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DOUBLY TRANSITIVE GROUPS OF EVEN DEGREE WHOSE ONE POINT STABILIZER HAS A NORMAL SUBGROUP ISOMORPHIC TO $PSL(3,2^n)$

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1. Introduction

Let G be a doubly transitive permutation group on a finite set Ω and $\alpha\in\Omega$. By [4], the product of all minimal normal subgroups of G_α is the direct product $A\times N$, where A is an abelian group and N is 1 or a nonabelian simple group.

In this paper we consider the case $N\simeq PSL(3,q)$ with q even and prove the following:

Theorem. *Let G be a doubly transitive permutation group on Ω of even degree and let α, $\beta\in\Omega$ ($\alpha\neq\beta$). If G_α has a normal subgroup N^α isomorphic to $PSL(3,q)$, $q=2^n$, then N^α is transitive on $\Omega-\{\alpha\}$ and one of the following holds:*

(i) *G has a regular normal subgroup E of order $q^3=2^{3n}$, where n is odd and G_α is isomorphic to a subgroup of $\Gamma L(3,q)$. Moreover there exists an element g in $Sym(\Omega)$ such that $\alpha^g=\alpha$, $(G_\alpha)^g$ normalizes E and $A\Gamma L(3,q)\geq(G_\alpha)^gE\geq ASL(3,q)$ in their natural doubly transitive permutation representation.*

(ii) $|\Omega|=22$, $G^\Omega=M_{22}$ and $N^\alpha\simeq PSL(3,4)$.

(iii) $|\Omega|=22$, *$G^\Omega=Aut(M_{22})$ and $N^\alpha\simeq PSL(3,4)$.*

We introduce some notations.

$V(n,q)$	:	a vector space of dimension n over $GF(q)$
$\Gamma L(n,q)$	:	the group of all semilinear automorphism of $V(n,q)$
$A\Gamma L(n,q)$	:	the semidirect product of $V(n,q)$ by $\Gamma L(n,q)$ in its natural action
$ASL(n,q)$	:	the semidirect product of $V(n,q)$ by $SL(n,q)$ in its natural action
$F(X)$	:	the set of fixed points of a nonempty subset X of G
$X(\Delta)$	:	the global stabilizer of a subset Δ ($\subseteq\Omega$) in X
X_Δ	:	the pointwise stabilizer of Δ in X
X^Δ	:	the restriction of X on Δ
$\mathrm{Sym}(\Delta)$	:	the symmetric group on Δ

X^H	:	the set of H-conjugates of X
$\|X\|_p$	:	the maximal power of a prime p dividing the order of X
$I(X)$	:	the set of involutions contained in X
E_m	:	an elementary abelian group of order m

Other notations are standard and taken from [1].

2. Preliminaries

Lemma 2.1 *Let G be a doubly transitive permutation group on Ω of even degree, $\alpha\in\Omega$ and N^α a normal subgroup of G_α isomorphic to $PSL(2,q)$, $Sz(q)$ or $PSU(3,q)$ with $q(>2)$ even. Then $N^\alpha\simeq PSL(2,q)$, $N^\alpha\not\simeq Sz(q)$, $PSU(3,q)$, N^α is transitive on $\Omega-\{\alpha\}$ and one of the following holds:*

(i) *G has a regular normal subgroup E of order q^2, $N^\alpha_\beta=N^\alpha\cap N^\beta\simeq E_q$ and G_α is isomorphic to a subgroup of $\Gamma L(2,q)$. Moreover there exists an element g in $Sym(\Omega)$ such that $\alpha^g=\alpha$, $(G_\alpha)^g$ normalizes E and $A\Gamma L(2,q)\geq(G_\alpha)^g E\geq ASL(2,q)$ in their natural doubly transitive permutation representation.*

(ii) *$|\Omega|=6$ and $G^\Omega=A_6$ or S_6.*

Proof. By Theorem 2 of [2], it suffices to consider the case that $N^\alpha_\beta=N^\alpha\cap N^\beta\simeq E_q$ and G has a regular normal subgroup of order q^2. Since $|N^\alpha: N^\alpha_\beta|=q^2-1$, N^α is transitive on $\Omega-\{\alpha\}$.

