1. Introduction

Let $G$ be a doubly transitive permutation group on a finite set $\Omega$ and $\alpha \in \Omega$. By [4], the product of all minimal normal subgroups of $G_\alpha$ is the direct product $A \times N$, where $A$ is an abelian group and $N$ is 1 or a nonabelian simple group.

In this paper we consider the case $N \cong PSL(3, q)$ with $q$ even and prove the following:

**Theorem.** Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and let $\alpha, \beta \in \Omega$ ($\alpha \neq \beta$). If $G_{\alpha}$ has a normal subgroup $N^{*}$ isomorphic to $PSL(3, q)$, $q = 2^n$, then $N^{*}$ is transitive on $\Omega - \{\alpha\}$ and one of the following holds:

(i) $G$ has a regular normal subgroup $E$ of order $q^3 = 2^{3n}$ (where $n$ is odd and $G_{\alpha}$ is isomorphic to a subgroup of $\Gamma L(3, q)$). Moreover, there exists an element $g$ in $\text{Sym}(\Omega)$ such that $\alpha^g = \alpha$, $(G_{\alpha})^g$ normalizes $E$ and $A\Gamma L(3, q) \geq (G_{\alpha})^g E \geq \text{ASL}(3, q)$ in their natural doubly transitive permutation representation.

(ii) $|\Omega| = 22$, $G^{o} = M_{22}$ and $N^{*} = PSL(3, 4)$.

(iii) $|\Omega| = 22$, $G^{o} = \text{Aut}(M_{22})$ and $N^{*} = PSL(3, 4)$.

We introduce some notations.

- $V(n, q)$: a vector space of dimension $n$ over $GF(q)$
- $\Gamma L(n, q)$: the group of all semilinear automorphism of $V(n, q)$
- $A\Gamma L(n, q)$: the semidirect product of $V(n, q)$ by $\Gamma L(n, q)$ in its natural action
- $\text{ASL}(n, q)$: the semidirect product of $V(n, q)$ by $SL(n, q)$ in its natural action
- $F(X)$: the set of fixed points of a nonempty subset $X$ of $G$
- $X(\Delta)$: the global stabilizer of a subset $\Delta (\subseteq \Omega)$ in $X$
- $X^\Delta$: the pointwise stabilizer of $\Delta$ in $X$
- $X^\Delta$: the restriction of $X$ on $\Delta$
- $\text{Sym}(\Delta)$: the symmetric group on $\Delta$
2. Preliminaries

Lemma 2.1 Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree, $\alpha \in \Omega$ and $N^a$ a normal subgroup of $G^a$ isomorphic to $\text{PSL}(2,q)$, $\text{Sz}(q)$ or $\text{PSU}(3,q)$ with $q(>2)$ even. Then $N^a=\text{PSL}(2,q)$, $N^a\neq\text{Sz}(q)$, $\text{PSU}(3,q)$, $N^a$ is transitive on $\Omega-\{\alpha\}$ and one of the following holds:

(i) $G$ has a regular normal subgroup $E$ of order $q^2$, $N^a=N^a\cap N^q=E_q$ and $G^a$ is isomorphic to a subgroup of $\text{TL}(2,q)$. Moreover there exists an element $g$ in $\text{Sym}(\Omega)$ such that $\alpha^g=\alpha$, $(G^a)^g\geqslant\text{ASL}(2,q)$ in their natural doubly transitive permutation representation.

(ii) $|\Omega|=6$ and $G^a=\text{A}_6$ or $\text{S}_6$.

Proof. By Theorem 2 of [2], it suffices to consider the case that $N^a=N^a\cap N^q=E_q$ and $G$ has a regular normal subgroup of order $q^2$. Since $|N^a|=q^2-1$, $N^a$ is transitive on $\Omega-\{\alpha\}$.

Let $E$ be the regular normal subgroup of $G$. Then we may assume $\Omega=E$, $\alpha=0\in E$ and the semidirect product $GL(E)E$ is a subgroup of $\text{Sym}(\Omega)$. There is a subgroup $H$ of $GL(E)$ such that $H\cong\text{TL}(2,q)$ and $HE\cong\text{ATL}(2,q)$. Let $L$ be the normal subgroup of $H$ isomorphic to $\text{SL}(2,q)$. Then $L\cong E_q$ for $\beta\in\Omega-\{\alpha\}$. Hence $(N^a)^{\beta\in\Omega-\{\alpha\}}=L$ and so there is an automorphism $f$ from $N^a$ to $L$ and $g\in\text{Sym}(\Omega)$ satisfying $\alpha^f=\alpha$ and $(\beta^s)^f=(\beta^s)^{\gamma(\beta)}$ for each $\beta\in \Omega-\{\alpha\}$ and $x\in N^a$. From this, $(\beta^s)^f=g^{-1}xg=f(x)$. Hence $g^{-1}Ng=L$.

Set $S=L_{\beta}$, $X=\text{Sym}(\Omega)\cap N(L), D=C_X(L)$ and $Y=N_{\beta}(S)$. By the properties of $\text{ATL}(2,q)$, $L$ is transitive on $\Omega-\{\alpha\}$, $|F(S)|=q$ and $Y/S=Z_{q-1}$. Hence $D$ is semi-simple on $\Omega-\{\alpha\}$ and $F(S)$ is regular on $F(S)-\{\alpha\}$ and so $D\cong D^{F(S)}\leq Y^{F(S)}$ because $[D,N^a]=1$. Therefore $D\leq Z_{q-1}$. Since $X/DL$ is isomorphic to a subgroup of the outer automorphism group of $SL(2,q)$, we have $|X|\leq|\text{TL}(2,q)|$, while $|\text{TL}(2,q)|H\geq X$. Hence $X=H$ and $X$ normalizes $E$. Therefore, as $(G_a)^f\geq (N^a)^f=L$, we have $(G_a)^f\leq H$. Thus Lemma 2.1 is proved.

Lemma 2.2 Let $G$ be a doubly transitive permutation group on $\Omega$ of even degree and $N^a$ a nonabelian simple normal subgroup of $G^a, \alpha \in \Omega$. If $C_G(N^a)\neq 1$, then $N^a\cong N^a\cap N^q$ for $\alpha \neq \beta \in \Omega$ and $C_G(N^a)$ is semi-simple on $\Omega-\{\alpha\}$. Moreover $C_G(N^a)=0(N^a)$.

Proof. See Lemma 2.1 of [2].
Lemma 2.3 Let $G$ be a transitive permutation group on a finite set $\Omega$, $H$ a stabilizer of a point of $\Omega$ and $M$ a nonempty subset of $G$. Then

$$|F(M)| = |N_G(M)| \times |\{M^g : M^g \subseteq H, g \in G\}| / |H|.$$ 

Proof. See Lemma 2.2 of [2].

Lemma 2.4 Let $H$ be a transitive permutation group on a finite set $\Delta$ and $N$ a normal subgroup of $H$. Assume that a subgroup $X$ of $N$ satisfies $X^H = X^N$. Then

(i) $|F(X) \cap \beta^N| = |F(X) \cap \gamma^N|$ for $\beta, \gamma \in \Delta$.
(ii) $|F(X)| = |F(X) \cap \beta^N| \times r$, where $r$ is the number of $N$-orbits on $\Delta$.

