Tsushima, Y.
Osaka J. Math.
15 (1978), 647-653

SOME NOTES ON THE RADICAL OF A FINITE GROUP RING

YUKIO TSUSHIMA

(Received September 6, 1977)

1. Introduction

Let p be a prime number and G a finite group with a Sylow p-subgroup P of order p^a. Let be $\mathfrak{R}$ the radical of the group ring kG of G taken over a field k of characteristic p. If $\mathfrak{Z}$ is the radical of the center of kG, then we see easily that $kG\cdot\mathfrak{Z}\subset\mathfrak{R}$. We shall show that $\mathfrak{R}=kG\cdot\mathfrak{Z}$ holds if and only if G is p-nilpotent and P is abelian.

The nilpotency index of $\mathfrak{R}$, which is denoted by $t(G)$, is the smallest integer t such that $\mathfrak{R}^t=0$. Suppose G is p-solvable, then it is known that $a(p-1)+1\leqq t(G)\leqq p^a$ (Passman [11], Tsushima [12], Wallace [16]). Furthermore if G has the p-length one, it holds that $t(G)=t(P)$ (Clarke [2]). We see easily from this that the first equality holds in the above if P is elementary, while the second holds if P is cyclic. However the equality $t(G)=a(p-1)+1$ does not necessarily imply that P is elementary, as is remarked by Motose (e.g. $G=S_4$ $p=2$, see Ninomiya [10]). In contrast with this, we shall show that if $t(G)=p^a$, then P is cyclic.

NOTATION: p is a fixed prime number. G is always a finite group and P a Sylow p-subgroup of order p^a. As usual, $|X|$ denotes the cardinality of a set X. Let K be an algebraic number field containing the $|G|$-th roots of unity and $\mathfrak{o}$ the ring of integers in K. We fix a prime divisor $\mathfrak{p}$ of p in $\mathfrak{o}$ and we let $k=\mathfrak{o}/\mathfrak{p}$. We denote by $\{\varphi_1, \cdots, \varphi_r\}$ and $\{\eta_1, \cdots, \eta_r\}$ the set of irreducible Brauer characters and principal indecomposable Brauer characters of G respectively, in which the arrangement is such that $(\eta_i, \varphi_j)=\delta_{ij}$ and φ_1 is the trivial character. We put $s(G)=\sum_{i=1}^{r}\varphi_i(1)^2$.

For a block B of kG, we denote by δ_B and ψ_B its block idempotent and the associated linear character respectively. $\mathfrak{R}(G)$ (or $\mathfrak{R}$ for brevity) denotes the radical of the group ring kG and $\mathfrak{Z}$ the radical of the center of kG. The nilpotency index of $\mathfrak{R}(G)$, which will be denoted by $t(G)$, is defined to be the smallest integer t such that $\mathfrak{R}(G)^t=0$. If $G\triangleright H$, then $kG\cdot\mathfrak{R}(H)=\mathfrak{R}(H)\cdot kG$ is a two sided ideal of kG contained in $\mathfrak{R}$, which will be denoted by $\mathfrak{L}_H$ (or $\mathfrak{L}$ for brevity). Other notations are standard.

We shall several times refer to the following Theorem of Green (Green [7], Dornhoff [4] § 52).

Theorem. *Let $G \triangleright H$ and G/H is a p-group.*

If V is a finitely generated absolutely indecomposable kH-module, then V^G is also absolutely indecomposable.

2. Square sum of the degrees of irreducible characters

In this section, we mention some remarks about the dimension of $\mathfrak{N}=\mathfrak{N}(G)$, most of which are direct consequences of our results [14].

Let S be the set of the p-elements of G and $c=\sum_{x\in S} x\in kG$. In [14], we have shown that $\mathfrak{N}\subset(0\colon c)$ and we have the equality provided G is p-solvable. For $\lambda=\sum_{x\in G} a_x x\in kG$, $a_x\in k$, we put $\sigma_p(\lambda)=\sum_{x\in S} a_x$. Note that $\sigma_p(\lambda)$ is the coefficient of the identity in $c\lambda$. Hence $c\lambda=0$ if and only if $\sigma_p(\lambda x)=0$ for any $x\in G$, or

$$(0\colon c) = \{\lambda\in kG \mid \ \sigma_p(x\lambda) = 0 \textit{ for any } x\in G\} \qquad \cdots\cdots\cdots(1)$$

