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THE RESIDUAL LIMIT SETS AND THE GENERATORS OF FINITELY GENERATED KLEINIAN GROUPS

TAKEHIKO SASAKI

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1. Introduction

1.1. Let G be a Kleinian group and denote by $\Omega(G)$ and $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. Throughout this paper, a Kleinian group means a non-elementary one. The residual limit set of G, which is denoted by $\Lambda_0(G)$, is the subset of $\Lambda(G)$ which consists of all the points not lying on the boundary of any component of $\Omega(G)$. Although the study of Kleinian groups has long history, the residual limit sets were not treated or were thought to be empty, until in 1971 Abikoff showed the existence of Kleinian groups with the non-empty residual limit sets [1]. In his paper [2], Abikoff also studied the properties of residual limit sets and showed their non-emptyness for all finitely generated Kleinian groups except for those of two classes which have clearly the empty residual limit set; one is a class of function groups and the other is a class of Z_2-extensions of quasi-Fuchsian groups.

In this paper we shall show the importance of the residual limit sets by proving the following.

Theorem 1.1. *Let G be a finitely generated Kleinian group and let S be a finite set of generators of G. If G is neither a function group nor a Z_2-extension of a quasi-Fuchsian group, then S can be changed into a set of generators S_0 of G with the following properties:*

i) *each element of S_0 is loxodromic and its fixed points lie on $\Lambda_0(G)$, and*

ii) *the number of elements of S_0 is not greater than that of S.*

Among the sets of generators of a finitely generated group, there is a set, the number of elements of which is minimum. We shall call it the minimal set of generators. Choosing S in Theorem 1.1 to be the minimal set of generators, we have the following.

Corollary 1.2. *Among the minimal sets of generators of a finitely generated Kleinian group G with the non-empty residual limit set, there is a set consisting of only loxodromic elements with the fixed points on $\Lambda_0(G)$.*

1.2. This paper is arranged with respect to steps of the proof of Theorem 1.1. In §2, we list up some known results which we shall need later and then define and discuss the rotation order of some loxodromic element. We change S into S_0 in three steps; in §3 into a set which consists of the loxodromic elements only, in §4 into another set which consists of the loxodromic elements only and contains at least one element which has the fixed points on $\Lambda_0(G)$ and in §5 into the desired S_0. In each step the changed set is a set of generators of G and the number of the elements of the set is not greater than that of the original set. In §6, S_0 is studied in detail for non-web groups. The author wishes to express his deep gratitude to professor T. Kuroda for his advices.

2. Known results and rotation order of a loxodromic element

2.1. Let G be a finitely generated Kleinian group and let Δ be a component of G. The component subgroup G_Δ for Δ is the maximal subgroup of G which leaves Δ invariant. For component subgroups of G, the followings are known.

Theorem 2.1 [3]. *G_Δ is a finitely generated function group with Δ as an invariant component.*

Theorem 2.2 [4]. *If G_Δ has an invariant component different from Δ, then G_Δ is a quasi-Fuchsian group with the invariant Jordan curve $\partial\Delta=\Lambda(G_\Delta)$.*

From these theorems we have the following.

Corollary 2.3. *Let Δ' be a component of G_Δ which is different from Δ. Then the component subgroup $G_{\Delta'}$ for Δ' of G_Δ is a quasi-Fuchsian group with the invariant Jordan curve $\partial\Delta'=\Lambda(G_{\Delta'})$.*

The Jordan curve $\partial\Delta'$ in this corollary is called a separator of G and the set of all such separators of G is called the set of separators of G.

Lemma 2.4 [2]. *Separators do not cross each other.*

Lemma 2.5 [2]. *If $\infty\in\Omega(G)$, then the diameters of separators of G form a null sequence.*

For common subgroups of component subgroups of G and for common boundary points of components of G, the followings are known.

Theorem 2.6 [5,7]. *Let $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ be an arbitrary collection of components of G. Then $\Lambda(\bigcap_{i=1}^n G_{\Delta_i})=\bigcap_{i=1}^n \partial\Delta_i$. If $n\geqq 3$, then $\bigcap_{i=1}^n \partial\Delta_i$ consists of at most two points.*

Theorem 2.7 [6]. *Let Δ', Δ'' be the non-invariant components of G_Δ. Then $\partial\Delta'\cap\partial\Delta''$ consists of at most one point. If it is not empty, then the point is* *the fixed point of a parabolic element of G.*

2.2. Here we shall recall the auxiliary domains. Let Δ_i, Δ_j be the components of G. Let Δ'_{ij} be the component of G_{Δ_i} containing Δ_j. The complement of the closure of Δ'_{ij} is called the auxiliary domain of Δ_i with respect to Δ_j and it is denoted by D_{ij} and, if there is no confusion, we write D instead of D_{ij}. By definition, the boundary ∂D is a separator of G. For auxiliary domains, we have followings.

Lemma 2.8 [6]. *Let Δ_i and Δ_j $(\neq \Delta_i)$ be the components of G. Then $D_{ij} \supset \Delta_i$, $\partial D_{ij} \subset \partial \Delta_i$, $D_{ij} \cap D_{ij} = \emptyset$ and $\partial \Delta_i \cap \partial \Delta_j = \partial D_{ij} \cap \partial D_{ji}$.*

Lemma 2.9 [7]. *Let $\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ $(n>2)$ be an arbitrary collection of components of G. If $\bigcap_{i=1}^{n} \partial \Delta_i$ consists of two points, then $D_{ij} = D_{ik}$ for any integers i, j, k.*

If Δ_i is a component of G containing ∞, then the auxiliary domain D_{ji} of any component Δ_j of G with respect to Δ_i is bounded. By Lemma 2.8, the diameter of Δ_j is identical with that of D_{ji}. By Theorem 2.6, the set ∂D_{ji} can be the subset of boundaries of at most two components of G. Hence by Lemma 2.5, we have the following.

Lemma 2.10. *If there is a component Δ of G containing ∞, then the diameters of components (excluding Δ) of G form a null sequence.*

2.3. For loxodromic elements of G and for component subgroups, the following is known.

Theorem 2.11 [5]. *Let γ be a loxodromic element of G with a fixed point on the boundary of a component Δ of F. Then there is a positive integer r such that $\gamma^r \in G_\Delta$. Hence the other fixed point of γ also lies on the boundary of the same component Δ.*

The minimum of r in the above theorem is called the rotation order of γ for Δ.

Lemma 2.12. *Let γ be a loxodromic element of G. If one fixed point of γ lies on a separator of G, then the other fixed point of γ also lies on the same separator.*

Proof. Let $\partial \Delta'$ be a separator, on which one fixed point of γ lies, and let Δ be a component such that Δ' is a component of G_Δ. Since $\partial \Delta' \subset \partial \Delta$, we see by Theorem 2.11 that both fixed points of γ lie on $\partial \Delta$ and that $\gamma^r \in G_\Delta$ for the rotation order r of γ for Δ. Hence we see by using Theorem 2.11 again that both fixed points of γ lie on $\partial \Delta'$.

Lemma 2.13. *Let γ be a loxodromic element of G with a fixed point on the* *boundary of a component Δ of G. Assume that the rotation order r of γ for Δ is greater than* 1. *If D denotes the auxiliary domain of Δ with respect to $\gamma(\Delta)$, then the followings hold:*

i) *the fixed points of γ lie on ∂D,*

ii) *$\gamma^i(D)$ is identical with the auxiliary domain of $\gamma^i(\Delta)$ with respect to Δ, $i \neq r$ (mod r), and*

iii) *$\gamma^i(D) \cap \gamma^j(D) = \emptyset$ for integers i, $j \neq i$ (mod r).*

Proof. i) Let D_1 be the auxiliary domain of $\gamma(\Delta)$ with respect to Δ. Since γ has a fixed point on the boundary of Δ, one fixed point of γ lies on $\partial\Delta \cap \partial\gamma(\Delta)$. By Lemma 2.8, we see that it also lies on $\partial D \cap \partial D_1$. Hence γ has one fixed point on the separator ∂D. Therefore the assertion follows from Lemma 2.12.

ii) Let D_i be the auxiliary domain of $\gamma^i(\Delta)$ with respect to Δ, $1 \leqq i < r$. Since $\gamma^i(D) \supset \gamma^i(\Delta)$ and $\partial\gamma^i(D) \subset \partial\gamma^i(\Delta)$, we see that the outside of $\gamma^i(D)$ is a component of the complement of $\gamma^i(\bar{\Delta})$. We assert that $\gamma^i(D) \cap D = \emptyset$. In fact, if $i=1$, then evidently $\gamma(D) \cap D = \emptyset$. If $1 < i$, then $\gamma^i(\Delta)$ is contained in a component of G_Δ different from Δ, and hence, if $\gamma^i(D) \cap D \neq \emptyset$, then $\gamma^i(D) \subset D$, so $\gamma^i(\Delta)$ lies in a non-invariant component of G_Δ which is different from the one containing $\gamma(\Delta)$. Since $\partial\gamma^i(\Delta) \cap \partial\gamma(\Delta)$ contains at least two fixed points of γ, this contardicts Theorem 2.7. Hence, in any case, we have the assertion that $\gamma^i(D) \cap D = \emptyset$. Therefore, the outside of $\gamma^i(D)$ is the component of the complement of $\gamma^i(\bar{\Delta})$ which contains Δ. So we have $\gamma^i(D) = D_i$.

iii) In the case of $r=2$, the assertion follows from Lemma 2.8 and ii). If $r > 2$, then by Lemma 2.9 and ii) we see that $\gamma^i(D)$ (or $\gamma^j(D)$) is the auxiliary domain of $\gamma^i(\Delta)$ (or $\gamma^j(\Delta)$) with respect to $\gamma^j(\Delta)$ (or $\gamma^i(\Delta)$). Hence the assertion follows from Lemma 2.8.

