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THE RIEMANN-ROCH THEOREM FOR COMPLEX V-MANIFOLDS

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Introduction and statement of theorem. This note is the sequel to our work [10]. We shall apply our method to the $\bar{\partial}$-operators over complex V-manifolds. Our result is a generalization of the Hirzebruch-Riemann-Roch Theorem (see Atiyah-Singer [4] and Hirzebruch [8]) to the case of complex V-manifolds and holomorphic vector V-bundles.

Let M be a compact complex manifold with a holomorphic action of a finite group G and let $E\to M$ be a G-equivariant holomorphic vector bundle. We denote by $\mathcal{O}(E)$ the sheaf of local holomorphic sections of E. Then Atiyah-Singer [4] proved: For each $g\in G$,

$$\text{(I)}\qquad \begin{aligned}\chi(g, M; \mathcal{O}(E)) &= \sum_i (-1)^i \operatorname{trace}_C[g\,|\,H^i(M; \mathcal{O}(E))]\\ &= \langle \mathcal{T}^g(M; E), [M^g]\rangle .\end{aligned}$$

Here $\mathcal{T}^g(M; E)$ is the equivariant Todd class.

Now the orbit space M/G has a structure of an analytic space and the local G-invariant holomorphic sections of E define a coherent anayltic sheaf $\mathcal{O}_V(E/G)$ over M/G. Then, by averaging (I) for all $g\in G$, we have:

$$\text{(II)}\qquad \begin{aligned}\chi(M/G; \mathcal{O}_V(E/G)) &= \sum_i (-1)^i \dim_C H^i(M/G; \mathcal{O}_V(E/G))\\ &= \frac{1}{|G|}\sum_{g\in G}\langle \mathcal{T}^g(M; E), [M^g]\rangle .\end{aligned}$$

We shall generalize this formula to the case of complex V-manifolds. The notion of V-manifold was introduced by Satake [11]. In [10] we have stated the precise definitions concerning V-manifold structures. So, here we put a brief description of complex V-manifolds and holomorphic vector V-bundles. Let X be an analytic space admitting only quotient singularities. A complex V-manifold structure $\mathcal{V}^c$ over X is the following: For each sufficiently small connected open set U in X, $\mathcal{V}^c(U)=$"$(G_U, \tilde{U})\to U$" is a ramified covering $\tilde{U}\to U$ such that $\tilde{U}$ is a connected complex manifold with an effective

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holomorphic action of a finite group G_U and the projection $\tilde{U}\to U$ gives an identification $U\simeq\tilde{U}/G_U$ of analytic spaces. For a connected open subset $V\subset U$, we assume also, that there is a biholomorphic open embedding φ: $\tilde{V}\to\tilde{U}$ that covers the inclusion $V\subset U$. Then the choice of φ is unique upto the action of G_U and each φ defines an injective group homomorphism λ_φ: $G_V\to G_U$ that makes φ be λ_φ-equivariant. Let p: $E\to X$ be a holomorphic map between analytic spaces. A holomorphic vector V-bundle structure $\mathscr{B}$ on "$E\to X$" is the following: For small $U\subset X$, $\mathscr{B}(U)=(G_U,\ \tilde{p}_U$: $\tilde{E}_U\to\tilde{U})$ is a G_U-equivariant holomorphic vector bundle with an identification "$p|p^{-1}(U)$: $p^{-1}(U)\to U$"$\simeq$"$\tilde{p}_U/G_U$: $\tilde{E}_U/G_U\to\tilde{U}/G_U$". For $V\subset U$, we assume that there is a holomorphic bundle map Φ: $\tilde{E}_V\to\tilde{E}_U$ over some open embedding φ: $\tilde{V}\to\tilde{U}$ that covers the inclusions $p^{-1}(V)\subset p^{-1}(U)$ and $V\subset U$. Then Φ becomes a λ_φ-equivariant bundle map. (In the terminology of [10], $(E, \mathscr{B})$ is a "proper" holomorphic vector V-bundle).

