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A NOTE ON QUASI-CORATIONAL EXTENSIONS

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In his paper [3] Bland has introduced quasi-corational completions of modules and by means of this concept he has given a characterization of finite direct sums of complete matrix rings over perfect completely primary rings.

In this note we shall look at his results again from the point of view of torsion theories. By this we can see his results in perspective and give simple proofs of some of his results. In section 1 we shall define an idempotent preradical t_Q of mod-R for a fixed R-module Q. As is well-known, when Q is projective, t_Q is a radical. An example will be given to show that t_Q is not always a radical even if Q is quasi-projective. However we can show that, for submodules of a quasi-projective module Q, t_Q acts as a radical (Proposition 1.2).

Using this proposition, in section 2, we shall give a slightly simple characterization of a quasi-corational extension of an R-module (Theorem 2.4). We shall show that a maximal quasi-corational extension of an R-module can be constructed naturally by this theorem and the proposition.

In section 3 we shall treat quasi-corationally complete modules. In section 4 we shall only deal with R-modules M having projective covers $0\to K\to P\to M\to 0$. Wu and Jans [9] has shown how to construct a quasi-projective cover of M out of the projective cover of M. They showed that P modulo the unique maximal R-$\mathrm{End}_R(P)$-submodule contained in K is the desired quasi-projective module. We shall show that this submodule is precisely $P(0: M)$ (Proposition 4.1). We also show that the quasi-corationally completeness of M can be described by means of the radical t_P (Theorem 4.6). Finally in closing this note we shall give an example of an R-module which is quasi-corationally complete but not quasi-projective.

Throughout this note R will denote an associative ring with identity and all modules will be unital right R-modules. The category of unital right R-modules is denoted by mod-R. For the notions and terminologies about torsion theories we refer to Stenström [8].

1. Preliminaries

A subfunctor t of the identity functor of mod-R is called a *preradical* of mod-R, i.e., t assigns to each module M a submodule $t(M)$ of M in such a way that every homomorphism $M \to N$ induces $t(M) \to t(N)$ by restriction. A preradical t is *idempotent* if $t(t(M))=t(M)$ for every module M and is called a *radical* if $t(M/t(M))=0$ for every module M.

To each preradical t we can associate two classes of modules, namely

$$\boldsymbol{T}(t)=\{A_R\colon t(A)=A\} \quad \text{and} \quad \boldsymbol{F}(t)=\{B_R\colon t(B)=0\}\ .$$

$\boldsymbol{T}(t)$ is closed under homomorphic images and direct sums and dually $\boldsymbol{F}(t)$ is closed under submodules and direct products.

Now let Q be a module and define a preradical t_Q of mod-R as

$$t_Q(M)=\sum \{\mathrm{Im}(f)\colon f\in \mathrm{Hom}_R(Q, M)\}$$

for each module M. Then t_Q is idempotent and $t_Q(Q)=Q$. Moreover t_Q is a unique minimal one of those preradicals t of mod-R for which $t(Q)=Q$.

A module M belongs to $\boldsymbol{T}(t_Q)$ if and only if M is a homomorphic image of a direct sum of copies of Q. While M belongs to $\boldsymbol{F}(t_Q)$ if and only if $\mathrm{Hom}_R(Q, M)=0$. $\boldsymbol{F}(t_Q)$ is also closed under group extensions and becomes a torsion-free class.

If in particular Q is a projective module, then t_Q becomes a radical and $\boldsymbol{F}(t_Q)$ is closed under homomorphic images, as is easily seen. However this is not true in general. The following example shows that t_Q is not always a radical even if Q is a quasi-projective module.

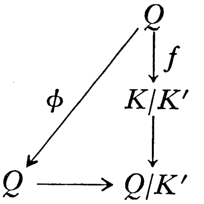
EXAMPLE 1.1. Let R be a right Artinian ring with identity and N $(\neq 0)$ its Jacobson radical. There exists an integer $n>0$ such that $N^n=0$ and $N^{n-1}\neq 0$. We put $Q=R/N^{n-1}$. Then, since $f(N)\subset N$ for all $f\in \mathrm{End}(R_R)$, by [9, Proposition 2.1] we see that Q is quasi-projective. Furthermore N^{n-1} is not a direct summand of R and so Q can not be a projective module. It is easy to see that $t_Q(R)=\mathrm{Ann}_R(N^{n-1})$, the left annihilator of N^{n-1} in R, and $t_Q(R)\neq R$. Since $N^n=0$, we have $N^{n-1}\subset N\subset t_Q(R)$ and so, by considering the canonical homomorphism $Q\to R/t_Q(R)$, we see that $t_Q(R/t_Q(R))=R/t_Q(R)\neq 0$.