Theorem 2.14. *Let γ and Δ_1, Δ_2 be a loxodromic element and two components of G, respectively. If the fixed points of γ lie on the common boundary of Δ_1 and Δ_2, then the rotation order of γ for Δ_1 is identical with that of γ for Δ_2.*

Proof. Assume that the rotation order of γ for Δ_1 be $r \geqq 2$. If $\Delta_2 = \gamma^i(\Delta_1)$ for some integer i, then we see at once that the rotation order of γ for Δ_2 is r. Therefore we assume that $\Delta_2 \neq \gamma^i(\Delta_1)$ for any integer i. Let D_{12} (or D_{21}) be the auxiliary domain of Δ_1 (or Δ_2) with respect to Δ_2 (or Δ_1). Then by Lemma 2.8, we see $D_{12} \cap D_{21} = \emptyset$. Let D be the auxiliary domain of Δ_1 with respect to $\gamma(\Delta_1)$. Then by Lemma 2.9, we see $D_{12} = D$. Hence by Lemma 2.13 and Lemma 2.9, we see $\gamma^i(D_{12}) \cap D_{21} = \emptyset$ for any integer i. Hence we can find two integers i and j such that the component of the complement of the closure of $\gamma^i(D_{12}) \cup \gamma^j(D_{12})$ including D_{21} does not include any $\gamma^k(D_{12})$, $1 \leqq k < r$. Since γ is an orientation preserving homeomorphism, $\gamma^l(D_{21})$ lies between $\gamma^{i+l}(D_{12})$ and $\gamma^{j+l}(D_{12})$ for any integr l. This and Lemma 2.13 imply that the rotation order r' of γ for Δ_2 is not less than r. Similarly, we see $r' \leqq r$. Hence we have $r'=r$. As a consequence of this, we see that the rotation order of γ for Δ_1 is 1 if and only if that of γ for Δ_2 is 1. Thus we have our theorem.

For the common subgroup $\cap_{i=1}^{n} G_{\Delta_i}$ in Theorem 2.6 we have the following.

Corollary 2.15. *Let γ and $\{\Delta_1, \Delta_2 \cdots, \Delta_n\}$ be a loxodromic element and an arbitrary collection of components of G, respectively. If the fixed points of γ lie on $\cap_{i=1}^{n} \partial\Delta_i$ and if $\gamma \in G_{\Delta_i}$, then $\gamma \in \cap_{i=1}^{n} G_{\Delta_i}$.*

Proof. If $\gamma \in G_{\Delta_i}$, then the rotation order of γ for Δ_i is 1. By Theorem 2.14, we see that the rotation order of γ for any Δ_j is 1. Hence our assertion follows.

For later use we also need a following form of Theorem 2.14.

Lemma 2.16. *Let D be an arbitrary auxiliary domain of a component Δ of G. If a loxodromic element γ of G_Δ has a fixed point on ∂D, then $\gamma(D)=D$.*

Proof. Let Δ' be a component of G_Δ whose complement is the closure of D. Then the fixed points of γ lie on $\partial\Delta'$. Applying Theorem 2.14 to Δ and Δ', we have $\gamma(\Delta')=\Delta'$, so that $\gamma(D)=D$.

3. Loxodromic generators

3.1. In the following three sections including this section, we assume that G satisfies the condition of Theorem 1.1. In this § we shall change a finite set S of generators of a given finitely generated Kleinian group G in Theorem 1.1 into the set of generators consisting of loxodromic elements only. Our process is repetition of the following three kinds of operations; γ_i is changed into one or γ_i^{-1}, $\gamma_i\gamma_j$ and $\gamma_j\gamma_i$, where γ_i, γ_j are elements of S or of the changed sets by this process. This operation does not increase the number of elements of the set of generators and the changed set is clearly a set of generators of the same group.

Let $S=\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$. Assume that there are elliptic elements in S with the same fixed points. Since G is Kleinian, we can replace them by a single elliptic element of G so that the changed set is also a set of generators of G and the number of elements of this changed set is not greater than that of S. Hence we may assume that S does not contain any two elliptic elements with the same fixed points. We consider three cases.

3.2. The case (I) where S contains at least one loxodromic element: Without loss of generality we may assume that γ_1 is loxodromic and its matrix representation has the form $\begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix}$, $|k|>1$. Consider an elliptic or a parabolic element $\gamma_i \in S$ with matrix representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$. We consider the element of the form $\gamma_1^m\gamma_i$ with the trace ak^m+dk^{-m}, where m is an integer. If $a\neq 0$ (or $d\neq 0$), then we can take m so large (or small) that $|ak^m+dk^{-m}|>2$. Hence $\gamma_1^m\gamma_i$ is loxodromic. With such an m we replace γ_i by $\gamma_1^m\gamma_i$ and after carrying out the above procedure for all such γ_i, we denote the new set of generators by the same S. If $a=d=0$, then γ_i is an elliptic element of order 2 and changes 0, ∞ into each other. Consider another loxodromic element γ_j $(\neq\gamma_1)$ of S whose fixed points are different from 0 and ∞. Since G is non-elementary, the existence of such a γ_j in S is assured. First we change γ_j into $\gamma_1^m\gamma_j\gamma_1^{-m}$ with so large integer m that $|\xi_j\xi_j'|>|b|^2$, where ξ_j, ξ_j' are the fixed points of $\gamma_1^m\gamma_j\gamma_1^{-m}$, which we also denote by the same γ_j. Let A be a linear transformation which maps ξ_j and ξ_j' to 0 and ∞, respectively. Then the conjugations of γ_i, γ_j by A have the forms

$$\gamma_i^* = A\gamma_i A^{-1} = \begin{pmatrix} \frac{1}{D} & \frac{-\xi_j}{D} \\ \frac{1}{D} & \frac{-\xi_j'}{D} \end{pmatrix} \begin{pmatrix} 0 & b \\ \frac{-1}{b} & 0 \end{pmatrix} \begin{pmatrix} \frac{-\xi_j'}{D} & \frac{\xi_j}{D} \\ \frac{-1}{D} & \frac{1}{D} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-(b+\xi_j\xi_j'b^{-1})}{D^2} & \frac{b+\xi_j^2 b^{-1}}{D^2} \\ \frac{-(b+(\xi_j')^2 b^{-1})}{D^2} & \frac{b+\xi_j\xi_j'b^{-1}}{D^2} \end{pmatrix}$$

and

$$\gamma_j^* = A\gamma_j A^{-1} = \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}, \quad |l|\neq 1,$$

respectively, where $D=(\xi_j-\xi_j')^{1/2}$. Since $(b+\xi_j\xi_j'b^{-1})/(\xi_j-\xi_j')\neq 0$, we see that $(\gamma_j^*)^m\gamma_i^*$ is loxodromic for some integer m and hence $\gamma_j^m\gamma_i$ is also loxodromic for some integer m. We replace γ_i by $\gamma_j^m\gamma_i$.

3.3. The case (II) where S contains at least one parabolic element: Without loss of generality we may assume that γ_1 is parabolic and its matrix representation has the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since G is non-elementary, there is an element γ_i of S with the matrix representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c\neq 0$. We consider the element of the form $\gamma_1^m\gamma_i$ with an integer m. Since the trace of $\gamma_1^m\gamma_i$ equals $a+d+cm$, we see that for a sufficiently large m, $\gamma_1^m\gamma_i$ is loxodromic. We replace γ_i by $\gamma_1^m\gamma_i$. Then S reduces to a set of generators in the previous case (I).

3.4. The case (III) where S consists of elliptic elements only: We shall first prove the following two lemmas.

Lemma 3.1. *Let γ and δ be linear transformations with the matrix representations of the forms $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, $0<|\theta|<\pi$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$, respectively. If*

$a+d$ is real and if $d\neq\bar{a}$, then $\gamma\delta$ is loxodromic.

