ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

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Let $A$ be a commutative ring with unity. A higher derivation $\Delta = \{1, \Delta_1, \Delta_2, \cdots\}$ of $A$ is called locally finite if for any $a \in A$ there exists an index $j$ such that $\Delta_n(a) = 0$ for all $n > j$. In a previous paper some properties of locally finite iterative higher derivations (abbreviated as lfih-derivations) and some applications of them were presented ([2]). In this paper the author gives another application of lfih-derivations, i.e., a characterization of two-dimensional polynomial ring. His proof supplies an alternative proof of Theorem 1 in [3], where the method is geometric while the present one is algebraic and elementary. As a Corollary a characterization of a line in an affine plane is given in terms of lfih-derivations where a line in an affine plane is meant a curve $C$ which can be taken as a coordinate axis of $A^2$. We call a curve $C: f(x, y) = 0$ a quasi-line if the coordinate ring $k[x, y]/(f)$ is isomorphic to one-parameter polynomial ring. It is known that if the ground field is the complex number field $C$, then a quasi-line is always a line (cf. [1]). Combined with the present investigation it turns out that if the plane curve $C: f(x, y) = 0$ is a quasi-line over $C$, then the derivation $D_f = (\partial f/\partial y) \frac{\partial}{\partial x} - (\partial f/\partial x) \frac{\partial}{\partial y}$ is locally nilpotent, i.e., the higher derivation $\left(1, D_f, \frac{1}{2!}D_f^2, \cdots\right)$ is a lfih-derivation and vice versa. The direct proof of this fact is expected very much.

Let $A$ be a commutative ring with unity. A higher derivation $\Delta = \{1, \Delta_1, \Delta_2, \cdots\}$ is a set of linear endomorphisms of $A$ into itself satisfying the conditions:

$\Delta_0(ab) = \sum_{i=0}^{\infty} \Delta_i(a)\Delta_{n-i}(b)$

where $\Delta_0$ denotes the identity mapping of $A$. Let $\Phi_\Delta$ be the homomorphism of the ring $A$ into $A[[T]]$ defined by

$\Phi_\Delta(a) = \sum_{i=0}^{\infty} D_i(a)T^i$.

We say that $\Delta$ is locally finite if $I_m \Phi_\Delta$ is contained in the polynomial ring $A[T]$, i.e., for any $a \in A$, there exists an integer $j$ such that $\Delta_n(a) = 0$ for all $n > j$. $\Delta$ is called an iterative higher derivation if the additional conditions
are satisfied by $\Delta$. Let $a$ be an element of the ring $A$. We say that $a$ is a $\Delta$-constant if $\Delta_i(a) = 0$ for all $i \geq 1$. This is equivalent to saying that $\Phi_a(a) = a$. Sometimes we use the notation $\Delta^{-1}(0)$ to denote the ring of $\Delta$-constants, and $\Delta(a) = 0$ to denote $a$ being a $\Delta$-constant.

**Lemma 1.** Let $\Delta$ be a locally finite higher derivation of an integral domain $A$. Then the constant ring $B = \Delta^{-1}(0)$ is inertly embedded in $A$.

Proof. Let $b$ be an element of $B$ and let $b = cd$ be a decomposition of $b$ in $A$. Then we have $\phi(b) = \phi(c)\phi(d)$ where $\phi = \Phi^\Delta$. By assumption $\phi(b)$ is in $A$ and $\phi(c)$, $\phi(d)$ are elements of a polynomial ring $A[T]$. Hence $\phi(c)$, $\phi(d)$ are also in $A$. It means that $\phi(c) = c$ and $\phi(d) = d$, i.e., $c$ and $d$ are in $B$.

**Theorem 1.** Let $k$ be an algebraically closed field of arbitrary characteristic and let $A$ be an integral domain containing $k$. Assume that $A$ satisfies the following conditions:

i) There exists a non-trivial left derivation $\Delta$ over $k$.

ii) The constant ring $A_0$ of $\Delta$ is a principal ideal domain finitely generated over $k$.

iii) Any prime element of $A_0$ remains prime in $A$.

Then $A$ is a polynomial ring in one variable over $A_0$.