Therefore, our result quoted above is written as

Proposition 1. *If $\lambda\in\mathfrak{N}$, then $\sigma_p(x\lambda)=0$ for any x of G.*

We next discuss the dimension of $(0\colon c)$. Let $M=M_G=(a_{g,h})$ be the $(|G|, |G|)$-matrix over k defined as

$$a_{g,h} = \begin{cases} 1, & \text{if } gh \text{ is a } p\text{-element} \\ 0, & \text{otherwise} \end{cases}$$

Then, we have

$$\mathit{dim}_k(0\colon c) = |G| - r(M), \textit{ where } r(M) \textit{ denotes the rank of } M \textit{ over } k. \cdots\cdots(2)$$

Indeed, for $\lambda=\sum a_x x\in kG$, we have $\sigma_p(x\lambda)=\sum_{y\in x^{-1}S} a_y$, that is $M\begin{pmatrix}\vdots\\ a_x\\ \vdots\end{pmatrix}=\begin{pmatrix}\vdots\\ \sigma_p(x\lambda)\\ \vdots\end{pmatrix}$ for $x\in G$. From this and (1), we get easily (2).

Furthermore from that $\mathfrak{N}\subset(0\colon c)$ and (2), we have

$$s(G) = |G| - \mathit{dim}_k N \geqq r(M) \qquad \cdots\cdots\cdots(3)$$

If H is a subgroup of G, then M_H appears in M_G as a submatrix. In particular $r(M_G)\geqq r(M_H)$. Now, recall that we have $\mathfrak{N}=(0\colon c)$ and hence $s(G)=r(M)$ provided G is p-solvable. Summarizing the aboves, we have

Proposition 2. *If G is p-solvable, then we have $s(G)\geqq s(H)$ for any subgroup H of G.*

REMARK 1. If H is a p'-subgroup, then $r(M_H)=|H|$. Hence we have from (3) that $s(G)\geqq |H|$ for any p'-subgroup H of G, which has been shown in Brauer and Nesbitt [1] by the inequalities $s(G)\geqq\frac{|G|}{u}\geqq |H|$, where $u=\eta_1(1)$.

In connection with the above remark, we give the following, which is essentially due to Wallace [15].

Proposition 3. *We have $s(G)=|H|$ for some p'-subgroup H of G if and only if $G\triangleright P$, in which case H is necessary a complement of P in G.*

Proof. "if part" is well known and easily shown (Curtis and Reiner [3] § 64 Exercise 1).

Suppose $s(G)=|H|$ for some p'-subgroup H of G. Then we have $s(G)=\frac{|G|}{u}$, which forces that $\eta_i=\varphi_i\eta_1$ for any i $(1\leqq i\leqq r)$ (see [1] pp. 580). We claim that $u=p^a$. If this would be shown, then H is necessary a complement of P and $\eta_1(x)$ is rational for any $x\in G$. Then the argument of Wallace [15] is valid, concluding $G\triangleright P$ (see also M.R. 22 ♯ 12146 No. 12 (1966)).

Let

$$\theta(x)=\begin{cases} p^a & \text{if } x \text{ is } p\text{-regular} \\ 0 & \text{otherwise} \end{cases}$$

As is well known, θ is an integral linear combination of η_i's: $\theta=\sum m_i\eta_i=\eta_1\sum m_i\varphi_i$, where each m_i is a rational integer. Comparing the degrees of both sides, we get $u=p^a$ as claimed. This completes the proof.

3. LC type

For convenience, we call a (finite dimensional) algebra A over a field to be LC if its (Jacobson) radical is generated over A by the radical of its center.

The objective of this section is to prove

Theorem 4. *The followings are equivalent to each other.*

(1) *kG is LC*

(2) *the principal block B_0 of kG is LC*

(3) *G is p-nilpotent and P is abelian*

"(1)$\Rightarrow$(2)" is trivial. On the other hand, we have already shown "(3)$\Rightarrow$(1)" in [13] assuming P is cyclic. The same argument, being simplified by virtue of the Green's Theorem quoted in the introduction, will be made below to yeild the present assertion.

We begin with

Lemma 5. *Let $G\triangleright H$ and b a block of kH. Let $B_1, \cdots, B_s$ be the blocks* *of kG which cover b. If a defect group of each B_i is contained in H, then we have $\mathfrak{N}B_i=\mathfrak{L}B_i$ for each i $(1\leqq i\leqq s)$.*

Proof. Let $b_1, \cdots, b_t$ be the blocks of kH which are conjugate to b under G and ε_i the block idempotent of b_i.