Proof. Set $a=a_1+ia_2$ and $d=d_1+id_2$ with real numbers a_1, a_2, d_1, d_2. Then $d_2=-a_2$ and $a_1\neq d_1$. Hence we see that the trace of $\gamma\delta$ is not real. Therefore, $\gamma\delta$ is loxodromic.

Lemma 3.2. *Let γ and δ be elliptic. If all four fixed points of them do not lie on a line nor on a circle, then $\gamma\delta$ is loxodromic.*

Proof. Without loss of generality we may assume that the fixed points of γ are 0 and ∞ and that $\delta=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1, b\neq 0$ and $c\neq 0$. Set $a=a_1+ia_2$ and $d=d_1+id_2$ with real numbers a_1, a_2, d_1, d_2. Since δ is elliptic, we see that $d_2=-a_2$. If $d=\bar{a}$, then, by writing $c=|c|e^{i\theta}$, we see that the fixed points of δ are

$$\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}=\frac{2a_2i\pm\sqrt{4-(a+d)^2}i}{2|c|e^{i\theta}}$$
$$=\frac{2a_2\pm\sqrt{4-(a+d)^2}}{2|c|}e^{i(\pi/2-\theta)},$$

because $(a+d)^2<4$. Hence all fixed points of γ and δ lie on the line which passes through 0 and makes an angle $\pi/2-\theta$ with the real axis. This contradicts our assumption. Hence we obtain that $d\neq\bar{a}$. By Lemma 3.1, $\gamma\delta$ is loxodromic.

If S contains two elements γ_i, γ_j whose all four fixed points do not lie on a circle nor on a line, then Lemma 3.2 implies that the changing γ_i into $\gamma_i\gamma_j$ takes S into a set of generators in the case (I). On the other hand, we shall see in the following that, under our assumption that G is neither a function group nor a Z_2-extention of a quasi-Fuchsian group, S or its changed set by our operation contains such γ_i and γ_j as stated above.

For the purpose, we assume that, for any two elements of S (or its changed set by our operations), all their fixed points lie on a circle or a line. Since this property is invariant under the conjugation by a linear transformation, we may assume that the fixed points of γ_1 are 0 and ∞. Let L_i be a line on which the fixed points of γ_i lie. Since G is non-elementary, there is an element γ_2 of S which does not leave ∞ invariant. Then L_2 passes through 0. If γ_i ($\neq\gamma_1$) has ∞ as a fixed point, then L_i must be identical with L_2.

3.5. We first treat the case where there is an element γ_{j_0} of S with $L_{j_0}\neq L_2$. Then γ_{j_0} has the finite fixed points and L_{j_0} passes through 0.

Lemma 3.3. *Under these circumferences, each element of S except for γ_1 has the finite and non-zero fixed points.*

Proof. If γ_2 has 0 as a fixed point, then γ_{j_0} fixed above must have 0 as a fixed point. This is also true for all γ_j with $L_j \neq L_2$. Further, since γ_{j_0} has another fixed point different from ∞ as already mentioned, we see γ_j satisfying $L_j = L_2$ must have 0 as a fixed point. This shows that every element of S and of G has 0 as the fixed point, so that G is elementary, a contradiction. Hence the fixed points of γ_2 are different from 0. So each γ_j with $L_j \neq L_2$ and, in particular, γ_{j_0} has not 0 as a fixed point. Therefore γ_j $(j \neq 1)$ satisfying $L_j = L_2$ has not 0 or ∞ as a fixed point. Thus we have our lemma.

Let ξ_2, ξ_2' be the fixed points of γ_2. Without loss of generality we may assume that $|\xi_2 \xi_2'| = 1$. By an elementary geometric consideration we see that if the line segment $\xi_2 \xi_2'$ includes (or does not include) 0, then the line segment $\xi_{j_0} \xi_{j_0}'$ includes (or does not include) 0 and $|\xi_{j_0} \xi_{j_0}'| = 1$, where ξ_{j_0}, ξ_{j_0}' are the fixed points of γ_{j_0}. Hence, it is not difficult to see that in both cases these are also true for each γ_j $(j \geqq 2)$ of S. In the case where the line segment $\xi_2 \xi_2'$ does not contain 0, this implies that the fixed points of each element of S lie in the mirror images with respect to the circle $C = \{z \mid |z| = 1\}$, so that C is invariant under the action of each element of S, hence, of G. Hence $\Lambda(G) \subset C$. This contradicts our assumption that G is neither a function group nor a Z_2-extension of a quasi-Fuchsian group. Hence this case does not occur. Before going to treat the case where the line segment $\xi_2 \xi_2'$ includes 0, we show the following.

Lemma 3.4. *Let γ_1 and γ_2 be elliptic transformations. If γ_j has the fixed points $r_j e^{i\Theta_j}$ and $-r_j^{-1} e^{i\Theta_j}$ $(j = 1, 2)$ and if these four points lie on a line or a circle, then $\gamma_1 \gamma_2$ is the identity or an elliptic transformation with the fixed points of the similar forms, where, if $r_j = 0$, then $r_j^{-1} e^{i\Theta_j}$ means ∞.*

Proof. If γ_1 and γ_2 have the same fixed points, then the assertion is clear. Hence we assume that the fixed points of γ_1 are different from those of γ_2. If the fixed points of γ_1 are finite, we may assume that $\Theta_1 = 0$ and we consider a transformation

$$A \colon z \mapsto \frac{1}{r_1} \frac{z - r_1}{z + r_1^{-1}}.$$

Then $A\gamma_1 A^{-1}$ has 0 and ∞ as the fixed points and the fixed points of $A\gamma_2 A^{-1}$ lie on a line passing through 0 and separate 0 and ∞ on the line. If $r_2 = 0$ or $= \infty$, then the fixed points of $A\gamma_2 A^{-1}$ are $-r_1$ and r_1^{-1}. If $0 < r_2 < \infty$, then the fixed points of $A\gamma_2 A^{-1}$ are

$$\frac{1}{r_1} \frac{r_2 e^{i\Theta_2} - r_1}{r_2 e^{i\Theta_2} + r_1^{-1}} \quad \text{and} \quad \frac{1}{r_1} \frac{-r_2^{-1} e^{i\Theta_2} - r_1}{-r_2^{-1} e^{i\Theta_2} + r_1^{-1}},$$

and clearly the absolute value of the product of these two numbers is equal to 1.

Thus we may assume without loss of generality that

$$\gamma_1=\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1}\end{pmatrix}$$

and

$$\gamma_2=\begin{pmatrix} \dfrac{r_2^{-1}e^{i\theta_2}+r_2e^{-i\theta_2}}{D} & \dfrac{-e^{\theta i_2}+e^{-i\theta_2}}{D} \\ \dfrac{-e^{i\theta_2}+e^{-i\theta_2}}{D} & \dfrac{r_2e^{i\theta_2}+r_2^{-1}e^{-i\theta_2}}{D}\end{pmatrix},$$

where $D=r_2+r_2^{-1}$ and θ_2 is not a multiple of π. Then the matrix representation of $\gamma_1\gamma_2$ is

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} =\begin{pmatrix} \dfrac{r_2^{-1}e^{i(\theta_1+\theta_2)}+r_2e^{-i(\theta_1-\theta_2)}}{D} & \dfrac{-e^{i(\theta_1+\theta_2)}+e^{i(\theta_1-\theta_2)}}{D} \\ \dfrac{-e^{-i(\theta_1-\theta_2)}+e^{-i(\theta_1+\theta_2)}}{D} & \dfrac{r_2e^{-i(\theta_1-\theta_2)}+r_2^{-1}e^{-i(\theta_1+\theta_2)}}{D}\end{pmatrix}$$

and the trace of $\gamma_1\gamma_2$ is

$$a+d=2\frac{\cos(\theta_1+\theta_2)+r_2^2\cos(\theta_1-\theta_2)}{1+r_2^2}.$$

Hence we have $-2\leqq \text{trace}\,\gamma_1\gamma_2\leqq 2$. The equalities occur only when $\cos(\theta_1+\theta_2)=\cos(\theta_1-\theta_2)=\pm 1$. These imply that $\theta_1+\theta_2=k\pi$ and $\theta_1-\theta_2=k\pi+2m\pi$, where k, m are integers, and hence θ_1 and θ_2 are multiples of π. Hence γ_1 and γ_2 are the identity transformations. Therefore, the equalities do not occur and $\gamma_1\gamma_2$ is elliptic. We see easily that the product of the fixed points of $\gamma_1\gamma_2$ has the absolute value 1. In order to complete the proof of our lemma we have only to show that the line segment connecting the fixed points of $\gamma_1\gamma_2$ includes 0. It is easy to see that the ratio of the fixed points of $\gamma_1\gamma_2$ is real. Hence it sufficies to show that the absolute value of the difference of the fixed points is greater than that of the sum of them, or equivalently, to show $|(a+d)^2-4|>|a-d|^2$, which can be easily verified. Hence the line segment connecting the fixed points of $\gamma_1\gamma_2$ includes 0. Thus we have completed the proof of our lemma.

