be a connection of Kaehler type. Then a curve $c(t)$ is an H-geodesic with respect to ∇ under a certain change of parameter, if the development of $c(t)$ with respect to the normal Cartan connection is an H-geodesic in $P^n(C)$.*

6. Proof of Theorem 6

In this section we shall prove Theorem 6.

Lemma 6.1. *Let c_1 and c_2 be the solutions of the following differential equation*

(6.1) $$u''-\sqrt{-1}\,bu'-ku = 0$$

with initial conditions

(6.2) $$c_1(0) = 1,\ c_1'(0) = 0 \ \textit{and}\ c_2(0) = 0,\ c_2'(0) = 1\,,$$

where b and k are real constants. Then we have the following:

a) *If* $-b^2+4k>0$, *then* $|\lim_{t\to\infty} c_2/c_1| = 1/\sqrt{k}$.

b) *If* $-b^2+4k<0$, *then* $\lim_{t\to\infty} c_2/c_1$ *does not exist.*

c) *If* $-b^2+4k=0$ *and* $k\neq 0$, *then* $|\lim_{t\to\infty} c_2/c_1|=1/\sqrt{k}$.

d) *If* $b=0$ *and* $k=0$, *then* $\lim_{t\to\infty} c_1/c_2=0$.

Proof. We have obtained the solutions c_1 and c_2 explicitly in the proof of Proposition 5.2. Lemma 6.1 follows directly from these results. q.e.d.

For the remainder of this section, let ∇ be an H-complete connection of Kaehler type on a complex manifold M whose Ricci tensor S is parallel. Let $Q(\nabla)$ and $s:C(M)\to Q(\nabla)$ be, as explained in Section 2, the L_0/(center)-structure and the injection corresponding to ∇ respectively. Let E_x: $\mathfrak{g}_{-1}\to\mathfrak{g}_1$ $(x\in C(M))$ be as in (4.1). Define a subset Φ_{E_x} of $\boldsymbol{P}^n(\boldsymbol{C})$ by

$$\Phi_{E_x} = \left\{\pi\begin{pmatrix} v^0 \\ v \end{pmatrix}\in \boldsymbol{P}^n(\boldsymbol{C})\,|\, -|v^0|^2+E_x(v)v = 0,\ v^0\in \boldsymbol{C},\ v\in \boldsymbol{C}^n\right\}. \tag{6.3}$$

Lemma 6.2. *Let* $c(t)$ *and* $x(t)$ *be an H-geodesic of* ∇ *and its horizontal lift in* $C(M)$ *respectively. Put* $x=x(0)$. *And let* $a(t)\in L$ *be as in* (2.6). *If* $\lim_{t\to\infty} a(t)0$ *exists, it belong to* Φ_{E_x}.

Proof. By Lemma 2.1

$$c'(t) = x(t) \text{ epx } \left(\int_0^t F(t)dt\right)v\,,$$

for a certain function F and a vector $v\in\boldsymbol{C}^n$. We see by the definition of an H-geodesic $F(t)=\sqrt{-1}\, b$, b being a constant. Thus $\theta(x'(t))=\exp(\sqrt{-1}\, bt)v$. On the ohter hand, by Lemma 4.1 and by the assumption that the Ricci tensor field is parallel, we easily see that $E_{x(t)}(u)w$ is constant for any u and $w\in\mathfrak{g}_{-1}$. Thus $f(t)=-E_{x(t)}(v)v$ in (5.1) is a constant, which we shall denote by $-k$.

Let a_0 denote the first column vector of $a(t)$. Then by (5.2) a_0 is the solution of

$$a_0''-\sqrt{-1}\, b\, a_0'-ka_0 = 0$$

with initial conditions

$$a_0(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad a_0'(0) = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Let c_1 and c_2 be the solutions of (6.1) with initial conditions (6.2), then

$$a_0(t) = \begin{pmatrix} c_1(t) \\ c_2(t)v \end{pmatrix}.$$

Thus

$$a(t)0 = \pi(a_0(t)) = \pi\begin{pmatrix} c_1(t) \\ c_2(t)v \end{pmatrix}.$$

Lemma 6.2 now follows from Lemma 6.1 and the definition of Φ_{E_x} in (6.3). q.e.d.

