REMARKS ON MULTIPLY TRANSITIVE PERMUTATION GROUPS

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1. Introduction

In [5], T. Oyama determined all 4-fold transitive permutation groups in which the stabilizer of four points has an orbit of length two. On the other hand, in Yoshizawa [8], 5-fold transitive permutation groups in which the stabilizer of five points has a normal Sylow 2-subgroup have been determined. In this note we give some analogous version of these results for any odd prime \( p \) on \( 2p \) (or \( 2p+1 \))-fold transitive permutation groups.

**Theorem 1.** Let \( p \) be an odd prime \( \geq 5 \). Let \( G \) be a \( 2p \)-fold transitive permutation group on \( \Omega = \{1, 2, \ldots, n\} \). If \( G_{1,2,\ldots,2p} \) has an orbit on \( \Omega - \{1, 2, \ldots, 2p\} \) whose length is less than \( p \), then \( G \) is one of \( S_n(2p+1 \leq n \leq 3p-1) \) and \( A_n(2p+2 \leq n \leq 3p-1) \).

**Corollary.** Let \( p \) be an odd prime \( \geq 5 \). Let \( D \) be a \( 2p \)-(\( v, k, 1 \)) design with \( 2p < k < 3p \). If an automorphism group \( G \) of \( D \) is \( 2p \)-fold transitive on the set of points of \( D \), then \( D \) is a \( 2p \)-(\( k, k_1 \)) design.

**Theorem 2.** Let \( p \) be an odd prime \( \geq 5 \). Let \( G \) be a \( 2p \)-fold transitive permutation group on \( \Omega = \{1, 2, \ldots, n\} \). Let \( P \) be a Sylow \( p \)-subgroup of \( G_{1,2,\ldots,2p} \). If \( P \) is a normal subgroup of \( G_{1,2,\ldots,2p} \), then \( G \) is one of \( S_n(2p+1 \leq n \leq 3p-1) \) and \( A_n(2p+2 \leq n \leq 3p-1) \).

**Theorem 3.** Let \( G \) be a 7-fold transitive permutation group on \( \Omega = \{1, 2, \ldots, n\} \). Let \( P \) be a Sylow 3-subgroup of \( G_{1,2,\ldots,7} \). If \( P \) is a normal subgroup of \( G_{1,2,\ldots,7} \), then \( G \) is \( S_7, S_8, S_9, S_{10}, A_9 \) or \( A_{10} \).

We shall use the same notation as in [4].

2. Proof of Theorem 1

Let \( G \) be a group satisfying the assumption of Theorem 1. By [4] and [5], if \( G_{1,2,\ldots,2p} \) has an orbit on \( \Omega - \{1, 2, \ldots, 2p\} \) whose length is one or two, then \( G \) is \( S_{2p+1}, S_{2p+2} \) or \( A_{2p+2} \). Hence we may assume that \( G_{1,2,\ldots,2p} \) has an orbit \( \Delta \)
such that $3 \leq |\Delta| \leq p-1$.

Let $P$ be a Sylow $p$-subgroup of $G_{1,2,\ldots,2p}$. If $P=1$, then $G$ is one of $S_s (2p+3 \leq s \leq 3p-1)$ and $A_s (2p+3 \leq s \leq 3p-1)$ by [1]. From now on we assume that $P \neq 1$, and prove that this case does not occur. Since $3 \leq |\Delta| \leq p-1$, we have $I(P) \supseteq \Delta \cup \{1, 2, \ldots, 2p\}$ and $N_G(I(P)^{I(P)}) = S_{2p+3}, \ldots, S_{3p-1}, A_{2p+3}, \ldots, A_{3p-1}$ by [1]. Therefore $N_G(I(P)^{I(P)}) = S_{3p}, S_{2p}, A_{3p}, \ldots, A_{p-1}$, and $I(P) = \Delta \cup \{1, 2, \ldots, 2p\}$. This shows that $I(P)$ is independent of the choice of Sylow $p$-subgroup $P$ of $G_{1,2,\ldots,2p}$ and is uniquely determined by $G_{1,2,\ldots,2p}$.

Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing more than $|I(P)|$ points. Set $N = N_G(Q)^{I(Q)}$, and $r = |\Delta|$. $N$ has an element $a$ of order $p$ fixing $2p+r$ points. We may assume that

$$a = (1)(2)\cdots(2p+r)(2p+r+1, \ldots, 2p+r+p)\cdots.$$  

Set $\mathcal{T} = C_N(a)^{I(Q)} = C_N(a)^{I(Q)}_{2p+r+1, \ldots, 2p+r+p}$ and $\Lambda = I(a)$. Then $T$ satisfies the following two properties:

(i) $T$ is a permutation group on $\Lambda$. $|\Lambda| = 2p+r$ and $3 \leq r \leq p-1$.

(ii) For any $p$ points $\alpha_1, \alpha_2, \ldots, \alpha_p$ in $\Lambda$, a Sylow $p$-subgroup $S$ of $T_{\alpha_1,\ldots,\alpha_p}$ is a cyclic group of order $p$ generated by a $p$-cycle, and $|I(S)| = p+r$. Moreover $I(S)$ is independent of the choice of Sylow $p$-subgroup $S$ of $T_{\alpha_1,\ldots,\alpha_p}$ and is uniquely determined by $T_{\alpha_1,\ldots,\alpha_p}$.

Suppose that $T$ is primitive. Since $r \geq 3$ and $T$ has a $p$-cycle, $T \supseteq A_{2p+r}$ by Theorem 13.9 in [7]. This contradicts (ii).

Suppose that $T$ is imprimitive, and let the set $\{\Delta_1, \ldots, \Delta_s\}$ be a nontrivial complete block system. Assume $|\Delta_i| \leq p$. For each $i \in \{1, \ldots, s\}$, let $\delta_i$ be a point of $\Delta_i$. By considering $T_{\delta_1,\ldots,\delta_s}(s \geq p)$ or $T_{\delta_1,\ldots,\delta_s}(s < p)$, we have a contradiction by (ii). Assume $|\Delta_i| > p$. Then $s = 2$ and $\Delta_1 \cup \Delta_2 = \Lambda$ by (i). Let $\Gamma_1$ be a subset of $\Delta_1$ with $|\Delta_1 - \Gamma_1| = p$, and let $\delta$ be a point of $\Delta_1 - \Gamma_1$. Since $|\Delta_1 - (\Gamma_1 \cup \{\delta\})| = p - 1$, for every subset $\Gamma_2$ of $\Delta_2$ with $|\Delta_2 - \Gamma_2| = p$, $T_{\Gamma_1 \cup \{\delta\} \cup \Gamma_2}$ has a $p$-cycle on $\Delta_2 - \Gamma_2$, contrary to (ii).

Therefore $T$ is intransitive on $\Lambda$. Moreover by (ii), $T$ has an orbit whose length is not less than $p$. If $T$ has two orbits $\Delta_1$ and $\Delta_2$ such that $|\Delta_1| > p$ and $|\Delta_2| > p$, then we have a contradiction by the similar argument to the above. Hence $T$ has a unique orbit $\Sigma$ with $|\Sigma| > p$. By (ii), we have $2p \leq |\Sigma| < |\Delta|$. Let $\Pi$ be a subset of $\Sigma$ with $|\Pi| = |\Lambda - \Sigma| = p$. Since $|\Lambda - \Sigma| < p$, for every subset $\Gamma$ of $\Sigma - \Pi$ with $|\Gamma| = p - |\Pi|$, $T_{\Pi \cup \Gamma}$ has a $p$-cycle on $(\Sigma - \Pi) - \Gamma$, contrary to (ii).

Thus we complete the proof of Theorem 1.