We note that, for a module Q, t_Q is a radical if and only if $\boldsymbol{T}(t_Q)$ is closed under group extensions (see e.g. [2, Proposition 3] or [7, Proposition 1.1]). So the above example also shows that, in contrast with $\boldsymbol{F}(t_Q)$, $\boldsymbol{T}(t_Q)$ is not always a torsion class in general.

As we have shown above, t_Q is not always a radical, but for submodules of Q it acts as a radical.

Proposition 1.2. *Let Q be a quasi-projective module and K and K' submodules of Q such that $K'\subset K$. Then $t_Q(K/K')=0$ if and only if $t_Q(K)\subset K'$. In particular we have $t_Q(K/t_Q(K))=0$.*

Proof. Suppose that $t_Q(K/K')=0$. Then we have $(t_Q(K)+K')/K' \in \boldsymbol{F}(t_Q)$. On the other hand, since t_Q is idempotent, $t_Q(K) \in \boldsymbol{T}(t_Q)$ and so $(t_Q(K)+K')/K' \in \boldsymbol{T}(t_Q)$. Thus we have $(t_Q(K)+K')/K'=0$ and $t_Q(K) \subset K'$. Conversely suppose that $t_Q(K) \subset K'$ and claim that $\mathrm{Hom}_R(Q, K/K')=0$. Take $f \in \mathrm{Hom}_R(Q, K/K')$. Then there exists a homomorphism ϕ: $Q \to Q$ making the diagram



commutative, where $Q \to Q/K'$ and $K/K' \to Q/K'$ are the canonical map and the inclusion map respectively. For any $x \in Q$, $\phi(x)+K'=f(x)=x'+K'$ for some $x' \in K$. From this it follows that $\phi(x) \in K$ and hence $\phi(x) \in t_Q(K)$. Therefore $f(x)=\phi(x)+K'=0$ by assumption. Thus we have $f=0$.

2. Quasi-corational extensions

We now record some required definitions.

A *quasi-projective cover* of a module M is an exact sequence $Q \xrightarrow{\beta} M \to 0$ with the properties that Q is quasi-projective, $\mathrm{Ker}(\beta)$ is small in Q and, for every nonzero submodule Q' of $\mathrm{Ker}(\beta)$, Q/Q' is not quasi-projective.

An exact sequence $N \xrightarrow{f} M \to 0$ of modules is called *corational by M* if every factor module of $\mathrm{Ker}(f)$ belongs to $\boldsymbol{F}(t_N)$. It is called *quasi-corational by M* if it is corational by M, besides M has a quasi-projective cover $Q \xrightarrow{\beta} M \to 0$ and there exists a homomorphism ϕ: $Q \to N$ such that $f\phi=\beta$. Note that, since $\mathrm{Ker}(f)$ is small in N (see [4, Theorem 2.3]), ϕ must be an epimorphism.

First we shall quote [2, Proposition 2] in a slightly general form.

Lemma 2.1. *Let $0 \to L \to N \xrightarrow{f} M \to 0$ be an exact sequence of modules such that L is small in N and let A be a module. If every factor module of A belongs to $\boldsymbol{F}(t_M)$, then A belongs to $\boldsymbol{F}(t_N)$. In the case where $\boldsymbol{F}(t_N)$ is closed under factor modules of A, the converse is also true.*

Proof. We claim that $\mathrm{Hom}_R(N, A)=0$. Take any $g \in \mathrm{Hom}_R(N, A)$. Then g induces the mapping g^*: $M \to A/g(L)$ given by $g^*(f(x))=g(x)+g(L)$ for $x \in N$. Since $g^*=0$ by assumption, $g(N)=g(L)$ and hence $N=L+\mathrm{Ker}(g)$. However L is small in N and so $N=\mathrm{Ker}(g)$. Thus we have $g=0$. The latter half of the lemma follows from the fact that $\boldsymbol{F}(t_N) \subset \boldsymbol{F}(t_M)$.