Lemma 6.3. *For any $\tilde{v} \in \Phi_{E_x}$, there exists a geodesic $c(t)$ with $c(0)=\pi^1(x)$ such that*

$$\lim_{t\to\infty} a(t)0 = \tilde{v},$$

$a(t)$ being defined in (2.6).

Proof. By the difinition of Φ_{E_x},

$$\tilde{v} = \pi\begin{pmatrix} v^0 \\ v \end{pmatrix}$$

for some $v^0 \in \boldsymbol{C}$ and $v \in \boldsymbol{C}^n$ with $-|v^0|^2 + E_x(v)v = 0$. In the case when $E_x(v)v > 0$, take a geodesic with initial conditions $c(0)=\pi^1(x)$, $c'(0)=x(v/v^0)$. Then by the same argument as in Lemma 6.2,

$$a(t)0 = \pi\begin{pmatrix} c_1(t) \\ c_2(t)v/v^0 \end{pmatrix}, \tag{6.4}$$

where c_1 and c_2 are the solutions of $u''-ku=0$ ($k=E_x(v/v^0)v/v^0$) with initial conditions (6.2). By i) with $b=0$ in the proof of Proposition 5.2,

$$\lim_{t\to\infty} c_2/c_1 = 1/\sqrt{k} = |v^0|/\sqrt{E_x(v)v} = 1.$$

Thus we have

$$\lim_{t\to\infty} a(t)0 = \pi\begin{pmatrix} v^0 \\ v \end{pmatrix}.$$

In the case when $E_x(v)v=0$, i.e., $v^0=0$, take a geodesic with initial conditions $c(0)=\pi^1(x)$, $c'(0)=xv$. Then by the same argument as above

$$a(t)0 = \pi\begin{pmatrix} c_1(t) \\ c_2(t)v \end{pmatrix}, \tag{6.5}$$

where c_1 and c_2 are solutions of $u''=0$ with initial conditions (6.2). By d) in Lemma 6.1,

$$\lim_{t\to\infty} c_1/c_2 = 0.$$

Hence

$$\lim_{t\to\infty} a(t)0 = \pi\begin{pmatrix} 0 \\ v \end{pmatrix}.$$

This completes the proof of Lemma 6.3.

Define a subset $\Phi(p)$ of $Q(\nabla)\times_{L_0}\boldsymbol{P}^n(\boldsymbol{C})$ for $p\in M$ by $\Phi(p)=s(x)\Phi_{E_x}$ with $\pi^1(x)=p$. This is independent of the cohice of $x\in C(M)$.

Let $\bar{\nabla}$ be another H-complete connection of Kaehler type on M whose Ricci tensor $\bar{S}$ is parallel. Then $\bar{s}$: $C(M)\to Q(\bar{\nabla})$, $\bar{E}_x$: $\mathfrak{g}_{-1}\to\mathfrak{g}_1$, $\Phi_{\bar{E}_x}\subset\boldsymbol{P}^n(\boldsymbol{C})$ and $\bar{\Phi}(p)$ can be defined in the same way as above. Assume that $\bar{\nabla}$ is H-projectively equivalent to ∇. Then $Q(\nabla)=Q(\bar{\nabla})$ by Theorem 1. Further we obtain the following:

Lemma 6.4. $\Phi(p)=\bar{\Phi}(p)$.

Proof. Let q^* be an arbitrary element in $\Phi(p)$. Then, by Lemma 6.3, there exists a geodesic $c(t)$ with respect to ∇ such that the limit point of its development is q^*. By Proposition 5.2 and Corollary 5.1 we see that $c(t)$ is an H-geodesic of $\bar{\nabla}$ under a certain change of parameter. Taking into consideration (5.0) which shows how to change parameter, we have $q^*\in\bar{\Phi}(p)$ by lemma 6.2. Thus $\Phi(p)\subset\bar{\Phi}(p)$. In a similar fashion we have $\bar{\Phi}(p)\subset\Phi(p)$, and the proof is complete.

In view of (1.4) we can define F: $C(M)\to\mathfrak{g}_1$ by $\bar{s}(x)=s(x)\exp(F(x))$. Then we have

Lemma 6.5. *$(v^0, Y)\in\boldsymbol{C}\times T_p(M)$ satisfies*

(A) $$|v^0|^2+S_p(Y, Y)/2(n+1)=0$$

if and only if it satisfies

(B) $$|v^0-F(y)v|^2+\bar{S}_p(Y, Y)/2(n+1)=0\,,$$

for $y\in C(M)$ and $v\in\boldsymbol{C}^n$ such that $Y=yv$.

