Harada, K.
Osaka J. Math.
15 (1978), 637-646

ON YOSHIDA'S TRANSFER

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(Received August 8, 1977)

1. Introduction

An important new work on transfer has recently been done by Yoshida [3]. The author discussed some of Yoshida's main results together with some applications of them at the Duluth Conference (Duluth, Minn., Aug., 1976). This paper is based on the lecture given by him at the conference.

In his paper [3], Yoshida introduced the following important notion.

DEFINITION. Let p be a prime. Let G be a finite group, H a subgroup, K a normal subgroup of H and let x be an element of G. The quadruple (G, H, K, x) is said to be *singular* if the following conditions hold:

(a) $|H: K|=p$,

(b) x is a p-element,

(c) if V is the transfer from G to H, then $V(x)\not\equiv 1$, mod K, and

(d) no element of $H-K$ is conjugate in G to an element of $\langle x^p\rangle$.

Moreover, if $G\neq H$, then the quadruple (G, H, K, x) is said to be *a proper singularity*. H is called *a singular subgroup* of G and x *a singular element*.[1)]

One of the main results due to Yoshida is:

Theorem. *If a p-group P has a proper singularity, then P is homomorphic to the wreathed product* $Z_p \wr Z_p$.

The main result of this paper (Theorem 9) is to classify all quadruples (P, S, M, x) with P a 2-group and $|P: S|\geq 4$. We end the paper with an application of Theorem 9 to a very special case.

Yoshida first introduced his transfer argument by using character theory. M. Isaac, however, has observed that one could obtain most of Yoshida's main theorems without character theory. Some of the proofs of Yoshida's theorems quoted in this paper are based on his note (unpublished) circulated at the Duluth Conference.

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1) Setting (a′) H/K *is cyclic p-group*, Yoshida calls a quadruple (G,H,K,x) a weak singularity if it satisfies (a)′, (b), (c) and (d).

2) This research was supported in part by NSF Grant MCS 76-07253.

2. Yoshida's transfer theorems

For convenience of the reader, we collect here some of Yoshida's transfer theorems.

NOTATION. Let G be a finite group, H a subgroup of G and let T be a left coset representatives for H; i.e., $G=\sum_{t\in T} Ht$. Every element x of G can be expressed uniquely as $x=ht$ where $h\in H$, $t\in T$. We define a map η_T from G to H by $\eta_T(x)=h$. Set

$$V_{G\to H}(x)=\prod_{t\in T}\eta_T(tx)H',$$

then $V_{G\to H}=V_{G\to H/H'}$ is a homomorphism. More generally, if $K\supseteq H'$, then $V_{G\to H/K}$: $x\to V_{G\to H}(x)K$ is a homomorphism. Write $H'(p)=O^p(H)H'$.

Unless otherwise stated, all results in this section are due to Yoshida [3]. Some of them are slightly generalized or specialized. Some of the proofs are revised by Isaac, Glauberman [1] or the author.

Lemma 1. *Let G be a finite group and let H be a subgroup of G with $p\not| |G: H|$. Let $X=V_{G\to H}(G)H'(p)$. Then $H/H'(p)=X/H'(p)\times H\cap G'(p)/H'(p)$.*

Proof. ([3, Lemma 2.4]).

Corollary 2. *Under the same condition as in Lemma* 1, *if $G^pG'\cap H=H^pH'$, then $G'(p)\cap H=H'(p)$ and $XH'(p)=H$.*

(This is the "first" part of Tate theorem.)

Lemma 3. (Mackey decomposition theorem for transfer.) *Let H, K be subgroups of a finite group G and let T be a set of representatives for the (H, K) double cosets of G*; i.e., $G=\sum_{t\in T} HtK$. *Let $x\in K$, then*

$$V_{G\to H}(x)\equiv\prod_{t\in T} t(V_{K\to K\cap t^{-1}Ht}(x))t^{-1}, \qquad \mathrm{mod}\, H'.$$

Proof. ([3, Lemma 2.3] or [1, Proposition 6.2]).