Proof. By the same argument as in the proof of Lemma 2.4 of [3], we obtain Lemma 2.4.

2.5 Properties of $PSL(3,q)$, $q=2^n$.

Let $N = SL(3,q)$, $S_1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $A_i = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $B_i = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $b, c \in GF(q)$ and $Z = \left\{ \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \right\}$, $d \in GF(q)$, $d^3 = 1$.

Then $|Z| = (3, q - 1)$ and $N_1 = N_1/Z$ is isomorphic to $PSL(3, q)$. Set $N = N_1$, $S = S_1$, $A = A_1$ and $B = B_1$. Then the following hold.

(i) $N$ is a nonabelian simple group of order $q^3(q - 1)(q + 1)(q^2 + q + 1)/\left(3, q - 1\right)$.
(ii) $|S| = q^3$, $S' = \Phi(S) = Z(S) = \{x^2 : x \in S\} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $b \in GF(q)$, $S/S' \simeq E_4$, and $S$ is a Sylow 2-subgroup of $N$.
(iii) $S = \langle A, B \rangle$, $A \cap B = Z(S)$, $I(S) \subseteq A \cup B$ and each elementary abelian subgroup of $S$ is contained in $A$ or $B$. Let $z \in I(S) - Z(S)$. Then $C_z(z) = A$ or $B$.
(iv) Set $M_1 = A^N$, $M_2 = B^N$. Then $M_1 \neq M_2$ and $M_1 \cup M_2$ is the set of all subgroup of $N$ isomorphic to $E_4$.
(v) Let $z$ be an involution of $N$. Then $I(N) = z^N$ and $|C_z(z)| = (q - 1)q^3/\left(3, q - 1\right)$.
(vi) Let $E$ denote $A$ or $B$. Then $|N_2(E)| = (q - 1)^2(q^2 + q^3)(3, q - 1)$, $N_2(E)/E \simeq Z_4 \times PSL(2, q)$, where $k = (q - 1)/(3, q - 1)$ and $N_2(E)$ is a maximal subgroup of $N$.
(vii) Set $M = \langle N_2(E) \rangle$. If $q > 2$, then $M = M'$, $M \supseteq E$, $M/E \simeq PSL(2, q)$ and $M$ acts irreducibly on $E$.
(viii) Set $\Delta = E^N$. Then $|\Delta| = q^2 + q + 1$ and by conjugation $N$ is doubly transitive on $\Delta$, which is an usual doubly transitive permutation representation.
of $N$. If $C \subseteq \{A, B\} - \{E\}$, $|F(C)| = q + 1$, $C$ is a Sylow 2-subgroup of $N_{F(C)}$ and $C$ is semi-regular on $\Delta - F(C)$.

**Lemma 2.6** ([6]). Let notations be as in (2.5) and set $G = \text{Aut}(N)$. Then the following hold.

(i) There exist in $G$ a diagonal automorphism $d$, a field automorphism $f$ and a graph automorphism $g$ and satisfy the following:

\begin{align*}
G &= \langle g, f, d \rangle N \triangleright H_1 = \langle f, d \rangle N \triangleright H_2 = \langle d \rangle N, \\
H_1 &= \Gamma L(3, q), \\
H_2 &= \text{PGL}(3, q) \\
H_2/N &= Z_r, \text{ where } r = (3, q - 1), \\
G|H_1 &= Z_2, \\
H_1|H_2 &= Z_2 \\
G/H_2 &= Z_2 \times Z_2.
\end{align*}

(ii) $M_1 = A^d, M_2 = B^g$, and $A^g = B$.

**Lemma 2.7** Let $N = \text{PSL}(3, q)$, where $q = 2^n$. Let $R$ be a cyclic subgroup of $N$ of order $q + 1$ and $Q$ a nontrivial subgroup of $R$. Then $N_N(Q) = N_N(R) - Z_k \times D_{2(q+1)}$, where $k = (q - 1)/(3, q - 1)$ and $D_{2(q+1)}$ is a dihedral group of order $2(q + 1)$.

Proof. We consider the group $N$ as a doubly transitive permutation group on $\Delta = \text{PG}(2, q)$ with $q^2 + q + 1$ points. By (2.5) (i), $R$ is a cyclic Hall subgroup of $N$ and so we may assume $R \leq N_{a}$, where $a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \text{PG}(2, q)$. Since $|N_{a}b| = (q - 1)^2 q^{2}/(q + 1, q + (q - 1)^2 q^{2}) = 1$, $R$ is semiregular on $\Delta - \{a\}$. Hence $N_{a}(Q) \leq N_{a}$. Put $E = \text{PG}(2, q)$. Then $N_2 = N_2(E)$ by (2.5) (viii) and $N_2(Q)E/E = Z_2 \times D_{2(q+1)}$ by (2.5) (vii). Since $N_{N}(Q) \cap E = C_2^2(Q) = 1$ by (2.5) (v). Hence $N_{N}(Q) = Z_2 \times D_{2(q+1)}$. As $R$ is cyclic, $N_{N}(R) \leq N_{N}(Q)$. Thus $N_{N}(Q) = N_{N}(R) = Z_2 \times D_{2(q+1)}$.

**Lemma 2.8** Let $N = \text{PSL}(3, q)$, $q = 2^n$ and let $H(\pm N)$ be a subgroup of $N$ of odd index. Then $H \leq N_{N_{b}}(E)$ for an elementary abelian subgroup $E$ of $N$ of order $q^2$.

Proof. Let $S, A$ and $B$ be as in (2.5) and let $\Delta$ be as in Lemma 2.7. Since $|N|H$ is odd, $H$ contains a Sylow 2-subgroup of $N$ and so we may assume $S \leq H$. Set $\alpha = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $S \leq N_{a} = N_{a}(A), S_{b} = B, S_{r} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ and hence $|\alpha^g| = 1, |\beta^g| = q$ and $|\gamma^g| = q^2$.

If $\alpha^g = \{a\}, H \leq N_{a} = N_{a}(A)$ and the lemma holds. By (2.5) (i), $(q^2 + 1,$ $|N|) = 1$. Hence $\alpha^g = \{a\} \cup \gamma^g$, so that we may assume either $\alpha^g = \{a\} \cup \beta^g$ or $\alpha^g = \Delta$.

If $\alpha^g = \{a\} \cup \beta^g, \alpha^g = F(B)$ and $B$ is a unique Sylow 2-subgroup of $H_{F(B)}$ by (2.5) (viii). Hence $H \geq B = E_{x}^g$ and the lemma holds.
If \( \alpha^H = \Delta \), by (2.5) (iv), \( N \) is transitive and so \( |H| \) is divisible by \( q+1 \). Since \((q^2 + q + 1, q+1) = 1\), \( |H| \) is divisible by \( q+1 \). By (2.5) (vi) and by the structure of \( PSL(2, q) \), \( Z_\Delta \times PSL(2, q) = H_\Delta/A \leq N \), where \( m \) is a divisor of \((n-1)/(3, n-1)\). Therefore \( |N : H| \leq q-1 \). We now consider the action of \( N \) on the coset \( \Gamma = NH \). As \( |\Gamma| \neq 1 \) and \( N \) is a simple group, \( N^\Gamma \) is faithful. But \( N \) has a cyclic subgroup of order \( q+1 \) and so \( |\Gamma| > q+1 \), which implies \( |N : H| > q+1 \), a contradiction.