Proof. Let $A\xi$ be the set of elements $\xi$ in $A$ such that $\Delta_n(\xi) = 0$ for $n > i$. $A_0$ is the ring of $\Delta$-constants and $A_i$'s are $A_0$-modules. It is proved in [2] that there exists an integer $s \geq 0$ such that

$$A_0 = A_1 = \cdots A_{s-1} \subset A_s = \cdots = A_{2s-1} \subset A_{2s} = \cdots$$

where $\subset$ denotes proper containment. The integer $mp^s$ is called the $m$-th jump index ($m = 1, 2, \cdots$). For simplicity we set $q = p^s$ and $M_s = A_n$. It is also proved in [2] that for any element $\xi$ in $M_1$, we have

$$\phi(\xi) = \xi + \alpha_0 T + \alpha_1 T^p + \cdots + \alpha_s T^{q^s}$$

where $\alpha$'s are in $A_0$ and $\phi = \phi^\Delta$. Let $I_1$ be the set of elements in $A_0$ which appear as coefficients of $T^s$ in $\phi(\xi)$ for some $\xi \in M_1$. It is easily seen that $I_1$ is an ideal of $A_0$. Similarly let $I_s$ be the set of elements which appear as coefficients of $T^{qs}$ in $\phi(\xi)$ for some $\xi \in M_n$. Then $I_s$ is also an ideal of $A_0$. Let $a_s$ be a generator of the $I_s$ and let $x$ be an element of $M_1$ such that

$$\phi(x) = x + \cdots + a_1 T^{q^s}.$$
We shall prove simultaneously the following

\[(1)_n \quad (a_n) = (a^n),\]
\[(2)_n \quad M_n = A_0 + A_0 x + \cdots + A_0 x^n, \quad (n=1, 2, \ldots)\]

by induction on \(n\). First we shall remark that \((1)_n\) implies \((2)_n\). In fact let \(\xi\) be in \(M_n\). Then \(\Delta_{a_n}(\xi)\) is in \(I_n=(a_n)\). From \((1)_n\) it follows that there exists a constant \(c\) in \(A_0\) such that \(\Delta_{a_n}(\xi)=ca_n^q\). Then \(\phi(\xi-cx^n)\) is of degree \(<nq\), hence \(\xi-cx^n \in M_{n-1}\). Now assume \((1)_n, (2)_n\) and we shall prove \((1)_{n+1}\). Since \(a_{n+1}^q \in I_{n+1}=(a_{n+1})\), there is a constant \(c\) in \(A_0\) such that \(a_{n+1}^q = ca_{n+1}\). Let \(\xi\) be an element of \(M_{n+1}\) such that

\[\phi(\xi) = \sum_{i=0}^{n+1} a_{n+1} x^i.\]

Then \(\phi(c\xi-x^{n+1})\) is of degree \(<(n+1)q\), hence \(c\xi-x^{n+1} \in M_n\). By \((2)_n\) there are \(b_i\)'s in \(A_0\) such that

\[c\xi = x^{n+1} + \sum_{i=0}^{n+1} b_i x^i.\]

We shall show that \(c\) is a unit of \(A_0\). Assume that \(c\) is a non-unit in \(A_0\). Let \(f\) be a prime element which divides \(c\). Taking the residue class modulo \(fA\) we get an algebraic relation

\[x^{n+1} + \tilde{b}_i x^i = 0.\]

By assumption (iii) \(f\) is also a prime element of \(A\). Hence \(A/fA\) is an integral domain. Since \(k\) is algebraically closed and \(A_0\) is finitely generated over \(k\), we have \(A_0/fA_0=k\). Hence there exists \(\gamma\) in \(k\) such that \(x=\gamma\). It means that \(x-\gamma=fy\) with some \(y \in A\). Then we have \(\phi(x-\gamma)=f\phi(y)\), i.e., \(\Delta_\phi(x)=f\Delta_\phi(y)\). Since \(\Delta_\phi(y) \in I=I=(a_i)=(\Delta_\phi(x))\) we get a contradiction. Thus we have proven \((1)_{n+1}\). Since \(A= \bigcup_{n=1}^\infty M_n\), we obtain the desired result \(A=A_0[x]\).