Now let X be a compact complex V-manifold and let $E\to X$ be a holomorphic vector V-bundle. The local G_U-invariant holomorphic sections of $\tilde{E}_U\to\tilde{U}$ define a coherent analytic sheaf $\mathcal{O}_V(E)$ over an analytic space X. Then we have the arithemtic genus $\chi(X;\ \mathcal{O}_V(E))=\sum_i(-1)^i\dim_C H^i(X;\ \mathcal{O}_V(E))$. We can choose invariant smooth linear connections on complex vector bundles $\tilde{E}_U\to\tilde{U}$, complex tangent bundles $T\tilde{U}\to\tilde{U}$ and complex normal bundles $\nu(\tilde{U}^g\subset\tilde{U})\to\tilde{U}^g$ for all U and for all $g\in G_U$, such that they are compatible with open embeddings Φ's and φ's. Then, by the Weil homomorphism, we have the equivariant Todd form $\mathcal{J}^g(\tilde{U};\ \tilde{E}_U)$ for each $\tilde{U}^g$. Then we can state our theroem in the following form. Let $\{f_U\}$ be a (smooth or continuous) partition of unity on X, then,

$$\text{(III)}\qquad \chi(X;\ \mathcal{O}_V(E))=\sum_U\frac{1}{|G_U|}\sum_{g\in G_U}\int_{\tilde{U}^g}f_U\mathcal{J}^g(\tilde{U};\ \tilde{E}_U)\,.$$

For each local coordinate $(G_U,\ \tilde{U})$ and for each $g\in G_U$, we consider $\tilde{U}^g$ as a complex manifold on which the centralizer $Z_{G_U}(g)$ acts. For $V\subset U$, the open embedding φ: $\tilde{V}\to\tilde{U}$ defines a natural open embedding $\tilde{V}^h/Z_{G_V}(h)\to\tilde{U}^g/Z_{G_U}(g)$ of analytic spaces, where $g=\lambda_\varphi(h)$. This embedding is unique for a fixed pair (g, h). We patch all $\tilde{U}^g/Z_{G_U}(g)$'s together by these identifications. Then we get a disjoint union of complex V-manifolds of various dimensions:

$$X\amalg\tilde{\tilde{\Sigma}}X=\bigcup_{(G_U,\tilde{U}),\,g\in G_U}\tilde{U}^g/Z_{G_U}(g)\,,$$

(X corresponds to the portion defined by $g=1$).

We have a canonical map $\tilde{\tilde{\Sigma}}X\to X$ covered locally by the inclusion $\tilde{U}^g\subset\tilde{U}$. For each $x\in X$, we can choose a coordinate neighbourhood $(G_x,\ \tilde{U}_x)$ such that $x\in\tilde{U}_x$ is a fixed point of G_x. G_x is unique upto isomorphisms. Then the number of pieces of $\tilde{\tilde{\Sigma}}X$ over x is equal to the number of the conjugacy classes of G_x other than the identity class.

Let $\tilde{\tilde{\tilde{\Sigma}}}X_1, \tilde{\tilde{\tilde{\Sigma}}}X_2, \cdots, \tilde{\tilde{\tilde{\Sigma}}}X_c$ be all the connected components of $\tilde{\tilde{\tilde{\Sigma}}}X$. To each $\tilde{\tilde{\tilde{\Sigma}}}X_i$, we assign a number m_i, defined by:

$$m_i = |\operatorname{kernel}[Z_{G_{\tilde{U}}}(g) \to \operatorname{Aut}(\tilde{U}^g)]| ,$$
$$(\tilde{U}^g/Z_{G_{\tilde{U}}}(g) \subset \tilde{\tilde{\tilde{\Sigma}}}X_i) .$$

Now the formal sum $\sum_{g\in G_{\tilde{U}}} \mathcal{I}^g(U; E_U)$ defines a "differential form" on $X \amalg \tilde{\tilde{\tilde{\Sigma}}}X$. It represents a cohomology class $\mathcal{I}(X; E) + \mathcal{I}^{\Sigma}(X; E)$ in $H^*(X \amalg \tilde{\tilde{\tilde{\Sigma}}}X; \boldsymbol{C})$. This class is independent of the choice of the connections. Then we get the following theorem:

Theorem. *Let X be a compact complex V-manifold and let $E \to X$ be a holomorphic vector V-bundle. Then*:

$$\text{(IV)} \qquad \chi(X; \mathcal{O}_V(E)) = \langle \mathcal{I}(X; E), [X] \rangle + \sum_{i=1}^{c} \frac{1}{m_i} \langle \mathcal{I}^{\Sigma}(X; E), [\tilde{\tilde{\tilde{\Sigma}}}X_i] \rangle .$$

REMARK 1. Since the class $\mathcal{I}(X; E)$ is defined over rationals, the term $\langle \mathcal{I}(X; E), [X] \rangle$ is a rational number.