As a consequence of this lemma we have

Corollary 2.2. *Let* $0 \to L \to N \to M \to 0$ *be an exact sequence with* L *small in* N. *If* $\boldsymbol{F}(t_M)$ *is closed under factor modules, then* $\boldsymbol{F}(t_N)=\boldsymbol{F}(t_M)$ *holds.*

From Proposition 1.2 we have

Corollary 2.3. *Let* Q *be a quasi-projective module and* K *a submodule of* Q. *Then every factor module of* K *belongs to* $\boldsymbol{F}(t_Q)$ *if and only if* K *itself belongs to* $\boldsymbol{F}(t_Q)$.

Proof. Note that if $K \in \boldsymbol{F}(t_Q)$, then $\boldsymbol{F}(t_Q)$ is closed under factor modules of K by Proposition 1.2.

We now obtain a characterization of quasi-corational extensions.

Theorem 2.4. *An exact sequence* $N \xrightarrow{f} M \to 0$ *is quasi-corational by* M *if and only if*

(1) *M has a quasi-projective cover* $0 \to K \to Q \xrightarrow{\beta} M \to 0$,

(2) *there exists an epimorphism* $\phi\colon Q \to N$ *such that* $f\phi=\beta$, *and*

(3) $\mathrm{Ker}(f) \in \boldsymbol{F}(t_Q)$.

Proof. The "only if" part follows from the definition and Lemma 2.1, so we only show the "if" part. Assume that (1), (2) and (3) hold. Then $\mathrm{Ker}(\phi)$ is a submodule of K and the diagram

$$\begin{array}{ccc} Q/\mathrm{Ker}(\phi) & \xrightarrow{\beta^*} & M \\ {\scriptstyle\phi^*}\downarrow & & \| \\ N & \xrightarrow[f]{} & M \end{array}$$

commutes, where β^* and ϕ^* are homomorphisms induced by β and ϕ respectively. Hence $\mathrm{Ker}(f)$ is isomorphic to $K/\mathrm{Ker}(\phi)$ and by Proposition 1.2 $t_Q(K) \subset \mathrm{Ker}(\phi)$. Every factor module of $\mathrm{Ker}(f)$ is also isomorphic to K/K' for some submodule K' of K containing $\mathrm{Ker}(\phi)$. Again by Proposition 1.2 $K/K' \in \boldsymbol{F}(t_Q)$ and thus every factor module of $\mathrm{Ker}(f)$ belongs to $\boldsymbol{F}(t_Q)$. Since $\boldsymbol{F}(t_Q) \subset \boldsymbol{F}(t_N)$, this implies that $N \xrightarrow{f} M \to 0$ is quasi-corational by M.

As is seen from the proof of the theorem, every exact sequence which is quasi-corational by M is of the form

$$Q/K' \xrightarrow{\beta^*} M \to 0$$

for some submodule K' of K, where $0 \to K \to Q \xrightarrow{\beta} M \to 0$ is a quasi-projective cover. Conversely for a module M having a quasi-projective cover $0 \to K \to Q \xrightarrow{\beta} M \to 0$, by Theorem 2.4 an exact sequence $Q/K' \xrightarrow{\beta^*} M \to 0$ with K' a submodule of K is quasi-coratinoal by M if and only if $K/K'=\mathrm{Ker}(\beta^*)\in \boldsymbol{F}(t_Q)$, or equivalently, by Proposition 1.2, $t_Q(K)\subset K'$.

Consequently, in particular,

$$Q/t_Q(K) \xrightarrow{\bar{\beta}} M \to 0$$

is surely quasi-corational by M, where $\bar{\beta}$ denotes the homomorphism induced by β, and moreover, for each exact sequence $Q/K' \xrightarrow{\beta^*} M \to 0$ which is quasi-corational by M, the mapping ϕ: $Q/t_Q(K) \to Q/K'$ defined by $\phi(x+t_Q(K))= x+K'$ makes the diagram

$$\begin{array}{ccc} Q/t_Q(K) & \xrightarrow{\bar{\beta}} & M \\ {\scriptstyle\phi}\downarrow & & \| \\ Q/K' & \xrightarrow[\beta^*]{} & M \end{array}$$

commutative, i.e.,

$$Q/t_Q(K) \xrightarrow{\bar{\beta}} M \to 0$$

is maximal quasi-corational by M in the following sense.

An exact sequence $N \xrightarrow{f} M \to 0$ of modules is called *maximal quasi-corational by M* if it is quasi-corational by M and, for each exact sequence $N' \xrightarrow{f'} M \to 0$ which is quasi-corational by M, there exists a homomorphism ϕ: $N \to N'$ such that $f'\phi=f$.