Lemma 4. *Let P be a p-group, A a normal elementary abelian subgroup of P, $x\in A$, and let $y\in P-A$. Suppose $|P: A|=p$ and $\prod_{i=0}^{p-1} y^{-i}xy^i\neq 1$. Then P is homomorphic to $Z_p \wr Z_p$.*

Proof. ([3, Lemma 3.5] or [1, Lemma 6.4]).

Lemma 5. *Let P be a p-group and let (P, S, M, x) be a proper singularity. Then S contains a conjugate of x. Moreover for every maximal subgroup A of P containing S, the following holds:*

(i) *$x\in A$, and*

(ii) *if $y\in P-A$, then* $V_{P\to S}(x)\equiv \prod_{i=0}^{p-1} x^{y^i}\equiv [x, y, \overset{p-1}{\cdots}, y]\not\equiv 1$, *mod* $\Phi(A)$.

Proof. ([3, Lemma 3.8(3) and 3.9]).

REMARK. Lemmas 4 and 5 yield the theorem of Yoshida mentioned in the introduction.

DEFINITION. *Let H, Q be subgroups of a finite group G. Suppose that $H\supseteq Q$. Q is said to be of Sylow type in H (with respect to G) if for $g\in G$, $Q^g\subseteq H$ implies $Q^g=Q^h$ with $h\in H$.*

Theorem 6. *Let H be a subgroup of a finite group G such that $p\not|\,|G: H|$. Let Q be a subgroup of a Sylow p-subgroup P of H. Suppose that Q is of Sylow type in H and $H\supseteq N(Q)$. Suppose moreover that $G'(p)\cap H\supset H'(p)$. Then*
(a) *for every maximal subgroup P_1 of P such that $|H: P_1V_{G\to H}(G)H'(p)|=p$ and for every element x of minimal order in $P-P_1$, there exists a proper singularity $(\langle x, Q\rangle, S, M, x)$.*
(b) *with $(\langle x, Q\rangle, S, M, x)$ as in* (a), *for every maximal subgroup A of $\langle x, Q\rangle$ containing S and for every element $y\in\langle x, Q\rangle-A$, there exists an element $g\in G$ and an integer k such that* $([x, y, \overset{p-1}{\cdots}, y]^k)^g\in P-P_1$.

Proof. Set $X=V_{G\to H}(G)H'(p)$. By assumption, H/X is nontrivial p-group. Choose a maximal subgroup of P satisfying $|H: P_1X|=p$ and an element x of minimal order in $P-P_1$. Set $K=P_1X$. We have $V_{G\to H}(x)\subseteq K$. By Lemma 3, $V_{G\to H}(x)\equiv \prod_{t\in T} t(V_{R\to R\cap H^t}(x))t^{-1}$, mod H', where $R=\langle x, Q\rangle$ and T is a complete set of the double coset representatives for $H\backslash G/R$. Since Q is of Sylow type in H, if $R\subseteq H^{t^{-1}}$, then $Q^t\subseteq H$ and so $Q^t=Q^h$ for some $h\in H$. Since $N(Q)\subseteq H$ by assumption, we have $t\in H$. $V_{R\to R}(x)=xR'\not\subseteq K$ forces that $T-H$ contains an element t such that $V_{R\to R\cap H^t}(x)\not\subseteq K^t$ and $R\supset R\cap H^t$. As $|H^t: K^t|=p$, $|R\cap H^t: R\cap K^t|\leq p$. But $V_{R\to R\cap H^t}(x)\not\subseteq K^t$ implies that the equality holds. Setting $S=R\cap H^t$ and $M=R\cap K^t$, we obtain (a) of the theorem. Note that we have used the fact: no element of $S-M$ has order less than $|x|$.