REMARK 2. For the case when $X = \Gamma \backslash \tilde{X}$, where $\tilde{X}$ is a complex manifold and Γ is a properly discontinuous group acting holomorphically on $\tilde{X}$, the number $\langle \mathcal{I}(X; E), [X] \rangle$ is just the Γ-index $\operatorname{ind}_\Gamma((\bar{\partial} + \bar{\partial}^*)_E^{0,ev})$ defined by Atiyah [1]. (Though Γ acts freely in [1], the similar argument holds for the case when Γ has finite isotropies, see III) below).

The proof of our theorem is a combination of our work [10] and Gilkey's result [7] on the Lefschetz fixed point formula for the Dolbeault complexes. Here we shall place a complete proof.

Proof of Theorem. In this proof, we use the "heat kernel-zeta function" method. We reivew the results briefly. (See Seeley [12], Atiyah-Bott-Patodi [2], Gilkey [6], [7], Donnelly-Patodi [5] and Kawasaki [10]).

Let U be a germ of a Riemannian manifold and let $E_U \to U$ be a smooth complex vector bundle with a smooth Hermitian fibre metric. Let $g: E_U \to E_U$ be an isometry of the pair (U, E_U). Let $A: C^\infty(U; E_U) \to C^\infty(U: E_U)$ be a g-invariant, formally self-adjoint, positive semi-definite, elliptic differential operator. Then we have a smooth measure Z_A^g on the fixed point set U^g. Z_A^g is a local invariant of the action of g and of the operator A. It is given by a universal expression in g and A. The explicit form of Z_A^g is given in [10]. Z_A^g has the following properties:

I) Let M be a compact Riemannian manifold and let $g: M \to M$ be an isometry. Let E, F be two g-equivariant smooth complex vector bundles over M with g-invariant Hermitian fibre metrics. Let $D: C^\infty(M; E) \to C^\infty(M; F)$ be a g-invariant elliptic differential operator. Then we have the adjoint operator $D^*: C^\infty(M; F) \to C^\infty(M; E)$ and two g-invariant, self-adjoint, positive semi-definite, elliptic differential operators D^*D and DD^*. Pur $\mu_D^g = Z_{D^*D}^g - Z_{DD^*}^g$. Then the equivariant index $\mathrm{ind}(g, D)$ is given by:

$$\mathrm{ind}(g, D) = \int_{M^g} \mathrm{d}\mu_D^g \,.$$

II) (Kawasaki [10]). Let X be a compact Riemannian V-manifold and let E, F be two "proper" differentiable complex vector V-bundles over X. Let $D: C_V^\infty(X; E) \to C_V^\infty(X; F)$ be an elliptic differntial operator, that is, a family $\{\tilde{D}_U: C^\infty(\tilde{U}; \tilde{E}_U) \to C^\infty(\tilde{U}; \tilde{F}_U\}_{(G_U, \tilde{U})}$ of invariant elliptic differential operators that are compatible with attaching maps $\{\Phi\}: \tilde{E}_V \to \tilde{E}_U$ and $\{\Psi\}: \tilde{F}_V \to \tilde{F}_U$. Then D operates on the differentiable V-sections and the kernel and the cokernel of the operator D are finite dimensional. We define the V-index $\mathrm{ind}_V(D)$ of the operator D by:

$$\begin{aligned} \mathrm{ind}_V(D) &= \dim_C \mathrm{kernel}[D: C_V^\infty(X; E) \to C_V^\infty(X; F)] \\ &\quad - \dim_C \mathrm{cokernel}[D: C_V^\infty(X; E) \to C_V^\infty(X; F)] \,. \end{aligned}$$

For each coordinate neighbourhood $(G_U, \tilde{U})$, we have a formal sum of measures:

$$\sum_{g \in G_U} \mu_{\tilde{D}_U}^g = \sum_{g \in G_U} (Z_{\tilde{D}_U^* \tilde{D}_U}^g - Z_{\tilde{D}_U \tilde{D}_U^*}^g) \,.$$

