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ON GROUPS $H^n(S/R)$ RELATED TO THE AMITSUR COHOMOLOGY AND THE BRAUER GROUP OF COMMUTATIVE RINGS

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The Amitsur cohomology with respect to the unit functor has been studied by many authors. One of the most interesting features of the theory is that its second cohomology group $H^2(S/R, U)$ gives a description of the Brauer group $Br(S/R)$ in far general cases beyond Galois extensions ([1], [13]). But in ring case the extension S/R must satisfy some restrictive condition for the validity of the isomorphism, and Chase and Rosenberg established an exact sequence which is comprised of the unit cohomology, Pic cohomology and the Brauer group, instead of the direct description of $Br(S/R)$ ([4]).

In a preceding paper, we attached a series of abelian groups $\boldsymbol{H}^n(S, G)$ to a commutative ring S and a group G operating on S, which are defined in close connection with the Pic-valued group cohomology, and we showed that if S is a finite Galois extension of R with G as the Galois group, $\boldsymbol{H}^2(S, G)$ is isomorphic to $Br(S/R)$ ([9]), see also [8]).

In this paper, we shall develop a parallel theory in the framework of the Amitsur cohomology, and prove among others that if S is finite projective and faithful as an R-module, our second group is isomorphic to $Br(S/R)$. This extends both the above mentioned case of Galois extensions, and the description by means of the unit-valued cohomology so far established.

In § 1 we shall define the groups $\boldsymbol{H}^n(S/R)$ and prove a long exact sequence which, combined with the interpretation of $\boldsymbol{H}^2(S/R)$ as the Brauer group, yields the Chase-Rosenberg sequence. This part is an immediate transcription of the corresponding part of [9]. The theory of faithfully flat descent precisely fits to the situation around $\boldsymbol{H}^1(S/R)$, and is applied to prove an isomorphism $\boldsymbol{H}^1(S/R)\simeq \mathrm{Pic}(R)$ (§ 2). After some analysis of '2-cocycles' in § 3, we introduce and study a class of algebras denoted by (A, P, p) in § 4. This may be considered as a far more generalized version of the concept of crossed products, and indeed covers the known constructions so far treated in various context. Further, it is immediately observed that the multiplication alteration of Sweedler [15] (hence in particular the construction of Rosenberg and Zelinsky [13] as noted by Sweedler) is nothing but the unit-valued case of our construction. We then prove in § 5 that this construction leads to the isomorphism $\boldsymbol{H}^2(S/R) \simeq Br(S/R)$ stated above. In § 6 we establish a long exact sequence concerning a homomorphism of extensions $S/R \to S'/R'$, in which appear a certain kind of relative Amitsur cohomology groups as relative terms. This section is parallel to [9] § 4. The paper closes by § 7 dealing with the case of Galois extensions. (See also Hattori [23].)

We owe to a recent paper of Yokogawa [18], which we have had an opportunity to read before publication. It gives a direct proof to the Chase-Rosenberg exact sequence, by attaching a Pic-valued 1-cocycle P to an S/R-Azumaya algebra, a U-valued 3-cocycle u to P, and by constructing an algebra related to P, which may be interpreted as our (E, P, p).

After this work was completed, we have got access to a recent paper of Villamayor and Zelinsky [16]. It deals with similar problems as ours, and establishes a description of the Brauer group in somewhat more general case. The basic ideas seem to be near to each other, but in contrast with their categorical approach, we proceed concretely by making use of the construction of crossed product nature. (See also Ulbrich [20], Hattori [22].)

M. Takeuchi informs us that he has also obtained several results on the Brauer group, including Theorem 5.2. His paper is in preparation. (Cf. [19].)

We shall treat in a subsequent paper the case where S is operated by a finite group G without being Galois over the fixed subring R. (Published as [21].)

1. $\boldsymbol{H}^n(S/R)$ and an exact sequence

1.1. Let R be a commutative ring with identity. R is the base ring of various algebras considered in this paper, and an unspecified $\otimes$ means $\otimes_R$ unless otherwise stated. Let S be a commutative algebra over R, and denote by S^n the tensor product $S \otimes \cdots \otimes S$ of n copies of S. Its identity $1 \otimes \cdots \otimes 1$ is denoted by 1^n. As customary, let $\varepsilon_i \colon S^n \to S^{n+1}$ $(i=1, \cdots, n+1)$ denote the algebra homomorphisms defined by

$$\varepsilon_i(s_1 \otimes \cdots \otimes s_n) = s_1 \otimes \cdots \otimes s_{i-1} \otimes 1 \otimes s_i \otimes \cdots \otimes s_n$$

They satisfy the following identities:

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \qquad (i \leq j) \tag{1.1}$$

Each ε_i defines a functor $S^{n+1} \otimes_{S^n}$ of the module categories. We prefer the notation $\varepsilon_i M$ to denote the module $S^{n+1} \otimes_{S^n} M$ thus obtained. We also use the notation $\varepsilon_i x$ to denote the image $1^{n+1} \otimes x$ of $x \in M$ by the canonical map $M \to \varepsilon_i M$. This is compatible with the original definition of $\varepsilon_i \colon S^n \to S^{n+1} = \varepsilon_i S^n$. $\varepsilon_i M$ is generated as S^{n+1}-module by the set of $\varepsilon_i x$ $(x \in M)$. For $f \in \mathrm{Hom}_{S^n}(M, N)$, $\varepsilon_i f \in \mathrm{Hom}_{S^{n+1}}(\varepsilon_i M, \varepsilon_i N)$ is determined by the condition $\varepsilon_i f(\varepsilon_i x) = \varepsilon_i f(x)$ $(x \in M)$.

Let $\mathcal{P}ic(S^n)$ be the category of projective S^n-modules of rank one ($n=1, 2, \cdots$). This is a category with product $\otimes_{S^n}$. For $P\in\mathcal{P}ic(S^n)$, P^* denotes the dual module of P as an S^n-module unless otherwise stated. Hence $P^*\in\mathcal{P}ic(S^n)$, and there is a canonical pairing $\langle\ \rangle$: $P\otimes_{S^n}P^*\xrightarrow{\sim}S^n$. This pairing satisfies the commutativity of the diagram

$$\begin{array}{ccc} P\otimes_{S^n}P^*\otimes_{S^n}P & \xrightarrow{\langle\ \rangle\otimes 1} & S^n\otimes_{S^n}P \\ \downarrow{\scriptstyle 1\otimes\langle\ \rangle} & & \| \\ P\otimes_{S^n}S^n & = & P \end{array} \tag{1.2}$$

This property will be utilized quite often in the sequel (cf. [9] § 1). In particular, if $\sum\langle x_i, \xi_i\rangle=1$, we have $x=\sum\langle x, \xi_i\rangle x_i$ for every $x\in P$. In this case we say that $\{x_i\}$ and $\{\xi_i\}$ are a pair of dual bases of P and P^*. An isomorphism f: $P\xrightarrow{\sim}Q$ has its dual $f^*=({}^tf)^{-1}$: $P^*\xrightarrow{\sim}Q^*$.

ε_i: $S^n\to S^{n+1}$ yields a functor $\mathcal{P}ic(S^n)\to\mathcal{P}ic(S^{n+1})$, which preserves the product and the dual. The latter means that there exists a natural isomorphism $\varepsilon_i(P^*)\simeq(\varepsilon_iP)^*$, where the convention on the usage of * is as explained above. We define d_n: $\mathcal{P}ic(S^n)\to\mathcal{P}ic(S^{n+1})$ as the 'alternate sum' of ε_i, i.e.

$$d_nP=\varepsilon_1P\otimes_{S^{n+1}}\varepsilon_2P^*\otimes_{S^{n+1}}\cdots$$

and also for f: $P\xrightarrow{\sim}Q$ in $\mathcal{P}ic(S^n)$,

$$d_nf=\varepsilon_1f\otimes\varepsilon_2f^*\otimes\cdots\ :d_nP\xrightarrow{\sim}d_nQ$$

There exists a canonical isomorphism I_{n+1}: $d_nS^n\xrightarrow{\sim}S^{n+1}$, through which we identify d_nS^n with S^{n+1}. An automorphism of $P\in\mathcal{P}ic(S^n)$ is given by the multiplication of a unit $u\in S^n$, which will be written as $\underline{u}$ in this paper. Then we have $d\underline{u}=\underline{du}$, where du denotes the coboundary of u in the U-valued cohomology.

We denote the isomorphism class of P by $|P|$. The set of all $|P|$ ($P\in\mathcal{P}ic(S^n)$) constitutes an abelian group $\mathrm{Pic}(S^n)$. d_n induces a homomorphism $\mathrm{Pic}(S^n)\to\mathrm{Pic}(S^{n+1})$, satisfying $d_{n+1}d_n=0$, and we have the Pic-valued Amitsur cohomology groups:

$$H^n(S/R, \mathrm{Pic})=\mathrm{Ker}(d_{n+1})/\mathrm{Im}(d_n).$$

In the sequel d_n will be denoted as d, unless specific mention to the degree n is needed.

1.2. We now proceed parallel with [9] § 2 toward the definition of groups $\boldsymbol{H}^n(S/R)$. For any $P\in\mathcal{P}ic(S^n)$, we have a canonical isomorphism $d^2P\xrightarrow{\sim}S^{n+2}$, given by contracting all dual pairs appearing in the expression of d^2P. We use the notation c_P or *can* to denote this isomorphism. For f: $P\xrightarrow{\sim}Q$, the following diagram is commutative:

$$(1.3)\qquad \begin{array}{ccc} d^2P & \xrightarrow{c_P} & S^{n+2} \\ \downarrow{\scriptstyle d^2f} & & \| \\ d^2Q & \xrightarrow{c_Q} & S^{n+2} \end{array}$$

In particular, $c_{S^n}: d^2S^n \xrightarrow{\sim} S^{n+2}$ coincides with the composite of $dI_{n+1}: d^2S^n \to dS^{n+1}$ and $I_{n+2}: dS^{n+1} \xrightarrow{\sim} S^{n+2}$, and we use this isomorphism to identify d^2S^n with S^{n+2}.

Let $n \geq 1$. (P, p) denotes a pair of a module $P \in \mathcal{P}ic(S^n)$ such that $|P| \in Z^{n-1}(S/R, \text{Pic})$ and an isomorphism $p: dP \xrightarrow{\sim} S^{n+1}$. An isomorphism $(P, p) \xrightarrow{\sim} (P', p')$ is an isomorphism $f: P \xrightarrow{\sim} P'$ satisfying $p = p'df$. We denote the category of these pairs and their isomorphisms by $\mathcal{P}^n(S/R)$. This is a category with product defined naturally by $(P, p)(Q, q) = (P \otimes_{S^n} Q, p \otimes_{S^{n+1}} q)$. The set of isomorphism classes $((P, p))$ of $(P, p) \in \mathcal{P}^n(S/R)$ forms an abelian group, which we write $\boldsymbol{P}^n(S/R)$. We denote by $\boldsymbol{Z}^n(S/R)$ the subgroup of $\boldsymbol{P}^n(S/R)$ consisting of all $((P, p))$ satisfying $dp = c_P$ (we are identifying dS^n with S^{n+1} via I_{n+1}), and by $\boldsymbol{B}^n(S/R)$ the set of all $((dP, c_P))$ $(P \in \mathcal{P}ic(S^{n-1}))$. For $n=1$, we put $\boldsymbol{B}^1(S/R) = \{((S, I_2))\}$. Since $dc_P = c_{dP}$, $\boldsymbol{B}^n(S/R)$ is a subgroup of $\boldsymbol{Z}^n(S/R)$, and we have the groups

$$\boldsymbol{H}^n(S/R) = \boldsymbol{Z}^n(S/R)/\boldsymbol{B}^n(S/R)$$

For $n=0$, we put $\boldsymbol{Z}^0(S/R) = \{u \in U(S) \mid du = u^{-1} \otimes u = 1\}$ and $\boldsymbol{B}^0(S/R) = \{1\}$. Hence $\boldsymbol{H}^0(S/R) = H^0(S/R, U)$.