Let E be the regular normal subgroup of G. Then we may assume $\Omega=E$, $\alpha=0\in E$ and the semidirect product $GL(E)E$ is a subgroup of $Sym(\Omega)$. There is a subgroup H of $GL(E)$ such that $H\simeq\Gamma L(2,q)$ and $HE\simeq A\Gamma L(2,q)$. Let L be the normal subgroup of H isomorphic to $SL(2,q)$. Then $L_\beta\simeq E_q$ for $\beta\in\Omega-\{\alpha\}$. Hence $(N^\alpha)^{\Omega-\{\alpha\}}\simeq L^{\Omega-\{\alpha\}}$ and so there are an automorphism f from N^α to L and $g\in Sym(\Omega)$ satisfying $\alpha^g=\alpha$ and $(\beta^x)^g=(\beta^g)^{f(x)}$ for each $\beta\in\Omega-\{\alpha\}$ and $x\in N^\alpha$. From this, $(\beta^g)^{g^{-1}xg}=(\beta^x)^g=(\beta^g)^{f(x)}$, so that $g^{-1}xg=f(x)$. Hence $g^{-1}N^\alpha g=L$.

Set $S=L_\beta$, $X=Sym(\Omega)\cap N(L)$, $D=C_X(L)$ and $Y=N_L(S)$. By the properties of $A\Gamma L(2,q)$, L is transitive on $\Omega-\{\alpha\}$, $|F(S)|=q$ and $Y/S\simeq Z_{q-1}$. Hence D is semi-regular on $\Omega-\{\alpha\}$ and $Y^{F(S)}$ is regular on $F(S)-\{\alpha\}$ and so $D\simeq D^{F(S)}\leq Y^{F(S)}$ because $[D,N^\alpha]=1$. Therefore $D\leq Z_{q-1}$. Since X/DL is isomorphic to a subgroup of the outer automorphism group of $SL(2,q)$, we have $|X|\leq|\Gamma L(2,q)|$, while $\Gamma L(2,q)\simeq H\leq X$. Hence $X=H$ and X normalizes E. Therefore, as $(G_\alpha)^g\trianglerighteq(N^\alpha)^g=L$, we have $(G_\alpha)^g\leq H$. Thus Lemma 2.1 is proved.

Lemma 2.2 *Let G be a doubly transitive permutation group on Ω of even degree and N^α a nonabelian simple normal subgroup of G_α, $\alpha\in\Omega$. If $C_G(N^\alpha)\neq1$, then $N^\alpha_\beta=N^\alpha\cap N^\beta$ for $\alpha\neq\beta\in\Omega$ and $C_G(N^\alpha)$ is semi-regular on $\Omega-\{\alpha\}$. Moreover $C_G(N^\alpha)=0(N^\alpha)$.*

Proof. See Lemma 2.1 of [2].

Lemma 2.3 *Let G be a transitive permutation group on a finite set Ω, H a stabilizer of a point of Ω and M a nonempty subset of G. Then*

$$|F(M)| = |N_G(M)| \times |\{M^g \mid M^g \subseteq H, g \in G\}| / |H| .$$

Proof. See Lemma 2.2 of [2].

Lemma 2.4 *Let H be a transitive permutation group on a finite set Δ and N a normal subgroup of H. Assume that a subgroup X of N satisfies $X^H = X^N$. Then*

(i) $|F(X) \cap \beta^N| = |F(X) \cap \gamma^N|$ *for* $\beta, \gamma \in \Delta$.

(ii) $|F(X)| = |F(X) \cap \beta^N| \times r$, *where r is the number of N-orbits on Δ.*

Proof. By the same argument as in the proof of Lemma 2.4 of [3], we obtain Lemma 2.4.