**Lemma 2.9** Let \( N = PSL(3, q) \), where \( q = 2^m \) and \( t \) a field automorphism of \( N \) of order \( 2 \). Let \( S \) be a \( t \)-invariant Sylow 2-subgroup of \( N \). Then the following hold.

(i) \( Z(\langle t \rangle S) = E_{q^2} \).

(ii) If \( S_1 \) is a subgroup of \( \langle t \rangle S \) isomorphic to \( S \), then \( S_1 = S \).

Proof. Since \( C_{S}(t) \) is isomorphic to a Sylow 2-subgroup of \( PSL(3, \sqrt{q}) \), \( Z(C_{S}(t)) = E_{q^2} \) and \( Z(C_{S}(t)) \leq Z(S) \) by (2.5) (ii). Hence \( Z(\langle t \rangle S) = Z(\langle t \rangle S) \cap \langle t \rangle C_{S}(t) \cup \langle t \rangle C_{S}(t) \cap C(Z(S)) = Z(\langle t \rangle S) \cap C_{S}(t) = Z(C_{S}(t)) = E_{q^2} \). Thus we have (i).

Suppose \( S_1 \neq S \). Then \( \langle t \rangle S = S_1 S \geq S_1 \) and \( |\langle t \rangle S : S| = |S_1 : S_1 \cap S| = 2 \). If \( Z(S_1) \leq S \), we have \( S_1 = \langle z \rangle \times (S_1 \cap S) \) for an involution \( z \) in \( Z(S_1) - S \). By (2.5) (ii), \( z \in \Phi(S_1) \) and so \( S_1 = \langle z \rangle, S_1 \cap S \rangle = S_1 \cap S \), a contradiction. Hence \( Z(S_1) \not\leq S \).

If \( Z(S_1) = Z(S) \), \( E_{q^2} = Z(S) \leq Z(S_1 S) = Z(\langle t \rangle S) = E_{q^2} \) by (i), which is a contradiction. Hence \( Z(S_1) \neq Z(S) \).

Let \( z \) be an involution in \( Z(S_1) - Z(S) \). Then \( C_{S}(z) = E_{q^2} \) by (2.5) (iii). On the other hand, \( S_1 \leq C_{\langle t \rangle S}(z) \) and \( [C_{\langle t \rangle S}(z) : C_{S}(z)] = 1 \) or \( 2 \). From this \( S_1 \) has an elementary abelian subgroup of index 2. Hence \( q = 2 \), a contradiction. Thus we have (ii).

### 3. Proof of the theorem

Throughout the rest of the paper, \( G^\alpha \) always denote a doubly transitive permutation group satisfying the hypotheses of the theorem.

Since \( G_\alpha \geq N^\alpha, |\beta^{N^\alpha}| = |\gamma^{N^\alpha}| \) for \( \beta, \gamma \in \Omega - \{\alpha\} \) and so \( |\Omega| = 1 + r |\beta^{N^\alpha}| \), where \( r \) is the number of \( N^\alpha \)-orbits on \( \Omega - \{\alpha\} \). Hence \( r \) is odd and \( N^\alpha \) is a proper subgroup of \( N^\alpha \) of odd index for \( \alpha \neq \beta \in \Omega \). Therefore, by Lemma 2.8 \( N^\alpha \geq A \) for some elementary abelian subgroup \( A \) of order \( q^2 \). Let \( S \) be a Sylow 2-subgroup of \( N^\alpha \). Then, by (2.5) (iii) there exists a unique elementary abelian 2-subgroup \( B \) of \( S \) such that \( A = B = E_{q^2} \) and \( A = B \). Set \( M_1 = A^{N^\alpha}, M_2 = B^{N^\alpha} \) and \( K = G_\alpha(M_1) = G_\alpha(M_2) \). By (2.5) (iv), \( M_1 \cup M_2 \) is the set of all elementary abelian 2-subgroups of \( N^\alpha \) of order \( q^2 \) and \( G_\alpha \) acts on \( \{M_1, M_2\} \), so that \( G_\alpha / K \leq Z_2 \). Hence \( K \) is transitive on \( \Omega - \{\alpha\} \).

(3.1) Let \( E = A \) or \( B \). Then \( N_{G_\alpha}(E) \) is transitive on \( F(E) - \{\alpha\} \).
Proof. If $E^h \leq K_\beta$ for some $h \in K$, $E^h \leq N^\alpha \cap K_\beta = N^\alpha_\beta$. Since $E^{N^\alpha_\beta} = E^K$ and $A^K = B^K$, $E^h$ is conjugate to $E$ in $N^\alpha_\beta$. By a Witt's theorem $N_\kappa(E)$ is transitive on $F(E)-\{\alpha\}$. Thus $N_{\alpha\kappa}(E)$ is transitive on $F(E)-\{\alpha\}$.

(3.2) If $q=2$, $G^\alpha$ is of type (i) of the theorem.

Proof. Assume $q=2$. We note that $PSL(3,2)$ is isomorphic to $PSL(2,7)$. It follows from [3] that $G$ has a regular normal subgroup $R$.

Since $K$ is transitive on $\Omega-\{\alpha\}$, by Lemmas 2.3 and 2.4

$$|F(A)| = 1 + \frac{|N^\alpha_\beta \cap N(A)|}{|N^\alpha_\beta|} r = \frac{24r}{|N^\alpha_\beta|} + 1$$

and

$$|F(B)| = 1 + \frac{|N^\alpha_\beta \cap N(A)|}{|N^\alpha_\beta|} r = \frac{24r}{|N^\alpha_\beta|} + 1.$$  

Let $E=A$ or $B$. As $N_\kappa(E)+1$, $N_\kappa(E)^F(E)$ is doubly transitive by (3.1). Hence $E \leq N^\alpha_\beta$ and $|F(A)|=2^r$, $|F(B)|=2^b$ for some integers $a, b$. From this $S=\langle A, B \rangle \leq N^\alpha_\beta \cap N^\alpha_\beta$ and $N^\alpha_\beta$: $N^\alpha_\beta \cap N^\alpha_\beta$ is odd. Hence, if $S^\alpha \leq G_{a\beta}$, $S^\gamma \leq N^\alpha_\beta \cap N^\alpha_\beta$, where $\gamma = \alpha^\alpha$ and so $S^\alpha \leq N^\alpha_\beta \cap N^\alpha_\beta$. Since $Q$ and $S^\alpha$ are Sylow 2-subgroups of $N^\alpha_\beta \cap N^\alpha_\beta$, $S^\alpha$ is conjugate to $S$ in $N^\alpha_\beta \cap N^\alpha_\beta$. By a Witt's theorem $N_\kappa(S)^F(S)$ is a doubly transitive permutation group with a regular normal subgroup $N_\kappa(S)$. Hence $|F(S)|=2^c$ for an integer $c$. By Lemmas 2.3 and 2.4,

$$|F(S)| = 1 + 8 \times \frac{|N^\alpha_\beta : S|}{|N^\alpha_\beta|} r = r+1 = 2^c.$$  