Remark. If \(A\) is a \(UFD\), then the condition (iii) is automatically satisfied.

**Theorem 2.** Let \(k\) be as in Theorem 1, and let \(A\) be a finitely generated normal integral domain over \(k\) such that

(i) \(\dim A=2\)
(ii) \(A^*=k^*\) where \(*\) denotes the set of units.
(iii) Either \(A\) is \(UFD\) or \(\mathbb{Q}(A)\) is unirational over \(k\).

Let \(\Delta\) be a non-trivial \(\text{iff-derivation of} A\) over \(k\). Then the constant ring \(A_0\) of \(\Delta\) is a polynomial ring over \(k\). More precisely let \(f\) be an irreducible element in \(A_0\). Then \(A_0=k[f]\).

Proof. \(A_0\) is not reduced to \(k\) because there exists an element \(u\) in \(A_0\) and
a variable \( t \) over \( A_0 \) such that \( A[ u^{-1}] = A_0[ u^{-1}][t] \). (cf. Appendix, [2]). Let \( f \) be an element of \( A_0 \setminus k \) which is irreducible in \( A \). The existence of such an element \( f \) is assured by the Lemma 1. We shall show that \( A_0 = k[f] \). Since \( A_0[ u^{-1}] = A[ u^{-1}]/tA[ u^{-1}], A_0[ u^{-1}] \) is a finitely generated integral domain over \( k \). In case \( A \) is a \( UFD, A_0[ u^{-1}] \) is also a \( UFD \) owing to the Lemma 1. Moreover the transcendence degree of the quotient field \( K \) of \( A_0 \) is 1. Hence \( K \) is a purely transcendental extension of \( k \). If \( A \) is not a \( UFD \) we assumed that \( Q(A) \) is unirational. Then by the generalized Lüroth's theorem \( K \) is also a one-dimensional purely transcendental extension of \( k \). Let \( B \) be the integral closure of \( k[f] \) in \( K \). Then \( B \) is also finitely generated over \( k \) and \( B^* = k^* \) because \( B \) is contained in \( A \). Hence there exists an element \( t \) in \( B \) such that \( B = k[t] \). Since \( f \) is contained in \( B \) we can write \( f = \lambda(t) \). But \( f \) is irreducible in \( A \), hence degree of \( \lambda \) in \( t \) must be 1. It proves that \( k[t] = k[f] = B \). Now assume \( A_0 \neq B \). Since \( A_0 \) and \( B \) have the same quotient field, \( A_0 \) contains an element of the form \( \gamma(f)/s(f) \) where \( (\gamma(f), s(f)) = 1 \) and \( \deg s(f) \geq 1 \). Then \( A_0 \) must contain a non-constant unit. This is against the assumption (ii).

Combining these theorems we have the following

**Theorem 3.** Let \( k \) be an algebraically closed field of arbitrary characteristic and let \( A \) be a finitely generated integral domain over \( k \). Assume that \( A \) satisfies the following conditions:

(i) \( \dim A = 2 \)
(ii) \( A^* = k^* \)
(iii) \( A \) is \( UFD. \)

Assume that \( A \) has a non-trivial \( k \)-derivation \( \Delta \) over \( k \). Then \( A \) is a two-dimensional polynomial ring over \( k \). More precisely if the constant ring \( A_0 \) of \( \Delta \) is written as \( k[f] \), then \( A = k[f, g] \) for some other element \( g \) in \( A \).

The assumption (iii) is essential as is shown in the following

**Example 1.** Let \( A = C[x, y, \frac{y(y-1)}{x}] \). Then as is easily seen \( A^* = C^* \) and \( A \) has a locally nilpotent derivation \( D \) such that

\[
Dx = 2y - 1, \quad Dy = \frac{y(y-1)}{x}.
\]

By a simple calculation we see \( D^{-1}(0) = k[\frac{y(y-1)}{x}] \). The element \( \frac{y(y-1)}{x} \) is not a prime element in \( A \). Hence \( A \) is neither \( UFD \) nor a polynomial ring.