These formal sums define a measure $\mu_D + \mu_{\tilde{D}}^{\vee}$ over $X \amalg \tilde{\tilde{\Sigma}} X$. Then the V-index $\mathrm{ind}_V(D)$ is given by:

$$\mathrm{ind}_V(D) = \int_X \mathrm{d}\mu_D + \sum_{i=1}^{c} \frac{1}{m_i} \int_{\tilde{\tilde{\Sigma}} X_i} \mathrm{d}\mu_{\tilde{D}}^{\vee} \,.$$

III) (See Aityah [1]). Let $\tilde{X}$ be a (non-compact) Riemannian manifold and let Γ be a properly discontinuous group acting on $\tilde{X}$ as isometries. We assume that the orbit V-manifold $X = \Gamma \backslash \tilde{X}$ is compact. Let $\tilde{E}$, $\tilde{F}$ be two Γ-equivariant complex vector bundles over $\tilde{X}$ with Γ-invariant Hermitian fibre metrics. Let $\tilde{D}: C^\infty(\tilde{X}; \tilde{E}) \to C^\infty(\tilde{X}; \tilde{F})$ be a Γ-invariant elliptic differential operator. Then we consider the completions $\mathcal{L}^2(\tilde{X}; \tilde{E})$, $\mathcal{L}^2(\tilde{X}; \tilde{F})$ and the unbounded operators $\tilde{D}: \mathcal{L}^2(\tilde{X}; \tilde{E}) \to \mathcal{L}^2(\tilde{X}; \tilde{F})$, $\tilde{D}^*: \mathcal{L}^2(\tilde{X}; \tilde{F}) \to \mathcal{L}^2(\tilde{X}; \tilde{E})$. (In this case the formal adjoint coincides with the Hilbert space adjoint). We put:

$$\begin{aligned} \mathcal{H}_0 &= \{f \in \mathcal{L}^2(\tilde{X}; \tilde{E}) \mid \tilde{D} f = 0\} \subset \mathcal{L}^2(\tilde{X}; \tilde{E}) \,, \\ \mathcal{H}_1 &= \{g \in \mathcal{L}^2(\tilde{X}; \tilde{F}) \mid \tilde{D}^* g = 0\} \subset \mathcal{L}^2(\tilde{X}; \tilde{F}) \,. \end{aligned}$$

Then $\mathcal{H}_i$ becomes a Γ-invariant closed subspace ($i=0, 1$). Let H_i be the orthogonal projection onto $\mathcal{H}_i$. Then H_i has a smooth kernel $H_i(\tilde{x}, \tilde{y})$ and we get a smooth measure $\operatorname{trace}_C[H_i(\tilde{x}, \tilde{x})]$ over $\tilde{X}$. Since the operator H_i is Γ-invariant, we may consider $\operatorname{trace}_C[H_i(\tilde{x}, \tilde{x})]$ as a measure over $X=\Gamma\backslash\tilde{X}$. Then the Γ-index of the operator $\tilde{D}$ is defined by:

$$\operatorname{ind}_\Gamma(\tilde{D}) = \int_X \mathrm{d}\,(\operatorname{trace}_C[H_0(\tilde{x}, \tilde{x})] - \operatorname{trace}_C[H_1(\tilde{x}, \tilde{x})])\,.$$

Now the elliptic differential operator $\tilde{D}$ over $\tilde{X}$ defines an elliptic differential operator $D\colon C_V^\infty(X; E)\to C_V^\infty(X; F)$ over a V-manifold X and we have a measure μ_D over X. Then $\operatorname{ind}_\Gamma(\tilde{D})$ is given by:

$$\operatorname{ind}_\Gamma(\tilde{D}) = \int_X \mathrm{d}\mu_D\,.$$

Now we return to our problem: Let X be a compact complex V-manifold and let $E\to X$ be a holomorphic vector V-bundle. We denote by $\boldsymbol{T}$ the holomorphic part of the complexified cotangent vector V-bundle. Consider the sheaf $\mathfrak{A}_V^{p,q}(E)=C_V^\infty(\Lambda^p \boldsymbol{T}\otimes\Lambda^q \bar{\boldsymbol{T}}\otimes E)$ of germs of E-valued (p, q)-forms over X. Then we have the $\bar\partial$-operators $\bar\partial\colon \mathfrak{A}_V^{p,q}(E)\to\mathfrak{A}_V^{p,q+1}(E)$ and a soft resolution:

$$0\to \mathcal{O}_V(\Lambda^p \boldsymbol{T}\otimes E)\hookrightarrow \mathfrak{A}_V^{p,0}(E)\xrightarrow{\bar\partial}\mathfrak{A}_V^{p,1}(E)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathfrak{A}_V^{p,n}(E)\to 0\,.$$