There is another way to describe these groups $\boldsymbol{H}^n(S/R)$. Let $\mathcal{P}_h^n(S/R)$ be the category of triples (P, f, Q) where $P, Q \in \mathcal{P}ic(S^n)$ and $f: dP \xrightarrow{\sim} dQ$, and isomorphisms $(P, f, Q) \xrightarrow{\sim} (P', f', Q')$ which is a pair of isomorphisms $p: P \xrightarrow{\sim} P'$ and $q: Q \xrightarrow{\sim} Q'$ satisfying $f'dp = dqf$. This is a category with product, and this product induces on the set of isomorphism classes $((P, f, Q))$ the structure of an abelian group. We write $\boldsymbol{P}_h^n(S/R)$ the factor group of this abelian group by the relation

$$((P, f, Q))((Q, g, R)) = (P, gf, R))$$

Then this group is isomorphic to $\boldsymbol{P}^n(S/R)$, since the map $((P, p) \mapsto ((P, p, S^n))$ has an inverse given by

$$((P, f, Q)) \mapsto (P \otimes_{S^n} Q^*, f^\#)\,,$$

where

$$f^\# = \langle\quad\rangle (f \otimes 1): dP \otimes_{S^{n+1}} dQ^* \to dQ \otimes_{S^{n+1}} dQ^* \to S^{n+1}$$

In this correspondence, $\boldsymbol{Z}^n(S/R)$ corresponds to the subgroup $\boldsymbol{Z}_h^n(S/R)$ consisting of $((P, f, Q))$ such that $df = c_Q^{-1} c_P$, and $\boldsymbol{B}^n(S/R)$ to $\boldsymbol{B}_h^n(S/R)$ consisting of

$((dP, c_Q^{-1}c_P, dQ))(P, Q\in \mathcal{P}ic(S^{n-1}))$, Thus $\boldsymbol{H}^n(S/R)$ is isomorphic to $\boldsymbol{Z}_h^n(S/R)/\boldsymbol{B}_h^n(S/R)$. The subscript h means the homogeneous description.

1.3. This part is an adaptation of [9] § 3 to the present case. For details the reader is referred to that part.

Every $u\in U(S^{n+1})$ determines a pair $(S^n, \underline{u})$ where $\underline{u}$: $dS^n=S^{n+1}\rightarrow S^{n+1}$, and $((S^n, \underline{u}))\in \boldsymbol{Z}^n(S/R)$ if and only if $u\in Z^n(S/R, U)$. Since $(S^n, \underline{dv})\simeq(S^n, \underline{1})$ $(\simeq(dS^{n-1}, c_{S^{n-1}})$ if $n\geqq 1)$, we have a homomorphism

$$\alpha^n\colon H^n(S/R, U)\rightarrow \boldsymbol{H}^n(S/R);\ cl(u)\mapsto cl((S^n, \underline{u}))$$

For $n=0$, α^0 is defined to be the identity map $u\mapsto u$.

The definability of the following map is clear.

$$\beta^n\colon \boldsymbol{H}^n(S/R)\rightarrow H^{n-1}(S/R, \mathrm{Pic});\ cl((P, p))\mapsto cl\,|P|\,.$$

Let $|P|\in Z^{n-1}(S/R, \mathrm{Pic})$, and take any p: $dP\overset{\sim}{\rightarrow} S^{n+1}$. There exists a unit $u\in S^{n+2}$ such that

(1.4)
$$\begin{tikzcd} d^2P \arrow[r, "c_P"] \arrow[d, equal] & S^{n+2} \arrow[d, "\underline{u}"] \\ d^2P \arrow[r, "dp"] & S^{n+2} \end{tikzcd}$$

is commutative, and we see easily that $du=1^{n+3}$. Changing P to an isomorphic module P' does not affect the cohomology class of u. Hence we have the following homomorphism.

$$\gamma^n\colon H^{n-1}(S/R, \mathrm{Pic})\rightarrow H^{n+1}(S/R, U);\ cl\,|P|\mapsto cl(u)\,.$$

Theorem 1.1. *The following sequence is exact*:

$$0\rightarrow H^1(S/R, U)\xrightarrow{\alpha^1}\boldsymbol{H}^1(S/R)\xrightarrow{\beta^1}H^0(S/R, \mathrm{Pic})\xrightarrow{\gamma^1}\cdots$$
$$\cdots\xrightarrow{\gamma^{n-1}}H^n(S/R, U)\xrightarrow{\alpha^n}\boldsymbol{H}^n(S/R)\xrightarrow{\beta^n}H^{n-1}(S/R, \mathrm{Pic})\xrightarrow{\gamma^n}H^{n+1}(S/R, U)\rightarrow\cdots$$

Outline of Proof. It is easily verified from the definition of maps that the composite of any two consecutive maps reduces to 0. Let $cl((P, p))\in \mathrm{Ker}(\beta^n)$. We may assume that $P=dQ$ with some $Q\in\mathcal{P}ic(S^{n-1})$. Then there exists $u\in U(S^{n+1})$ such that $p=\underline{u}c_Q$, and it must satisfy $du=1$. Since we have

$$(dQ, p)=(dQ, c_Q)(S^n, \underline{u})\,,$$

$((P, p))=((dQ, p))\in \mathrm{Im}(\alpha^n)$. Here we treated the case $n>1$. But the case $n=1$ is easy. If $cl|P|\in\mathrm{Ker}(\gamma^n)$, we have $dp=c_P$ with a suitably chosen p: $dP\overset{\sim}{\rightarrow}S^{n+1}$. This means that $cl|P|\in\mathrm{Im}(\beta^n)$. If $\mathrm{cl}(u)\in\mathrm{Ker}(\alpha^{n+1})$, there exists $P\in\mathcal{P}ic(S^{n-1})$ such that $(S^n, \underline{u})\simeq(dP, c_P)$. This means that there exists $p\colon dP\xrightarrow{\sim} S^n$ satisfying $c_P=\underline{u}\,dp$. Hence $u^{-1}\in\mathrm{Im}(\gamma^n)$, and therefore $u\in\mathrm{Im}(\gamma^n)$.

2. Interpretation of $\boldsymbol{H}^0(S/R)$ and $\boldsymbol{H}^1(S/R)$

Proposition 2.1. *If S is faithfully flat over R, then $\boldsymbol{H}^0(S/R)\simeq U(R)$.*

This is clear by [12] II.2.2. We shall proceed to $\boldsymbol{H}^1(S/R)$. We denote the unit map $R\to S$ by ε_0.

Theorem 2.2. *If S is faithfully flat over R, then $\boldsymbol{H}^1(S/R)\simeq\mathrm{Pic}(R)$.*

Proof. $P_0\in\mathcal{P}ic(R)$ determines a pair (P, p) defined as follows:

$$P=\varepsilon_0P_0=S\otimes P_0$$
$$p\colon \varepsilon_1P\otimes_{S^2}\varepsilon_2P^*\xrightarrow{\sim} S^2;\ \varepsilon_1\varepsilon_0x\otimes\varepsilon_2\varepsilon_0\xi\mapsto\langle x, \xi\rangle 1^2$$

where we identified P^* with $\varepsilon_0P_0{}^*$. We shall compare dp with c_P. The image of

$$(\varepsilon_{110}x_1\otimes\varepsilon_{120}\xi_1)\otimes(\varepsilon_{210}\xi_2\otimes\varepsilon_{220}x_2)\otimes(\varepsilon_{310}x_3\otimes\varepsilon_{320}\xi_3)\in d^2P\,,$$

(where $x_i\in P_0$, $\xi_i\in P_0{}^*$, $\varepsilon_{ijk}=\varepsilon_i\varepsilon_j\varepsilon_k$, and $\otimes=\otimes_{S^3}$) by the map dp is

$$\langle x_1, \xi_1\rangle\langle x_2, \xi_2\rangle\langle x_3, \xi_3\rangle 1^3\,,$$

while its image by c_P is

$$\langle x_1, \xi_2\rangle\langle x_2, \xi_3\rangle\langle x_3, \xi_1\rangle 1^3$$

But by the commutativity of (1.2) these two elements of S^3 are identical. Namely (P, p) satisfies the $\boldsymbol{Z}^1$-condition. Clearly the correspondence $P_0\mapsto(P, p)$ is multiplicative and preserves the isomorphism of objects. Hence we have a homomorphism $\mathrm{Pic}(R)\to\boldsymbol{Z}^1(S/R)=\boldsymbol{H}^1(S/R)$. We shall show that this homomorphism admits an inverse mapping. To this purpose, let $((P, p))\in\boldsymbol{Z}^1(S/R)$. We convert p to the following isomorphism:

$$\tilde{p}\colon \varepsilon_1P\xrightarrow{\sim}\varepsilon_1P\otimes_{S^3}\varepsilon_2P^*\otimes_{S^3}\varepsilon_2P\xrightarrow[\sim]{p\otimes 1}\varepsilon_2P$$
$$\varepsilon_1x\mapsto\sum p(\varepsilon_1x\otimes\varepsilon_2\xi_i)\varepsilon_2x_i$$

where $\{x_i\}$ and $\{\xi_i\}$ are a pair of dual bases. On the other hand, the $\boldsymbol{Z}^1$-condition $dp=c_P$ can be expressed as the commutativity of the following diagram (where $\otimes=\otimes_{S^3}$):

$$\begin{array}{ccccc} (\varepsilon_1\varepsilon_1P\otimes\varepsilon_1\varepsilon_2P^*)\otimes(\varepsilon_3\varepsilon_1P\otimes\varepsilon_3\varepsilon_2P^*) & & \overset{1\otimes\langle\ \rangle\otimes 1}{\simeq} & & \varepsilon_2\varepsilon_1P\otimes\varepsilon_2\varepsilon_2P^* \\ \downarrow\varepsilon_1p \qquad\qquad \downarrow\varepsilon_3p & & & & \downarrow\varepsilon_2p \\ \varepsilon_1S^2\quad\otimes\quad\varepsilon_3S^2 & & \simeq & & \varepsilon_2S^2 \end{array}\tag{2.1}$$

We examine the composite of maps $\varepsilon_1\tilde{p}$ and $\varepsilon_3\tilde{p}$:

$$\varepsilon_1\tilde{p}\colon \varepsilon_{11}x \mapsto \sum \varepsilon_1 p(\varepsilon_{11}x\otimes\varepsilon_{12}\xi_i)\varepsilon_{12}x_i$$
$$\varepsilon_3\tilde{p}\colon \qquad \mapsto \sum \varepsilon_1 p(\varepsilon_{11}x\otimes\varepsilon_{12}\xi_i)\sum \varepsilon_3 p(\varepsilon_{31}x_i\otimes\varepsilon_{32}\xi_j)\varepsilon_{32}x_j$$

By the commutativity of (2.1), this last element is identical with

$$\sum \varepsilon_2 p(\varepsilon_{11}x\otimes\sum\langle\varepsilon_{31}x_i,\ \varepsilon_{12}\xi_i\rangle\otimes\varepsilon_{32}\xi_j)\varepsilon_{32}x_j$$
$$=\sum \varepsilon_2 p(\varepsilon_{21}x\otimes\varepsilon_{22}\xi_j)\varepsilon_{22}x_j$$

Thus we have $\varepsilon_3\tilde{p}\circ\varepsilon_1\tilde{p}=\varepsilon_2\tilde{p}$. It follows from the descent theory that there exist $P_0\in\mathcal{P}ic(R)$ and an S-isomorphism $p_0\colon S\otimes P_0\xrightarrow{\sim}P$ such that $\tilde{p}\varepsilon_i p_0=\varepsilon_2 p_0$, and the pair (P_0, p_0) is determined up to isomorphism by the condition ([12]) II Theorem 3.2). Hence we have a well-defined map $((P, p))\mapsto P_0$ which is the inverse of the map defined at the first part of the proof.