Now we return to the case where the line segment $\xi_2\xi_2'$ includes 0. As was already mentioned, the line segment $\xi_j\xi_j'$ has the same property ($j\geqq 2$), where ξ_j and ξ_j' are the fixed points of $\gamma_j\in S$. By Lemma 3.4, we see that any product of a finite number of elements of S is an elliptic transformation or the identity, so that G is a finite group, a contradiction. Hence this case also does not occur. Therefore, we have shown that if there is an element γ_i of S with $L_i \neq L_2$, then S contains two elements whose four fixed points do not lie on a circle nor a line.

3.6. We next treat the case where the fixed points of each element of S lie on L_2. If the order of each element of S is two, then L_2 is invariant under the actions of S and of G, so that $\Lambda(G) \subset L_2$. This contradicts our assumption that G is neither a function group nor a Z_2-extension of a quasi-Fuchsian group. Hence there is an element of S whose order is greater than two. We may assume that the order of γ_1 is greater than two. We shall show that S contains at least one more element which is different from γ_1 and γ_2. Assume contrary that $S=\{\gamma_1, \gamma_2\}$. Since G is non-elementary, the fixed points of γ_2 are finite and different from 0. If the line segment connecting the fixed points of γ_2 does not contain 0, then there is a circle with the center 0 being invariant under γ_2. This circle is also invariant under γ_1. Hence the limit set of G is contained in the circle, a contradiction. If the line segment connecting the fixed points of γ_2 contains 0, then, by Lemma 3.4, we see that G is elementary, a contradiction. Thus we have shown that S contains an element γ_i $(i>2)$ with the fixed points on L_2. We change S into $\{\gamma_1, \gamma_2, \cdots, \gamma_1\gamma_i\gamma_1^{-1}, \cdots, \gamma_n\}$. Since the order of γ_1 is greater than 2, the line on which the fixed points of $\gamma_1\gamma_i\gamma_1^{-1}$ $(i>2)$ lie is different from L_2. Hence this case reduces to the case discussed already.

Therefore the case where each element of S is elliptic can be reduced to the case where S contains at least one loxodromic element. Thus we have completed to change S into the set of generators of G, which consists of loxodromic elements only and the number of elements of which is not greater than that of S.

4. Loxodromic elements with the fixed points on $\Lambda_0(G)$

4.1. Let S be a set of generators of G consisting of loxodromic elements only. As the second step of the proof of Theorem 1.1 we shall change S into a set of generators of G, which consists of loxodromic elements only and containing at least one element with fixed points on $\Lambda_0(G)$ and the number of elements of which is not greater than that of S. Without loss of generality we may assume that $\infty \in \Omega(G)$. We shall first prove the following four lemmas.

Lemma 4.1. *Let γ_i and γ_j be loxodromic elements of G with no common fixed point and let ξ_j, ξ_j' be the repelling and the attractive fixed points of γ_j, respectively. Then, for a sufficiently large integer m, $\gamma_i\gamma_j^m$ is loxodromic and the repelling and the attractive fixed points of $\gamma_i\gamma_j^m$ converge to ξ_j and to $\gamma_i(\xi_j')$, respectively, as m tends to ∞.*

Proof. For an arbitrary positive number $\varepsilon>0$, there is a neighbourhood

U of $\gamma_i(\xi_j')$ such that $U \cap \gamma_i^{-1}(U) = \emptyset$, $\bar{U} \cap \xi_j = \emptyset$ and the diameter of U is smaller than ε. Let m be an integer and let C_m be the isometric circle of γ_j^m. For a sufficiently large m, we see that the diameter of C_m is smaller than ε and that C_m is contained in an open disc with center ξ_j and radius ε and $\gamma_j^m(C_m) \subset \gamma_i^{-1}(U)$. Then $\gamma_i\gamma_j^m$ sends the exterior of C_m into U. Hence, for a large m, $\gamma_i\gamma_j^m$ is loxodromic and the distance between the repelling (or the attractive) fixed point of $\gamma_i\gamma_j^m$ and ξ_j (or $\gamma_i(\xi_j')$) is smaller than ε, which are to be shown.

Lemma 4.2. *Let γ_i and γ_j be loxodromic elements of G whose fixed points are different from each other and do not lie on $\Lambda_0(G)$. If there is no component of G, on whose boundary all the fixed points of γ_i and γ_j lie, then there is an integer m such that $\gamma_i\gamma_j^m$ is a loxodromic element with the fixed points on $\Lambda_0(G)$.*

Proof. Let ξ_j and ξ_j' be the repelling and the attractive fixed points of γ_j, respectively. Then, by our assumption and by Theorem 2.11, ξ_j and ξ_j' lie on the boundary of a component of G. Since G is Kleinian, $\gamma_i(\xi_j') \neq \xi_j$, so that $d = |\xi_j - \gamma_i(\xi_j')| > 0$. By Lemma 2.10, there is a finite number of components of G whose diameters exceed $d/2$. Let δ_1 (or δ_2) be the minimum of the distances between ξ_j (or $\gamma_i(\xi_j')$) and the components whose diameters exceed $d/2$ and whose boundaries do not contain ξ_j (or $\gamma_i(\xi_j')$). Let δ be a positive number smaller than min $(\delta_1, \delta_2, d/4)$. Lemma 4.1 implies that we can find an integer m sufficiently large such that the distances between ξ_j and the repelling fixed point of $\gamma_i\gamma_j^m$ and between $\gamma_i(\xi_j')$ and the attractive fixed point of $\gamma_i\gamma_j^m$ are smaller than δ and such that $\gamma_i\gamma_j^m$ is loxodromic. If there is a component on whose boundary the fixed points of $\gamma_i\gamma_j^m$ lie, then we see from the definition of δ that ξ_j and $\gamma_i(\xi_j')$ must lie on the boundary of that component. Since ξ_j and $\gamma_i(\xi_j')$ are the fixed points of γ_j and $\gamma_i\gamma_j\gamma_i^{-1}$, respectively, we see by Theorem 2.11 that ξ_j' and $\gamma_i(\xi_j)$ also lie on the same boundary. Hence, by Theorem 2.6, there are at most two components of G on whose boundaries the fixed points of $\gamma_i\gamma_j^m$ lie. Let Δ be such a one. Then the rotation order of $\gamma_i\gamma_j^m$ for Δ is at most 2. Let r be the rotation order of γ_j for Δ and take m as a multiple of r. Then rotation order of $\gamma_i\gamma_j^m$ for Δ must be 2. In fact, otherwise, $\gamma_i\gamma_j^m \in G_\Delta$ or $\gamma_i \in G_\Delta$, so that the fixed points of γ_i lie on the boundary of Δ, which contradicts our assumption. Hence $\gamma_i\gamma_j^m\gamma_i\gamma_j^m \in G_\Delta$ or $\gamma_i\gamma_j^m\gamma_i \in G_\Delta$. On the other hand, since $\gamma_i\gamma_j^r\gamma_i^{-1}$ is an element of $G_{\gamma_i(\Delta)}$ and has the fixed points on the boundary of Δ, we see by Theorem 2.14 that $\gamma_i\gamma_j^r\gamma_i^{-1} \in G_\Delta$. Hence we have $\gamma_i^2 \in G_\Delta$ so that the fixed points of γ_i lie on the boundary of Δ. This contradicts our assumption that four fixed points of γ_i, γ_j do not lie on boundary of a single component of G. Therefore, for a large integer m there is no component on whose boundary the fixed points of $\gamma_i\gamma_j^m$ lie. Thus $\gamma_i\gamma_j^m$ is a loxodromic element with the fixed points on $\Lambda_0(G)$.

Lemma 4.3. *Let γ_i and γ_j be loxodromic elements of G, let Δ be a compo-*

nent of G on whose boundary the fixed points of γ_i lie and let D be the auxiliary domain of Δ with respect to $\gamma_i(\Delta)$. If the rotation order of γ_i for Δ is greater than 1 *and if the fixed points of γ_j lie in D, then, for a large integer m, $\gamma_i\gamma_j^m$ is a loxodromic element with the fixed points on $\Lambda_0(G)$.*

Proof. Let ξ_j, ξ_j' be the repelling and the attractive fixed points of γ_j, respectively. By Lemma 2.13, D and $\gamma_i(D)$ lie outside of each other and contain the points ξ_j and $\gamma_i(\xi_j')$, respectively. Lemma 4.1 shows that, for a large integer m, the repelling and the attractive fixed points of $\gamma_i\gamma_j^m$ lie in D and in $\gamma_i(D)$, respectively. Hence the fixed points of $\gamma_i\gamma_j^m$ are separated by the separator ∂D. Therefore we see easily that the fixed points of $\gamma_i\gamma_j^m$ lie on $\Lambda_0(G)$ (cf. [1]).