Let A be a maximal subgroup of R containing S. By Lemma 5, we have that $x\in A$ and $\prod_{i=0}^{p-1} x^{y^i}\notin\Phi(A)$ for every $y\in R-A$ One also has that $\prod_{i=0}^{p-1} x^{y^i}\equiv [x, y, \overset{p-1}{\cdots}, y]$, mod $\Phi(A)$. By the transitivity of transfer we obtain $V_{R\to S}(x)=V_{A\to S}(V_{R\to A}(x))$. Thus $V_{R\to S}(x)\equiv V_{A\to S}([x, y, \cdots, y])$, mod M. Since $V_{A\to S}([x, y, \cdots, y])\equiv \prod_j a_j[x, y, \cdots, y]^{k_j}\cdot a_j^{-1}$, mod S' for a suitable subset $\{a_j\}$ of a complete coset representatives of $S\backslash P$, there must exist j such that $a_j[x,y, \cdots,y]^{k_j}a_j^{-1}\in$

$S-M$. This in turn implies that $ta_j[x, y, \cdots, y]^{k_j}a_j^{-1}t^{-1} \in H-K$. Hence (b) holds.

Theorem 7. *Let H be a subgroup of a finite group G with $p \nmid |G: H|$. Let Q be a subgroup of Sylow type in a Sylow p-subgroup P of H. Suppose further that $H \supseteq N(Q)$. Set $P_1 = \langle [x, y, \overset{p-1}{\cdots}, y] \mid x \in P, y \in Q \rangle$, $F_1 = \langle u^{-1}u^g \mid u \in P_1, u^g \in P, g \in G \rangle$. Then $H \cap G'(p) = F_1 H'(p)$.*

Proof. Suppose that $H \cap G'(p) \supset F_1 H'(p)$. Then one can choose a maximal subgroup P_1 of P such that $|H: P_1 F_1 V_{G \to H}(G) H'(p)| = p$. As in the previous theorem, choose an element x of $P-P_1$, of minimal order. Theorem 6 is now applicable, and so there must exist a proper singularity (R, S, M, x) with $R = \langle Q, x \rangle$. Let A be a maximal subgroup of R containing S. Since $x \in A$, $Q \cap A \subset Q$. So we may choose $y \in Q - Q \cap A$. By Theorem 6(b), there exist an element $g \in G$ and an integer k such that $([x, y, \overset{p-1}{\cdots}, y]^k)^g \in P - P_1$. But $[x, y, \cdots, y]^k \equiv ([x, y, \cdots,]^k)^g$, mod $F_1 P'$. Since $P_1 \supseteq F_1$ and $[x, y, \cdots, y] \in P_1$, we have $([x, y, \cdots, y]^k)^g \in P_1$, which is a contradiction.

Remark. The preceding theorems are the main results of Yoshida [3], with which one can easily obtain a generalization of a theorem of Wielandt or that of Hall-Wielandt (See [3] or [1] for the detail).

3. Some properties of singular subgroups

Yoshida has proved the following basic property of the singularity.

Theorem 8. *Let P be a p-group and (P, S, M, x) a singularity. Then*
(a) *for every subgroup R of P with $R \supseteq S$, $(P, R, \ker V_{R \to S/M}, x)$ is a singularity, and (R, S, M, x^u) is also a singularity for some conjugate x^u of x, $u \in P$.*
(b) $N_P(M) = S$,
(c) *S contains a conjugate of $N_P(\langle x \rangle)$,*
(d) *if $N \lhd P$ and $N \subseteq M$, then $(\bar{P}, \bar{S}, \bar{M}, \bar{x})$ is a singularity, where $\bar{P} = P/N$,*
(e) *if $|P: S| = p^n$, then the nilpotent class of P is at least $n(p-1)+1$*
(f) *if $S \lhd P$ and M does not contain a nontrivial normal subgroup of P, then $P \cong Z_p \wr (P/S)$, where P/S is regarded as a regular permutation group on $|P/S|$ letters.*

Proof. (See [3, Lemma 3.2, 3.4, 3.8, 3.9]).

There are many more properties found in [3]. But those listed above are all we need in this paper.

We now classify 2-groups P having a singularity (P, S, M, x) with $|P: S| \geq 4$.

Definition. *Let X be a finite group and Y a subgroup of X. Y_X denotes* *the largest normal subgroup of* X *contained in* Y; i.e., $Y_X=\bigcap_{x\in X} Y^x$.