3. Preliminary considerations on $\mathcal{P}^2(S/R)$

3.1. From this section on, we deal with S^2-modules and S^3-modules of various type. Sometimes (but not always) we regard an S^2-module X as a left S- and right S-module. Then the notation ${}_1X_2$ means that $s_1\otimes s_2\in S^2$ acts on X as s_1xs_2. $\mathrm{End}_{S,-}(X)$ means the endomorphism ring of X regarded as a left S-module. ${}_1X_2\otimes_{S_2}{}_2Y_3$ means that we form the tensor product of X and Y satisfying the condition $xs\otimes y=x\otimes sy$, and then regard it as an S^3-module under the operation $(s_1\otimes s_2\otimes s_3,\ x\otimes y)\mapsto s_1xs_2\otimes ys_3$. X^S denotes the subset $\{x\in X\mid sx=xs,\ s\in S\}$ of X, which is isomorphic to $\mathrm{Hom}_{S^2}(S, X)$. Thus e.g. $({}_1X_2\otimes_{S_1}{}_1Y_2)^{S_2}$ means that this is an S^2-module consisting of elements $\sum x_i\otimes y_i$ of ${}_1X\otimes_{S_1}{}_1Y$ which satisfy $\sum x_is\otimes y_i=\sum x_i\otimes y_is$ $(s\in S)$, with the S^2-operation given by $\sum s_1x_i\otimes y_is_2$.

We denote the twist map $S^2\to S^2\colon s_1\otimes s_2\mapsto s_2\otimes s_1$ by τ. For an S^2-module M, we denote the module τM by M^0, which is derived from M by exchanging the left and right S-operations. We use the notation $\pi\colon S^n\to S$ to denote the map defined by $s_1\otimes\cdots\otimes s_n\mapsto s_1\cdots s_n$. We further introduce the notations $\pi_i\colon S^3\to S^2$ $(i=1, 2, 3)$ to denote the contraction maps defined by

$$\pi_1(s_1\otimes s_2\otimes s_3)=s_1\otimes s_2s_3$$
$$\pi_2(s_1\otimes s_2\otimes s_3)=s_2\otimes s_1s_3$$
$$\pi_3(s_1\otimes s_2\otimes s_3)=s_3\otimes s_1s_2$$

The composite of π_i with the ε_j is given by the following table:

(3.1)

	ε_1	ε_2	ε_3
π_1	$1\otimes\pi$	1	1
π_2	1	$1\otimes\pi$	τ
π_3	τ	τ	$1\otimes\pi$

Now an object (P, p) of $\mathcal{P}^2(S/R)$ consists of an S^2-module $P \in \mathcal{P}ic(S^2)$ and an S^3-isomorphism $p: dP \xrightarrow{\sim} S^3$. The isomorphism p can be transformed to the following form:

$$\varepsilon_3 P \otimes_{S^3} \varepsilon_1 P \xrightarrow{\sim} \varepsilon_2 P, \tag{3.2}$$

or in another expression to

$$\tilde{p}: {}_1P_2 \otimes_{S_2} {}_2P_3 \xrightarrow{\sim} S_2 \otimes {}_1P_3 \tag{3.3}$$

We introduce the notation

$$\tilde{p}(x, y) = \sum_{(P,p)} \tilde{p}_S(x, y) \otimes \tilde{p}_P(x, y) \tag{3.4}$$

to denote the image of $x \otimes y$ by the isomorphism (3.3). The relation with the S^3-operation is expressed as

$$\tilde{p}(s_1 x s_2, y s_3) = \tilde{p}(s_1 x, s_2 y s_3) = \sum_{(P,p)} s_2 \tilde{p}_S(x, y) \otimes s_1 \tilde{p}_P(x, y) s_3 .$$

Conversely, any S^3-isomorphism (3.3) gives rise to an S^3-isomorphism $p: dP \xrightarrow{\sim} S^3$ by putting

$$p(\varepsilon_1 y \otimes \varepsilon_2 \zeta \otimes \varepsilon_3 x) = \langle \tilde{p}(x, y), \varepsilon_2 \zeta \rangle \qquad (x, y \in P, \zeta \in P^*) . \tag{3.5}$$

Proposition 3.1. *Let $P \in \mathcal{P}ic(S^2)$ be such that $|P| \in Z^1(S/R, \mathrm{Pic})$. Then we have the following isomorphisms:*

$$\pi P \simeq S , \tag{3.6}$$

$$P^0 \otimes_{S^2} P \simeq S^2 . \tag{3.7}$$

Hence P^0 is isomorphic to the dual module P^*.

Proof. Take any $p: dP \xrightarrow{\sim} S^3$, and apply π_1 to both terms of the isomorphism (3.2). Then, in view of (3.1), we have

$$P \otimes_{S^2} (S \otimes \pi P) \simeq P$$

Multiplying P^*, we have $S \otimes \pi P \simeq S^2$, and the contraction π: $S \otimes_{S^2}$ yields the first isomorphism $\pi P \simeq S$. If we apply π_2 to the isomorphism (3.2), we get

$$P^0 \otimes_{S^2} P \simeq S \otimes \pi P \simeq S \otimes S .$$

These isomorphisms obviously depend on the choice of $p: dP \xrightarrow{\sim} S^3$. We denote the induced map $P \to \pi P \to S$ by π_p, whose explicit form is given by

$$\pi_p(y) = \pi \sum \langle \pi_1 \tilde{p}(x_i, y), \xi_i \rangle$$

where $\{x_i\}$ and $\{\xi_i\}$ are a pair of dual bases of P and P^*. Using this map π_p, the isomorphism (3.7) is expressed as

$$x^0 \otimes y \mapsto \sum_{(P,p)} \tilde{p}_S(x, y) \otimes \pi_p \tilde{p}_P(x, y) \tag{3.8}$$

Lemma 3.2. *Let $P \in \mathcal{P}ic(T)$. If an isomorphism $f: X \xrightarrow{\sim} Y$ of T-modules is derived from $g: P \otimes_T X \xrightarrow{\sim} P \otimes_T Y$ by the commutativity of*

$$\begin{array}{ccc} P^* \otimes_T P \otimes_T X & \xrightarrow[\sim]{1 \otimes g} & P^* \otimes_T P \otimes_T Y \\ \downarrow{\scriptstyle \langle\ \rangle \otimes 1} & & \downarrow{\scriptstyle \langle\ \rangle \otimes 1} \\ X & \xrightarrow{f} & Y \end{array}$$

then conversely we have $g = 1 \otimes f$.

This follows immediately by tensoring P and applying the commutativity concerning $\langle\ \ \rangle$ (1.2).

We apply this Lemma to the following maps:

$$f: S \otimes \pi P \xrightarrow{\sim} S^2;\ s \otimes \pi y \mapsto s \otimes \pi_p(y)$$
$$g: P \otimes_{S^2} (S \otimes \pi P) \xrightarrow{\sim} P;\ x \otimes (1 \otimes \pi y) \mapsto \pi_1 \tilde{p}(x, y)$$

Then we get

$$x\pi_p(y) = \pi_1 \tilde{p}(x, y) = \sum_{(P,p)} \tilde{p}_P(x, y) \tilde{p}_S(x, y) \qquad (x, y \in P)\,. \tag{3.9}$$

If we apply π_3 to the isomorphism (3.2), we have the following isomorphism:

$$(S \otimes \pi P) \otimes P^0 \xrightarrow{\sim} P^0;\ (1 \otimes \pi x) \otimes y^0 \mapsto \pi_3 \tilde{p}(x, y)$$

Arguing as in the proof of Proposition 3.1, we obtain a new isomorphism $\pi P \xrightarrow{\sim} S$. For a moment we denote the map $P \to \pi P \to S$ thus obtained by π'_p, i.e.

$$\pi'_p(x) = \pi(\sum \langle \pi_3 p(x, x_j), \xi_j^0 \rangle)\,.$$

Lemma 3.3. *π'_p coincides with π_p.*

Proof. In view of (3.5), we can express π_p as follows:

$$\pi_p(x) = \pi \sum p(\varepsilon_1 x \otimes \varepsilon_2 \xi_i \otimes \varepsilon_3 x_i)$$

where $\{x_i\}$ and $\{\xi_i\}$ are as above. Since $\varepsilon_1 x = \sum (1 \otimes \langle x, \xi_j \rangle) \varepsilon_1 x_j$, we have

$$\begin{aligned} \pi_p(x) &= \sum_j \pi \langle x, \xi_j \rangle \pi \sum_i p(\varepsilon_1 x_j \otimes \varepsilon_2 \xi_i \otimes \varepsilon_3 x_i) \\ &= \sum_{i,j} p(\varepsilon_1 x_j \otimes \varepsilon_2 (\langle x, \xi_j \rangle \xi_i) \otimes \varepsilon_3 x_i) \end{aligned}$$

A similar computation shows that

$$\pi'_p(x) = \pi \sum_{i,j} p(\varepsilon_1 x_j \otimes \varepsilon_2 (\langle x, \xi_i \rangle \xi_j) \otimes \varepsilon_3 x_i)$$

But, since $\langle x, \xi_j\rangle\xi_i=\langle x, \xi_i\rangle\xi_j$, we obtain that $\pi_p(x)=\pi'_p(x)$.

Applying Lemma 3.2 to this $\pi'_p=\pi_p$, we have

$$\pi_p(x)y = (\pi_3\tilde{p}(p(x, y)))^0 = \sum_{(P,p)}\tilde{p}_S(x, y)\tilde{p}_P(x, y) \tag{3.10}$$

3.2. Now we shall make explicit what is meant by the $\boldsymbol{Z}^2$-condition $dp=c_P$. This identity may be read as the commutativity of the following diagram (where $\otimes=\otimes_{S^4}$):

$$\begin{array}{ccc}
(\varepsilon_1\varepsilon_1P\otimes\varepsilon_1\varepsilon_2P^*\otimes\varepsilon_1\varepsilon_3P)\otimes(\varepsilon_3\varepsilon_1P\otimes\varepsilon_3\varepsilon_2P^*\otimes\varepsilon_3\varepsilon_3P) & \overset{(p)}{\simeq} & \varepsilon_1S^3\otimes\varepsilon_3S^3 \\
\wr\!\wr \text{ canonical} & & \wr\!\wr \text{ canonical} \\
(\varepsilon_2\varepsilon_1P\otimes\varepsilon_2\varepsilon_2P^*\otimes\varepsilon_2\varepsilon_3P)\otimes(\varepsilon_4\varepsilon_1P\otimes\varepsilon_4\varepsilon_2P^*\otimes\varepsilon_4\varepsilon_3P) & \overset{(p)}{\cong} & \varepsilon_2S^3\otimes\varepsilon_4S^3
\end{array}$$

Tensoring $\varepsilon_3\varepsilon_2P\simeq\varepsilon_2\varepsilon_2P$ to every term, and cancelling several pairs of dual modules, we have the following commutative diagram:

$$\begin{array}{ccc}
\varepsilon_3\varepsilon_3P\otimes\varepsilon_1\varepsilon_3P\otimes\varepsilon_1\varepsilon_1P & \simeq & \varepsilon_3\varepsilon_2P \\
\wr\!\wr & & \wr\!\wr \\
\varepsilon_4\varepsilon_3P\otimes\varepsilon_4\varepsilon_1P\otimes\varepsilon_2\varepsilon_1P & \simeq & \varepsilon_2\varepsilon_2P
\end{array}$$

where the vertical isomorphisms are those derived from the identity (1.1). The horizontal isomorphisms are given respectively by the following maps:

$$\varepsilon_3\varepsilon_3x\otimes\varepsilon_1\varepsilon_3y\otimes\varepsilon_1\varepsilon_1z \mapsto \sum_{(P,p)}\tilde{p}_S(y, z)1_3\otimes\tilde{p}(x, \tilde{p}_P(y, z))\in S_3\otimes S_2\otimes P$$

$$\varepsilon_4\varepsilon_3x\otimes\varepsilon_4\varepsilon_1y\otimes\varepsilon_2\varepsilon_1z \mapsto \sum_{(P,p)}\tilde{p}_S(x, y)1_2\otimes\tilde{p}(\tilde{p}_P(x, y), z))\in S_2\otimes S_3\otimes P$$

Therefore we have

Lemma 3.4. *$((P, p))\in\boldsymbol{Z}^2(S/R)$ if and only if the following identity holds in $S_2\otimes S_3\otimes_1 P_4$ for every $x, y, z\in P$:*

$$\begin{aligned}
&\sum_{(P,p)}\sum_{(P,p)}\tilde{p}_S(x, \tilde{p}_P(y, z))1_2\otimes\tilde{p}_S(y, z)1_3\otimes\tilde{p}_P(x, \tilde{p}_P(y, z)) \\
&= \sum_{(P,p)}\sum_{(P,p)}\tilde{p}_S(x, y)1_2\otimes\tilde{p}_S(\tilde{p}_P(x, y), z)1_3\otimes\tilde{p}_P(\tilde{p}_P(x, y), z)
\end{aligned}$$

Later, this identity will be interpreted as the associativity of algebras constructed using (P, p).