From the choice of B_j's we have

$$\varepsilon=\varepsilon_1+\cdots+\varepsilon_t=\delta_1+\cdots+\delta_s\,, \qquad \text{where } \delta_i=\delta_{B_i}$$

Let $\Lambda=kG\varepsilon/\mathfrak{L}\varepsilon\supset\Gamma=kH\varepsilon/\mathfrak{N}(H)\varepsilon$. We show that Λ is semisimple. Let M be a Λ-module and N any submodule of M. The inclusion map $N\rightarrow M$ splits as Γ-modules, since Γ is semisimple and then it does as Λ-modules, since M is (G, H) projective by the assumption. Therefore Λ is semisimple and our assertion is clear.

The following remark is useful.

REMARK 2.

(1) (well known) If G/H is a p'-group, then the assumption of Lemma 5 is always satisfied and hence we have $\mathfrak{N}=\mathfrak{L}_H$.

(2) (Feit [5] pp. 268) If G/H is a p-group, then there is a unique block which covers b.

The following Lemma is not so essential here, but we write down it for its own interest. The result is noticed by Y. Nobusato.

Lemma 6. *Suppose $G\triangleright H$ and G/H is a p-group. Then for any simple kH-module N, N^G has the composition length $[I\colon H]$, where I is the inertia group of N in G.*

Proof. Clear from the Green's Theorem and the orthogonality relations $(\eta_i, \varphi_j)=\delta_{ij}$.

The following result has been shown in our previous paper [13].

Lemma 7. *Let $G\triangleright H$ and $[G\colon H]=p$. Let B be a block of kG. Suppose there is a conjugate class C of G such that $C\not\subset H$ and $\psi_B(\tilde{C})\neq 0$, where $\tilde{C}=\sum_{x\in C}x$. Then, we have $\mathfrak{N}B=\mathfrak{L}B+kG(\tilde{C}-\psi_B(\tilde{C}))\delta_B$.*

Proof. We put $\delta=\delta_B$ and $\psi=\psi_B$ for brevity. Let $\delta=\sum e$ be a decomposition into the sum of primitive idempotents. We may assume each e is contained in kH by the Green's Theorem. It suffices to show that $\mathfrak{N}e=\mathfrak{L}e+kG(\tilde{C}-\psi(\tilde{C}))e$. Let $a\in G$ be any element not contained in H. We have

$$(\tilde{C}-\psi(\tilde{C}))^{p-1}e=a^{p-1}\lambda_1+\cdots+a\lambda_{p-1}-\psi(\tilde{C})^{p-1}e\,, \qquad \text{where } \lambda_i\in kH\,.$$

Since $\psi(\tilde{C})\neq 0$, this implies that $(\tilde{C}-\psi(\tilde{C}))^{p-1}e$ is not contained in $\mathfrak{L}e=a^{p-1}\mathfrak{N}(H)e\oplus\cdots\oplus\mathfrak{N}(H)e$. Therefore we have a sequence (note that $(\tilde{C}-\psi(\tilde{C}))\delta\in\mathfrak{N}$)

$kG\bar{e} \supsetneqq (\tilde{C}-\psi(\tilde{C}))kG\bar{e} \supsetneqq \cdots \supsetneqq (\tilde{C}-\psi(\tilde{C}))^{p-1}kG\bar{e} \supsetneqq 0$, where $kG\bar{e}=kGe/\mathfrak{L}e \cong kG\otimes_{kH}kHe/\mathfrak{N}(H)e$.

However, since $kG\bar{e}$ has at most p composition factors by Lemma 6, we have $(\tilde{C}-\psi(\tilde{C}))kG\bar{e}=\mathfrak{N}\bar{e}$, that is $\mathfrak{N}e=\mathfrak{L}e+kG(\tilde{C}-\psi(\tilde{C}))e$ as required. This completes the proof.

Before proceeding, we mention a remark. If B is a block of kG of full defect, then there is an ordinary irreducible character χ belonging to B whose degree is not divisible by p. If x is a p-element, then $\chi(x)\equiv\chi(1) \bmod \mathfrak{p}$. Hence it follows that if C is a conjugate class of a p-element, then $\psi_B(\tilde{C})=|C|$.

The following proposition proves "(3)$\Rightarrow$(1)" of Theorem 4.