Lemma 4.4. *Let γ_i and γ_j be loxodromic elements of G, let Δ be a component of G on whose boundary the fixed points of γ_i lie and let D be the auxiliary domain of Δ with respect to $\gamma_i(\Delta)$. If the rotation order of γ_i for Δ is greater than* 2 *and if the fixed points of γ_j lie on ∂D and are different from the fixed points of γ_i, then, for a large integer m, $\gamma_i\gamma_j^m$ is a loxodromic element with the fixed points on $\Lambda_0(G)$.*

Proof. We assume that the conclusion of the lemma is false. By Lemma 2.13, we see that D, $\gamma_i(D)$, $\cdots$, $\gamma_i^{r-1}(D)$ lie outside of each other and have the fixed points of γ_i as the common boundary points, where r is the rotation order of γ_i for Δ. As we have seen in the proof of Lemma 4.2, for a sufficiently large integer m, the components which have the fixed points of $\gamma_i\gamma_j^m$ on the boundaries, must have the four fixed points of γ_j and $\gamma_i\gamma_j\gamma_i^{-1}$ on the boundaries. Let C_1 (or C_2) be the subarc of ∂D which has the end points at the fixed points of γ_i and on which the repelling fixed point of γ_j lies (or does not lie). If the attractive fixed point of γ_j lies on C_2, then the component which has the fixed points of $\gamma_i\gamma_j^m$ on the boundary, must be identical with Δ, so that the rotation order of $\gamma_i\gamma_j^m$ for Δ is 1. Take $m=sk$ with an integer k, where s is the rotation order of γ_j for Δ. Then $\gamma_i\gamma_j^m=\gamma_i(\gamma_j^s)^k\in G_\Delta$ or $\gamma_i\in G_\Delta$, which contradicts the assumption that the rotation order of γ_i for Δ is greater than 2. Hence the attractive fixed point of γ_j lies on C_1. Here we shall say that $\gamma_i(C_1)$ (or $\gamma_i(C_2)$) faces to C_2 (or C_1) if the component of the complement of $\gamma_i(C_1)\cup C_2$ (or $\gamma_i(C_2)\cup C_1$) containing D includes all $\gamma_i^l(D)$, $0\leqq l\leqq r-1$. Note that $\gamma_i(C_1)$ and $\gamma_i(C_2)$ are the subboundaries of $\gamma_i(D)$ with the common terminal points being the fixed points of γ_i. We see easily that if $\gamma_i(C_1)$ (or $\gamma_i(C_2)$) does not face to C_2 (or C_1), then there is no component of G on whose boundary the fixed points of $\gamma_i\gamma_j^m$ lie. If $\gamma_i(C_1)$ faces to C_2, then, since the fixed points of $\gamma_i\gamma_j\gamma_i^{-1}$ lie on $\gamma_i(C_1)$, the component which has the fixed points of $\gamma_i\gamma_j^m$ on the boundary, must be identical with Δ. So we have the same contradiction stated just above. Hence $\gamma_i(C_2)$ must face to C_1, so that the component which has the fixed points of $\gamma_i\gamma_j^m$ on the boundary, must be identical with $\gamma_i(\Delta)$ and the rotation order of

$\gamma_i\gamma_j^m$ for $\gamma_i(\Delta)$ is 1. Then $\gamma_i^{-1}\gamma_i\gamma_j^m\gamma_i\in G_\Delta$ or $\gamma_j^m\gamma_i\in G_\Delta$. Taking $m=sk$ with integers k and s being the rotation order of γ_i for Δ, we have the same contradiction as before. Thus we have proved our lemma.

4.2. If there are elements of S with the common fixed points, then we shall replace one of them by another one as follows: Let γ_i and γ_j be elements of S with the common fixed points and let γ_k be an element of S with no common fixed point with γ_i. Lemma 4.1 implies that, for a large integer m, $\gamma_i\gamma_k^m$ is a loxodromic element with no common fixed point with γ_j and γ_k. The new set, S with the replacement of γ_i by $\gamma_i\gamma_k^m$, has the same property as S stated at the beginning of this §. Making such replacements, if necessary, we may assume that the fixed points set of each element of S is different from that of any other element of S. If there are elements γ_i, γ_j of S satisfying one of the conditions of Lemmas 4.2～4.4, then we change γ_i into $\gamma_i\gamma_j^m$ with a suitable integer m so that we have the desired conclusion in this §. Hence we assume that there are no elements γ_i, γ_j in S satisfying one of the conditions of Lemmas 4.2～4.4. So we may consider the case where, for any two elements of S, there is a component of G on whose boundary all the fixed points of them lie.

4.3. We shall first treat the case where there is a component Δ of G on whose boundary the fixed points of each element of S lie. Since G is not a function group, there is at least one element, say γ_i, of S whose rotation order for Δ is not 1. Let D be the auxiliary domain of Δ with respect to $\gamma_i(\Delta)$. Since, by Lemma 2.12 and Lemma 2.8, each element of S has both fixed points in D or on ∂D simultaneously and since we have assumed that there are no elements γ_i, γ_j in S satisfying one of conditions of Lemma 4.2～4.4, we see that each element of S has both fixed points on ∂D and that the rotation order of γ_i for Δ is 2. Let γ_j be an element of S different from γ_i. By the same reasoning as above, the rotation order of γ_j for Δ is equal to 1 or 2. We assert that if, for any large inetger m, the fixed points of $\gamma_i\gamma_j^m$ do not lie on $\Lambda_0(G)$, then either $\gamma_j(D)=D$ and $\gamma_j(\gamma_i(D))=\gamma_i(D)$ or $\gamma_j(D)=\gamma_i(D)$ and $\gamma_j(\gamma_i(D))=D$.

To prove our assertion, we recall the following facts: As we have seen in the proof of Lemma 4.2, the components, whose boundary contains the fixed points of $\gamma_i\gamma_j^m$ for a large integer m, have four fixed points of γ_j and $\gamma_i\gamma_j\gamma_i^{-1}$ on the boundary and the rotation order of $\gamma_i\gamma_j^m$ for such a component is at most two. Moreover, as we have seen in the proof of Lemma 4.4, such a component is Δ or $\gamma_i(\Delta)$.

First we consider the case where the rotation order of γ_j for Δ is 1. If the fixed points of $\gamma_i\gamma_j^m$ lie on $\partial\Delta$, then $\gamma_i\gamma_j^m\gamma_i\gamma_j^m(\Delta)=\Delta$ or $\gamma_i\gamma_j^m\gamma_i(\Delta)=\Delta$. If the fixed points of $\gamma_i\gamma_j^m$ lie on $\partial\gamma_i(\Delta)$, then $\gamma_i\gamma_j^m\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_i(\Delta)$ or $\gamma_j^m\gamma_i\gamma_j^m\gamma_i(\Delta)=\Delta$ or, equivalently, $\gamma_i\gamma_j^m\gamma_i(\Delta)=\Delta$. In both cases we have that $\gamma_i\gamma_j^m\gamma_i(\Delta)=\Delta$. Since $\gamma_i^2(\Delta)=\Delta$, we see that $\gamma_j^m\gamma_i(\Delta)=\gamma_i^{-1}(\Delta)=\gamma_i(\Delta)$, so that

γ_j has the fixed points on $\partial\gamma_i(\Delta)$. By Theorem 2.14, we see that the roation order of γ_j for $\gamma_i(\Delta)$ is 1. Applying Lemma 2.16 to Δ (or $\gamma_i(\Delta)$) and the auxiliary domain D (or $\gamma_i(D)$) of Δ (or $\gamma_i(\Delta)$), we obtain that $\gamma_j(D)=D$ and $\gamma_j(\gamma_i(D))=\gamma_i(D)$.

Next we consider the case where the rotation order of γ_j for Δ is 2. Let m be an odd number. If the fixed points of $\gamma_i\gamma_j^m$ lie on $\partial\Delta$, then $\gamma_i\gamma_j^m(\Delta)=\gamma_i\gamma_j(\Delta)\neq\gamma_i(\Delta)$ so that $\gamma_i\gamma_j^m(\Delta)=\Delta$. Hence we obtain from $\gamma_i\gamma_j(\Delta)=\Delta$ that $\gamma_j(\Delta)=\gamma_i(\Delta)$. If the fixed points of $\gamma_i\gamma_j^m$ lie on $\partial\gamma_i(\Delta)$, then either $\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_i(\Delta)$ or $\gamma_i\gamma_j^m\gamma_i(\Delta)=\Delta$ holds. The latter case corresponds to the case where the rotation order of $\gamma_i\gamma_j^m$ for $\gamma_i(\Delta)$ is 2. So, if this case occurs, we see $\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_j(\Delta)$ for an odd m, which is absord. Hence the latter case does not occur. Therefore, it holds that $\gamma_i\gamma_j^m\gamma_i(\Delta)=\gamma_i(\Delta)$. This implies that $\gamma_j^m\gamma_i(\Delta)=\Delta$ so that $\gamma_j\gamma_i(\Delta)=\Delta$ and $\gamma_j(\Delta)=\gamma_i(\Delta)$. In both cases we have $\gamma_j(\Delta)=\gamma_i(\Delta)$. Since the auxiliary domain of Δ with respect to $\gamma_j(\Delta)$ is then identical with D, we obtain $\gamma_j(D)=\gamma_i(D)$ and $\gamma_j\gamma_i(D)=\gamma_j^2(D)=D$ by Lemma 2.13 and Lemma 2.16. Thus our assertion is established.