Next we shall prove a proposition concerning the splitting of $\boldsymbol{Z}^2$-elements. Let R' be a commutative algebra over R, and denote the R'-algebra $R'\otimes S$ by $S_{R'}$, or S'. Then the R'-algebra $(S_{R'})^n$ is canonically isomorphic to $(S^n)_{R'}$, for every n. Similarly, for an S^n-module P, the S'^n-module $P'=S'^n\otimes_{S^n}P$ is isomorphic to $P_{R'}$. An S^n-homomorphism f: $M\to N$ yields an S'^n-homomorphism $f'=f_{R'}$: $M'\to N'$. For an S^n-isomorphism f: $M\xrightarrow{\sim} N$, we have $(df)'=df'$: $dM'\xrightarrow{\sim} dN'$. Hence a pair $(P, p)\in\mathcal{P}^n(S/R)$ yields $(P', p')\in\mathcal{P}^n(S'/R')$. (P, p) is said to be *split* by R' if $(P', p')\simeq(dQ, c_Q)$ with some $Q\in\mathcal{P}ic(S'^{n-1})$.

Proposition 3.5. *Every element of* $Z^2(S/R)$ *is split by* S. *More precisely, we have* $(P_S, p_S)\simeq(dP, c_P)$ *for every* (P, p) *satisfying* Z^2*-condition, where* P *of the right hand side is considered as an element of* $\mathcal{P}ic(S_S)$ *(not of* $\mathcal{P}ic(S^2)$*).*

Proof. We treat S^2_S-modules as S^3-modules via the canonical isomorphism $S^2_S\simeq S\otimes S^2\simeq S^3$, where we put $R'=S$ as the first factor. Then $P_S=S_1\otimes {}_2P_3\simeq\varepsilon_1P$. The isomorphism p is applied to yield

$$P_S\simeq\varepsilon_1P\simeq\varepsilon_2P\otimes_{S^3}\varepsilon_3P^* \tag{3.11}$$

On the other hand, we have for an S'-module M,

$$dM=(S'_1\otimes_{R'}M)\otimes_{S'^2}(S'_2\otimes_{S'}M^*)\simeq(S_1\otimes M)\otimes_{S'^2}(S_2\otimes M)^*$$

Adapting to the present case, we observe

$$dP\simeq(S_2\otimes_1P_3)\otimes_{S^3}(S_3\otimes_1P_2)^*\simeq\varepsilon_2P\otimes\varepsilon_3P^* \tag{3.12}$$

Combining this to (3.11), we have an S^3-isomorphism $P_S\simeq dP$. Next we shall show $p_S=c_P$. c_P: $d^2P\xrightarrow{\sim}(S_S)^3\simeq S^4$ is described, in view of the isomorphism (3.12), as the following isomorphism which takes place by the canonical pairings (where $\otimes=\otimes_{S^4}$):

$$(\varepsilon_2\varepsilon_2P\otimes\varepsilon_2\varepsilon_3P^*)\otimes(\varepsilon_3\varepsilon_2P^*\otimes\varepsilon_3\varepsilon_3P)\otimes(\varepsilon_4\varepsilon_2P^*\otimes\varepsilon_4\varepsilon_3P^*)\simeq S^4 \tag{3.13}$$

while p_S: $dP_S\xrightarrow{\sim}S^4$ is the following isomorphism derived from p: $dP\xrightarrow{\sim}S^3$:

$$\varepsilon_2\varepsilon_1P\otimes\varepsilon_3\varepsilon_1P^*\otimes\varepsilon_4\varepsilon_1P\simeq S^4$$

which is converted to an isomorphism of the same type as (3.13) by applying the isomorphism (3.11). Now the Z^2-condition $dp=c_P$ can be expressed in the following form:

$$\begin{array}{ccc}
S^4\otimes S^4\otimes S^4 & \overset{(p)}{\simeq} & \varepsilon_2dP^*\otimes\varepsilon_3dP\otimes\varepsilon_4dP^* \\
\wr\,\text{canonical} & & \wr\,\text{canonical} \\
S^4 & \overset{(p)}{\simeq} & \varepsilon_1dP^*
\end{array}$$

Multiplying three factors appearing in ε_1dP on all the four terms, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\varepsilon_2\varepsilon_1P\otimes\varepsilon_3\varepsilon_1P^*\otimes\varepsilon_4\varepsilon_1P & \overset{(3.11)}{\simeq} & \text{left hand side of (3.13)} \\
\| & & \wr\,\text{canonical} \\
\varepsilon_1\varepsilon_1P\otimes\varepsilon_1\varepsilon_2P^*\otimes\varepsilon_1\varepsilon_3P & \overset{p_S}{\simeq} & S^4
\end{array}$$

This commutativity means that the maps p_S and c_P as expressed in the form (3.13) exactly coincide.

4. The algebra (A, P, p)

4.1. In treating the crossed product-like construction, it is convenient to call a pair of an R-algebra A and an algebra embedding ι: $S\to A$ an S/R-*algebra*. In this case A has a natural S^2-module structure. An S-isomorphism of S/R-algebras is an algebra isomorphism which preserves the embedding of S.

Let A be an S/R-algebra, and let $(P, p)\in\mathcal{P}^2(S/R)$. We define a multiplication in $D=A\otimes_{S^2}P$ by putting

$$(a\otimes x)(b\otimes y)=\sum_{(P,p)}a\tilde{p}_S(x, y)b\otimes\tilde{p}_P(x, y) \qquad (a, b\in A,\ x, y\in P) \tag{4.1}$$

Lemma 4.1. *If $((P, p))\in Z^2(S/R)$, then the multiplication in $D=A\otimes_{S^2}P$ defined above satisfies the associativity for any S/R-algebra A. Conversely, if* $\mathrm{End}(S)\otimes_{S^2}P$ *satisfies the associativity and S is R-projective, then we have* $((P, p))\in Z^2(S/R)$.

Proof. The first half follows immediately from Lemma 3.4. Now the equality of $((1\otimes x)(\alpha\otimes y))(1\otimes z)$ and $(1\otimes x)((\alpha\otimes y)(1\otimes z))$, where $\alpha\in\mathrm{Hom}_R(S, R\cdot 1)$, assumed to hold in $\mathrm{End}(S)\otimes_{S^2}P$, is expressed as $(1\otimes\alpha\otimes 1)(L-R)=0$, where L resp. R denotes the left resp. right hand side of the equality of Lemma 3.4. But under the assumption of R-projectivity of S, this certainly implies that $L-R=0$, as desired.

Henceforth we assume that $((P, p))\in Z^2(S/R)$. We define a map ι_D: $S\to D$ by the commutativity of

$$\begin{array}{ccc} S\otimes_{S^2}P & \xrightarrow{\iota_A\otimes 1} & A\otimes_{S^2}P \\ \wr\downarrow & & \| \\ S & \xrightarrow{\iota_D} & D \end{array} \tag{4.2}$$

where the left vertical map is the isomorphism of (3.6). ι_D is a monomorphism of S^2-modules. We shall show that this is actually an algebra homomorphism, and that the left resp. right multiplication of $\iota_D(s)$ in D yields the left resp. right action of $s\in S$ on the S^2-module D, namely that

$$\iota_D(s)d=sd\,,\quad d\iota_D(s)=ds \qquad (s\in S,\ d\in D) \tag{4.3}$$

Indeed, if $e_P\in P$ be such that $\pi_p(e_P)=1$, then $\iota_D(s)=s\otimes e_P$, and we have

$$\begin{aligned}(s\otimes e_P)(b\otimes y)&=\sum_{(P,p)}s\tilde{p}_S(e_P, y)b\otimes\tilde{p}_P(e_P, y)\\ &=sb\otimes\sum_{(P,p)}\tilde{p}_S(e_P, y)\tilde{p}_P(e_P, y)=sb\otimes y\end{aligned}$$

in view of (3.10), proving the first half of (4.3). The second half is shown similarly. In particular we have $\iota_D(s)\iota_D(t)=s\iota_D(t)=\iota_D(st)$, which shows that ι_D is an algebra embedding. Hence (D, ι_D) is an S/R-algebra, whose identity element is given by $1\otimes e_P$. We denote this S/R-algebra by $D=(A, P, p)$. Clearly if $(P, p)\simeq(Q, q)$, then (A, P, p) and (A, Q, q) are isomorphic.

Lemma 4.2. *If $B=(A, P, p)$ and $C=(B, Q, q)$, then we have $C\simeq(A, P\otimes_{S^2}Q, p\otimes q)$. In particular, if $B=(A, P, p)$, then $A\simeq(B, P^*, p^*)$.*

Proof. Clearly we have

$$\begin{aligned}\widetilde{p\otimes q}(x\otimes y, x'\otimes y') &= \tilde{p}(x, x')\otimes\tilde{q}(y, y')\\ &=\sum_{(P,p)}\sum_{(Q,q)}\tilde{p}_S(x, x')\tilde{q}_S(y, y')\otimes(\tilde{p}_P(x, x')\otimes\tilde{q}_Q(y, y'))\end{aligned}$$

Hence the multiplication in $(A, P\otimes_{S^2}Q, p\otimes q)$ is given by

$$\begin{aligned}&(a\otimes(x\otimes y))(a'\otimes(x'\otimes y'))\\ &\quad=\sum_{(P,p)}\sum_{(Q,q)}a\tilde{p}_S(x, x')\tilde{q}_S(y, y')a'\otimes(\tilde{p}_P(x, x')\otimes\tilde{q}_Q(y, y'))\\ &\quad=\sum_{(Q,q)}(a\otimes x)(\tilde{q}_S(y, y')a'\otimes x')\otimes\tilde{q}_Q(y, y')\\ &\quad=((a\otimes x)\otimes y)((a'\otimes x')\otimes y') \qquad \text{(product in } (B, Q, q))\end{aligned}$$

A direct computation shows that $\pi_{p\otimes q}\colon P\otimes_{S^2}Q\to S$ is given by $\pi_{p\otimes q}(x\otimes y)=\pi_p(x)\pi_q(y)$. Hence $e_P\otimes e_Q$ serves as an $e_{P\otimes Q}$. It follows that the way of embedding of S in $(A, P\otimes_{S^2}Q, p\otimes q)$ agrees with that in (B, Q, q).

By this Lemma, the set of isomorphism classes of S/R-algebras is partitioned to orbits with respect to the operation of $Z^2(S/R)$. The following is the most degenerate case.

Proposition 4.3. *If S is central in A, we have $(A, P, p)\simeq A$ for every $((P, p))\in Z^2(S/R)$.*

Proof. In this case, the multiplication in (A, P, p) reduces to the form

$$(a\otimes x)(b\otimes y) = ab\otimes\pi_p(x)y \tag{4.4}$$

Further, the isomorphism $A\otimes_S(S\otimes_{S^2}P)\xrightarrow{\sim}A\otimes_S S$ may be expressed in the following form:

$$D = A\otimes_{S^2}P \xrightarrow{\sim} A;\ a\otimes x\mapsto a\pi_p(x)$$

and the identity (4.4) shows that this map gives an algebra isomorphism.