Proposition 8. *Suppose G is p-nilpotent and P is abelian. Let $\{C_1,\cdots,C_v\}$ be the set of the conjugate classes of p-elements of G. For a (normal) subgroup H of G containing $O_{p'}(G)$, let Δ_H be the sum of the block idempotents of kH of full defect and for any C_i such that $C_i\subset H$, let $\Delta(C_i, H)=(\tilde{C}_i-|C_i|)\Delta_H$.*

Then we have $\mathfrak{N}=\sum_{i,H} kG\Delta(C_i, H)$, where H is taken over the subgroups of G containing $O_{p'}(G)$. In particular, kG is LC.

Proof. Let B be any block of kG. If B has the defect smaller than a, then there is a normal subgroup H of index p which contains a defect group of B. Then by Lemma 5 and Remark 2, we have $\mathfrak{N}B=\mathfrak{L}_H B$. On the other hand, assume B has full defect. Let H be any normal subgroup of G of index p. There is some C_i such that $C_i\not\subset H$ and $\psi_B(\tilde{C}_i)=|C_i|\neq 0$, since P is abelian. Hence by Lemma 7, we have $\mathfrak{N}B=\mathfrak{L}_H B+kG(\tilde{C}_i-|C_i|)\delta_B$. From the aboves, we have $\mathfrak{N}=\sum_H \mathfrak{L}_H+\sum_{i=1}^{v} kG\Delta(C_i, G)$, where H is taken over the normal subgroups of G of index p and thus the result will follow by the induction on the order of G (note that if $H\supset C_i$, where $H\supset O_{p'}(G)$, then C_i is also a conjugate class of H).

We next go into the proof of "(2)$\Rightarrow$(3)".

Lemma 9. *Let I be the augumentation ideal of kG and δ_0 the block idempotent of the principal block B_0 of kG. If $I\mathfrak{N}\delta_0=\mathfrak{N}I\delta_0$, then G is p-nilpotent.*

Proof. Let e be a primitive idempotent of kG such that $kGe/\mathfrak{N}e$ is the trivial G-module. It is easy to see that $Ie=\mathfrak{N}e$. Hence we have $I\mathfrak{N}e=I\mathfrak{N}\delta_0 e=\mathfrak{N}I\delta_0 e=\mathfrak{N}Ie=\mathfrak{N}^2e$. Recurring this, we get $I\mathfrak{N}^s e=\mathfrak{N}^{s+1}e$ for any $s\geqq 0$. This implies that G acts trivially on each factor of the series,

$kGe\supset\mathfrak{N}e\supset\cdots\supset\mathfrak{N}^s e=0$, in other words, kGe has the only (non isomorphic) simple constituent, the trivial one. Hence G is p-nilpotent.

Lemma 10. *Suppose G is a p-group. If kG is LC, then G is abelian.*

Proof. We prove by the induction on the order of G. It is clear that if kG is LC, then $k(G/H)$ is also LC for any normal subgroup H of G.

Let Z be the center of G and let z be an element of Z of order p. We may assume $G/\langle z\rangle$ is abelian by the induction hypothesis. Assume G is not abelian. Then we have $G'=[G, G]=\langle z\rangle$. Since $|gG'|=p$, gG' is the conjugate class of g unless g is central. Therefore, $\mathfrak{Z}$ is spanned over k by the set $\{u-1, x\sigma\,|\,u\in Z, x\in G-Z\}$, where $\sigma=\sum_{x\in G'} x$. Let $t=t(Z)$ be the nilpotency index of $\mathfrak{N}(Z)$. We show that $\mathfrak{Z}^t=0$. This will be deduced from the following observations.

(1) $x\sigma\cdot y\sigma=xy\sigma^2=0$.

(2) $(x\sigma)\prod_{i=1}^{t-1}(z_i-1)\in(x\sigma)\mathfrak{N}(Z)^{t-1}=(x\sigma)k\tau=0$, where $\tau=\sum_{z\in Z} z$. In fact, $\mathfrak{N}(Z)^{t-1}=k\tau$, as is easily shown (for any p-group Z) and $\sigma\tau=p\tau=0$, since $G'\subset Z$.

(3) $\prod_{i=1}^{t}(z_i-1)=0$, since $t=t(Z)$, where $z_1, \cdots, z_t$ are arbitrary elements of Z.

Now, from the assumption, we conclude that $\mathfrak{N}^t=0$. Take $y\in G-Z$. Then $(y-1)\tau$ is not zero and is contained in $(y-1)\mathfrak{N}(Z)^{t-1}\subset\mathfrak{N}^t=0$, a contradiction. This completes the proof.