Let $\pi$ be an involution of $Z(S)$ and assume $\pi^S \in G_\alpha$ for some $g \in G$. Then $\pi^S \in N^\alpha_\beta$, where $\gamma = \alpha^\alpha$. Since $|N^\alpha_\beta : N^\gamma \cap N^\alpha_\beta|$ is odd, $\pi^S$ is contained in $N^\alpha_\beta$. By (2.5) (v), $\pi^S$ is conjugate to $\pi$ in $N^\alpha_\beta$. Hence $C_G(\pi)$ is transitive on $F(\pi)$ and by Lemmas 2.3 and 2.4,

$$|F(\pi)| = 1 + 8 \times \frac{|I(N^\alpha_\beta)|}{|N^\alpha_\beta|} r.$$  

Suppose $N^\alpha_\beta = S$. Then $|F(A)| = 3r + 1 = 2^r + 2^r$ and $|F(\pi)|=5r+1$. Hence $r=1$. Since $N^\alpha_\beta(A) = C_G(\alpha) \leq C_G(\pi)$ and $N_\kappa(A) = E_{11}$, $|F(\pi)|$ is divisible by 4. But $|F(\pi)|=5r+1=6$. This is a contradiction.

Suppose $N^\alpha_\beta = S$. Then $N^\alpha_\beta = N^\alpha_\beta(\pi)$ as $N^\kappa(\pi) = S_4$. From this, $|F(B)| = 2^b = 2^2 + 2^2$ and so $r=1$. Hence $|\Omega| = 1 + |N^\alpha_\beta : N^\alpha_\beta| = 8$. Thus $|R|=8$ and $G_\alpha = GL(3,2)$, hence $G=AL(3,2)$.

By (3.2), it suffices to consider the case $q > 2$ to prove the theorem. From now on we assume the following.

Hypothesis (*): $q = 2^a \geq 4

(3.3) The following hold.
(i) $|N_\alpha^*/N_\alpha^* \cap N^\beta| \text{ is odd.}$

(ii) Let $\gamma \in \Omega$ and $S_0$ a 2-subgroup of $N^\gamma$. Then $F(S_0) = \{ \delta \in \Omega \mid S_0 \leq N^\delta \}$.

Proof. Suppose false and let $T$ be a Sylow 2-subgroup of $N_\alpha^*N_\beta^*$ such that $T \supsetneq S$. Then $T \neq S$. Set $S_1 = T \cap N_\alpha^*$ and $S_2 = T \cap N_\alpha^* \cap N^\beta$. Then $S_1$ is a Sylow 2-subgroup of $N_\alpha^*$, $S_1 \supseteq S$ and $S_1$, $S_2$ and $S$ are normal subgroups of $T$. By Lemma 2.2, $S_1N_\alpha^*/N_\alpha^*$ is isomorphic to a subgroup of the outer automorphism group of $N^\alpha$. It follows from Lemma 2.6 that $S_1N_\alpha^*/N_\alpha^*$ is abelian of 2-rank at most 2. Since $S_1N_\alpha^*/N_\alpha^* = S_1/S_2$ and $S_1 \supseteq S$, we have $S_1/S_2 \leq E_4$ by (2.5) (ii).

Let $A_1, B_1$ be the subgroups of $S_1$ such that $A_1 = B_1 = E_4$ and $A_1 \cap S_2 \leq A, B_1 \cap S_2 \leq B$. Since $A_1 \cap S_2 = A_1S_2/S_2 \leq S_1/S_1 \leq E_4$ and by the hypothesis (*), $q \geq 4$, we have $|A_1 \cap S_2| \geq q^2/4$. Therefore, if $A_1 \cap S_2 \leq Z(S)$, then $q = 4$, $A_1 \cap S_2 = Z(S)$ and $T = A_1S$ and so $Z(S) \leq Z(T)$, contrary to Lemma 2.9. Hence $A_1 \cap S_2 \not\subseteq Z(S)$. Similarly $B_1 \cap S_2 \not\subseteq Z(S)$.

Let $x \in A_1 \cap S_2 \not\subseteq Z(S)$. Then $x \in A \leq S$ for each $y \in A_1$ and so $A_1$ normalizes $A$. Hence $A_1$ normalizes $B$. Similarly $B_1$ normalizes $A$ and $B$. From this $T = \langle A_1, B_1 \rangle \supseteq A, B$ and so $S_1N_\alpha^* \leq K$. Hence $S_1N_\alpha^*/N_\alpha^* = S_1/S_2 \leq Z_2$, so that there exists a field automorphism $\tau$ of order 2 such that $T = \langle \tau \rangle \cdot S \supsetneq S$. Since $S_1 \leq T$ and $S_1 \supseteq S$, we have $S_1 = S$ by Lemma 2.9, a contradiction. Thus (i) holds.

Let $\delta \in F(S_0) = \{ \gamma \}$. Then $S_0 \leq N_\alpha^*$. Since $N_\gamma^* \supseteq N_\gamma \cap N^\beta$ and $|N_\gamma^* / N_\gamma \cap N^\beta|$ is odd by (i), $S_0 \leq N_\alpha^*/N_\alpha^* \leq N^\beta$. Hence $F(S_0) = \{ \delta \in \Omega \mid S_0 \leq N^\delta \}$. The converse implication is clear. Thus (ii) holds.

(3.4) The following hold.

(i) $N_\alpha(B)^F(B)$ is doubly transitive.

(ii) If $F(A) = \{ \alpha, \beta \}$, $N_\alpha(A)^F(A)$ is doubly transitive.

Proof. Let $E = A$ or $B$. By (3.3) (i), $S$ is a Sylow 2-subgroup of $N_\beta^*$. Therefore, by a similar argument as in (3.1), $N_\gamma^*(E)$ is transitive on $F(E) = \{ \beta \}$. Suppose $N_\gamma(E)^F(E)$ is not doubly transitive. Then, $F(E) = \{ \alpha, \beta \}$ by (3.1) and (3.3). Since $N_\gamma^*(E)$ acts on $F(E)$ and fixes $\{ \alpha \}$, we have $N_\gamma^*(E) \leq N_\gamma^*$. On the other hand $N_\gamma^*(E)$ is a maximal subgroup of $N^\gamma$ by (2.5) (vi). Hence $N_\gamma^*(E) = N_\gamma^*$. If $E = B$, then $N_\beta^* \supsetneq A$, a contradiction. Thus $E = A$ and (3.4) follows.

(3.5) The following hold.

(i) Put $M = (N_\gamma^*(A))^*$. Then $F(M) = F(A)$.

(ii) $N_\gamma^* = N_\alpha^*$ for each $\gamma \in F(A) = \{ \alpha \}$.