This Proposition suggests that the opposite extreme case where S is maximally commutative in A will be most interesting.

In [15], Sweedler studied the multiplication alteration of S/R-algebras by U-valued Amitsur 2-cocycles. Namely, let A be an S/R-algebra, and $u=\sum_{(u)} u_{(1)}\otimes u_{(2)}\otimes u_{(3)}\in Z^2(S/R, U)$. On a copy $A^{(u)}=\{a^u\}$ of A, one defines a multiplication by

$$a^u b^u = (\sum_{(u)} u_{(1)} a u_{(2)} b u_{(3)})^u$$

Then $A^{(u)}$ becomes a new S/R-algebra. Now, u defines $((S^2, \underline{u}))\in \boldsymbol{Z}^2(S/R)$, and we have

Proposition 4.4. *$(A, S^2, \underline{u})$ is isomorphic to Sweedler's $A^{(u)}$.*

Proof. The isomorphism $\tilde{p}$ associated with $p=\underline{u}$ is given by

$$\tilde{p}(1^2, 1^2) = 1^3 u = \sum_{(u)} u_{(2)}\otimes(u_{(1)}\otimes u_{(3)})\in S\otimes S^2$$

Hence the multiplication in $(A, S^2, u)=A\otimes_{S^2} S^2=A\otimes_{S^2} 1^2$ is given by

$$(a\otimes 1^2)(b\otimes 1^2) = \sum_{(u)} a u_{(2)} b\otimes(u_{(1)}\otimes u_{(3)}) = \sum_{(u)} u_{(1)} a u_{(2)} b u_{(3)}\otimes 1^2$$

which coincides with that of $A^{(u)}$ given above.

REMARK. Construction of an inverse isomorphic algebra. We define the opposite pair (P^0, p^0) of (P, p) as follows. $P^0=\tau P$ as in §3. p^0: $dp^0 \xrightarrow{\sim} S^3$ is defined by

$$p^0(\varepsilon_1 x^0\otimes\varepsilon_2\zeta^0\otimes\varepsilon_3 y^0) = p(\varepsilon_1 y\otimes\varepsilon_2\zeta\otimes\varepsilon_3 x)^0 \qquad (x, y\in P,\ \zeta\in P^*)$$

where ^{0} means the involution of S^3 defined by $r\otimes s\otimes t\mapsto t\otimes s\otimes r$. Then $\tilde{p}^0$ is given by

$$\tilde{p}^0(y^0, x^0) = \tilde{p}(x, y)^0 = \sum_{(P,p)} \tilde{p}_S(x, y)\otimes\tilde{p}_P(x, y)^0$$

If $\pi_p(e_P)=1$, we have $\pi_{p^0}(e_P^0)=1$. Now let A be an S/R-algebra, and A^0 its opposite algebra. Then (A^0, P^0, p^0) is an opposite algebra of (A, P, p). The embedding of S is certainly preserved, since $s\otimes e_P$ is mapped to $s\otimes e_P^0$.

4.2. Next we consider the case where $((P, p))\in \boldsymbol{B}^2(S/R)$.

Proposition 4.5. *For $M\in\mathcal{P}ic(S)$, the S/R-algebra (A, dM^*, c_{M^*}) is isomorphic to $\mathrm{End}_A(M\otimes_S A)$ in which S embeds as left operations.*

Proof. We have $dM^*\simeq M_1\otimes M_2^*$, and $\tilde{c}_{M^*}$: $dM^*\otimes dM^*\xrightarrow{\sim} S\otimes dM^*$ is given by $(x\otimes\xi)\otimes(y\otimes\eta)\mapsto\langle y, \xi\rangle\otimes x\otimes\eta$. Hence the multiplication of $D=A\otimes_{S^2} dM^*$ is expressed in the following form:

$$[a\otimes(x\otimes\xi)][b\otimes(y\otimes\eta)] = a\langle y, \xi\rangle b\otimes(x\otimes\eta)$$

Now we have an isomorphism

$$A\otimes_{S^2}dM^* \xrightarrow{\sim} \mathrm{End}_A(M\otimes_S A) \tag{4.5}$$
$$a\otimes(x\otimes\xi)\mapsto f, \text{ where } f(z\otimes c) = x\otimes a\langle z, \xi\rangle c$$

and it is immediate to see that the composition in $\mathrm{End}_A(M\otimes_S A)$ precisely corresponds to the multiplication of $A\otimes_{S^2}dM^*$ given above. Next, the isomorphism π_{can}: $\pi(dM^*)\xrightarrow{\sim} S$ is given by the map $M\otimes M^*\to M\otimes_S M^*\xrightarrow{\sim} S$. Hence an element e_{M^*} such that $\pi_{can}(e_{M^*})=1$ is given by $e_{M^*}=\sum m_i\otimes\mu_i$ where $\{m_i\}$ and $\{\mu_i\}$ are a pair of dual bases of S-modules M and M^*. Hence the embedding $S\to D$ is given by $s\mapsto s\otimes\sum(m_i\otimes\mu_i)$. Then the corresponding embedding $S\to \mathrm{End}_A(M\otimes_S A)$ is as follows:

$$\begin{aligned} s\mapsto f;\ f(x\otimes a) &= \sum m_i\otimes s\langle x, \mu_i\rangle a \\ &= \sum m_i\langle x, \mu_i\rangle\otimes sa = x\otimes sa \end{aligned}$$

which coincides with the natural embedding.

Proposition 4.6. *If $A=\mathrm{End}_R(N)$, where N is an S-module and is R-projective, then we have*

$$(A, dM^*, c_{M^*})\simeq \mathrm{End}_R(M\otimes_S N)$$

Proof. It suffices to show that $\mathrm{End}_A(M\otimes_S A)\simeq\mathrm{End}_R(M\otimes_S N)$ in view of the preceding Proposition. This isomorphism is established by the following correspondence:

$$f\mapsto f':\ f'(m\otimes n)=f(m\otimes 1)(n)$$

where we regard elements of $M\otimes_S\mathrm{End}_R(N)$ as inducing maps $N\to M\otimes_S N$ in a natural way. Indeed, this is the composite of the following isomorphisms:

$$\begin{aligned} \mathrm{End}_A(M\otimes_S A) &\simeq \mathrm{Hom}_S(M, M\otimes_S A)\simeq \mathrm{Hom}_S(M, (M\otimes_S N)\otimes N^*) \\ &\simeq \mathrm{Hom}_R(M\otimes_S N, M\otimes_S N) \end{aligned}$$

A simple calculation shows that the map $f\mapsto f'$ is multiplicative, and it clearly preserves the embedding of S.

The explicit form of the isomorphism of the Proposition is given by

$$\mathrm{End}_R(N)\otimes_{S^2}dM^* \to \mathrm{End}_R(M\otimes_S N) \tag{4.6}$$
$$\lambda\otimes(x\otimes\xi)\mapsto f,\ \text{ where } f(z\otimes y) = x\otimes\lambda(\langle z, \xi\rangle y)$$

As a particular case, we have

Corollary 4.7. *If S is R-projective, we have*

$$(\mathrm{End}_R(S), dM^*, c_{M^*})\simeq \mathrm{End}_R(M)\,.$$

Let R' be a commutative R-algebra, and we use the notations such as

$S_{R'}$, $M_{R'}$, to denote the change of the base ring (cf. § 3.2). Then we clearly have $(A, P, p)_{R'} \simeq (A_{R'}, P_{R'}, p_{R'})$. If, in particular, $R'=S$, every 2-cocycle splits (Proposition 3.5), and we can apply Proposition 4.5 to the extended algebra D_S. Fortunately several properties can be descended to D, and lead to

Proposition 4.8. *Assume that the unit map $R \to S$ R-splits, and A is finite projective and faithful as an R-module. Then $D=(A, P, p)$ is semi-simple, separable or central separable if and only if A is so respectively. In the central separable case, D_S belongs to the same algebra class as A_S.*

Proof. In view of Lemma 4.2, it suffices to prove the if part. In the split case where $(P, p)=(dM, c_M)$ we have $D \simeq \mathrm{End}_A(M^* \otimes_S A)$. By the assumption on A, $M^* \otimes_S A$ is an R-progenerator, so that D is the commuter of A^0 in the central separable algebra $\mathrm{End}_R(M^* \otimes_S A)$. Hence the result follows from the commuter theory ([2] for central separable, [10] for separable, and [6] [7] for semi-simple). In the general case, the above facts can be applied to D_S, and we know that these properties of algebras are descended to D under the assumption of R-splitness of $R \to S$.

This assumption on S is equivalent to saying that S is an R-generator, and is satisfied if S is R-finite projective and faithful as is well known.

REMARK. In order to derive the splitting property of D_S, we can argue more simply, without employing Proposition 3.5, as follows. We consider the case $A=\mathrm{End}_R(S)=\mathrm{Hom}_R(S_2, S_1)$. Then $S \otimes A \simeq \mathrm{End}_S(S \otimes S)$. We express the isomorphism $p\colon dP \xrightarrow{\sim} S^3$ in the form $\varepsilon_1 P \simeq \mathrm{Hom}_{S^3}(\varepsilon_2 P^*, \varepsilon_3 P^*)$, and we have

$$
\begin{aligned}
S_1 \otimes (A \otimes_{S^2} P) &\simeq (S_1 \otimes A) \otimes_{S^3} (S_1 \otimes P) \simeq \mathrm{Hom}_{S^3}(\varepsilon_2 P^*, \mathrm{End}_S(S \otimes S) \otimes_{S^3} \varepsilon_3 P^*) \\
&\simeq \mathrm{Hom}_{S_1 \otimes S_3}(P^*, \mathrm{Hom}_{S_1}(S_1 \otimes S_3, (S_1 \otimes S_2) \otimes_{S^3} (S_3 \otimes P^*))) \\
&\simeq \mathrm{Hom}_{S_1}(P^*, P^*) = \mathrm{End}_S(P^*)
\end{aligned}
$$

5. $H^2(S/R) \simeq Br(S/R)$

5.1. In this section we assume that S is R-finite projective and faithful. An S/R-algebra (A, ι) is called a *left* (resp. *right*) *S/R-Azumaya algebra,* if A is central separable over R, $\iota(S)$ is a maximally commutative subalgebra of A, and A is left (resp. right) S-projective. The set $\hat{B}r(S/R)$ ($A(S, R)$ in the notation of [4]) of all S/R-isomorphism classes of left S/R-Azumaya algebras has the structure of an abelian group ([4] § 2). An expression of the product of A and B is given by

$$
A * B = ({}_1A_2 \otimes_{S_1} {}_1B_2)^{S_2} \tag{5.1}
$$

whose multiplication is the one naturally induced from that of $A \otimes B$. (Notice that in $A \otimes_{S_1} B$ itself this natural multiplication can not be spoken of.) The embedding of S in $A*B$ is given by $s\mapsto s\otimes 1=1\otimes s$. We know that by forgetting the embedding of S we have an epimorphism $\hat{Br}(S/R)\to Br(S/R)$, whose kernel $Pr(S/R)$ consists of $\mathrm{End}_R(M)$, $M\in\mathcal{P}ic(S)$ ([4] § 2).

Proposition 5.1. *(A, P, p) is a left S/R-Azumaya algebra if and only if A is.*

Proof. It suffices to prove the if part. Already we know that $D=(A, P, p)$ is central separable. The left S-projectivity of $D=A\otimes_{S^2}P$ is clear. Suppose $d=\sum a_i\otimes x_i$ commutes with every $s(=s\otimes e_P)\in S$. Then $d\otimes\xi(\xi\in P^*)$ commutes with $s(=s\otimes e_P\otimes e_{P^*})$ in (D, P^*, p^*). By Lemma 4.2, this means that $\sum\langle x_i, \xi\rangle a_i \in A$ commutes with every $s\in S$. Hence $\sum\langle x_i, \xi\rangle a_i\in S$. Let $\{p_j\}$ and $\{\xi_j\}$ be a pair of dual bases of P and P^*. Then we have

$$\sum a_i\otimes x_i=\sum_j\sum_i\langle x_i, \xi_j\rangle a_i\otimes p_j\in S\otimes_{S^2}P(=S\text{ in }D).$$

REMARK. The part of proof concerning the maximal commutative embedding of S is independent of the separability, and valid for general S/R-algebras.