Proof of "(2)⇒(3)". Let $\delta_0=\delta_{B_0}$. Since by the assumption $\mathfrak{N}\delta_0$ is generated by central elements over kG, we have $\mathfrak{N}I\delta_0=\mathfrak{N}I\delta_0$ and hence G is p-nilpotent by Lemma 9. In particular, B_0 is isomorphic to $k(G/O_{p'}(G))\cong kP$. Hence kP is also LC, implying P is abelian by Lemma 10. This completes the proof of Theorem 4.

4. Application of a result of Clarke

In this section we shall show,

Theorem 11. *Suppose G is p-solvable. If $t(G)=p^a$, then P is cyclic.*

To prove this, the following Theorem is essential.

Theorem (Clarke [2]). *If G is a p-solvable group of p-length one, then $t(G)=t(P)$.*

Proof (of Theorem 11). We prove by the induction on the order of G. If G is a p-group, then our result follows from the Theorem 3.7 of Jennings [9]. If G has a proper normal subgroup H of index prime to p, then we have $\mathfrak{N}=\mathfrak{L}_H$ and the result follows from the induction hypothesis on H. Hence we may assume G has no proper normal subgroup of index prime to p. Furthermore, by the Theorem of Clarke, it suffices to show that G is p-nilpotent.

Let H be a normal subgroup of index p. Since $\mathfrak{N}^p\subset\mathfrak{L}_H$ ([11] or [12]), we find $t(H)=p^{a-1}$. Hence a Sylow p-subgroup Q of H is cyclic by the induction hypothesis. In particular H has the p-length one. Let $K=O_{p'}(G)=O_{p'}(H)$. Then $G/K\triangleright QK/K=O_p(H/K)$. Now, assume $G\neq PK$. Then we have $O_p(G/K)=QK/K$ and $C_{G/K}(QK/K)=QK/K$, as is well known (Hall and Higman [8]).

Therefore, G/QK is isomorphic to a subgroup of $Aut(QK/K)$, whence G/QK is abelian, since the automorphism group of a cyclic group is abelian. Since we have assumed that G has no normal subgroup of index prime to p, G/QK must be a p-group, contradicting that $G \neq PK$. This completes the proof.

OSAKA CITY UNIVERSITY

References

[1] R. Brauer and C. Nesbitt: *On the modular characters of groups*, Ann. of Math. **42** (1941), 556–590.
[2] R.J. Clarke: *On the radical of the group algebra of a p-nilpotent* group, J. Australian Math. Soc. **13** (1972), 119–123.
[3] C.W. Curtis and I. Reiner: Representation theory of finite groups and associative algebras, Wiley, New York, 1962.
[4] L. Dornhoff: Group representation theory Part B, Marcel Dekker, New York, 1972.
[5] W. Feit: Representations of finite groups, Lecture notes, Yale University.
[6] D. Gorenstein: Finite groups, Harper and Row, New York, 1968.
[7] J.A. Green: *On the indecomposable representation of a finite group*, Math. Z. **70** (1959), 430–445.
[8] P. Hall and G. Higman: *On the p-length of p-solvable groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. **6** (1965), 1–42.
[9] S.A. Jennings: *The structure of the group ring of a p-group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185.
[10] Y. Ninomiya: *On the nilpotency index of the radical of a group algebra*, Symposium on algebra (1974) (in Japanese).
[11] D.S. Passman: *Radicals of twisted group rings*, Proc. London Math. Soc. **20** (1970), 409–437.
[12] Y. Tsushima: *Radicals of group algebras*, Osaka J. Math. **4** (1967), 179–182.
[13] ———: *A group algebra of a p-solvable group*, *ibid*, **5** (1968), 89–98.
[14] ———: *On the annihilator ideals of the radical of group algebra*, *ibid*, **8** (1971), 91–97.
[15] D.A.R. Wallace: *On the radical of a group algebra*, Proc. Amer. Math. Soc. **12** (1961), 133–137.
[16] ———: *Lower bounds for the radical of the group algebra of a p-soluble group*, Proc. Edinburgh Math. Soc. **16** (1968/69), 127–134.

Added in proof.
Lemma 5 has been obtained in

[17] R. Knörr: *Blocks, vertices and normal subgroups*, Math. Z. **148** (1976), 53–60.