Proof. Suppose $F(M) \neq F(A)$. Then $M \not\subseteq N_\alpha(A)^F(A)$. It follows from (3.4) that $F(A) = \{ \alpha, \beta \}$ and $N_\alpha(A)^F(A)$ is doubly transitive. Moreover by (2.5) (vii) $N_\alpha(A)^F(A) \supseteq M^F(A) \iso PSL(2, q)$ as $q > 2$. By Lemma 2.1, $r = 1$ and either (1) $q =$
4 and \( N_G(A)F(A) = A_6 \) or \( S_6 \) or (2) \( |F(A)| = q^2 \).

If (1) holds, \( |F(A)| = 1 + |N_{N^A}(A): N_{N^A}^A| = 1 + 2^6 \cdot 3 \cdot 5 = 1 + 2^{25} \cdot 3 \cdot 5 \cdot 7 / 2^6 \cdot 3 = 2 \cdot 53 \). Let \( z \) be an involution of \( N^A \cap N^B \). Then, by (2.5) (v) and (3.3), \( z^0 \cap G = z^0 A \), so that \( C_G(z)F(z) \) is transitive by a Witt’s theorem. On the other hand \( |F(A)| = 1 + |N_{N^A}(z) \times |I(N_{N^A}^A)| = 1 + 2^6 \cdot 3^3 / 2^6 \cdot 3 = 10. \) In particular \( |C_G(z)| \) is divisible by \( |N_{N^A}^A| \).

5. Let \( R \) be a Sylow 5-subgroup of \( C_G(z) \). Then \( |\Omega|, |G_a: N^A| and |N_{N^A}^A| are not divisible by 5 and so \( F(R) = \{ \gamma \} \) and \( R \leq N^\gamma \) for some \( \gamma \in \Omega \). Therefore \( <\gamma> \times R \leq N^\gamma \) by (3.3) (ii). But \( |C_N^\gamma(z)| = 2^6 \) by (2.5) (v). This is a contradiction.

If (2) holds, \( q^2 = |F(A)| = 1 + |N_{N^A}(A): N_{N^A}^A|, \) hence \( |N_{N^A}^A| = (q - 1)q^3 / (q - 1) = q - 1 + q + 1 = q(q^3 - 1). \) Hence \( |G| = |\Omega| = 1 + |N^A: N_{N^A}^A| = 1 + (q - 1)(q + 1)(q^2 + 1) = q(q^2 + 1). \) Hence \( |G| = |\Omega| \times |G_a: K| \times |K| \geq 2 q^2 K |K| = 2 q - 1 \). On the other hand \( |N_G(A)|_2 = |F(A)| \times |N_{G_a(A)}| = q^2 |K| \times |K| \geq 2 q^2 |K| = 2 q^2 |K|, \) because \( K = N_{G_a(A)} N_{N^A} \). Therefore \( q^2 |K| = q^2 |K| \leq |G| \leq q^2 |K| \times |K| \geq 2 q^2 |K| \) and we obtain \( q = 2 \), contrary to the hypothesis \( B \).

Thus we have (i).

Let \( \gamma \in F(A) \setminus \{ \alpha \} \). By (i) and (3.4) (ii), \( N_a^\gamma \supset A \) and \( M \leq N_a^\gamma \). Since \( N_{N^A}(A)/M = Z_k \), where \( k = (q - 1) / (3, q - 1) \) and \( |N_{N^A}^A| / |M| = |N_{N^A}^A| / |M| \), we have \( N_a^\gamma = N_a^\gamma \).

Thus (ii) holds.

(3.6) \( B \subseteq A^6 \) and \( G_a = K \).

Proof. If \( B \subseteq A^6 \), by (3.4) (i), there is an element \( g \in G_a B \) such that \( B = A^\varepsilon \). Hence \( N^6_a = g^{-1} N_{N^A}^a g \supset g^{-1} A g = B \) and so \( M \) normalizes \( <A, B> = S \), a contradiction.

(3.7) Set \( L = (N_{N^A}(B))' \). Then \( r = 1, L_{F(B)} = B, L_{F(B)} = L = B \equiv PSL(2, q), L_{B} = S \) and one of the following holds.

(i) \( C_G(N^A) = 1, |F(B)| = 6, q = 4 \) and \( N_G(B)F(B) = A_4 \) or \( S_6 \).

(ii) \( C_G(N^A) \leq Z_{q - 1}, |F(B)| = q^2 \) and \( N_G(B)F(B) \) has a regular normal subgroup.

Proof. By (3.4) (i), \( N_G(B)F(B) \) is doubly transitive. If \( L \leq G_a B \), then \( L \leq N_a^\gamma \) and so \( B \leq L = L' \leq (N_a^\gamma)' = M \). Therefore \( L = M \) and \( M \geq <A, B> = S \), a contradiction. Hence \( L \leq G_a B \). From this \( N_{G_a}(B)F(B) \geq L_{F(B)} = PSL(2, q) \) and (3.7) follows from Lemmas 2.1 and 2.2.

(3.8) If (i) of (3.7) occurs, then we have (ii) or (iii) of the theorem.

Proof. Since \( |F(B)| = 1 + |N_{N^A}(B): N_{N^A}^A(B)| = 6 \) and \( N_{N^A}^A(N_{N^A}(B)) = |N_{N^A}^A| = 5 \), we have \( |N_{N^A}^A| = 2^6 \cdot 3 \cdot 5 \). Hence \( |N_{N^A}^A(N_{N^A}(A))| = 5 \) and so \( |\Omega - \{ \alpha \}| = |N^A: N_{N^A}^A| = 21 \). By (3.6), \( PSL(3, 4) \leq (G_a)^{Q(\theta)} \leq PTL(3, 4) \) in their natural doubly transitive permutation representation and hence (3.8) follows from Satz 7 of [7].

In the rest of this paper, we consider the case (ii) of (3.7). From now on
we assume the following.

Hypothesis (**): \( r=1, q=2^s > 2, |F(B)| = q^2 \) and \( N_G(B)^{F(B)} \) is a doubly transitive permutation group with a regular normal subgroup.

(3.9) **The following hold.**

(i) \( N^a_B = N^a \cap N^B = M \) and \( |N^a_B| = (q-1)(q+1)q^3 \).

(ii) \( n \) is odd.

(iii) \( |F(A)| = q \).

Proof. Since \( q^2 = |F(B)| = 1 + |N_{Na}(B): N_{Na}(B)| \) by (3.7), we have \( |N_{Na}(B)| = |N_{Na}(B)| / (q^2 - 1) = (q-1)q^3 / (3q-1) \). As \( N_{Na}(B) \supseteq A \), \( N_{Na}(B) = N_{Na}(A, B) = N_{Na}(S) \). On the other hand, from (2.5) (vi) \( |N_{Na}(S)| = |N^a_B: M| \times |N_M(S)| = |N^a_B: M| \times (q-1)q^3 \). Therefore \( (3q-1) = 1 \) and \( |N^a_B: M| = 1 \). Hence \( N^a_B = M \) and \( n \) is odd. By (3.3) (i) and (2.5) (vii), \( N^a_B = N^a \cap N^B \). Hence \( F(A) = 1 + |N_{Na}(A)| / |N^a| = q \). Thus we have (3.9).