2.5 *Properties of $PSL(3,q)$, $q=2^n$.*

Let $N_1 = SL(3,q)$, $S_1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \ a, b, c \in GF(q) \right\}$, $A_1 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ a, b \in GF(q) \right\}$, $B_1 = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \ b, c \in GF(q) \right\}$ and $Z = \left\{ \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \middle| \ d \in GF(q),\ d^3 = 1 \right\}$. Then $|Z| = (3, q-1)$ and $\bar{N}_1 = N_1/Z$ is isomorphic to $PSL(3,q)$. Set $N = \bar{N}_1$, $S = \bar{S}_1$, $A = \bar{A}_1$ and $B = \bar{B}_1$. Then the following hold.

(i) N is a nonabelian simple group of order $q^3(q-1)^3(q+1)(q^2+q+1)/(3,q-1)$.

(ii) $|S| = q^3$, $S' = \Phi(S) = Z(S) = \{x^2 \mid x \in S\} = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ b \in GF(q) \right\} \simeq E_q$, $S/S' \simeq E_{q^2}$ and S is a Sylow 2-subgroup of N.

(iii) $S = \langle A, B \rangle$, $A \cap B = Z(S)$, $I(S) \subseteq A \cup B$ and each elementary abelian subgroup of S is contained in A or B. Let $z \in I(S) - Z(S)$. Then $C_S(z) = A$ or B.

(iv) Set $M_1 = A^N$, $M_2 = B^N$. Then $M_1 \neq M_2$ and $M_1 \cup M_2$ is the set of all subgroup of N isomorphic to E_{q^2}.

(v) Let z be an involution of N. Then $I(N) = z^N$ and $|C_N(z)| = (q-1)q^3/(3,q-1)$.

(vi) Let E denote A or B. Then $|N_N(E)| = (q-1)^2(q+1)q^3/(3,q-1)$, $N_N(E)/E \simeq Z_k \times PSL(2,q)$, where $k = (q-1)/(3,q-1)$ and $N_N(E)$ is a maximal subgroup of N.

(vii) Set $M = (N_N(E))'$. If $q > 2$, then $M = M'$, $M \trianglerighteq E$, $M/E \simeq PSL(2,q)$ and M acts irreducibly on E.

(viii) Set $\Delta = E^N$. Then $|\Delta| = q^2+q+1$ and by conjugation N is doubly transitive on Δ, which is an usual doubly transitive permuation representation of N. If $C\in\{A,B\}-\{E\}$, $|F(C)|=q+1$, C is a Sylow 2-subgroup of $N_{F(C)}$ and C is semi-regular on $\Delta-F(C)$.

Lemma 2.6 ([6]). *Let notations be as in* (2.5) *and set* $G=Aut(N)$. *Then the following hold.*

(i) *There exist in* G *a diagonal automorphism* d, *a field automorphism* f *and a graph automorphism* g *and satisfy the following*:

$$G=\langle g,f,d\rangle N\trianglerighteq H_1=\langle f,d\rangle N\trianglerighteq H_2=\langle d\rangle N,\ H_1=P\Gamma L(3,q),\ H_2=PGL(3,q)$$
$$H_2/N\simeq Z_r,\ \text{where } r=(3,q-1),\ G/H_1\simeq Z_2,\ H_1/H_2\simeq Z_n \text{ and } G/H_2\simeq Z_2\times Z_n\,.$$

(ii) $M_1=A^{H_1}$, $M_2=B^{H_1}$ *and* $A^g=B$.

Lemma 2.7 *Let* $N=PSL(3,q)$, *where* $q=2^n$. *Let* R *be a cyclic subgroup of* N *of order* $q+1$ *and* Q *a nontrivial subgroup of* R. *Then* $N_N(Q)=N_N(R)\simeq Z_k\times D_{2(q+1)}$, *where* $k=(q-1)/(3,q-1)$ *and* $D_{2(q+1)}$ *is a dihedral group of order* $2(q+1)$.