5.2. We now argue in reversed direction. Let A be a left S/R-Azumaya algebra. Since A is left S-projective, ${}_SA_A$ is right $S\otimes A$-projective by the separability of A. Hence the dual module

$$P=\mathrm{Hom}_{-,S\otimes A}(A, S\otimes A)$$

is a left $S\otimes A$-projective module. Since the S^2-module $S_2\otimes_1 A$ is S^2-projective, P has the structure of an S^2-projective module, and we are interested in this S^2-module P. An element f of P is determined by $x=f(1)\in S\otimes A$ which should satisfy $sx=xs$ $(s\in S)$. Hence P may be identified with $(S_2\otimes_1 A_2)^{S_2}$.

We begin by showing that an isomorphism of S^3-modules

$$\tilde{p}\colon {}_1P_2\otimes_{S_2}{}_2P_3\xrightarrow{\sim} S_2\otimes_1 P_3 \tag{5.2}$$

is established by the correspondence

$$(\sum_i s_i\otimes a_i)\otimes(\sum_j t_j\otimes b_j)\mapsto\sum_i s_i\otimes(\sum_j t_j\otimes a_ib_j) \tag{5.3}$$

Since the canonical pairing

$$\mathrm{Hom}_{S\otimes A}(A, S\otimes A)\otimes_S A\to S\otimes A$$

is an isomorphism and A is $S\otimes A$-finite projective, we have a series of isomorphisms of S^3-modules as follows:

$$\begin{aligned}
&\mathrm{Hom}_{-,S\otimes A}(A, S_2\otimes_1 A)\otimes_{S_2}\mathrm{Hom}_{-,S\otimes A}(A, S_3\otimes_2 A)\\
&\simeq\mathrm{Hom}_{S_3\otimes A}(A, S_3\otimes\mathrm{Hom}_{S_2\otimes A}(A, S_2\otimes_1 A)\otimes_{S_2}A)\\
&\simeq\mathrm{Hom}_{S_3\otimes A}(A, S_3\otimes(S_2\otimes A))\\
&\simeq S_2\otimes\mathrm{Hom}_{S_3\otimes A}(A, S_3\otimes A)=S_2\otimes_1 P_3\,.
\end{aligned}$$

An examination of maps shows that the isomorphism (5.2) thus obtained is given by the correspondence (5.3).

Next we shall show that

$$\pi P = S\otimes_{S^2}P \simeq S;\ \pi(\sum s_i\otimes a_i)\mapsto \sum s_i a_i \tag{5.4}$$

This is verified as follows:

$$S\otimes_{S^2}\mathrm{Hom}_{S\otimes A}(A,\ S\otimes A)\simeq \mathrm{Hom}_{S\otimes A}(A,\ S\otimes_{S^2}(S\otimes A))\simeq \mathrm{Hom}_{S\otimes A}(A,\ A)\simeq S$$

The isomorphism (5.2) may be interpreted as $\varepsilon_3 P\otimes_{S^3}\varepsilon_1 P\simeq \varepsilon_2 P$. So, applying π_2 as in the proof of Proposition 3.1, we have (using (5.4)):

$$P^0\otimes_{S^2}P\simeq S^2\,. \tag{5.5}$$

It follows that $P\in \mathcal{P}ic(S^2)$, and $P^0\simeq P^*$. The isomorphism (5.2) then gives rise to

$$p\colon\ \varepsilon_1 P\otimes_{S^3}\varepsilon_2 P^*\otimes_{S^3}\varepsilon_3 P\xrightarrow{\sim} S^3$$

Thus a left S/R-Azumaya algebra A determines a pair $(P, p)\in \mathcal{P}^2(S/R)$. Clearly any algebra S-isomorphic to A yields a pair isomorphic to (P, p).

Now we shall examine the algebra (E, P, p), where $E=\mathrm{End}_R(S)$ and (P, p) is the pair derived from A. Since

$$E\otimes_{S^2}(S_2\otimes A)\simeq (S_1\otimes S_2^*)\otimes_{S_1}A\simeq S_2^*\otimes A\simeq \mathrm{Hom}_R(S_2,\ A)$$

we have

$$\begin{aligned}E\otimes_{S^2}\mathrm{Hom}_{S\otimes A}(A,\ S_2\otimes A)&\simeq \mathrm{Hom}_{S\otimes A}(A,\ E\otimes_{S^2}(S_2\otimes A))\\&\simeq \mathrm{Hom}_{S\otimes A}(A,\ \mathrm{Hom}_R(S,\ A))\simeq \mathrm{Hom}_S(S,\ A)\simeq A\end{aligned}$$

Namely we have

$$(E,\ P,\ p)\simeq A;\ \lambda\otimes(\sum s_i\otimes a_i)\mapsto \sum \lambda(s_i)a_i \tag{5.6}$$

This is actually an algebra isomorphism, since we have

$$\sum \lambda(s_i\mu(t_j))a_i b_j = \sum \lambda(s_i)a_i\mu(t_j)b_j \qquad (\lambda,\ \mu\in E)$$

in view of $\sum s_i\otimes a_i\in (S\otimes A)^S$. Since $(E, P, p)\simeq A$ satisfies the associativity, $((P, p))$ is contained in $\boldsymbol{Z}^2(S/R)$ by Lemma 4.1. As an element $e_P\in P$ such that $\pi_p(e_P)=1$, we can take $1\otimes 1\in (S\otimes A)^S$. Hence the embedding of S in (E, P, p) is given by $s\mapsto \underline{s}\otimes e_P$, and this corresponds to the original embedding $S\to A$. Hence (5.6) is an isomorphism of S/R-algebras.

REMARK. If S/R is a quasi-Frobenius extension, $\mathrm{End}_R(S)$ and A are S^2-projective (cf. [17] Theorem 2.1). In this case, the situation is simpler, since $P=\mathrm{End}_R(S)^*\otimes_{S^2}A\simeq \mathrm{Hom}_{S^2}(\mathrm{End}_R(S),\ A)$ works.

We now start with a pair (P, p) satisfying the Z^2-condition, and construct $A=(E, P, p)$. A, in turn, determines a pair, say (P', p'), by the above procedure. We shall show that this pair is isomorphic to the original (P, p). At first, we observe

$$\begin{aligned} P' &= \mathrm{Hom}_{S\otimes A}(A,\ S_2\otimes(S_3^*\otimes_{S_3}{}_1P_3)) \simeq \mathrm{Hom}_{S_2\otimes S_3}({}_2S_3,\ (S_2\otimes S_3^*)\otimes_{S_3}{}_1P_3) \\ &\simeq \mathrm{Hom}_{S_2\otimes S_3}({}_2S_3,\ S_2\otimes S_3^*)\otimes_{S_3}{}_1P_3 \simeq {}_2S_3\otimes_{S_3}{}_1P_3 \simeq P \end{aligned}$$

In this isomorphism, $x\in P$ corresponds to $\sum s_i\otimes(\alpha_i\otimes x)\in P'$, where $\{s_i\}$ and $\{\alpha_i\}$ are a pair of dual R-bases of S and S^*. Next we examine p'. This is defined by (5.3) as follows:

$$\begin{aligned} [\textstyle\sum s_i\otimes(\alpha_i\otimes x)]\otimes[\sum s_j\otimes(\alpha_j\otimes y)] &\mapsto \sum_i s_i\otimes(\sum_j s_j\otimes(\alpha_i\otimes x)(\alpha_j\otimes y)) \\ &= \sum_i s_i\otimes(\sum_j s_j\otimes \sum_{(P,p)} \alpha_i\tilde{p}_S(x, y)\alpha_j\otimes\tilde{p}_P(x, y)) \end{aligned}$$

Since $\alpha_i t\alpha_j=\alpha_i(t)\alpha_j$ for $t\in S$, this is identical with

$$\sum_i s_i\otimes(\sum_j s_j\otimes\alpha_j\otimes \sum_{(P,p)} \alpha_i(\tilde{p}_S(x, y))\tilde{p}_P(x, y))$$

In terms of the above isomorphism $P'\xrightarrow{\sim}P$, this last element of $S\otimes P'$ corresponds to

$$\sum s_i\otimes\sum_{(P,p)}\alpha_i(\tilde{p}_S(x, y))\tilde{p}_P(x, y) = \sum_{(P,p)}\tilde{p}_S(x, y)\otimes\tilde{p}_P(x, y) = \tilde{p}(x, y)$$

Thus the isomorphism of (P', p') with (P, p) is verified. Summing up, *we have established a bijective correspondence between* $\hat{B}r(S/R)$ *and* $Z^2(S/R)$.

Finally we shall show that this correspondence preserves the product, namely that, if $A\simeq(E, P, p)$ and $B\simeq(E, Q, q)$ then $A*B\simeq(E, P\otimes_{S^2}Q, p\otimes q)$. By Lemma 4.2, it suffices to show that $A*B\simeq(A, Q, q)$. Now, as S^2-modules we have

$$A\otimes_{S^2}Q \simeq A\otimes_{S^2}\mathrm{Hom}_{-,S\otimes B}(B,\ S_2\otimes_1 B) \simeq \mathrm{Hom}_{-,S\otimes B}(B,\ A\otimes_{S^2}(S\otimes B))$$

Since $A\otimes_{S^2}(S\otimes B)$ is isomorphic to $A\otimes_{S_1}B$ as a left S^2-and right $S\otimes B$-module, further we have

$$A\otimes_{S^2}Q \simeq \mathrm{Hom}_{S\otimes B}(B,\ A\otimes_{S_1}B) \simeq A*B$$

The explicit correspondence of this isomorphism is given by

$$a\otimes y\mapsto \sum as_i\otimes b_i\,, \quad \text{where} \quad y=\sum s_i\otimes b_i\in(S\otimes A)^S$$

The multiplication in (A, Q, q) is given by

$$[a\otimes(\textstyle\sum s_i\otimes b_i)]\cdot[a'\otimes(\sum t_j\otimes c_j)] = \sum_i as_ia'\otimes\sum_j(t_j\otimes b_ic_j)$$

which precisely corresponds to the multiplication in $A*B$:

$$(\sum as_i\otimes b_i)(\sum a't_j\otimes c_j)=\sum as_ia't_j\otimes b_ic_j$$

This verifies the assertion.

These considerations, combined with Proposition 5.1 and Corollary 4.7, proves the following theorem.

Theorem 5.2. *Assume that S is R-finite projective and faithful. Then the correspondence $(P, p)\mapsto(E, P, p)$, where $E=\mathrm{End}_R(S)$, yields an isomorphism $\boldsymbol{Z}^2(S/R)\simeq\hat{B}r(S/R)$. It induces an isomorphism $\boldsymbol{H}^2(S/R)\simeq Br(S/R)$.*

As will be shown in § 7, this generalizes [9] Theorem 3, which treats the case of finite Galois extensions.

Remark. The opposite algebra of a left S/R-Azumaya algebra is right S/R-Azumaya. The isomorphism classes of right S/R-Azumaya algebras form an abelian group $\hat{B}r'(S/R)$ isomorphic to $\hat{B}r(S/R)$. If S/R is quasi-Frobenius, they coincide; if not, they are two distinct orbits with respect to $\boldsymbol{Z}^2(S/R)$.

By this Theorem, the exact sequence of Theorem 1.1 yields that of Chase and Rosenberg [4], [18]:

$$\begin{aligned}(5.7)\qquad & 0\to H^1(S/R, U)\to \mathrm{Pic}(R)\to H^0(S/R, \mathrm{Pic})\\ &\to H^2(S/R, U)\to \mathrm{Br}(S/R)\to H^1(S/R, \mathrm{Pic})\to H^3(S/R, U)\to\end{aligned}$$

The assumption on S/R is satisfied, if R is a Dedekind domain and S is R-finite and torsion-free. If, in particular, L/K is a finite extension of fields, the homomorphism α^2: $H^2(L/K, U)\to\boldsymbol{H}^2(L/K)$ is actually an isomorphism. Hence the isomorphism $H^2(L/K, U)\simeq Br(L/K)$ so far established in various context may be considered as a special case of Theorem 5.2. An important general case is the Hopf Galois extensions of Sweedler [14]. We briefly recall the relevant part of his theory, adapted to the ring case [5], [17].