Put $A_V^{p,q}(X; E)=\Gamma(X; \mathfrak{A}_V^{p,q}(E))$, then we have a complex:

$$0\to A_V^{p,0}(X; E)\xrightarrow{\bar\partial} A_V^{p,1}(X; E)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial} A_V^{p,n}(X; E)\to 0\,,$$

whose i-th cohomology group is $H^i(X; \mathcal{O}_V(\Lambda^p \boldsymbol{T}\otimes E))$. Choose a Hermitian metric h on X and a Hermitian fibre metric h_E on E. Then we have the adjoint operator $\bar\partial^*\colon A_V^{p,q}(X; E)\to A_V^{p,q-1}(X; E)$ of $\bar\partial$. Consider a differential operator:

$$(\bar\partial+\bar\partial^*)_E^{0,ev} = \bar\partial+\bar\partial^*\,|\,A_V^{0,ev}\colon A_V^{0,ev}(X; E)\to A_V^{0,od}(X; E)\,,$$

$$\Big(A_V^{0,\substack{ev\\od}}(X; E) = \bigoplus_{q\,:\,\substack{\text{even}\\\text{odd}}} A_V^{0,q}(X; E)\Big)\,.$$

Then $(\bar\partial+\bar\partial^*)_E^{0,ev}$ is an elliptic operator and:

$$\operatorname{ind}_V((\bar\partial+\bar\partial^*)_E^{0,ev}) = \chi(X; \mathcal{O}_V(E))\,.$$

Thus we can express the arithmetic genus as the V-index of an elliptic operator $(\bar\partial+\bar\partial^*)_E^{0,ev}$. Then, by II) above, we have a measure $\mu_{(\bar\partial+\bar\partial^*)_E^{0,ev}}+\mu^{\Sigma}_{(\bar\partial+\bar\partial^*)_{\tilde{E}}^{0,ev}}$ over $X\amalg\tilde{\tilde{\Sigma}}X$ that gives the arithmetic genus. But this measure is not equal to the Todd class in general. So we use the *Spin*^{c} Dirac operator instead, which gives the arithmetic genus for complex V-manifolds and is defined over more general V-manifolds.

Now let (X, h) and (E, h_E) be as before. Consider the almost complex structure (TX, J). (TX, J) is a holomorphic vector V-bundle. The Hermitian metric h define a reduction $U(n)\,(TX)$ of the principal tangent V-bundle. We consider $U(n)$ as a subgroup of $Spin^c(2n)=Spin(2n)\times_{Z_2}U(1)$. (See Atiyah-Bott-Shapiro [2]). Let $Spin^c(2n)\,(TX)$ be the associated $Spin^c(2n)$-principal tangen V-bundle. We construct a connection ∇^c on $Spin^c(2n)\,(TX)$ as follows: We have a Riemannian connection ∇_{SO} on $SO(2n)\,(TX)$ and a Hermitian connection ∇_L on $L=\Lambda^n((TX,J)$. Then ∇^c is a unique lift of $\nabla_{SO}\times\nabla_L$ on $(SO(2n)\times U(1))(TX)$ by the double covering $Spin^c(2n)\to SO(2n)\times U(1)$. Let $\Delta^{\pm,c}$ be the half $Spin^c$-representations. Then we have two complex vector V-bundles:

$$\Delta^{\pm,c}(TX) = Spin^c(2n)\,(TX)\times_{Spin^c(2n)}\Delta^{\pm,c}\,,$$

with induced connections $\nabla^{\pm,c}$. The Clifford module structures on $\Delta^{\pm,c}$ define the Clifford multiplications:

$$m\colon TX\otimes_R\Delta^{\pm,c}(TX)\to\Delta^{\mp,c}(TX)\,.$$

On (E, h_E) we have the Hermitian connection ∇_E. Then the $Spin^c$ Dirac operator $d_E^{+,c}$ is defined by:

$$\begin{aligned} d_E^{+,c}\colon\, & C_V^\infty(X;\,\Delta^{+,c}(TX)\otimes_C E)\\ & \xrightarrow{\nabla^{+,c}\otimes 1+1\otimes\nabla_E} C_V^\infty(X;\,T^*X\otimes_R\Delta^{+,c}(TX)\otimes_C E)\\ & \xrightarrow{\ m\ } C_V^\infty(X;\,\Delta^{-,c}(TX)\otimes_C E)\,.\end{aligned}$$

Here we identify $TX=T^*X$ by the real Hermitian metric $\mathcal{R}e\,h$.