(3.10) **Put** \( m = |Ga: N^a| \). **Then the following hold.**

(i) \( m \) is odd and \( S \) is a Sylow 2-subgroup of \( G_a \).

(ii) \( |\Omega| = q^2 \) and \( |G| = q^2(q-1)(q+1)(q^2+q+1)m \).

Proof. Set \( C^a = C_G(N^a) \). By (3.6), (3.9) (ii) and Lemma 2.6, \( |G_a : C^a N^a| \) is odd. Since \( C^a \cap N^a = 1, m = |G_a : C^a N^a| \cdot |C^a| \) and so \( m \) is odd by Lemma 2.2. Therefore \( S \) is a Sylow 2-subgroup of \( G_a \) and so (i) holds.

Since \( |\Omega| = 1 + |N^a: N^a_B|, |\Omega| = q^3 \) by (3.9). From this \( |G| = |\Omega| \times |G_a| = q^5(m \times |N^a|) = q^5(q-1)(q+1)(q^2+q+1)m \). Thus (ii) holds.

(3.11) **Let** \( z \) **be an involution of** \( G_a \). **Then** \( |F(z)| = q^2 \). **In particular** \( B \) **is semi-regular on** \( \Omega - F(B) \).

Proof. By (3.10) (ii), \( z \) is contained in \( N^a \). By (2.5) (vii) and (3.9) (ii), \( |I(N^a_B)| = |N^a_B: N_{Na}(S)| \times (q^2 - q) + q^2 - 1 = (q-1)(q^2 - q) + q^2 - 1 = (q-1)(q+1)^3 \), hence \( |F(z)| = 1 + q^3(q-1)^2(q+1)^2(q^2 - q)(q+1) = q^2 \) by Lemma 2.3. As \( |F(B)| = q^2, B \) is semi-regular on \( \Omega - F(B) \).

(3.12) **Set** \( \Delta = F(B) \). **Then the following hold.**

(i) \( G_\Delta \supset B \) and \( B \) is a Sylow 2-subgroup of \( G_\Delta \).

(ii) \( G(\Delta) = N_G(B) \) and \( |N_G(B)| = q^5(q-1)(q+1)m \).

Proof. Since \( N_{Na}(B) \leq N^a(\Delta) \neq N^a \) and \( N_{Na}(B) \) is a maximal subgroup of \( N^a \), we have \( N_{Na}(B) = N^a(\Delta) \). By (3.7), \( B \) is a normal Sylow 2-subgroup of \( (N^a)_\Delta \) and (i) follows immediately from (3.10) (i).

Since \( G(\Delta) \supseteq G_\Delta \) and \( B \) is a characteristic subgroup of \( G_\Delta \) by (i), we have \( G(\Delta) \leq N_G(B) \). The converse implication is clear. Thus \( G(\Delta) = N_G(B) \). By (3.6), \( G_a = N_G(B)N^a \) and so \( |N_G(B): N^a(B)| = |G_a : N^a| = m \). Hence \( |N_G(B)| \)
\[ |F(B)| \times |N_G(B)| = q^2m \times |N_G(B)| = q^5(q-1)^2(q+1)m. \] Thus we have (ii).

(3.13) Let \( T_1 \) be a Sylow 2-subgroup of \( N_G(B) \) and \( T_2 \) a Sylow 2-subgroup of \( N_G(T_1) \). Then \( T_1 \cong T_2 \). Let \( x \) be an element of \( T_2 - T_1 \) and set \( U = BB^x \). Then \( U = E_4 \) and for each \( \gamma \in \Omega \), \( U_\gamma = E_4^{\gamma} \), \( U_\gamma \in B^G \), \( \gamma^U = F(U_\gamma) \) and \( |\gamma^U| = q^2 \). Moreover \( U_\gamma = U_8 \) for all \( \delta \in \gamma^U \).

Proof. If \( B \cap B^x \neq 1 \), by (3.11) and (3.12) (i), we have \( B = B^x \) and so \( x \in T_1 \), contrary to the choice of \( x \). Hence \( B \cap B^x = 1 \). As \( T_1 \supseteq B \) and \( T_1 = T_1^s \supseteq B^s \), \( U = B \times B^s \) and \( U = E_4^s \).

Let \( \gamma \in \Omega \) and put \( D = U_\gamma \). Then \( F(D) \cong \gamma^U \) as \( U \) is abelian. Therefore \( |U : D| = |\gamma^U| \leq q^2 \) by (3.11), while \( |D| \leq q^2 \) because \( D \) is an elementary abelian subgroup of \( N_\gamma \). Hence \( D = E_4^s \) and \( |F(D)| = |\gamma^U| = q^2 \). By (3.6) and (3.9) (iii), \( D \subseteq B^G \). Since \( U_\gamma \leq U_8 = E_4^s \) for each \( \delta \in \gamma^U \), we have \( U_\gamma = U_8 \).

(3.14) Let \( U \) be as in (3.13). Let \( \Gamma = \{X_i|1 \leq i \leq s\} \) be the set of \( U \)-orbits on \( \Omega \) and set \( B_i = U_\gamma \) for \( \gamma \in X_i \) with \( 1 \leq i \leq s \). Then the following hold.

(i) \( s = q, \Omega = \bigcup_{i=1}^s X_i \) and \( |X_i| = q^2 \).

(ii) \( B_i \) is semi-regular on \( \Omega - X_i \) and \( B_i \cap B_j = 1 \) for each \( i, j \) with \( i \neq j \).

Proof. By (3.10) (ii) and (3.13), \( |X_i| = q^2 \) and \( |\Omega| = q^3 \), hence \( s = q \). Cleary \( \Omega = \bigcup_{i=1}^s X_i \). Thus we have (i).

By (3.13) (ii), \( B_i \) is conjugate to \( B \) for each \( i \). Hence \( B_i \) is semi-regular on \( \Omega - X_i \) by (3.11). Therefore, if \( B_i \cap B_j \neq 1 \), then \( X_i = F(B_i) = F(B_j) = X_j \), so that \( i = j \). Thus we have (ii).

(3.15) Set \( Y = \{B_i|1 \leq i \leq q\} \) and let \( D \subseteq Y \). Then \( N_0(D) \leq N_0(U) \) and \( U \) is a unique Sylow 2-subgroup of \( C_0(D) \).