\((*) \) This example is due to K. Yoshida.
As an application of Theorem 3 we give a necessary and sufficient condition for a plane curve $C: f(x, y)=0$ to be a line. We recollect here some definitions. A plane curve $C: f(x, y)=0$ defined over a field $k$ is called a quasi-line over $k$ if the coordinate ring $A=k[x, y]/(f)$ is isomorphic to a polynomial ring in one variable. $C$ is called a line if there exists another curve $\Gamma: g(x, y)=0$ such that we have $k[x, y]=k[f, g]$.

**Theorem 4.** Let $k$ be an algebraically closed field and let $C: f(x, y)=0$ be an irreducible curve over $k$. Then the following conditions are equivalent to each other.

(i) $C$ is a line
(ii) There is a $\delta^{ih}$-derivation $\Delta$ such that $\Delta(f)=0$.
(iii) $C_u: f(x, y)—u=0$ is a quasi-line over $k(u)$ where $u$ is an indeterminate.

Proof. The implication (i)$\implies$(ii), (i)$\implies$(iii) is obvious (ii)$\implies$(i) follows from Theorem 2 and 3. It remains to show that (iii) implies (i). Assume (iii). Since $k(u)[x, y]/(f—u)$ is isomorphic to $k(f)[x, y]$, there exists an element $t$ in $k[x, y]$ such that $k(f)[x, y]=k(f)[t]$. Let $\Delta'$ be the $\delta^{ih}$-derivation of $k(f)[t]$ over $k(f)$ such that

$$\Delta'(t^n) = \binom{m}{n}t^{n-s}$$

Then there exists an element $a$ in $k[f]$ such that $a\Delta'=\Delta$ sends $k[x, y]$ into itself, where $a\Delta'$ is higher derivation

$$a\Delta' = (1, a\Delta', a^2\Delta', \ldots, a^s\Delta', \ldots).$$

Clearly $\Delta(f)=0$ and $f$ is a prime element in $k[x, y]$. Hence $\Delta^{-1}(0)=k[f]$ and by Theorem 3, $f$ is a line.

In case where the characteristic of $k$ is zero we can say more. First we prove a Lemma.

**Lemma 2.** Let $C: f(x, y)=0$ be a line in a plane. Then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}=1$.

Proof. Since $C$ is a line, there exists a curve $\Gamma: g(x, y)=0$ such that $k[x, y]=k[f, g]$. Then there exists $F(X, Y)$ and $G(X, Y)$ in $k[X, Y]$ such that

$$F(f, g) = x$$
$$G(f, g) = y.$$
\[
\frac{\partial F}{\partial f} \frac{\partial f}{\partial x} + \frac{\partial F}{\partial g} \frac{\partial g}{\partial x} = 1 \hspace{1cm} (1)
\]
\[
\frac{\partial F}{\partial f} \frac{\partial f}{\partial y} + \frac{\partial F}{\partial g} \frac{\partial g}{\partial y} = 0 \hspace{1cm} (2)
\]
\[
\frac{\partial G}{\partial f} \frac{\partial f}{\partial x} + \frac{\partial G}{\partial g} \frac{\partial g}{\partial x} = 0 \hspace{1cm} (3)
\]
\[
\frac{\partial G}{\partial f} \frac{\partial f}{\partial y} + \frac{\partial G}{\partial g} \frac{\partial g}{\partial y} = 1 \hspace{1cm} (4)
\]

Now assume \((\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \subseteq m\) for some maximal ideal \(m\). Then from (2) either \(\frac{\partial F}{\partial g}\) or \(\frac{\partial g}{\partial y}\) is contained in \(m\). The first case cannot occur because of (1) and the second case contradicts (4). Thus \((\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\) is a unit ideal.

**Theorem 5.** Let \(k\) be an algebraically closed field of characteristic zero and let \(C: f(x, y)=0\) be an irreducible curve over \(k\). Then \(C\) is a line if and only if the derivation

\[
D_f = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}
\]

is locally nilpotent.

**Proof.** Assume that \(C: f(x, y)=0\) is a line. Let \(\Gamma: g(x, y)=0\) be a curve such that \(k[f, g]=k[x, y]\). Then there exists a locally nilpotent derivation \(\Delta\) of \(k[x, y]\) such that \(\Delta f=0\) and \(\Delta g=1\). Since \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\) form a basis of derivations of \(k[x, y]\) we can write

\[
\Delta = a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y}
\]

with \(a, b \in k[x, y]\).