Theorem 9. *Let* P *be a 2-group having a singularity* (P, S, M, x) *with* $|P: S|\geq 4$. *Then* P *has a singularity* (P, S_1, M_1, x) *with* $|P: S_1|=4$. *Set* $\bar{P}=P/(M_1)_P$. *Then*

(a) $(\bar{P}, \bar{S}_1, \bar{M}_1, \bar{x})$ *is singular,*

(b) $\bar{P}$ *is isomorphic to a subgroup of* S_8; *the symmetric group of degree* 8, *and*

(c) *the quadruple* $(\bar{P}, \bar{S}_1, \bar{M}_1, \bar{x})$ *satisfies one of the possibilities in the following list: (conversely, every quadruple in the list is singular).*

$\lvert\bar{P}\rvert$	$\bar{P}$	$\bar{S}_1$	$\bar{M}_1$	$\bar{x}$
16	dihedral	$Z_2\times Z_2$ [3]	Z_2	involutions in $\bar{M}_1$
	quasi-dihedral	[2]		
32	$gp\langle a, b, m, y \mid a^2=b^2=m^2=y^2=1,\ \langle a, b\rangle\cong D_8,\ \langle m, a, b\rangle=\langle m\rangle\times\langle a, b\rangle,\ m^y=m[a, b],\ a^y=b\rangle$	$\langle m, a, [a, b]\rangle\cong Z_2\times Z_2\times Z_2$ [2]	$\langle m, a\rangle\cong Z_2\times Z_2$	involutions in $\bar{S}_1-\langle m, [a, b]\rangle$
64	$Z_2 \wr Z_2\times Z_2$	$Z_2\times Z_2\times Z_2\times Z_2$ [1]	$Z_2\times Z_2\times Z_2$	involutions in $\bar{S}_1-Z_2(\bar{P})$
	$Z_2 \wr Z_4$	$Z_2\times Z_2\times Z_2\times Z_2$ [1]	$Z_2\times Z_2\times Z_2$	involutions in $\bar{S}_1-Z_3(\bar{P})$
	$gp\langle a, b, t, u \mid a^4=b^4=t^2=u^2=1,\ a^t=b,\ a^u=a^{-1},\ b^u=b^{-1},\ \langle a, b\rangle\cong Z_4\times Z_4,\ \langle t,u\rangle\cong Z_2\times Z_2\rangle$	$\langle a, b^2, u\rangle$ [3]	D_8	involutions in $\bar{S}_1-(\langle a^2, b^2, u)\rangle$
	$gp\langle a, b, c \mid a^4=b^4=c^4=1,\ \langle a, b\rangle\cong Z_4\times Z_4,\ a^c=b,\ b^c=a^{-1}\rangle$	$\langle a, b^2, c^2\rangle$ [3]	D_8	involutions in $\bar{S}_1-\langle a^2, b^2, c^2\rangle$
128	$D_8 \wr Z_2$	$Z_2\times Z_2\times D_8$ [3]	$Z_2\times D_8$	some involutions

Proof. By Theorem 8, we know the existence of singularities (P, S_1, M_1, x) and $(\bar{P}, \bar{S}_1, \bar{M}_1, \bar{x})$ where $\bar{P}=P/(M_1)_P$. Since $|P: M_1|=8$, (b) is trivial. So all we need to show is (c).

For simplicity we drop the subscript 1 from S_1 and M_1 and we also drop the bar from $\bar{P}$, $\bar{S}$, $\bar{M}$, and $\bar{x}$. Thus we have
(a)′ (P, S, M, x) is singular with $|P: S|=4$,
(b)′ $M_P=1$ and so P is isomorphic to a subgroup of S_8.
Furthermore, since $C_P(x)^u \subseteq S$ for some $u \in P$, $Z(P) \subseteq S$. As $M_P=1$, $|Z(P)|=2$ and $S=Z(P)\times M$ hold. Choosing x^u instead of x, we may assume that $C_P(x)\subseteq S$. Since $cl(P)\geq 3$, $|P|\geq 16$. Thus we have four cases to consider.

Case (i). $|P|=16$, $|S|=4$, $|M|=2$.