In this definition, if a homomorphism ϕ': $N \to N'$ also satisfies that $f'\phi'=f$, then we have $\mathrm{Im}(\phi-\phi')\subset \mathrm{Ker}(f')$. Since $N' \xrightarrow{f'} M \to 0$ is (quasi-) corational, every factor module of $\mathrm{Ker}(f')$ belongs to $\boldsymbol{F}(t_{N'})$ and whence to $\boldsymbol{F}(t_M)$. By Lemma 2.1 $\mathrm{Ker}(f')$ itself belongs to $\boldsymbol{F}(t_N)$. Therefore $\phi-\phi'$: $N \to N'$ is a zero mapping and thus we have $\phi=\phi'$. From this it follows that if $N \xrightarrow{f} M \to 0$ and $N' \xrightarrow{f'} M \to 0$ are both maximal quasi-corational by M, then there exists an isomorphism ϕ: $N \to N'$ for which the diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ {\scriptstyle\phi}\downarrow & & \| \\ N' & \xrightarrow[f']{} & M \end{array}$$

is commutative.

Corollary 2.5 ([3, Theorem 2.1]). *For a module M having a quaisi-projective cover* $0\to K\to Q\overset{\beta}{\to} M\to 0$, *the sequence*

$$Q/t_Q(K)\xrightarrow{\bar{\beta}} M\to 0$$

is maximal quasi-corational by M.

3. Quasi-corationally complete modules

A module M is called *quasi-corationally complete* if $N\overset{f}{\to} M\to 0$ is quasi-corational by M, then f is an isomorphism.

In case M has a quasi-projective cover, the exact sequence $M\overset{1}{\to} M\to 0$ is certainly quasi-corational by M, where 1 means the identity map of M. Hence we have

Theorem 3.1. *For a module M having a quasi-projective cover, the following conditions are equivalent:*

(1) *M is quasi-corationally complete.*

(2) $M\overset{1}{\to} M\to 0$ *is maximal quasi-corational by M.*

(3) $\mathrm{Hom}_R(M, -)$ *is right exact on all exact sequences of the form* $0\to L\to N\to M\to 0$ *which is quasi-corational by M.*

(4) *Every exact sequence* $N\to M\to 0$ *which is quasi-corational by M splits.*

Proof. (1)$\Rightarrow$(2). If $N\overset{f}{\to} M\to 0$ is quasi-corational by M, then f must be an isomorphism by assumption. Hence the diagram

$$\begin{array}{ccc} M & \xrightarrow{1} & M \\ {\scriptstyle f^{-1}}\downarrow & & \| \\ N & \xrightarrow[f]{} & M \end{array}$$

is commutative. This shows that $M\overset{1}{\to} M\to 0$ is maximal quasi-corational by M.

(2)$\Rightarrow$(3). If $N\overset{f}{\to} M\to 0$ is quasi-corational by M, then by the maximality of $M\overset{1}{\to} M\to 0$ we can find a homomorphism ϕ: $M\to N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{1} & M \\ {\scriptstyle \phi}\downarrow & & \| \\ N & \xrightarrow[f]{} & M \end{array}$$

is commutative. Therefore, for any $g \in \mathrm{Hom}_R(M, M)$, $f(\phi g) = g$ and thus $\mathrm{Hom}_R(M, -)$ is right exact on $0 \to L \to N \xrightarrow{f} M \to 0$.

(3)⇒(4). This is clear.

(4)⇒(1). If $N \xrightarrow{f} M \to 0$ is quasi-corational by M, then by assumption $\mathrm{Ker}(f)$ is a direct summand of N. However $\mathrm{Ker}(f)$ is small, it must be zero, i.e., M is quasi-corationally complete.

We also have another characterization of quasi-corationally complete modules by means of t_Q.

Theorem 3.2. *For a module M having a quasi-projective cover* $0 \to K \to Q \xrightarrow{\beta} M \to 0$, *the following conditions are equivalent*:

(1) *M is quasi-corationally complete.*

(2) $Q/t_Q(K) \cong^{\bar{\beta}} M$.

(3) $K \in \boldsymbol{T}(t_Q)$.