Proof. Lemma 4.1 shows that (A) (resp. (B)) is equivalent to

$$\pi\begin{pmatrix} v^0 \\ v \end{pmatrix}\in\Phi_{E_y} \tag{6.6}$$

$$\left(\text{resp. } \pi\begin{pmatrix} v^0-F(y)v \\ v \end{pmatrix}\in\bar{\Phi}_{\bar{E}_y}\right). \tag{6.7}$$

We have by Lemma 6.4

$$\exp(-F(y))\Phi_{E_y}=\bar{\Phi}_{\bar{E}_y} \tag{6.8}$$

Since

$$\exp(-F(y))\pi\begin{pmatrix} v^0 \\ v \end{pmatrix}=\pi\begin{pmatrix} 1 & -F(y) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^0 \\ v \end{pmatrix}=\pi\begin{pmatrix} v^0 & -F(y)v \\ & v \end{pmatrix},$$

(A) is equivalent to (B) by (6.6), (6.7) and (6.8). q.e.d.

Proof of Theorem 6. Let p be an arbitrary point in M. In the case when $S \neq 0$, $S_p(Y, Y) < 0$ for some $Y \in T_p(M)$. Choose $v^0 \in \boldsymbol{R}$ such that

$$(v^0)^2 + S_p(Y, Y)/2(n+1) = 0 . \tag{6.9}$$

Then we have also

$$(v^0)^2 + S_p(-Y, -Y)/2(n+1) = 0 . \tag{6.10}$$

Applying Lemma 6.5 to (6.9) and (6.10), we obtain

$$|v^0 - F(y)v|^2 + \bar{S}_p(Y, Y)/2(n+1) = 0$$
$$|v^0 + F(y)v|^2 + \bar{S}_p(-Y, -Y)/2(n+1) = 0$$

for $y \in C(M)$ and $v \in \boldsymbol{C}^n$ such that $Y = yv$. By these two formulas $Re(F(y)v) = 0$. On the other hand, the set

$$\{v \in \mathfrak{g}_{-1} \mid S_p(yv, yv) < 0\}$$

is open in $\mathfrak{g}_{-1}$. Thus the $\boldsymbol{R}$-linear map $L: \mathfrak{g}_1 \to \boldsymbol{R}$ defined by $L(v) = \mathrm{Re}(F(y)v)$ is zero. Since $F(y)v = \mathrm{Re}(F(y)v) - \sqrt{-1}\ \mathrm{Re}(F(y)\sqrt{-1}\, v)$, the map $N: \mathfrak{g}_{-1} \to \boldsymbol{C}$ definedby $N(v) = F(y)v$ is zero. Thus $F = 0$, because p is an arbitrary point. Also in the case when $S = 0$, we obtain $F = 0$ in a similar fashion. This completes the proof of Theorem 6.

MITSUBISHI ELECTRIC CORPORATION

References

[1] S. Ishihara and S. Tachibana: *A note on holomorphic projective transformations of a Kaehlerian space with parallel Ricci tensor*, Tohoku Math. J. **13** (1961), 193–200.
[2] S. Kobayashi: *Canonical forms on frame bundles of higher order contact*, Proc. Symp. Pure Math. Vol. **3**, Amer. Math. Soc. (1961), 186–193.
[3] S. Kobayashi: *Theory of connections*, Ann. Math. Pura Appl. **43** (1957), 119–194.
[4] T. Ochiai: *Geometry associated with semi-simple flat homogeneous spaces*, Trans. Amer. Math. Soc. **152** (1970), 159–193.
[5] T. Ōtsuki and Y. Tashiro: *On curves in Kaehlerian spaces*, Math. J. Okayama Univ. **4** (1954), 57–78.
[6] N. Tanaka: *On the equivalence problems associated with a certain class of homogeneous spaces*, J. Math. Soc. Japan **17** (1965), 103–139.
[7] N. Tanaka: *Projective connections and projective transformations*, Nagoya Math. J. **12** (1957), 1–24.
[8] Y. Tashiro: *On a holomorphically projective correspondence in an almost complex space*, Math. J. Okayama Univ. **6** (1957), 147–152.