In this case, P is of maximal class and P has a noncyclic subgroup of order 4. Hence P is dihedral or quasi-dihedral. Moreover, x is an involution not conjugate to a central involution of P. Thus $P=gp\langle x, a \mid x^2=1, a^8=1, a^x=a^{-1}$ or $a^3\rangle$, $S=\langle x, a^4\rangle$. Clearly P is a disjoint union of S, Sa, Sa^2 and Sa^3. We compute that $V(x)=V_{P\to S}(x)=a^4 \notin M$. Therefore, P has indeed a singularity.

Case (ii). $|P|=32$, $|S|=8$, $|M|=4$.

By Theorem 8(f), S is not normal in P. Therefore $R=N_P(S)$ is the unique maximal subgroup of P containing S. Suppose $M\cong Z_4$. Then $S=Z(P)\times M\cong Z_2\times Z_4$. Let $\langle m\rangle=\Phi(M)=\Phi(S)$. Then $C_P(m)=R$. Hence $m\sim mz$ where $\langle z\rangle=Z(P)$. Since $m\in \ker V_{P\to S/M}$, x is of order 4. Clearly then $x^2=m$ is conjugate in P to $mz\in S-M$. As (P, S, M, x) is singular, we conclude that $M\cong Z_2\times Z_2$. So $S\cong Z_2\times Z_2\times Z_2$. $S \ntriangleleft P$ implies that R contains another elementary abelian group S^y, $y\in P-R$. $C_P(x)\subseteq S$ implies that R is nonabelian and so $R\cong Z_2\times D_8$. As $S\ntriangleleft P$, $P/Z(R)\cong D_8$. We may assume that $y^2\in Z(R)$. As y acts nontrivially on $Z(R)$, we may further assume that $y^2=1$. As y interchanges two elementary abelian subgroups of order 8 of R, the structure of P is uniquely determined up to isomorphism: i.e., $P=gp\langle m, a, b, y \mid m^2=a^2=b^2=y^2=1, \langle a, b\rangle\cong D_8, \langle m, a, b\rangle=\langle m\rangle\times\langle a, b\rangle\cong Z_2\times D_8, m^y=m[a, b], a^y=b\rangle$, with $S=\langle m, [a, b], a\rangle$. Conversely, it is easy to show that the configuration given above is indeed a singularity.

Case (iii). $|P|=64$, $|S|=16$, $|M|=8$.

Firstly, if $S\triangleleft P$, then by Theorem 8 (f), $P\cong Z_2 \wr Z_2\times Z_2$ or $Z_2 \wr Z_4$, and S is elementary of order 16. M, x can readily be determined. Conversely, it is easy to show that the configuration so obtained is a singularity.

Next we assume that $S\ntriangleleft P$. Then $R=N_P(S)$ is the unique maximal subgroup of P. Suppose S is abelian. Then, as S is embedded in S_8, $S\ncong Z_2\times Z_8$ or Z_{16}. It is convenient for us to have a full description of a Sylow 2-group T of S_8. The isomorphic type of T is $D_8 \wr Z_2$. More precisely, $T=\langle a_1, b_1, a_2, b_2, u \mid a_1^2=b_1^2=a_2^2=b_2^2=u^2=1, \langle a_1, b_1\rangle\cong\langle a_2, b_2\rangle\cong D_8, [\langle a_1, b_1\rangle, \langle a_2, b_2\rangle]=1, a_1^u=a_2, b_1^u=b_2\rangle$. Set $D=\langle a_1, b_1\rangle\times\langle a_2, b_2\rangle$.

If $S \cong Z_4 \times Z_4$, then S is uniquely determined in T. Namely $S = \langle a_1 b_1 \rangle \times \langle a_2 b_2 \rangle$. Clearly then $S \lhd P$, which is against our assumption. T contains four elementary abelian subgroups of order 16, two of them are normal, the remaining two are conjugate in T. Moreover all four are contained in D and the last two which are not normal generate D. Since $S \ntriangleleft P$, if S is elementary, then $P \supset D$. Hence $P = T$, which is absurd.