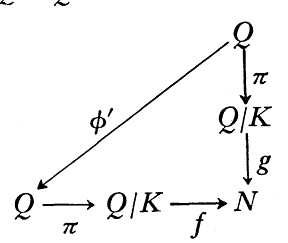
Proof. By Corollary 2.5, $0 \to K/t_Q(K) \to Q/t_Q(K) \xrightarrow{\bar{\beta}} M \to 0$ is maximal quasi-corational by M. In this case, $K \in \boldsymbol{T}(t_Q)$ if and only if $\bar{\beta}$ is an isomorphism, and moreover this is so if and only if $M \xrightarrow{1} M \to 0$ is maximal quasi-corational by M since the maximal quasi-corational extension is unique up to an isomorphism. Thus (1), (2) and (3) are equivalent by Theorem 3.1.

It follows from this theorem that a quasi-projective module is quasi-corationally complete ([3, Theorem 1.1]), but the converse of this fact is not always true in general, as we shall show later.

The following is a generalization of [3, Theorem 2.3].

Proposition 3.3. *Let Q be a quasi-projective module and K its submodule such that K is small in Q and* $K \in \boldsymbol{T}(t_Q)$. *Then* $\mathrm{Hom}_R(Q/K, -)$ *is right exact on all exact sequences of the form* $0 \to L \to Q/K \xrightarrow{f} N \to 0$ *which is corational by N.*

Proof. Take any $g \in \mathrm{Hom}_R(Q/K, N)$. Since Q is quasi-projective, there exists a homomorphism $\phi' : Q \to Q$ for which



is commutative, where π denotes the canonical homomorphism $Q \to Q/K$. We put $\phi'' = \pi\phi'$. Then to prove the proposition it is sufficient to show that

$\phi''(K)=0$. Since every factor module of $\mathrm{Ker}(f)$ belongs to $\boldsymbol{F}(t_{Q/K})$, $\mathrm{Ker}(f)$ always belongs to $\boldsymbol{F}(t_Q)$ by Lemma 2.1. On the other hand, $f(\phi''(K))=g(\pi(K))=0$ and $\phi''(K)\subset\mathrm{Ker}(f)$. Hence $\phi''(K)\in\boldsymbol{T}(t_Q)\cap\boldsymbol{F}(t_Q)=0$.

Combining Theorem 3.2 with this proposition, we have

Corollary 3.4 ([3, Corollary 2.4]). *For a quasi-corationally complete module M having a quasi-projective cover,* $\mathrm{Hom}_R(M, -)$ *is right exact on all exact sequences of the form* $0\to L\to M\to N\to 0$ *which is corational by N.*

We do not know if this property of $\mathrm{Hom}_R(M, -)$ characterize the quasi-corationally completeness of M conversely.

4. Modules having projective covers

Let $0\to K\to P\overset{\alpha}{\to}M\to 0$ be a projective cover of a module M, i.e., it is an exact sequence with P projective and K small in P. Throughout this section we assume that M always has a projective cover and denote it by $0\to K\to P\overset{\alpha}{\to}M\to 0$. Wu and Jans ([9, Proposition 2.6]) has shown how to construct a quasi-projective cover out of a projective cover. They showed that P modulo the unique maximal R-$\mathrm{End}_R(P)$-submodule contained in K is the desired quasi-projective module. Here we shall claim that this submodule is precisely $P(0: M)$, where $(0: M)$ denotes the right annihilator of M in R.

Proposition 4.1. *The unique maximal R-$\mathrm{End}_R(P)$-submodule of P contained in K is $P(0: M)$.*

Proof. First we shall show that $P(0: M)$ coincides with $\cap\{\mathrm{Ker}(f): f\in\mathrm{Hom}_R(P, M)\}$. This is a result due to Azumaya ([1, Proposition 7]). But, for the sake of completeness, we give here its proof. Since P is projective, as is well-known, there exist homomorphisms $f_\lambda\in\mathrm{Hom}_R(P, R)$ and elements $x_\lambda\in P$ for $\lambda\in\Lambda$ such that, for each $x\in P$, $f_\lambda(x)=0$ for almost all $\lambda\in\Lambda$ and $x=\sum x_\lambda f_\lambda(x)$. Put $A=\cap\{\mathrm{Ker}(f): f\in\mathrm{Hom}_R(P, M)\}$, and take $x'\in A$ and $u\in M$. Then, for each λ, the mapping $P\to M$ defined by $x\to uf_\lambda(x)$ for $x\in P$ is a homomorphism and so $uf_\lambda(x')=0$. Therefore $f_\lambda(x')\in(0: M)$ and thus we have $x'=\sum x_\lambda f_\lambda(x')\in P(0: M)$. This shows that $A\subset P(0: M)$ and, since the reverse inclusion is clear, we have the desired equality.