Proof. Suppose \( N_0(D) \leq N_0(U) \). Since \( [N_0(D), U] \leq U \), there exist \( g \in N_0(D) \) and \( B_i \in Y - \{D\} \) such that \( (B_i)^g \subseteq U \). Set \( D_i = (B_i)^g \). Since \( [D_i, D] = [B_i, D]^g = 1 \), it follows from (3.10) (i) that \( F(D_i) \cap F(D) = \phi \) and so \( D \) is regular on \( F(D_i) \) by (3.11). Hence \( F(D_i) = \gamma^D = \gamma^U \) for \( \gamma \in F(D_i) \). By (3.14), \( F(D_i) = F(B_j) \) for some \( B_j \in Y \). By (3.12) (i), \( D_i = B_j \), so that \( D_i \leq U \), a contradiction. Thus we have \( N_0(D) \leq N_0(U) \). Hence \( U \leq 0_2(C_0(D)) \). Since \( U \leq C_0(B) \), \( C_0(B) \) is transitive on \( F(B) \). Hence \( |C_0(B)| = |F(B)| \times |C_0(B)| = q^4 \) by (3.10) (i). Therefore \( |C_0(D)| = q^4 \) as \( D \subseteq B^G \) and so \( U \) is a unique Sylow 2-subgroup of \( C_0(D) \).

(3.16) \( |N_0(U)| = q^5(q-1)^2(q+1)m \).

Proof. Let \( S_1 \) be a Sylow 2-subgroup of \( N_0(U) \) and \( S_2 \) be a Sylow 2-subgroup of \( N_0(S_1) \). Suppose \( S_1 \neq S_2 \) and let \( w \) be an element of \( S_2 - S_1 \).
Set $\gamma = \alpha^{w-1}$. Then $(U_\gamma)^w \leq B^G$ by (3.13) and $(U_\gamma)^w \leq (G_\gamma)^w = G_\alpha$. Since $U$ and $U^w$ are normal subgroups of $S_1$, $(B, (U_\gamma)^w)$ is 2-subgroup of $G_\alpha \cap S_1 = S$. Hence $B = (U_\gamma)^w$ by (2.5) (iii) and (3.6). Therefore $U, U^w \leq C_G(B)$, so that $U = U^w$ by (3.15) and $w \in S_2 \cap N_\alpha(U) = S_1$, contrary to the choice of $w$. Hence $S_1 = S_2$ and $S_1$ is a Sylow 2-subgroup of $G$. It follows from (3.10) that $|S_1| = q^2$.

We now consider the action of $N_\alpha(U)$ on $\Gamma = \{x_i \mid 1 \leq i \leq q\}$. Set $B = (U_\gamma)^w$ by (2.5) (iii) and (3.6). Therefore $U, U^w \leq C(G(B))$, so that $U = U^w$ by (3.15) and $w \in S_2 \cap N_\alpha(U) = S_1$, contrary to the choice of $w$. Hence $S_1 = S_2$ and $S_1$ is a Sylow 2-subgroup of $G$. It follows from (3.10) that $|S_1| = q^2$.

(3.17) Let $R$ be a cyclic subgroup of $N_\alpha^a$ of order $q+1$. Then $|F(R)| = q$ and $R$ is semi-regular on $\Omega - F(R)$.

Proof. Since $N_\alpha^a/A \cong PSL(2, q)$, there exists a cyclic subgroup $R$ of $N_\alpha^a$ of order $q+1$. Let $Q$ be a subgroup of $R$. Then, by Lemma 2.7

$$|F(Q)| = 1 + \frac{|N_{N_\alpha^a}(Q)| \times |N_{N_\alpha^a}(Q)|}{|N_{N_\alpha^a}|} = 1 + \frac{2(q-1)(q+1)}{2(q+1)} = q.$$ 

Thus (3.17) holds.

(3.18) Let $V \leq U^G$. If $V \neq U$, then $|F(U_\gamma) \cap F(V_\gamma)| = 1$ or $q$ for $\gamma \in \Omega$.

Proof. Suppose $\gamma \neq \delta \in F(U_\gamma) \cap F(V_\gamma)$. By (3.13), $U_\gamma, V_\gamma \leq B^G$ and so by (3.3) (ii), $U_\gamma, V_\gamma \leq N^a \cap N^a$. Set $H = C_G(N_\gamma^a)$. Then, by (3.6) and (3.9) (i), $U_\gamma H$ and $V_\gamma H$ are Sylow 2-subgroups of $N_\gamma^a$. If $U_\gamma H = V_\gamma H$, then $U_\gamma = V_\gamma$ and $U, V \leq C_G(U_\gamma)$. By (3.15) we have $U = V$, a contradiction. Therefore $U_\gamma H \neq V_\gamma H$. Set $X = \langle U_\gamma, V_\gamma \rangle$. Then $XH = N_\gamma^a$ because $N_\gamma^a/H \cong PSL(2, q)$, $q = 2^a$, and $PSL(2, q)$ is generated by its two distinct Sylow 2-subgroups. Hence $N_\gamma^a \geq X \cap H$. By (2.5) (vii), $E_\gamma \leq U_\gamma \cap H \leq X \cap H$. Since $N_\gamma^a$ acts irreducibly on $H$ by (2.5) (vii), $X \cap H = H$ and hence $X \leq H$. From this $X = N_\gamma^a$. Thus, by (3.5) (i) and (3.9), $|F(U_\gamma) \cap F(V_\gamma)| = |F(X)| = |F(N_\gamma^a)| = q$.

(3.19) Let $Q$ be a cyclic subgroup of $N_{N^a}(B)$ of order $q+1, V \leq U^G$ and set $P = N_\alpha(U_\gamma)$. Then the following hold.

(i) $Q$ is semi-regular on $\Omega - F(Q)$ and $|F(Q)| = q$.

(ii) If $P \neq 1$ and $V \geq D \leq B^G$, then $P$ normalizes $D$ and $|F(P) \cap F(D)| = 1$.

Proof. Since $N_{N^a}(B)/B \cong PSL(2, q)$, there exists a cyclic subgroup $Q$ of $N_{N^a}(B)$ of order $q+1$. Clearly $Q$ is a cyclic Hall subgroup of $N_{N^a}$, hence $Q$ is conjugate to $R$ defined in (3.17). By (3.17), $Q$ is semi-regular on $\Omega - F(Q)$ and $|F(Q)| = q$. Thus (i) holds.

Suppose $P \neq 1$ and let $\gamma \in F(P)$. Then, by (3.9) (i), $P \leq N^\gamma$ and hence $P$ normalizes $N^\gamma \cap V$. By (3.10) (i) and (3.13), $N^\gamma \cap V = V_\gamma$ and $V_\gamma \leq B^G$ and so $P \leq N_{N^a}(V_\gamma)$ and $N_{N^a}(V_\gamma)^{\gamma} = N_{N^a}(B)^{\gamma}$. Hence we have $F(P) \cap F(V_\gamma) = \{\gamma\}$ by (3.7). As $|F(P)| = q$ by (i), (ii) holds.
Let $V \in U^G - \{U\}$ and let $Q$ be a cyclic subgroup of $N_{p^q}(B)$ of order $q+1$. Then $N_Q(V) = 1$.

Proof. Set $P = N_Q(V)$ and assume $P \neq 1$. Let $\gamma \in \Omega - F(Q)$ and set $B_1 = U \gamma$, $B_2 = V$. By (3.15), $Q$ normalizes $U$ and so by (3.19) $Q$ normalizes $B_1$. Similarly $P$ normalizes $B_2$. Therefore $F(B_1) \cap F(B_2) \geq \gamma^p \neq \{\gamma\}$ as $P \neq 1$ and $P$ is semi-regular on $\Omega - F(Q)$. By (3.18), we have $|F(B_1) \cap F(B_2)| = q$. Since $P$ acts on $F(B_1) \cap F(B_2)$ and $|P|$ divides $q+1$, $P$ fixes at least two points of $F(B_1) \cap F(B_2)$, which contradicts to (3.19).