By what just has been proved, we see that if, for each element γ_j of S and for any large integer m, the fixed points of $\gamma_i\gamma_j^m$ does not lie on $\Lambda_0(G)$, then $\partial D\cup\partial\gamma_i(D)$ is invariant under S and hence under G, so that $\Lambda(G)=\partial D\cup\partial\gamma_i(D)$. But, we shall show that this does not occur. Since $D\cap\gamma_i(D)=\emptyset$ by Lemma 2.13, the equality $\Lambda(G)=\partial D\cup\partial\gamma_i(D)$ implies $D\cap\Lambda(G)=\emptyset$. Therefore, we have $D=\Delta$ by Lemma 2.8, so that G_Δ is a quasi-Fuchsian group with the invariant curve ∂D. If $\partial D=\partial\gamma_i(D)$, then $\Lambda(G)=\partial\Delta$. This contradicts our assumption that G is not a Z_2-extension of a quasi-Fuchsian group. Hence $\partial D\neq\partial\gamma_i(D)$. Since $\Lambda(G)$ and ∂D are invariant under G_Δ, the set $\partial\gamma_i(D)\backslash\partial D$ is invariant under G_Δ. Let p be a point on ∂D not lying on $\partial\gamma_i(D)$ and let d be the distance between p and $\partial\gamma_i(D)$. It is well known that there is a loxodromic element γ of G_Δ such that the distance between p and the attractive fixed point of γ is smaller than d. Then, for a sufficiently large integer m, the distance between p and $\gamma^m(\partial\gamma_i(D)\backslash\partial D)$ is smaller than d. Since $\gamma^m(\partial\gamma_i(D)\backslash\partial D)=\partial\gamma_i(D)\backslash\partial D$, we have a contradiction. Thus the equality $\Lambda(G)=\partial D\cup\partial\gamma_i(D)$ does not occur. Therefore, there are an element γ_j of S and an integer m such that $\gamma_i\gamma_j^m$ is a loxodromic element with the fixed points on $\Lambda_0(G)$. Thus we can change S into the desired one in the beginning of this section.

4.4. Now we shall treat the case where there is no component of G on whose boundary the fixed points of all elements of S lie. First we shall show that there are elements γ_i, γ_j, γ_k of S and components Δ_i, Δ_j, Δ_k of G such that the fixed points of γ_i, γ_j and γ_k lie on $(\partial\Delta_j\cap\partial\Delta_k)\backslash\partial\Delta_i$, $(\partial\Delta_k\cap\partial\Delta_i)\backslash\partial\Delta_j$ and $(\partial\Delta_i\cap\partial\Delta_j)\backslash\partial\Delta_k$, respectively. Let Δ be a component of G on whose boundary the fixed points of γ_1, $\gamma_2\in S$ lie. By Theorem 2.6, there is, except for Δ, at most one component Δ' on whose boundary the fixed points of γ_1 and γ_2 lie.

If Δ' does not exist, then let $\gamma_i=\gamma_1$, $\gamma_j=\gamma_2$, γ_k be an element of S whose fixed points do not lie on $\partial\Delta$, Δ_i (or Δ_j) be a component of G on whose boundary the fixed points of γ_j, γ_k (or γ_k, γ_i) lie, and $\Delta_k=\Delta$. If Δ' exists and if there is an element γ_l of S whose fixed points do not lie on $\partial\Delta\cup\partial\Delta'$, then let $\gamma_i=\gamma_1$, $\gamma_j=\gamma_2$, $\gamma_k=\gamma_l$, Δ_i (or Δ_j) be a component of G on whose boundary the fixed points of γ_2, γ_l (or γ_1, γ_l) lie, and $\Delta_k=\Delta$. If Δ' exists and if the fixed points of all elements of S lie on $\partial\Delta\cup\partial\Delta'$, then there are elements $\gamma_p, \gamma_q\in S$ whose fixed points lie on $\partial\Delta'\backslash\partial\Delta$ and on $\partial\Delta\backslash\partial\Delta'$, respectively. Let Δ'' be a component of G on whose boundary the fixed points of γ_p and γ_q lie. It is easy to see that $\Delta''\neq\Delta$ and $\Delta''\neq\Delta'$. Hence the fixed points of γ_1 and γ_2 do not lie on $\partial\Delta''$ simultaneously. Let γ_i be either γ_1 or γ_2 whose fixed points do not lie on $\partial\Delta''$, $\gamma_j=\gamma_p$, $\gamma_k=\gamma_q$, $\Delta_i=\Delta''$, $\Delta_j=\Delta$ and $\Delta_k=\Delta'$. It is easy to see that, in each case stated just above, these γ_i, γ_j, γ_k, Δ_i, Δ_j and Δ_k have the desired property.

Next, by using γ_i, γ_j, γ_k, we shall change S into a set of generators of G, which satisfies the property stated in the beginning of this section. Let D_{pq} be the auxiliary domain of Δ_p with respect to Δ_q, $p, q=i, j, k$. Since $\partial\Delta_j\cap\partial\Delta_k$ contains both fixed points of γ_i, it follows from Theorem 2.7 that Δ_j and Δ_k are not included in the distinct components of G_{Δ_i}. Hence we see that $D_{ij}=D_{ik}$. By the same reasoning as above, we see that $D_{ji}=D_{jk}$ and $D_{ki}=D_{kj}$. For simplicity, we shall denote D_{pq} by D_p, $p=i, j, k$. Let ξ_p and ξ_p' be the attractive and the repelling fixed points of γ_p, respectively, $p=i, j, k$. Then, we see by Lemma 2.8 that ξ_i and ξ_i' lie on $(\partial D_j\cap\partial D_k)\backslash\partial D_i$. Let r be the rotation order of γ_i for Δ_j. Clearly $\gamma_i^r\in G_{\Delta_j}$. Then by Theorem 2.14, $\gamma_i^r\in G_{\Delta_k}$. By Lemma 2.16, we see that $\gamma_i^r(D_j)=D_j$ and $\gamma_i^r(D_k)=D_k$. We consider the element of the form $\gamma_i^{rm}\gamma_k\gamma_i^{-rm}$ with a positive integer m. Since $\gamma_i^{rm}(\partial D_j)=\partial D_j$ and $\gamma_i^{rm}(\partial D_k)=\partial D_k$, we see that, for a sufficiently large m, $\gamma_i^{rm}(\partial D_i)$ lies near ξ_i and meets to ∂D_j (or ∂D_k) at $\gamma_i^{rm}(\xi_k)$ (or $\gamma_i^{rm}(\xi_j)$), and that the fixed points of $\gamma_i^{rm}\gamma_k\gamma_i^{-rm}$ lie on $(\partial D_j\cap\gamma_i^{rm}(\partial D_i))\backslash\partial D_k$. On the other hand, we can easily verify that, for any integer l, there is a Jordan curve lying in $D_j\cup D_k\cup\gamma_i^{rl}(D_i)\cup\{\xi_i, \gamma_i^{rl}(\xi_j), \gamma_i^{rl}(\xi_k)\}$. Therefore, there is no component of G on whose boundary the fixed points of both γ_j and $\gamma_i^{rm}\gamma_k\gamma_i^{-rm}$ lie. Hence γ_j and $\gamma_i^{rm}\gamma_k\gamma_i^{-rm}$ satisfy the assumption of Lemma 4.2. Changing γ_k into $\gamma_i^{rm}\gamma_k\gamma_i^{-rm}$ and applying Lemma 4.2, we can change S into a desired set of generators of G.