Let H be a finite cocommutative Hopf algebra over R which is R-projective, and let S be a commutative R-algebra which is an H-module algebra. S is called an *H-Hopf Galois extension* of R if S is R-finite projective and faithful, and the natural map from the smash product $S\#H$ to $\mathrm{End}_R(S)$ is an isomorphism. The cohomology is defined with respect to a semi-simplicial complex, composed of H^n and the appropriate face operators. If S/R is H-Hopf Galois, then S^{n+1} is isomorphic to $\mathrm{Hom}_R(H^n, S)$ and yields an isomorphism of U-valued Amitsur cohomology with the Hopf Galois cohomology. Sweedler defined the crossed product $S\underset{\sigma}{\#}H$ with respect to a 2-cocycle σ, and proved that if L/K is an H-Galois field extension this construction leads to an isomorphism $H^2(H, L)\simeq Br(L/K)$ ([14] Theorem 9.7). Now let $u\in S^3$ be the 2-Amitsur cocycle corresponding to σ. Then a simple computation shows that $S \underset{\sigma}{\#} H$ is isomorphic to $(S \# H)^{(u)}$ in the notation of [15]. Hence it is isomorphic to $(E, S^2, \underline{u})$ by Proposition 4.4. Therefore the present theory may be considered as a natural generalization of his theory [14]. The Brauer group of Hopf Galois extensions *in ring case* was dealt with by Yokogawa [17], where he considered certain construction which generalised Sweedler's crossed products, and employed it to a direct proof of Chase-Rosenberg exact sequence. Yokogawa [18] then extended his theory further to the general case, and some constructions of the present paper are anticipated by these works.

6. Homomorphism $S/R \to S'/R'$

Let φ_0: $R \to R'$ be a homomorphism of commutative rings and S resp. S' be an algebra over R resp. R'. By a homomorphism of algebras φ: $S/R \to S'/R'$ we mean a ring homomorphism φ: $S \to S'$ satisfying $\varphi(rs)=\varphi_0(r)\varphi(s)$ $(r \in R, s \in S)$. φ induces a homorphism φ_n: $S^n/R \to S'^n/R'$ for every n.

We introduce a complex $\mathrm{Am}(\varphi, \mathrm{Pic})$ as follows. $\mathrm{Am}^n(\varphi, \mathrm{Pic})=\mathrm{Pic}(\varphi_n)$ $(n \geq 1)$, which consists of isomorphism classes of pairs $[P, f]$ such that $P \in \mathcal{P}ic(S^n)$ and f: $\varphi_n P \xrightarrow{\sim} S'^n$ (in $\mathcal{P}ic(S'^n)$) (cf. [9] § 4). We identify $\varepsilon_i(\varphi_n P)$ and $\varphi_{n+1}(\varepsilon_i P)$ canonically. Hence we can define $\varepsilon_i f$: $\varphi_{n+1}\varepsilon_i P=\varepsilon_i\varphi_n P \to S'^{n+1}$, and thereupon

$$df\colon \varphi_{n+1}dP = d\varphi_n P \to S'^{n+1}$$

Hence we have a homomorphism

$$d_n\colon \mathrm{Pic}(\varphi_n) \to \mathrm{Pic}(\varphi_{n+1});\ [P, f] \mapsto [dP, df] \qquad (n \geq 1)$$

For convenience, we put $d_0=0$. The commutativity of

$$\begin{array}{ccc} \varphi_{n+2}d^2P & = & d^2\varphi_n P \\ \downarrow{\scriptstyle \varphi_{n+2}c_P} & & \downarrow{\scriptstyle c_{\varphi_n P}=d^2 f} \\ \varphi_{n+2}S^{n+2} & = & S'^{n+2} \end{array}$$

means that c_P: $d^2P \xrightarrow{\sim} S^{n+2}$ defines an isomorphism $[d^2P, d^2f] \simeq [S^{n+2}, 1]$. Thus $\mathrm{Am}(\varphi, \mathrm{Pic})=\{\mathrm{Pic}(\varphi_n), d\}$ constitutes a complex, and we shall denote

$$H^n(\varphi, \mathrm{Pic}) = \mathrm{Ker}(d_{n+1})/\mathrm{Im}(d_n) \qquad (n \geq 0)$$

$\mathrm{Pic}(\varphi_n)$ can be treated homogeneously as the group of isomorphism classes of triples $[P, f, Q]$, where $P, Q \in \mathcal{P}ic(S^n)$ and f: $\varphi_n P \xrightarrow{\sim} \varphi_n Q$, subject to the condition $[P, f, Q][Q, g, R]=[P, gf, R]$ (cf. [3], [9]). In this description, the coboundary operator is defined by $d[P, f, Q]=[dP, df, dQ]$.

Now, φ_n: $S^n \to S'^n$ induces a homomorphism of Amitsur cohomology groups, and also of our groups:

$$\varphi^n\colon \boldsymbol{H}^n(S/R) \to \boldsymbol{H}^n(S'/R');\ cl((P, p)) \mapsto cl((\varphi_n P, \varphi_{n+1} p))$$

For $n=0$, φ^0 is defined to be the restriction of φ to $\boldsymbol{H}^0(S/R)$ ($\subset U(S)$). The kernel and cokernel of this map are connected with the relative Amitsur groups $H^n(\varphi, \mathrm{Pic})$ defined above, and what follows proceeds completely parallel to [9] § 4. We say that φ satisfies the U_r-injectivity resp. the Pic_r-surjectivity, if $U(S^r) \to U(S'^r)$ is injective, resp. if $\mathrm{Pic}(S^r) \to \mathrm{Pic}(S'^r)$ is surjective. Notice that if φ satisfies the Pic_r-surjectivity for some r, then φ necessarily satisfies the Pic_k-surjectivity for every $k \leq r$, since the homomorphisms $\varepsilon_1 \cdots \varepsilon_1\colon S^k \to S^r$ and $S'^k \to S'^r$ are simultaneously split by contraction homomorphisms $S^r \to S^k$, $S'^r \to S'^k$. Similary for the U-injectivity.

First, *we assume the* Pic_n-*surjectivity,* and define the following map:

$$\chi^n\colon \boldsymbol{H}^n(S'/R') \to H^n(\varphi, \mathrm{Pic});\ cl((\varphi_n P, p)) \mapsto cl[dP, p]$$

For $n=0$, χ^0 is defined by $u' \mapsto [S, \underline{u'}]$ where $u' \in U(S')$ and satisfies $1 \otimes u' = u' \otimes 1$. The definability of this map ($n \geq 1$) is verified as follows. Since $dp = c_{\varphi_n P}$, we have $d[dP, p] \simeq [S^{n+2}, 1]$. If $(\varphi_n P, p) \simeq (\varphi_n Q, q)$, there exists $f\colon \varphi_n P \xrightarrow{\sim} \varphi_n Q$ satisfying $p = qdf$. The equality $[dP, p, S^{n+1}] = [dP, df, dQ][dQ, q, S^{n+1}]$ shows that $[dP, p, S^{n+1}]$ and $[dQ, q, S^{n+1}]$ are cohomologous. Hence the corresponding non-homogeneous objects $[dP, p]$ and $[dQ, q]$ are cohomologous. Finally, $(\varphi_n dP, c_{\varphi_n P})$ corresponds to $[d^2 P, c_{\varphi_n P}] \simeq [S^{n+1}, 1]$.

Next *we assume the* U_{n+3}-*injectivity,* and will define

$$\psi^n\colon H^n(\varphi, \mathrm{Pic}) \to \boldsymbol{H}^{n+1}(S/R);\ cl[P, f] \mapsto cl((P, p))$$

Let $[P, f] \in \mathrm{Pic}(\varphi_{n+1})$, and assume that $d[P, f] \simeq [S^{n+2}, 1]$, i.e. there exists $p\colon dP \xrightarrow{\sim} S^{n+2}$ satisfying $\varphi_{n+2} p = df$. Such p is unique by the U_{n+2}-injectivity. We compare dp with c_P. We observe $\varphi_{n+3} dp = d\varphi_{n+2} p = d^2 f = c_{\varphi_{n+1} P} = \varphi_{n+3} c_P$, whence follows $dp = c_P$ by the U_{n+3}-injectivity. Hence $((P, p)) \in \boldsymbol{Z}^n(S/R)$. A simple computation using U-injectivity shows that isomorphic $[P, f]$'s yield cohomologous $((P, p))$'s. Since $d[P, f]$ yields $((d^2 P, c_P))$, we have a well-defined map ψ^n as above.

Theorem 6.1. *If* $\varphi\colon S/R \to S'/R'$ *satisfies the* U_{n+3}-*injectivity and the* Pic_n-*surjectivity, then the following sequence is exact:*

$$0 \to \boldsymbol{H}^0(S/R) \xrightarrow{\varphi^0} \boldsymbol{H}^0(S'/R') \xrightarrow{\chi^0} H^0(\varphi, \mathrm{Pic}) \xrightarrow{\psi^0} \cdots$$
$$\cdots \to \boldsymbol{H}^n(S/R) \xrightarrow{\varphi^n} \boldsymbol{H}^n(S'/R') \xrightarrow{\chi^n} H^n(\varphi, \mathrm{Pic}) \xrightarrow{\psi^n} \boldsymbol{H}^{n+1}(S/R) \xrightarrow{\varphi^{n+1}} \boldsymbol{H}^{n+1}(S'/R').$$

Proof. We reproduce the proof of [9] Theorem 2 almost word by word.

1) Clearly $\chi^0 \varphi^0 = 0$. If $[S, \underline{u'}] \simeq [S, 1]$ for $u' \in U(S')$ such that $du' = 1$, then there exists $u \in U(S)$ such that $\varphi(u) = u'$. By the U-injectivity, $d\varphi(u) = 1$

means $du=1$, i.e. $u'\in\mathrm{Im}(\varphi^0)$. For $n\geq 1$, $\chi^n\varphi^n$ maps $((P,p))$ to $[dP, \varphi_{n+1}p]\overset{p}{\simeq}[S^{n+1}, 1]$. Assume for $((\varphi_n P, p))\in \boldsymbol{Z}^n(S'/R')$, there exists $[Q, q]$ such that $[dP, p]\simeq d[Q, q]$. This means that there exists $f\colon dP\xrightarrow{\sim} dQ$ satisfying $p=dq\varphi_{n+1}f$. In homogeneous description of $\boldsymbol{P}^n(S'/R')$, we have

$$((\varphi_n P, p, S'^n)) = ((\varphi_n P\ \varphi_{n+1}f, \varphi_n Q))((\varphi_n Q, dg, S'^n))$$

From $c_{\varphi_n P}=dp=d^2q\varphi_{n+2}df$ and the U-injectivity, we deduce $c_P=c_Q df$, which means $((P, f, Q))\in \boldsymbol{Z}_h^n(S/R)$. Since $\{q, 1\}$ gives an isomorphism $(\varphi_n Q, dq, S'^n)\simeq(S'^n, 1, S'^n)$, we have $cl((\varphi_n P, p, S'^n))\in\mathrm{Im}(\varphi^n)$, which also means $cl((\varphi_n P, p))\in\mathrm{Im}(\varphi^n)$.