3. Proof of Theorem 2

Let $G$ be a group satisfying the assumption of Theorem 2. Let $P$ be a
Sylow $p$-subgroup of $G_{1,2,...,2p}$. If $P=1$, then $G$ is one of $S_n$ ($2p \leq n \leq 3p-1$) and $A_n(2p+2 \leq n \leq 3p-1)$ by [1]. From now on we assume that $P \neq 1$, and prove that this case does not occur. By [1] and Theorem 1, we have $N_G(P)^{I(P)} = S_{2p}$. By [2], we may assume that $P \neq 1$, and prove that this case does not occur. By [1] and Theorem 1, we have $N_G(P) = S_{2p}$.

Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing more than $2p$ points. By [3, Lemma 6] and [2], $N_G(Q)^{I(Q)} \geq A^{I(Q)} = A_{3p}$. Since $A_n$ is a simple group, we have a contradiction.

4. Proof of Theorem 3

Let $G$ be a group satisfying the assumption of Theorem 3. Let $P$ be a Sylow $3$-subgroup of $G_{1,2,...,7}$. If $P=1$, then $G$ is $S_7$, $S_8$, $S_9$, or $A_9$ by [1]. From now on we may assume that $P \neq 1$. Since $P \triangleleft G_{1,2,...,7}$, we have $N_G(P)^{I(P)} = S_7$ by [1], [4] and [5]. If $P$ is semiregular on $\Omega - I(P)$, then $G$ is $S_{10}$ or $A_{10}$ by [2]. Hereafter we assume that $P$ is not semiregular, and prove that this case does not occur.

Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing more than ten points. Let $N=N_G(Q)^{I(Q)}$ and $\Gamma=N(Q)$.

Then $N$ is a permutation group on $\Gamma$, and $|\Gamma| \geq 13$ and $3|\Gamma|-7$. If $N$ has no element of order three fixing ten points, then $N$ is $S_{10}$ or $A_{10}$ by [3, Lemma 6] and [2], which is a contradiction. Hence from now on we may assume that $N$ has an element $a$ of order three fixing exactly ten points. We may assume that


Set $T=C_N(a)^{I(a)}_{1,12,13}$.

Suppose that $T$ has an orbit of length one. Then we may assume that $\{1\}$ is a $T$-orbit. $T_{234}$ has an element $x_1$ of order three, and we may assume that $x_1=(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)$. $T_{234}$ has an element $x_2$ of order three. Since a Sylow $3$-subgroup of $T_{1234}$ is normal in $T_{1234}$, $x_1x_2$ is a $3$-element. Hence we may assume that $x_2=(1)(2)(3)(4)(5)(8)(9)(10)(5 6 7)$. $T_{2358}$ has an element $x_3$ of order three. Since a Sylow $3$-subgroup of $T_{12358}$ is normal in $T_{12358}$, $x_1x_3$ is a $3$-element. Hence we may assume that $x_3=(1)(2)(3)(5)(8)(9)(10)(4 6 7)$, and so $x_2x_3=(1)(2)(3)(5)(8)(9)(10)(4 6 5 7)$. On the other hand, since $x_2$ and $x_3$ are $3$-elements of $T_{12358}$, $x_1x_3$ is a $3$-element. So, we have a contradiction.

By the same argument as the above, we have that $G$ has no orbit of length two or three.

Suppose that $T$ has an orbit of length four. Then we may assume that $\{1, 2, 3, 4\}$ is a $T$-orbit. Since $T_{5678}$ has an element of order three, we may assume that $T$ has an element of order three of the form $(1, 2, 3)(4)(5)(6)(7)(8)(9)$ or $(10)$. Since $T^{I(1234)}$ is transitive, we have $T^{I(1,2,3,4)} \geq A_4$, which is a contradiction.

By the similar argument to the above, we have that $T$ is neither an intransi-
tive group with an orbit of length five nor an imprimitive group with two blocks of length five.

Finally, it is easily seen that $T$ is neither an imprimitive group with five blocks of length two nor a primitive group (cf. [6]), and we complete the proof.

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References