4.5. We can easily see that the results in this section give an alternative proof of the following.

Theorem [2]. *Let G be a finitely generated Kleinian group. Then $\Lambda_0(G)=\emptyset$ if and only if G is either a function group or a Z_2-extension of a quasi-Fuchsian group.*

5. Final step of the proof of Theorem 1.1

5.1. Let S be a set of generators of G which consists of loxodromic elements only and one of which has the fixed points on $\Lambda_0(G)$. We shall change S into S_0 in Theorem 1.1. Without loss of generality we may assume that $\infty\in\Omega(G)$ and $\gamma_1\in S$ has the fixed points on $\Lambda_0(G)$. Let ξ_1 and ξ_1' be the repelling and the attractive fixed points of γ_1, respectively, and let γ_i be an element of S whose fixed points do not lie on $\Lambda_0(G)$. By Lemma 4.1, for a sufficiently large integer m, $\gamma_i\gamma_1^m$ is loxodromic and the repelling and the attractive fixed points of $\gamma_i\gamma_1^m$ lie near ξ_1 and $\gamma_i(\xi_1')$, respectively. Let d be the distance between ξ_1 and $\gamma_i(\xi_1')$. By Lemma 2.10, there is a finite number of components of G whose diameters are greater than $d/2$. Let δ be the minimum of the distances of ξ_1 or $\gamma_i(\xi_1')$ from the components whose diameters are greater than $d/2$. Since ξ_1 and $\gamma_i(\xi_1')$ are the points on $\Lambda_0(G)$, δ is positive. Let m be so large that the distance between ξ_1 (or $\gamma_i(\xi_1')$) and the repelling (or the attractive) fixed point of $\gamma_i\gamma_1^m$ is smaller than δ. Then there is no component of G whose diameter is greater than $d/2$ and on whose boundary the fixed points of $\gamma_i\gamma_1^m$ lie. By Theorem 2.11, we see that there is no component on whose boundary the fixed points of $\gamma_i\gamma_1^m$ lie. Hence, for a large integer m, $\gamma_i\gamma_1^m$ is a loxodromic element with the fixed points on $\Lambda_0(G)$. Changing each γ_i of S, whose fixed points do not lie on $\Lambda_0(G)$, into $\gamma_i\gamma_1^m$, we obtain the desired S_0. Since our operations do not increase the number of elements of the set of generators, the second property of S_0 is clear. Thus we have completed the proof of Theorem 1.1.

6. Non-web groups

6.1. Among the set of finitely generated Kleinian groups with the non-empty residual limit sets there is a class of web groups. A finitely generated (non-elementary) Kleinian group G is called a web group if, for each component Δ of G, the component subgroup G_Δ is quasi-Fuchsian [2]. Usually those group which are themselves quasi-Fuchsian are excluded from the class. If G is a finitely generated Kleinian group with the non-empty residual limit set and is not a web group, then there is a subset $L_1(G)$ of $\Lambda_0(G)$ consisting of the points, to each of which there is a converging nest sequence of the separators of G [2]. A sequence $\{C_m\}_{m=1}^{\infty}$ of Jordan curves, which converges to a point p, is called a nest sequence if $p\notin C_m$ and C_{m+1} separates p from C_m for every m. In this § we shall improve Theorem 1.1 and Corollary 1.2 for those groups G with non-empty sets $L_1(G)$.

6.2. Later we need the followings.

Lemma 6.1. *Let G be a finitely generated Kleinian group and let D be a Jordan domain whose boundary is a separator of G. Assume that the fixed points* *of a loxodromic element* γ *of* G *lie on* $\Lambda_0(G)$. *If* $D\subset\gamma(D)$, *then the fixed points of* γ *lie on* $L_1(G)$.

Proof. The assumption that $D\subset\gamma(D)$ implies that the repelling and the attractive fixed points of γ lie in $\bar{D}$ and in the complement of $\gamma(D)$, respectively. Since the fixed points of γ do not lie on ∂D, they are separated by a separator ∂D. Then $\{\gamma^m(\partial D)\}_{m=1}^{\infty}$ (or $\{\gamma^{-m}(\partial D)\}_{m=1}^{\infty}$) forms a nest sequence of separators converging to the attractive (or the repelling) fixed point of γ. Hence the fixed points of γ lie on $L_1(G)$.

Lemma 6.2. *Let* G *be a finitely generated Kleinian group and let* Δ *and* γ *be a component and a loxodromic element* G, *respectively. Assume that the fixed points of* γ *lie on* $\Lambda_0(G)\backslash L_1(G)$ *and denote by* D *and* D' *the auxiliary domains of* Δ *and of* $\gamma(\Delta)$ *with respect to* $\gamma(\Delta)$ *and* Δ, *respectively. Then* $\gamma(D)=D'$ *so that* $D\cap\gamma(D)=\emptyset$.

Proof. Since both $\gamma(D)$ and D' contain $\gamma(\Delta)$, we have only to prove that $\gamma(\partial D)=\partial D'$. If it is not true, then $\gamma(\partial D)\cap D'\neq\emptyset$ and $\gamma(\partial D)$ lies in $\bar{D}'$, because $\bar{D}'\supset\gamma(\bar{\Delta})$. Hence either $\gamma(\bar{D})$ or the exterior of $\gamma(D)$ is contained in $\bar{D}'$. If $\gamma(\bar{D})\subsetneqq\bar{D}'$, then there are points of $\partial D'\backslash\gamma(\bar{D})$ $(\subset\gamma(\bar{\Delta})\backslash\gamma(\bar{D}))$. This contardicts the fact that $\gamma(\bar{\Delta})\subset\gamma(\bar{D})$. Hence the exterior of $\gamma(D)$ is contained in $\bar{D}'$. Therefore the exterior of D' is contained in $\gamma(D)$. Since $D\cap D'=\emptyset$, we have $D\subset\gamma(D)$. By Lemma 6.1, the fixed points of γ lie on $L_1(G)$, a contradiction. Hence we have $\gamma(\partial D)=\partial D'$ and our lemma.

Lemma 6.3. *Let* G *be a finitely generated Kleinian group and let* Δ *and* γ *be a component and a loxodromic element of* G, *respectively. If the fixed points of* γ *lie on* $\Lambda_0(G)\backslash L_1(G)$, *then* $\gamma^{-1}(\Delta)$ *is contained in the component of* G_Δ *which contains* $\gamma(\Delta)$.

Proof. Let D be the auxiliary domain of Δ with respect to $\gamma(\Delta)$. Note that the exterior of D is a component of G_Δ. We shall show that both $\gamma(\Delta)$ and $\gamma^{-1}(\Delta)$ are contained in the exterior of D. By Lemma 6.2, we have only to show this for $\gamma^{-1}(\Delta)$. If it is not true, then $\gamma^{-1}(\Delta)\subset D$. If $\gamma^{-1}(D)\subset D$, then by Lemma 6.1 we see that the fixed points of γ^{-1} lie on $L_1(G)$, which contradicts the assumption of the lemma. Hence $\gamma^{-1}(D)\not\subset D$. On the other hand, $\gamma^{-1}(\Delta)\subset D$ implies $\partial\gamma^{-1}(D)\subset\bar{D}$. This implies that $\gamma^{-1}(D)$ contains the exterior of D. Hence by Lemma 6.2, $\gamma^{-1}(D)\supset\gamma(D)$ or $D\subset\gamma^{-2}(D)$. By Lemma 6.1, the fixed points of γ lie on $L_1(G)$, a contradiction. Hence $\gamma^{-1}(\Delta)$ is contained in the exterior of D. Thus we have our lemma.

6.3. Now we shall prove the following.

Theorem 6.4. *Let* G *be a finitely generated Kleinian group and let* S *be a* *finite set of generators of G. If G is neither a function group nor a web group, then S can be changed into a set of generators S_1 of G with the following properties:*

i) *each element of S_1 is loxodromic and its fixed points lie on $L_1(G)$, and*

ii) *the number of elements of S_1 is not greater than that of S.*

To prove our theorem, we first change S into S_0 which has the properties i) and ii) in Theorem 1.1. We shall next change S_0 by our operation stated in §3.1 into a set which consists of loxodromic elements only and contains at least one element with the fixed points on $L_1(G)$. Assume that each element of S_0 has the fixed points on $\Lambda_0(G)\backslash L_1(G)$. Then we assert that there are elements γ_i, γ_j of S_0 and a component Δ of G such that the components $\gamma_i(\Delta)$ and $\gamma_j(\Delta)$ lie in the distinct components of the component subgroup G_Δ.