Finally assume that $S \cong Z_2 \times Z_2 \times Z_4$. Since all involutions of $T-D$ are conjugate to u in T and $C_T(u) \cong Z_2 \times D_8$, we have $\Omega_1(S) \subseteq D$. Clearly then $C_T(\Omega_1(S)) \subseteq D$ or $C_T(\Omega_1(S)) \cong Z_2 \times Z_2 \wr Z_2$. As $S \cong Z_2 \times Z_2 \times Z_4$, $S \subseteq D$ must holds. Since $\Phi(S) \subseteq M$ and $M \cap Z(P) = 1$, $\mho^1(S) \neq Z(T)$. Therefore P must contain an element of $T-D$. It is now trivial to show that $P = T$, which is again absurd. We have thus shown that S is nonabelian. As $Z_2 \times Q_8$ is not embedded in S_8 and so $S \cong Z_2 \times D_8$ and $M \cong D_8$, for we know $S = Z(P) \times M$. S_P is a maximal subgroup of S which is normal in P. Since $Z(P) \cong Z_2$, we must have $S_P \cong Z_2 \times Z_2 \times Z_2$. Furthermore $P/S_P \cong D_8$.

In order to obtain a full list of (P, S, M, x) for the case $|P| = 64$, we divide the proof into four subcases.

Subcase (1). P is of type A_8; i.e., $P \cong Z_2 \wr Z_2 \times Z_2$.
P is generated by involutions a, b, c, d, e, f together with the relations:

$$[c, e] = [b, f] = a\,, \qquad [d, e] = b\,, \qquad [d, f] = c$$

with all the other commutators of a pair of generators being trivial. P has precisely three normal elementary abelian subgroups A such that $P/A \cong D_8$. Those are $\langle a, b, e \rangle$, $\langle a, c, f \rangle$, and $\langle a, bc, ef \rangle$. Since they are permuted by an automorphism of P, we may assume that $S_P = \langle a, b, e \rangle$. P is a split extension of S_P by $\langle f, d \rangle \cong D_8$. As $|S : S_P| = 2$ and $S \ntriangleleft P$, $S = \langle a, b, e, f \rangle$, $\langle a, b, e, d \rangle$ or a conjugate of them. The first case must be ruled out as otherwise $Z(P) = \langle a \rangle = \Phi(S) \subseteq M$. Thus $S = \langle a, b, e, d \rangle \cong \langle a \rangle \times \langle e, d \rangle$. We are in a position to compute transfer. Note first that $P = S + Sc + Sf + Sfc$. An easy computation shows that $S \subseteq \ker V_{P \to S}$. Thus this subcase does not occur.

Subcase (2). $P \cong Z_2 \wr Z_4$.
P is generated by involutions a, b, c, d and an element e of order 4 with the relations:

$$[e, b] = a\,, \qquad [e, c] = b\,, \qquad [e, d] = c$$

with all the other commutators of pair of generators being trivial. P has only one normal elementary abelian subgroup such that the factor group is isomorphic to D_8. Hence $S_P = \langle a, b, e^2 \rangle$ and so $S = \langle a, b, e^2, e \rangle$ or $S = \langle a, b, e^2, d \rangle$ (up to conjugacy). The first case must be ruled out as otherwise $Z(P) = \langle a \rangle = \Phi(S) \subseteq M$. Thus $S = \langle a, b, e^2, d \rangle$. In a similar way as in the subcase (1), it can be shown very easily that $S\subseteq\ker V_{P\to S}$. Thus this subcase does not occur.

Subcase (3). *P is of type* M_{12}.

T has only one maximal subgroup of the type. P is generated by elements a, b, t, u with the relations:

$$a^4=b^4=t^2=u^2=1\,,\qquad a^t=b\,,\qquad a^u=a^{-1},\qquad b^u=b^{-1}$$

with all the other pair of generators commuting. P has precisely two normal elementary abelian subgroups of order 8; $\langle a^2, b^2, u\rangle$ and $\langle a^2, b^2, uab\rangle$. Clearly they are permuted by an automorphism of P. Hence without loss we may assume that $S_P=\langle a^2,b^2,u\rangle$. As in the previous two cases we conclude that $S=\langle a^2,b^2,u,a\rangle$ (up to conjugacy). We compute that $\langle a^2, b^2, u\rangle\subseteq\ker V_{P\to S}$ and $V(ua)\equiv a^2b^2$, mod S'. So if M is a dihedral group of order 8 in S and x is an involution in $S-\langle a^2, b^2, u\rangle$, then $V(x)\not\equiv 1$, mod M. Hence (P, S, M, x) is a singularity. If x' is an element of order 4 in S, then $x'^2=a^2$ and $a^2\sim b^2\in S-M$. Thus (P, S, M, x) is not singular. This completes the proof of this subcase.