Since $Spin^c(2n)\,(TX)$ has a reduction $U(n)\,(TX)$, we have:

$$\Delta^{\pm,c}(TX)\cong\Lambda^{\substack{ev\\od}}(TX, J)\,.$$

The Hermitian metric h defines a V-bundle isometry $\psi\colon (TX,J)\cong\bar{T}$. So we have a V-bundle isomorphism:

$$\psi^{\pm}\colon\,\Delta^{\pm,c}(TX)\underset{C}{\otimes}E\cong\Lambda^{\substack{ev\\od}}\bar{T}\underset{C}{\otimes}E\,.$$

By a standard computation (see Hitchin [9]), we have:

Proposition. *The two operators $(\bar\partial+\bar\partial^*)_E^{0,ev}$ and $d_E^{+,c}$ have the same principal symbol (via $\psi^{\pm}$) upto a constant factor.*

As a corollary, we have:

$$\begin{aligned}\chi(X;\,\mathcal{O}_V(E)) &= \mathrm{ind}_V(d_E^{+,c})\\ &= \int_X \mathrm{d}\mu_{d_E^{+,c}}+\sum_{i=1}^{c}\frac{1}{m_i}\int_{\tilde{\tilde{\Sigma}}X_i}\mathrm{d}\mu_{\tilde{d}_E^{+,c}}^{\Sigma}\,.\end{aligned}$$

Now the operator $d_E^{+,c}$ does not depend on the complex structure on X. It depends only on the $Spin^c$-structure $Spin^c(2n)$ (TX), the metric connection ∇_L and the Hermitian V-bundle (E, h_E, ∇_E). Its index $\mathrm{ind}_V(d_E^{+,c})$ does not depend on the choices of metrics h and h_E, nor the choices of connections ∇_L and ∇_E. So we can change metrics and connections.

We consider over a coordinate neighbourhood $(G_U, \tilde{U}) \to U$. Choose a metric h on $\tilde{U}$ so that, for each $g \in G_U$, on a neighbourhood of $\tilde{U}^g$ in $\tilde{U}$, h is equal to the Riemannian metric over the total space N_g of the normal bundle $\nu_g = \nu(\tilde{U}^g \subset \tilde{U})$ induced from a g-invariant Hermitian structure $(\nu_g, h_{\nu_g}, \nabla_{\nu_g})$. We identify N_g with a neighbourhood of $\tilde{U}^g$ in $\tilde{U}$. Then, over N_g, the principal bundle $Spin^c(2n)(T\tilde{U})$ reduces equivariantly to $\pi^*(Spin^c(2n_0)(T\tilde{U}^g) \times_{\tilde{U}^g} U(n-n_0)(\nu_g))$, where π: $N_g \to \tilde{U}^g$ is the projection of ν_g and $2n_0 = \dim_R U^g$. The associated line bundle L splits into a tensor product $\pi^*(L_0 \otimes \Lambda^{n-n_0}\nu_g)$, where L_0 is the associated line bundle of $Spin^c(2n_0)$ $(T\tilde{U}^g)$.

The actions of g on the first factors $Spin^c(2n_0)(T\tilde{U}^g))$ and L_0 are trivial. On L_0, we have the induced metric h_{L_0}. Choose a metric connection ∇_{L_0} on (L_0, h_{L_0}). Then we choose a metric connection ∇_L so that, over N_g, ∇_L is equal to the induced connection $\pi^*(\nabla_{L_0} \otimes \Lambda^{n-n_0}\nabla_{\nu_g})$. Also, we choose a Hermitian structure (E, h_E, ∇_E) so that, over N_g, it is equal to the induced structure $(\pi^*(E \mid \tilde{U}^g)$, $\pi^*(h_E \mid \tilde{U}^g)$, $\pi^*(\nabla_E \mid \tilde{U}^g))$.