Since \(\Delta f=0\) we have

\[
a \frac{\partial f}{\partial x} - b \frac{\partial f}{\partial y} = 0 \hspace{1cm} (1)
\]

Let \(a \frac{\partial f}{\partial y} = b \frac{\partial f}{\partial y} = \lambda\), i.e., \(a=\lambda \frac{\partial f}{\partial y}, b=\lambda \frac{\partial f}{\partial x}\). Then we have \(\Delta = \lambda D_f\).

We show that \(\lambda \in k[x, y]\). From Lemma 2 it follows that \(\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 1\) for some \(\alpha, \beta\) in \(k[x, y]\). Hence \(\lambda=b\alpha+a\beta \in k[x, y]\). On the other hand the existence of \(g \in k[x, y]\) such that \(\Delta g=1\) implies \((a, b)=1\). Since \(\lambda\) is a common
divisor of $a$ and $b$ we see that $\lambda \in k^*$. This means that $D_f$ is locally nilpotent. The "if" part of the Theorem is immediate from Theorem 3.

According to S. Abhyankar and T. Moh a quasi-line is a line in case of characteristic zero ([1]). In the case where the characteristic of $k$ is a positive prime integer $p$ there is a counter example.

**Example 2(***) A curve $C$: $f(x, y)=0$ such that

$$f(x, y) \equiv y^p - x - x^p$$

is a quasi-line but not a line where $p$ is the characteristic of $k$ and $q$ is an integer $\geq 2$ not divisible by $p$.

**Proof.** If we set

$$u = y - (y^p - x^p)^q$$

then $x \equiv u^p$ and $y \equiv u + u^p$ modulo $f(x, y)$. Hence $f(x, y)=0$ is a quasi-line. To see that $c$ is not a line it suffices to show that there is no locally finite higher derivation killing $f$. Assume the contrary and let $\Delta$ be a lfih-derivation killing $f$ and $\phi = \Phi_\Delta$. Let

$$\phi(x) = x + \sum a_i T^i$$

$$\phi(y) = y + \sum b_i T^i.$$ 

From $\phi(f) = f$ we get

$$(y^p + \sum b_i^p T^{pi}) - (x + \sum a_i T^i) - (x^p + \sum a_i^p T^{pi}) q = y^p - x - x^p \quad \cdots (1)$$

First we easily see that $a_i = 0$ if $i \equiv 0 \pmod{p}$. We set $a_{pi} = \alpha_i$. Then we have

$$(y^p + \sum b_i^p T^{pi}) - (x + \sum \alpha_i T^{pi}) - (x^p + \sum \alpha_i^p T^{pi}) q = y^p - x - x^p$$

First we remark that

$$\alpha_i \in A^p$$

for any $i$ where $A = k[x, y]$. Now assume that $n \geq 1$. We compute the coefficient of $T^{p^n(q-1)}$. Since $T^{p^n(q-1)}$ does not appear in the middle term we have the relation:

$$b_{pi(q-1)} = \sum \alpha_1^i \cdots \alpha_q^i + qx^q \alpha_n^{(q-1)}$$

From (2) $\alpha_1^i \cdots \alpha_q^i, \alpha_n^{(q-1)}$ are in $A^p$. Hence $x^q$ must also be in $A^p$. This is

(***) This example is a generalization of the one given in [4].
impossible. This proves \( n=0 \), i.e., \( x \) must be a \( \Delta \)-constant. Hence \( y \) is also a \( \Delta \)-constant. Thus there is no non-trivial \( \delta \)-derivation \( \Delta \) such that \( \Delta(f)=0 \).

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References


Added in Proof. In Theorem 3 we assumed that \( k \) is algebraically closed. This assumption is essential as is shown in the following Example. Let 
\[ B = \mathbb{R}[X, Y]/X^2 + Y^2 + 1. \] Then \( B \) is a UFD and satisfies \( B^*=\mathbb{R}^* \). The ring \( A=B[Z] \) satisfies all the requirement in Theorem 3, but \( A \) is not a polynomial ring of two variables over the field \( \mathbb{R} \).