Subcase (4). *P is a split extension of* $Z_4\times Z_4$ *by* Z_4.

P is generated by elements a, b, c of order 4 with relations:

$$a^c=b\,,\qquad b^c=a^{-1}$$

with all the other pairs of generators commuting. P has precisely two normal elementary abelian subgroups of order 8: $\langle a^2, b^2, abc^2\rangle$ and $\langle a^2, b^2, c^2\rangle$. Clearly the automorphism $(a\to b,\ b\to a,\ c\to cab)$ of P permute them. So we may assume that $S_P=\langle a^2, b^2, c^2\rangle$. We then have $S=\langle a^2, b^2, c^2, a\rangle$ again up to conjugacy. It is clear that $a^2, b^2, c^2\in\ker V_{P\to S/M}$. $V_{P\to S}(c^2a)\equiv a^2b^2$, mod S'. By the same reasoning as in the subcase (3), we conclude that (P, S, M, x) is a singularity, where M is a dihedral group of order 8 in S and x is an involution of $S-\langle a^2, b^2, c^2\rangle$.

Finally we consider:

Case (iv). $|P|=128$.

In this case, we use the generators and relations of $P=T=\langle a_1, b_1, a_2, b_2, u\rangle$ given before. As $S=Z(T)\times M$, S must contain an involution t of $T-Z(T)$ such that $C_T(t)\supseteq S$. One can check easily that $S\cong C_T(a_1)=\langle a_1, [a_1, b_1], a_2, b_2\rangle\cong Z_2\times Z_2\times D_8$. Renaming the generators of T if necessary we may assume that $S=C_T(a_1)$. Thus $T=S+Sb_1b_2+Su+Sub_1b_2$. By a direct computation, $\langle[a_1, b_1], [a_2, b_2], a_1a_2, b_2\rangle\in\ker V_{T\to S}$ and $V(a_1)\equiv[a_1, b_1][a_2, b_2]$, mod S'. So if we choose a maximal subgroup M with $M\cap Z(P)=1$ and an involution $x\in S-\ker V_{T\to S}$ then (T, S, M, x) is singular. As $[a_1, b_1]\sim[a_2, b_2]\in S-M$ an element of order 4 in S can not be singular.

Note. In Theorem 9, 1), 2), or 3) means that the choice of $\bar{S}_1$ is unique, unique up to conjugacy in $\bar{P}$, unique up to conjugacy in $\bar{P}$. Aut $(\bar{P})$, respectively.

4. Some application

We consider the following situation:

(H). *G is a finite group which contains an involution z. Let $H=C_G(z)$ and suppose that H satisfies*:

(i) *$E=O_2(H)$ is extra special, and*

(ii) *$C_G(E)\subseteq E$.*

Theorem 10 (F. Smith [2]). *Assume $G=O^2(G)$, (H) and the width of $E\geq 2$. Then E is contained in every normal subgroup of H of index 2.*

Proof. It is clear from (H) (ii) that E is weakly closed in H. Hence Theorem 6 is applicable. Suppose by way of contradiction that K is a normal subgroup of index 2 such that $K\not\supseteq E$. Then there exists an involution $x\in E-K$. By Theorem 6 (a), there exists a proper singularity (E, S, M, x). But E does not have a homomorphic image isomorphic to $Z_2 \wr Z_2\simeq D_8$. This is a contradiction.