In order to prove this assertion we assume that there is no triple $(\gamma_i, \gamma_j, \Delta)$ with the property stated just above. Let Δ be a component of G, for which the component subgroup G_Δ is not quasi-Fuchsian. Then, by Theorem 2.2, each component of G_Δ which is different from Δ is a non-invariant component of G_Δ. Let Δ' be the component of G_Δ which contains the component $\gamma_1(\Delta)$ of G, $\gamma_1\in S_0$. Then, from the assumption just stated above, Δ' contains each component $\gamma_i(\Delta)$ of G, $\gamma_i\in S_0$. Let D be the auxiliary domain of Δ with respect to $\gamma_i(\Delta)$, $\gamma_i\in S_0$. Clearly $\partial D=\partial\Delta'$. Since Δ' is a non-invariant component of G_Δ, there are a component Δ'' $(\neq\Delta')$ of G_Δ and an element g of G_Δ such that $g(\Delta')=\Delta''$. It is easy to see that $\Delta''\subsetneqq D$ and $\partial\Delta''\cap D\neq\emptyset$. Let $\delta=g\gamma_1 g$. Then δ maps Δ to a component $\delta(\Delta)=\Delta^*$ of G lying in Δ'' and we have $\delta(D)\cap D\neq\emptyset$. Let D^* be the auxiliary domain of Δ^* with respect to Δ. We can see that $\delta(\partial D)\cap g\gamma_1(D)=g\gamma_1(\partial\Delta'')\cap g\gamma_1(D)\neq\emptyset$. On the other hand, we have easily $g\gamma_1(D)=D^*$. So we obtain $\delta(\partial D)\neq\partial D^*$. Therefore, $\delta(D)$ is not contained in D and $\delta(D)$ does not contain D. Since δ is an element of G, we can represent it by elements of S_0 as $\delta=\delta_m\delta_{m-1}\cdots\delta_1$, where δ_i $(i=1, 2, \cdots, m)$ is an element of S_0 or its inverse and $\delta_i\delta_{i-1}$ is not identity $(2\leqq i\leqq m)$. We set $\varepsilon_k=\delta_k\delta_{k-1}\cdots\delta_1$ $(1\leqq k\leqq m)$.

Lemma 6.2 implies $\varepsilon_1(D)\cap D=\emptyset$. It may happen for some k $(2<k\leqq m)$ that $\varepsilon_{k-1}(D)\cap D\neq\emptyset$. By noting Lemma 2.4, we see that following three cases may occur:

$(1)_{k-1}$: $\varepsilon_{k-1}(D)\subset D$,

$(2)_{k-1}$: $\varepsilon_{k-1}(D)\supset D$,

and

$(3)_{k-1}$: $D^c\subset\varepsilon_{k-1}(D)$, where D^c is the complementary set of D.

We also denote by $(0)_k$ the property $\varepsilon_k(D)\cap D=\emptyset$.

[I] The property $(1)_{k-1}$ implies the property $(0)_k$. In fact, $(1)_{k-1}$ means $\varepsilon_{k-1}(D)\subset D$, so we have $\varepsilon_k(D)=\delta_k(\varepsilon_{k-1}(D))\subset\delta_k(D)$. On the other hand, Lemma 6.2 and Lemma 6.3 yield $\delta_k(D)\cap D=\emptyset$. Hence $\varepsilon_k(D)\cap D=\emptyset$.

[II] The property $(2)_{k-1}$ implies the property $(0)_k$. In fact, the property

$(0)_1$: $\mathcal{E}_1(D) \cap D = \emptyset$ shows $\mathcal{E}_1(\Delta) \cap D = \emptyset$. Hence $\delta_1(\mathcal{E}_1(\Delta)) \cap \delta_1(D) = \delta_1(\mathcal{E}_1(\Delta) \cap D) = \emptyset$. Lemma 6.3 and the assumption for elements of S_0 and for components of G imply that $\mathcal{E}_2(\Delta) \cap \mathcal{E}_1(D) = \emptyset$. By the same reasoning, we can see that $\mathcal{E}_2(\Delta) \cap \mathcal{E}_1(D) = \emptyset$ implies $\mathcal{E}_3(\Delta) \cap \mathcal{E}_2(D) = \emptyset$. Repeating this procedure, we obtain $\mathcal{E}_k(\Delta) \cap \mathcal{E}_{k-1}(D) = \emptyset$. Therefore, $\mathcal{E}_{k-1}(D)$ is an auxiliary domain of $\mathcal{E}_{k-1}(\Delta)$ with respect to $\mathcal{E}_k(\Delta)$. Lemma 6.2 yields $\mathcal{E}_k(D) \cap \mathcal{E}_{k-1}(D) = \emptyset$. Since $\mathcal{E}_{k-1}(D) \supset D$, we have $\mathcal{E}_k(D) \cap D = \phi$.

[III] If the property $(0)_{k-1}$ holds, then the property $(3)_k$ does not hold $(1 < k \leqq m)$. In fact, $(0)_{k-1}$ implies $D^c \supset \mathcal{E}_{k-1}(D)$, which contradicts $(3)_k$.

Now we recall that the property $(0)_1$ holds. The propositions [I], [II] and [III] show that $(3)_m$ does not occur. So we see that the one of two relations $\delta(D) \subset D$ and $\delta(D) \supset D$ must hold, becuase $\delta(D) \cap D \neq \emptyset$. This contradicts the fact obtained already. Thus we have the assertion that there is a triple $(\gamma_i, \gamma_j, \Delta)$ such that $\gamma_i(\Delta)$ and $\gamma_j(\Delta)$ lie in the distinct components of the component subgroups G_Δ.

6.4. Let D_i and D_j be the auxiliary domains of $\gamma_i(\Delta)$ and $\gamma_j(\Delta)$ with respect to Δ, respectively. Since they are included in the distinct components of G_Δ, we see by Theorem 2.7 that $D_i \cap D_j = \emptyset$ and that $\partial D_i \cap \partial D_j$ consists of at most one point. We shall show that $\gamma_i \gamma_j^{-1}$ is loxodromic and its repelling and attractive fixed points lie in D_j and in D_i, respectively. Since two fixed points of $\gamma_i \gamma_j^{-1}$ are separated from each other by a separator of G, the fixed points of $\gamma_i \gamma_j^{-1}$ lie on $L_1(G)$. Since $\gamma_1(\Delta) \subset D_i$ and $\gamma_j(\Delta) \subset D_j$, it suffices to show that $\gamma_i \gamma_j^{-1}(\partial D_j) \neq \partial D_i$ and that if $\partial D_i \cap \partial D_j = \{p\}$, then $\gamma_i \gamma_j^{-1}(p) \neq p$. In fact, from these properties, we see easily that $\gamma_i \gamma_j^{-1}(D_j) \supsetneqq D_j$ and that $\gamma_i \gamma_j^{-1}$ can be neither parabolic nor elliptic so that Lemma 6.1 implies the assertion. Let D_i^* and D_j^* be the auxiliary domains of Δ with respect to $\gamma_i(\Delta)$ and $\gamma_j(\Delta)$, respectively. Then $\gamma_i(D_i^*) = D_i$, $\gamma_j(D_j^*) = D_j$ and $\partial D_i^* \neq \partial D_j^*$. Lemma 6.2 shows $\gamma_i(D_i^*) = \partial D_i$ and $\gamma_i(\partial D_j^*) = \partial D_j$ and we see that $\gamma_i \gamma_j^{-1}(\partial D_j) = \gamma_i(\partial D_j^*) \neq \gamma_i(\partial D_i^*) = \partial D_i$. If $\partial D_i \cap \partial D_j = \{p\}$, then, by Theorem 2.7, p is the fixed point of a parabolic element of G. Since γ_j is loxodromic, we see $\gamma_j^{-1}(p) \neq p$ and $\gamma_j^{-1}(p) \in \partial D_j^*$. Hence, by Theorem 2.7, we have $\gamma_j^{-1}(p) \notin \partial D_i^*$. Hence we have $\gamma_i \gamma_j^{-1}(p) \notin \partial D_i$, so that $\gamma_i \gamma_j^{-1}(p) \neq p$.

Thus we have shown that $\gamma_i \gamma_j^{-1}$ is loxodromic and has the fixed points on $L_1(G)$. Changing γ_i by $\gamma_i \gamma_j^{-1}$, we obtain the desired set of generators, which we shall denote by S_0^*.

Lastly we shall change S_0^* into S_1. Without loss of generality we may assume that $\infty \in \Omega(G)$ and that the fixed points of γ_1 lie on $L_1(G)$. Let ξ_1 and ξ_1' be the repelling and the attractive fixed points of γ_1, respectively. Then there is a nest sequence of separators of G which converges to ξ_1. Let γ_i be an element of S_0^* and let m be an integer such that $\gamma_i \gamma_1^m$ is loxodromic and that the fixed points of $\gamma_i \gamma_1^m$ are separated by a separator. The existence of such

m is assured by Lemma 4.1. Then clearly $\gamma_i\gamma_1^m$ is the desired loxodromic element. Changing each element γ_i of S_0^* which has the fixed points on $\Lambda_0(G)\setminus L_1(G)$ by the element of the form $\gamma_i\gamma_1^m$, we obtain the desired S_1 and complete the proof of Theorem 6.4.

6.5. Choosing S in Theorem 6.4 to be the minimal set of generators, we have the following.

Corollary 6.5. *Among the minimal sets of generators of a finitely generated Kleinian group G which is neither a function group nor a web group, there is a set consisting of only loxodromic elements with the fixed points on $L_1(G)$.*

YAMAGATA UNIVERSITY

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