(3.21) Let $T$ be a Sylow 2-subgroup of $N_G(U)$. Then, for each $V \in U^G - \{U\}$, $|T: N_T(V)|$ is divisible by $q$.

Proof. Suppose $|T: N_T(V)| < q$ and set $T_1 = N_T(V)$. Then $|T_1| > q^5$ as $|T| = q^6$ by (3.16). Hence $q > |T_1: V| = |V: V \cap T_1|$ and so $|V \cap T_1| > q^3$. Therefore, for each $B_i \in B^G$ such that $B_i \leq V$, $q > |B_i(V \cap T_1)| = |B_i: B_i \cap T_1| = |B_i: B_i \cap T_1|$. Hence $|B_i: B_i \cap T| > q$. Let $\gamma \in F(B_i) \cap T$ and set $B_2 = U \gamma$. Then $\langle B_1 \cap T, B_2 \rangle \leq N_V \cap T$. As $|B_1 \cap T| > q$ by (2.5) (iii), $B_1 \cap T \cap B_2 \neq 1$. By (3.11), $\langle B_1 \cap T, B_2 \rangle \leq F_G(B_2)$. By (3.12) (i), we have $B_1 \cap T \leq B_2$, so that $F(B_1) = F(B_1 \cap T) = F(B_2)$. Again, by (3.12) (i), $B_1 = B_2$ and so $U \cap C_G(B_2)$. Therefore $U = V$ by (3.15), a contradiction.

(3.22) Put $W = U^G$. Then $|W| = q^2 + q + 1$ and $G_W$ is doubly transitive.

Proof. Set $H = N_G(U)$. By (3.10) (ii) and (3.16), $|W| = |G: H| = q^2 + q + 1$. Let $V \in W - \{U\}$ and let $Q$ be as defined in (3.20). By (3.15), $Q \leq H$ and by (2.5) (iii), $Q \leq N_V \cap T$. Hence $|V^H|$ is divisible by $q + 1$. On the other hand, by (3.21), $|V^H|$ is divisible by $q$ and so we have $|V^H| = q(q+1)$. Thus (3.22) holds.

(3.23) $G_w \cap U \neq 1$.

Proof. Suppose $G_w \cap U = 1$. Since $G \geq G_w$ and $H \geq U$, $[G_w, U] \leq G_w \cap U = 1$. Hence $G_w \leq C_{G}(U)$. By (3.15), $U$ is a unique Sylow 2-subgroup of $C_{G}(U)$ and so $G_w \leq 0(G)$. On the other hand, as $|\Omega|$ is even and $G$ is doubly transitive on $\Omega$, we have $0(G) = 1$. Therefore $G_w = 1$ and hence $G$ acts faithfully on $W$. Since $U$ is not semi-regular on $W - \{U\}$, by [4], $PSL(n_1, q_1) \leq G \leq PTL(n_1, q_1)$ for some $n_1 \geq 3$ and $q_1$ with $q_1$ even. As $|W| = q^2 + q + 1 = q_1^n + \cdots + q_1 + 1$, $q(q+1) = q_1(q_1^{-2} + \cdots + 1)$ and so $q = q_1$ and $n_1 = 3$. Therefore $PSL(3, q) \leq G \leq PTL(3, q)$. But $|PTL(3, q)| = q^3$ by (3.9) (ii) and Lemma 2.6. Hence $q^3 = q^6$ by (3.10) (ii). This is a contradiction. Thus $G_w \cap U \neq 1$.

(3.24) $G^q$ has a regular normal subgroup.

Proof. Since $G_w \leq N_G(U)$, $G_w \cap U$ is a normal subgroup of $G_w$. As $G_w \cap
DOUBLE TRANSITIVE GROUPS OF EVEN DEGREE 829

Let $E$ be a minimal normal subgroup of $G$ contained in $0_2(G_w)$. Then $E$ is an elementary abelian 2-subgroup of $G$ and acts regularly on $\Omega$.

(3.25) If (ii) of (3.7) occurs, we have (i) of the theorem.

Proof. By (3.9), (3.10) and (3.24), $G$ has a regular normal subgroup $E$ of order $q^3$, where $q=2^n$ and $n\equiv 1 \pmod{2}$ and $N^a$ is transitive on $\Omega - \{\alpha\}$. Moreover $G=G_aE$ and $G_a$ is isomorphic to a subgroup of $GL(E)$. As in the proof of Lemma 2.1, we may assume $\Omega = E$, $\alpha=0 \in E$ and $GL(E)E \leq Sym(\Omega)$. There exists a subgroup $H$ of $GL(E)$ such that $H=\Gamma L(3,q)$ and $HE=\Gamma L(3,q)$. Let $L$ be a normal subgroup of $H$ isomorphic to $SL(3,q)$. Since $q=2^n$ and $n\equiv 1 \pmod{2}$, $L$ is isomorphic to $PSL(3,q)$.

By (3.9) (i) and by the structure of $\Gamma L(3,q)$, there exist an automorphism $f$ from $N^a$ to $L$ and $g \in Sym(\Omega)$ such that $\alpha^g=\alpha$ and $(\beta^g)^x=(\beta^x)^g$ for each $\beta \in \Omega - \{\alpha\}$ and $x \in N^a$. From this $(\beta^g)^x=(\beta^x)^g$ for each $\beta \in \Omega - \{\alpha\}$ and so $g^{-1}xg=f(x)$. Hence $g^{-1}N^a \leq L$. Hence $g^{-1}N^a \leq L$.

Set $X=N(L) \cap Sym(\Omega)$ and $D=C_X(L)$. Then $D$ is semi-regular on $\Omega - \{\alpha\}$ as $L$ is transitive on $\Omega - \{\alpha\}$. Put $T=f(A)$. Then $N_2(T)^{f(T)}Z_{q-1}$ and it is semi-regular on $F(T) - \{\alpha\}$ by (3.5) (i) and (3.9) (i), (iii). It follows that $D \leq Z_{q-1}$. Since $X/DL$ is isomorphic to a subgroup of the outer automorphism group of $PSL(3,q)$ and $f(A)$ and $f(B)$ are not conjugate in $Sym(\Omega)$ by the hypothesis (***) and (3.9) (ii), it follows from Lemma 2.6 (i) that $|X/DL| \leq n$. Hence $|X| \leq n(q-1)|L|=|\Gamma L(3,q)|$. On the other hand $\Gamma L(3,q)=H \leq X$ and so $X=H$. Therefore $g^{-1}G_ag \leq g^{-1}N^ag=L$ and $g^{-1}G_ag \leq X=H$. Thus we have (3.25).

The conclusion of the theorem now follows immediately from steps (3.2), (3.7), (3.8) and (3.25).

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References

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