Theorem 11. *Assume $G=O^2(G)$, (H) and H has normal subgroup of index 2 and that either the width of $E\geq 3$ or $E\simeq D_8*Q_8$, then the involution z is conjugate in G to an involution of $E-\langle z\rangle$.*

Proof. Let K be a normal subgroup of index 2. Let x be a 2-element of minimal order in $H-K$. Then by Theorem 6, there exists a proper singularity (R, S, M, x) with $R=\langle E, x\rangle=E\langle x\rangle$. Suppose that $|R: S|\geq 4$, then by Theorem 9, R must have a homomorphic image $\bar{R}=R/M_R$ isomorphic to one of the groups listed in the theorem. By our assumption on E, $M_R\ni\langle z\rangle$. Hence $\bar{R}$ is an extension of an elementary abelian group $\bar{E}$ by $\langle \bar{x}\rangle$. But this group is not on the list. Thus we have shown $|R: S|=2$ and so $S\cap E$ is a maximal subgroup of E.

Now recall how we defined S in the proof of Theorem 6. Namely $S=R\cap H^t$, $t\in G-H$. Set $E_1=S\cap E=H^t\cap E$. Then $|E: E_1|=2$ and $E_1\subseteq C_G(z)^t$. It is well known that under this condition $z^t\in E-\langle z\rangle$ holds. This completes the proof.

REMARK. $PS_p(4, 3)$ is a counter example to Theorem 11 if $E\simeq Q_8*Q_8$.

Theorem 12. *Assume $G=O^2(G)$ and (H). Then the Sylow 2-subgroups of H/H' are elementary.*

Proof. Suppose false. Then the width of E is greater than 1. Let K be a normal subgroup of index 2 of H such that $K\subseteq\Omega_1(H \bmod H'(2))$. Let x be a 2-element of $H-K$ of minimal order. Then as in the previous theorem there is a proper singularity (R, S, M, x) with $R=E\langle x\rangle$. We also see that $|E: S\cap E|=2$ unless $E\cong Q_8*Q_8$. So suppose first $|E: S\cap E|=2$. Clearly then $S\cap E/(S\cap E)'$ is elementary. This implies that $S\cap E\subseteq K^t$ where $S=R\cap H^t$, $t\in G-H$. Since $M=R\cap K^t$, $S\cap E=M\cap E$. As $S\cap E\lhd R$, $M_R\supseteq M\cap E$. Clearly then R/M_R is abelian and hence does not involve $Z_2 \wr Z_2$. This is a contradiction.

Suppose next that $|E: S\cap E|>2$. Then $E\cong Q_8*Q_8$. Since $\mathrm{Out}(Q_8*Q_8)$ is an extension of $Z_3\times Z_3$ by D_8, we must have $|H|=2^7\cdot 3^2$ with $H/H'(2)\cong Z_4$. Moreover, $R=\langle E, x\rangle$ is a Sylow 2-subgroup of H. Since $|R: S|\geq 4$, there is a singularity (R, S_1, M_1, x) with $|R: S_1|=4$. If $M_R=1$ then x is of order at least 4 and so the quadruple (R, S_1, M_1, x) can not be singular by Theorem 9. So $M_R\supseteq Z(E)$. Clearly then $\bar{R}=R/M_R$ is an extension of an abelian group $\bar{E}$ by $\bar{x}$. Again the quadruple $(\bar{R}, \bar{S}_1, \bar{M}_1, \bar{x})$ does not appear in the list of Theorem 9. This completes the proof.

CONCLUDING REMARK. There are several ways to generalize Theorem 11 or 12. We have presented the simplest cases. It is hoped that Theorem 9 can be used to simplify the proof of the classification of groups of sectional rank at most 4 or the proofs of the characterization of simple groups of low 2-rank. If all one needs to prove are Theorem 11 and 12, then a much simpler result than Theorem 9 is sufficient. (see [3]).

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References

[1] G. Glauberman: *Factorizations in local subgroups of finite groups*, Amer. Math. Soc. CBMS/33 (1978).

[2] F. Smith: *On the centralizers of involutions in finite fusion-simple groups*, J. Algebra **38** (1976), 268–273.

[3] T. Yoshida: *Character-theoretic transfer*, J. Algebra **52** (1978), 1–38.