Now suppose that X is a submodule of P contained in K and is $\mathrm{End}_R(P)$-allowable. Then, for each $g: P\to M$, there exists $f: P\to P$ such that $\alpha f=g$. Take $x\in X$. Since $f(x)\in X$, $g(x)=\alpha(f(x))=0$. Thus we have $x\in A=P(0: M)$. Since $P(0: M)$ is clearly a submodule of P contained in K and is $\mathrm{End}_R(P)$-allowable, this completes the proof of the proposition.

We denote $K/P(0: M)$ and $P/P(0: M)$ simply by $\bar{K}$ and $\bar{P}$ respectively. Then we have

Corollary 4.2 ([5, Corollary 3.4]). $0\to\bar{K}\to\bar{P}\xrightarrow{\bar{\alpha}}M\to 0$ *is the quasi-projective cover of* M, *where* $\bar{\alpha}$ *is the map induced by* α.

Corollary 4.3 ([6, Theorem 2.3]). *M is quasi-projective if and only if* $K=P(0\colon M)$.

Proof. Since K is small in P, by [9, Proposition 2.1 and 2.2], M ($\cong P/K$) is quasi-projective if and only if K is $\mathrm{End}_R(P)$-allowable. Hence by Proposition 4.1 this is so if and only if $K=P(0\colon M)$.

Now we shall show that the quasi-corationally completeness of M can be described by means of the radical t_P. Before we do this, we claim the following

Lemma 4.4. $t_{\bar{P}}(\bar{K})=(t_P(K)+P(0\colon M))/P(0\colon M)$.

Proof. To show this it is enough to note that, for each $\bar{f}\colon \bar{P}\to\bar{K}$, by the projectivity of P there exists $f\colon P\to K$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi} & \bar{P} \\ f\downarrow & & \downarrow\bar{f} \\ K & \xrightarrow[\pi]{} & \bar{K} \end{array}$$

is commutative, and that conversely for each $f\colon P\to K$ the mapping $\bar{f}\colon \bar{P}\to\bar{K}$ defined by $\bar{f}\pi(x)=\pi f(x)$ for $x\in P$ is a well-defined homomorphism since $\mathrm{Ker}(\pi)=P(0\colon M)$ is $\mathrm{End}_R(P)$-allowable.

Using this lemma we see that $\bar{P}/t_{\bar{P}}(\bar{K})$ is isomorphic to $P/(t_P(K)+P(0\colon M))$ and so by Corollary 2.5 we have

Corollary 4.5 ([3, Theorem 3.4]). *The exact sequence* $P/(t_P(K)+P(0\colon M))\to M\to 0$ *is maximal quasi-corational by* M.

As another application of the lemma we have

Theorem 4.6. *M is quasi-corationally complete if and only if* $K=t_P(K)+P(0\colon M)$.

Proof. By Theorem 3.2 M is quasi-corationally complete if and only if $\bar{K}=t_{\bar{P}}(\bar{K})$. Hence the theorem follows from the lemma.

Recently Bican [2, Theorem 4] has shown that M is corationally complete (in the sense that it has no proper corational extensions by M) if and only if $K=t_P(K)$. Hence by Theorem 4.6 for a module having a projective cover we see that the corationally completeness implies the quasi-corationally completeness. Concerning the reverse implication, again by Theorem 4.6, we see that a quasi-corationally complete module M is corationally complete if and only if $P(0\colon M)\subset t_P(K)$. Examples of a module which satisfies this condition are provided by a faithful module having a projective cover and any module having a projective cover over a commutative ring (cf. [3, p. 158 and Corollary 3.5]).

Finally in closing this note we give an example of a module which is quasi-corationally complete but not quasi-projective.

EXAMPLE 4.7. Let N be the Jacobson radical ($\neq 0$) of R and let N' ($\neq 0$) be a right subideal of N but not a left subideal of N. Then, since N' is not $\mathrm{End}_R(R)$-allowable, R/N' is not quasi-projective. R/N' has a projective cover $0\to N'\to R\to R/N'\to 0$ and, since $\boldsymbol{T}(t_R)=$ mod-R, it is in fact (quasi-)corationally complete.

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