2) $\psi^0\chi^0(u')=((S, 1))$. Let $[P, f]\in\mathrm{Ker}(\psi^0)$. There exists $g\colon P\xrightarrow{\sim} S$ satisfying $dg=p$, where $p\colon dP\xrightarrow{\sim} S^2$ is defined by the condition $\varphi_2 p=df$. Put $f=u'\varphi g$ with $u'\in U(S')$. Then we have $du'=1$, since $df=\varphi_2 dg$, and φg provides an isomorphism $[P, f]\simeq[S, \underline{u}']$. Hence $[P, f]\in\mathrm{Im}(\chi^0)$. For $n\geq 1$, $\psi^n\chi^n$ maps $((\varphi_n P, p))$ to $((dP, c_P))$. Let $[P, f]\in\mathrm{Ker}(\psi^n)$. There exist $Q\in\mathcal{P}ic(S^n)$ and $g\colon dQ\xrightarrow{\sim} P$ such that $pdg=c_Q$. Since $\varphi_{n+1}p=df$, we have $dfd\varphi_{n+1}g=c_{\varphi_n Q}$. It follows that $((\varphi_n Q, f\varphi_{n+1}g))\in\boldsymbol{Z}^n(S'/R')$, and χ^n maps this pair to $[dQ, f\varphi_{n+1}g]\simeq[P, f]$.

3) $\varphi^{n+1}\psi^n$ maps $[P, f]$ to $((\varphi_{n+1}P, \varphi_{n+2}p))\simeq((S'^{n+1}, 1))$ (cf. definition of maps). Let $((P, p))\in\mathrm{Ker}(\varphi^{n+1})$. If $n=0$, there exists $f\colon \varphi P\xrightarrow{\sim} S'$ satisfying $df=\varphi_2 p$. This means $((P, p))=\psi^0[P, f]$. If $n\geq 1$, there exists $Q\in\mathcal{P}ic(S^n)$ such that $(\varphi_{n+1}P, \varphi_{n+2}p)\simeq(d\varphi_n Q, c_{\varphi_n Q})$. Hence there exists $g\colon \varphi_{n+1}P\xrightarrow{\sim} d\varphi_n Q$ satisfying $\varphi_{n+2}p=c_{\varphi_n Q}dg$. Then $\{p, c_Q\}$ defines an isomorphism $(dP, dg, d^2Q)\simeq(S^{n+2}, 1, S^{n+2})$, and we have $[P, g, dQ]\in Z^n(\varphi, \mathrm{Pic})$. This means $[P\otimes_{S^n} dQ^*, g^\sharp]\in Z^n(\varphi, \mathrm{Pic})$, where $g^\sharp=\langle\ \rangle(g\otimes 1)\colon \varphi_{n+1}P\otimes_{S'^{n+1}}d\varphi_n Q^*\to d\varphi_n Q\otimes_{S'^{n+1}}d\varphi_n Q^*\to S'^{n+1}$. Since the commutativity of

$$\begin{array}{ccccc}
d\varphi_{n+1}P\otimes_{S'^{n+2}}d^2\varphi_n Q^* & \xrightarrow{dg\otimes 1} & d^2\varphi_n Q\otimes_{S'^{n+2}}d^2\varphi_n Q^* & \xrightarrow{\langle\ \rangle} & S'^{n+2}\\
\Big\downarrow{\scriptstyle \varphi_{n+2}p\otimes 1} & & \Big\downarrow{\scriptstyle c_{\varphi_n Q}\otimes 1} & & \Big\|\\
d^2\varphi_n Q^* & = & d^2\varphi_n Q^* & \xrightarrow{c_{\varphi_n Q^*}} & S'^{n+2}
\end{array}$$

shows $dg^\sharp=(dg)^\sharp=\varphi_{n+2}p\otimes c_{\varphi_n Q^*}$, we have

$$\psi^n[P\otimes_{S^{n+1}}dQ^*, g^\sharp] = ((P\otimes_{S^{n+1}}dQ^*, p\otimes c_{Q^*}) = ((P, p))(dQ^*, c_{Q^*}))$$

Hence $cl((P, p))\in\mathrm{Im}(\psi^n)$.

Proposition 6.2. *If S resp. S' is faithfully flat over R resp. R' we have $H^0(\varphi, \mathrm{Pic})\simeq\mathrm{Pic}(\varphi_0)$.*

Proof. Let $[P, f]\in\mathrm{Pic}(\varphi)$ satisfy $d[P, f]\simeq[S^2, 1]$. Namely there exists $p\colon dP\xrightarrow{\sim} S^2$ such that $\varphi_2 p=df$. By Theorem 2.2, (P, p) determines up to isomorphism a pair of $P_0\in\mathcal{P}ic(R)$ and $p_0\colon S\otimes P_0\xrightarrow{\sim} P$ satisfying $\tilde{p}\varepsilon_1 p_0=\varepsilon_2 p_0$. df

is converted to $\widetilde{df}$: $\mathcal{E}_1\varphi P \underset{\rightarrow}{\sim} \mathcal{E}_2\varphi P$, and the commutativity of

$$\begin{array}{ccc} \mathcal{E}_1\varphi P \otimes_{S'^2} \mathcal{E}_2\varphi P^* \otimes_{S'^2} \mathcal{E}_2\varphi P & \longrightarrow & \mathcal{E}_1\varphi P \\ \Big\downarrow df \otimes 1 & & \Big\downarrow \mathcal{E}_1 f \\ S'^2 \otimes_{S'^2} \mathcal{E}_2\varphi P & \xrightarrow{1\otimes \mathcal{E}_2 f} & S'^2 \end{array}$$

shows $\mathcal{E}_1 f = \mathcal{E}_2 f \widetilde{df} = \mathcal{E}_2 f \widetilde{\varphi p}$. Put $P_0' = R' \otimes P_0$. Then we have the following isomorphism f' in $\mathcal{P}ic(S')$:

$$f'\colon\ S' \otimes_{R'} P_0' = S' \otimes_S (S \otimes P_0) \xrightarrow{\varphi p_0} S' \otimes_S P \xrightarrow{f} S' \otimes_{R'} R'$$

Since we have

$$\mathcal{E}_1 f' = \mathcal{E}_1 f \mathcal{E}_1 \varphi p_0 = \mathcal{E}_2 f \widetilde{\varphi p} \mathcal{E}_1 \varphi p_0 = \mathcal{E}_2 f \mathcal{E}_2 \varphi p_0 = \mathcal{E}_2 f'$$

f' is descended uniquely to f_0: $P_0' \underset{\rightarrow}{\sim} R'$ such that $f' = 1 \otimes f_0$ ([12] II Proposition 2.5). Thus $[P, f]$ is descended to $[P_0, f_0]$, and an isomorphism $H^0(\varphi, \mathrm{Pic}) \simeq \mathrm{Pic}(\varphi_0)$ immediately follows.

Combining Proposition 2.1, Theorem 2.2 and the above Proposition, the initial part of the exact sequence of Theorem 6.1 reduces to the following basic sequence:

$$0 \to U(R) \to U(R') \to \mathrm{Pic}(\varphi_0) \to \mathrm{Pic}(R) \to \mathrm{Pic}(R') \to$$

in case S/R is faithfully flat. (If we assume the exactness of Theorem 6.1, the above proposition can easily be proved by the 5-lemma technique. But the direct proof given above will be of some interest in itself.)

The following proposition is a generalization of [9] Proposition 5.1 (cf. § 7).

Proposition 6.3. *Let S/R be an extension of integral domains, and L/K the extension of the respective quotient fields. We assume that S is finite projective and faithful as an R-module. Then we have the following exact sequence:*

$$0 \to H^1(\varphi, \mathrm{Pic}) \to Br(S/R) \to Br(L/K) \to H^2(\varphi, \mathrm{Pic})$$

This is immediate applying Theorems 2.2, 5.2 and 6.1.

7. Galois extensions

Let S/R be a Galois extension with a finite group G as the Galois group. In this case the Amitsur cohomology is naturally isomorphic to the group cohomology, based on the isomorphism $S^{n+1} \simeq C^n(G, S)$ (n-th cochain group) (cf. [12] V). In particular it induces an equivalence of $\mathcal{P}ic(S^{n+1})$ to the category C^n of the maps $P(\sigma^n)$: $G^n \to \mathcal{P}ic(S)$ introduced in [9] § 2, in which an isomorphism $dP \underset{\rightarrow}{\sim} S^{n+2}$ corresponds to $\delta P(\sigma^{n+1}) \underset{\rightarrow}{\sim} S$. It follows that $\boldsymbol{H}^n(S/R)$ is isomorphic to $\boldsymbol{H}^n(S, G) = \boldsymbol{H}^n(S/R)$ defined in [9], and the exact sequence of

Theorem 1.1 agrees with that of [9] § 2 in case S/R is a Galois extension. The isomorphism of Theorem 5.2 extends that of [9] Theorem 3.

Moreover, in this case the algebra (E, P, p) can be described as a generalized crossed product $\Delta(J, j)$ of Kanzaki [11] in the following manner. Let $e=\sum u_i\otimes v_i$ be the separability idempotent of S and put $e_\sigma=\sum \sigma(u_i)\otimes v_i (\sigma\in G)$ as in [8]. $1^2=\sum e_\sigma$ gives a decomposition of the identity of S^2 into orthogonal idempotents. If we regard S^3 as ${}_1S_2^2\otimes_{S_2}{}_2S_3^2$ it yields a decomposition $1^3=\sum_{\sigma,\tau} e_\sigma\otimes e_\tau$. When we regard $S^3={}_1S_2^2\otimes_{S_1}{}_1S_3^2$, we have $e_\sigma\otimes_{S_2}e_\tau=e_\sigma\otimes_{S_1}e_{\sigma\tau}$. Now $P\in\mathcal{P}ic(S^2)$ has a decomposition

$$P=\coprod_\sigma J_\sigma\,,\quad J_\sigma=e_\sigma P\qquad (\in P(S,\sigma) \text{ in the notation of [8]})$$

and the isomorphism $p\colon dP\xrightarrow{\sim} S^3$, expressed in the form

$${}_1P_2\otimes_{S_2}{}_2P_3\simeq (S_1\otimes S_2)\otimes_{S_1}{}_1P_3$$

decomposes into

$$\begin{aligned} e_\sigma P\otimes_{S_2}e_\tau P &\simeq e_\sigma S_1\otimes_{S_1}e_{\sigma\tau}P\\ x\ \otimes\ y\ &= e_\sigma\otimes j_{\sigma,\tau}(x,y)\end{aligned}$$

Hence we have, for $x\in J_\sigma$, $y\in J_\tau$,

$$\tilde{p}(x,y)=\sum v_i\otimes\sigma(u_i)j_{\sigma,\tau}(x,y)$$

Kanzaki's construction is $\Delta(J,j)=\coprod_\sigma J_\sigma$, with $j_{\sigma,\tau}$ as multiplication, while

$$D=(E,P,p)=\mathrm{End}(S)\otimes_{S^2}P\simeq\coprod_\sigma S_2^*\otimes_{S_2}e_\sigma P=\coprod_\sigma J_\sigma$$

where $S^*\simeq S$ is given by $tr\leftrightarrow 1$. The multiplication of D is given as follows: for $x\in J_\sigma$, $y\in J_\tau$,

$$\begin{aligned}(tr\otimes x)(tr\otimes y)&=\sum tr\cdot v_i\cdot tr\otimes\sigma(u_i)j_{\sigma,\tau}(x,y)\\ &=tr\otimes\sum tr(v_i)\sigma(u_i)j_{\sigma,\tau}(x,y)\\ &=tr\otimes j_{\sigma,\tau}(x,y)\end{aligned}$$

This shows the coincidence of the multiplication of D and that of Δ.

If S/R and S'/R' are Galois extensions both with G as the Galois group, and $\varphi\colon S\to S'$ is a G-homomorphism, φ is a homomorphism of algebras in the sense of § 6, φ_0 being the restriction of φ to R. The equivalence of $\mathcal{P}ic(S^{n+1})$ resp. $\mathcal{P}ic(S'^{n+1})$ with $C^n(S, G)$ resp. $C^n(S', G)$ then gives rise to an isomorphism of $H^n(\varphi, \mathrm{Pic})$ to $H^n(G, \mathrm{Pic}(\varphi))$ for every n. Thus the exact sequence of Theorem 6.1 coincides with that of [9] Theorem 2 in this case of Galois extensions.

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