Then, over a neighbourhood N_g of $\tilde{U}^g$ in $\tilde{U}$, the operator $d_E^{+,c}$ is completely determined by the data over $\tilde{U}^g$, that is, the $Spin^c$-structure $Spin^c(2n_0)$ $(T\tilde{U}^g)$, the metric connection ∇_{L_0} and the g-equivariant Hermitian bundles $(g; \nu_g, h_{\nu_g}, \nabla_{\nu_g})$ and $(g; E \mid U^g, h_E \mid U^g, \nabla_E \mid U^g)$.

We remark here that we can choose a metric h, a metric connection ∇_L and a hermitian structure (E, h_E, ∇_E) over a V-manifold X so that the above conditions are satisfied for all coordinate neighbourhood $(G_U, \tilde{U}) \to U$ and for all $g \in G_U$ at the same time.

Now we consider differently: Let (U_0, h_0) be a germ of $(2n_0)$-dimensional Reimannian manofold with trivial g-action and assume that we are given a Hermitian line bundle $(L_0, h_{L_0}, \nabla_{L_0})$ with trivial g-action and two g-equivariant Hermitian bundles $(g; \nu, h_\nu, \nabla_\nu)$ $(\dim_C \nu = n - n_0)$ and $(g; E, h_E, \nabla_E)$ over U_0. So g acts on each fibre of ν and E. We assume that the fixed points in ν are all in the zero section. We may assume that U_0 is contractible. Then an orientation o, the Riemannian metric h_0 and the Hermitian line bundle $(L_0, h_{L_0}, \nabla_{L_0})$ define a unique $Spin^c$-structure $Spin^c(2n_0)(TU_0)$ upto $Spin^c$-isomorphisms. (There are two canonical isomorphisms). The Riemannian metric h_0 and the metric connection ∇_{L_0} define a connection ∇_0^c on $Spin^c(2n_0)$ (TU_0). Consider the total space N of ν. The Hermitian structure (ν, h_ν, ∇_ν) define a $Spin^c(2n_0) \times U(n-n_0)$-structure over N. Also we have the action of g that preserve the above structure. Then we have the associated $Spin^c(2n)$-structure with

g-action over N. Its associated line bundle is $\pi^*(L_0 \otimes \Lambda^{n-n_0}\nu)$ and the metric connection $\nabla_{L_0} \otimes \Lambda^{n-n_0}\nabla_\nu$ defines a connection ∇^c on $Spin^c(2n)\,(TN)$. Also we have an induced g-equivariant Hermitian bundle $(g;\, \pi^*E,\, \pi^*h_E,\, \pi^*\Delta_E)$ over N.

Then the $Spin^c$-structure $Spin^c(2n)\,(TN)$ with connection ∇^c and the Hermitian bundle $(\pi^*E,\, \pi^*h_E,\, \pi^*\nabla_E)$ define the $Spin^c$ Dirac operator $d^{+,c}_{\pi^*E}$. The operator $d^{+,c}_{\pi^*E}$ and the action of g define a measure $\mu^g_{d^{+,c}_{\pi^*E}}$ over U_0. The only ambiguity of this construction comes from the choice of the orientation o over U_0. If we change the orientation, then the measure $\mu^g_{d^{+,c}_{\pi^*E}}$ changes its sign. So the measure $\mu^g_{d^{+,c}_{\pi^*E}}$ defines a $2n_0$-form $d\mu^g_{d^{+,c}_{\pi^*E}}$ with no ambiguity.

Thus we have shown that the $2n_0$-form $d\mu^g_{d^{+,c}_{\pi^*E}}$ is a local invariant of a Riemannian structure $(U_0,\, h_0)$ and Hermitian bundles $(L_0,\, h_{L_0},\, \nabla_{L_0})$, $(g;\, \nu_\nu,\, h_\nu,\, \nabla_\nu)$ and $(g;\, E,\, h_E,\, \nabla_E)$. In [10], we have an explicit form of μ^g_D. Then we can see that the $2n_0$-form $d\mu^g_{d^{+,c}_{\pi^*E}}$ is a homogeneous regular local invariant of weight 0, in the terminology of Atiyah-Bott-Patodi [2]. Then, by Gilkey's Theorem (see [2]), we can conclude:

Proposition. *$d\mu^g_{d^{+,c}_{\pi^*E}}$ is expressed by a universal polynomial in the Pontrjagin forms of $(U_0,\, h_0)$, the first Chern form of $(L_0,\, h_{L_0},\, \nabla_{L_0})$, the equivariant Chern forms of $(g;\, \nu,\, h_\nu,\, \nabla_\nu)$ and the equivariant Chern forms of $(g;\, E,\, h_E,\, \nabla_E)$.*

We restrict ourselves to the case when TU_0 has an almost complex structure J_0 and $L_0 = \Lambda^{n_0}(TU_0,\, J_0)$. Let M be a compact complex manifold and let $E \to M$ be a holomorphic vector bundle. Let g be an automorphism of the pair $(M,\, E)$ that generates a compact transformation group. Then by Atiyah-Singer [4] we know:

$$\int_{M^g} \mathrm{d}\mu^g_{d^{+,c}_E} = \sum_i (-1)^i \operatorname{trace}_C[g \mid H^i(M;\, \mathcal{O}(E))]$$
$$= \langle \mathcal{I}^g(M;\, E),\, [M^g] \rangle .$$

The computations over the products of complex projective spaces with linear actions show that the expression of $d\mu^g_{d^{+,c}_{\pi^*E}}$ in the characteristic classes must be unique. This shows:

$$\mathrm{d}\mu^g_{d^{+,c}_E} = \mathcal{I}^g(M;\, E) .$$

Now we return to the original situation. Over a coordinate neighbourhood $(G_U,\, \tilde{U}) \to U$, we have:

$$\mu_{d^{+,c}_E} + \mu^{\Sigma}_{d^{+,c}_E} = \sum_{g \in G_U} \mu^g_{d^{+,c}_E} .$$

Then, by choosing suitable metrics and connections, we have:

$$\mathrm{d}\mu_{d^{+,c}_E} + \mathrm{d}\mu^{\Sigma}_{d^{+,c}_E} = \mathcal{I}(X;\, E) + \mathcal{I}^{\Sigma}(X;\, E) .$$

Hence we have:

$$\begin{aligned}\chi(X;\, \mathcal{O}_V(E)) &= \int_X d\mu_{d_E^+,c} + \sum_{i=1}^{c} \frac{1}{m_i} \int_{\widetilde{\widetilde{\Sigma}} X_i} d\mu^{\Sigma}_{d_E^+,c} \\ &= \langle \mathcal{I}(X;\, E),\, [X] \rangle \\ &\quad + \sum_{i=1}^{c} \frac{1}{m_i} \langle \mathcal{I}^{\Sigma}(X;\, E),\, [\widetilde{\widetilde{\Sigma}} X_i] \rangle\,.\end{aligned}$$

The both sides are independent of the metrics and connections.

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References

[1] M.F. Atiyah: *Elliptic operators, discrete groups and von Neumann algebras,* Société Mathématique de France, Astérisque **32–33** (1976), 43–72.

[2] M.F. Atiyah, R. Bott and V.K. Patodi: *On the heat equation and the index theorem,* Invent. Math. **19** (1973), 379–330.

[3] M.A. Atiyah, R. Bott and A. Shapiro: *Clifford modules,* Topology **3** (1964), Suppl. 1, 3–38.

[4] M.F. Atiyah and I.M. Singer: *The index of elliptic operators*: III, Ann. of Math. **87** (1968), 546–604.

[5] H. Donnelly and V.K. Patodi: *Spectrum and the fixed point sets of isometries* - II, Topology **16** (1977), 1–11.

[6] P.B. Gilkey: *Curvature and the eigenvalues of the Laplacian for elliptic complexes,* Advances in Math. **10** (1973), 344–382.

[7] P.B. Gilkey: *Spectral geometry and the Lefschetz formulas for a holomorphic isometry of an almost complex manifold,* to appear.

[8] F. Hirzebruch: Topological methods in algebraic geometry, Springer-Verlag, 1966.

[9] N. Hitchin: *Harmonic spinors,* Advances in Math. **14** (1974), 1–55.

[10] T. Kawasaki: *The signature theorem for V-manifolds,* Topology **17** (1978) 75–83.

[11] I. Satake: *The Gauss-Bonnet theorem for V-manifolds,* J. Math. Soc. Japan **9** (1957), 464–492.

[12] R.T. Seeley: *Complex powers of an elliptic operator,* Proc. Sympos. Pure Math. Math. 10, Amer. Math. Soc. (1967), 288–307.