Proof. We consider the group N as a doubly transitive permutation group on $\Delta=PG(2,q)$ with q^2+q+1 points. By (2.5) (i), R is a cyclic Hall subgroup of N and so we may assume $R\leq N_\alpha$, where $\alpha=\begin{pmatrix}1\\0\\0\end{pmatrix}\in PG(2,q)$. Since $|N_{\alpha\beta}|=(q-1)^2q^2/(3,q+1)$ for $\alpha\neq\beta\in\Delta$ and $(q+1,(q-1)^2q^2)=1$, R is semiregular on $\Delta-\{\alpha\}$. Hence $N_N(Q)\leq N_\alpha$. Put $E=0_2(N_\alpha)$. Then $N_\alpha=N_N(E)$ by (2.5) (viii) and $N_N(Q)E/E\simeq Z_k\times D_{2(q+1)}$ by (2.5) (vi). Since $N_N(Q)\cap E=C_E(Q)=1$ by (2.5) (v). Hence $N_N(Q)\simeq Z_k\times D_{2(q+1)}$. As R is cyclic, $N_N(R)\leq N_N(Q)$. Thus $N_N(Q)=N_N(R)\simeq Z_k\times D_{2(q+1)}$.

Lemma 2.8 *Let* $N=PSL(3,q)$, $q=2^n$ *and let* $H(\neq N)$ *be a subgroup of* N *of odd index. Then* $H\leq N_N(E)$ *for an elementary abelian subgroup* E *of* N *of order* q^2.

Proof. Let S, A and B be as in (2.5) and let Δ be as in Lemma 2.7. Since $|N\colon H|$ is odd, H contains a Sylow 2-subgroup of N and so we may assume $S\leq H$. Set $\alpha=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\beta=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\gamma=\begin{pmatrix}0\\0\\1\end{pmatrix}$. Then $S\leq N_\alpha=N_N(A)$, $S_\beta=B$, $S_\gamma=\left\{\begin{pmatrix}1&a&0\\0&1&0\\0&0&1\end{pmatrix}\middle|\, a\in GF(q)\right\}\simeq E_q$ and hence $|\alpha^S|=1$, $|\beta^S|=q$ and $|\gamma^S|=q^2$.

If $\alpha^H=\{\alpha\}$, $H\leq N_\alpha=N_N(A)$ and the lemma holds. By (2.5) (i), $(q^2+1, |N|)=1$. Hence $\alpha^H\neq\{\alpha\}\cup\gamma^S$, so that we may assume either $\alpha^H=\{\alpha\}\cup\beta^H$ or $\alpha^H=\Delta$.

If $\alpha^H=\{\alpha\}\cup\beta^H$, $\alpha^H=F(B)$ and B is a unique Sylow 2-subgroup of $H_{F(B)}$ by (2.5) (viii). Hence $H\trianglerighteq B\simeq E_{q^2}$ and the lemma holds.

If $\alpha^H=\Delta$, by (2.5) (iv), $N_H(A)^{F(A)}$ is transitive and so $|H|$ is divisible by $q+1$. Since $(q^2+q+1,q+1)=1$, $|H_\alpha|$ is divisible by $q+1$. By (2.5) (vi) and by the structure of $PSL(2,q)$, $Z_m\times PSL(2,q)\simeq H_\alpha/A\leq N_N(A)/A$, where m is a divisor of $(n-1)/(3,n-1)$. Therefore $|N: H|\leq q-1$. We now consider the action of N on the coset $\Gamma=N/H$. As $|\Gamma|\neq 1$ and N is a simple group, N^Γ is faithful. But N has a cyclic subgroup of order $q+1$ and so $|\Gamma|>q+1$, which implies $|N: H|>q+1$, a contradiction.

Lemma 2.9 *Let $N=PSL(3,q)$, where $q=2^{2m}$ and t a field automorphism of N of order 2. Let S be a t-invariant Sylow 2-subgroup of N. Then the following hold.*

(i) $Z(\langle t\rangle S)\simeq E_{\sqrt{q}}$.

(ii) *If S_1 is a subgroup of $\langle t\rangle S$ isomorphic to S, then $S_1=S$.*

Proof. Since $C_S(t)$ is isomorphic to a Sylow 2-subgroup of $PSL(3,\sqrt{q})$, $Z(C_S(t))\simeq E_{\sqrt{q}}$ and $Z(C_S(t))\leq Z(S)$ by (2.5) (ii). Hence $Z(\langle t\rangle S)=Z(\langle t\rangle S)\cap \langle t\rangle C_S(t)\cap C(Z(S))=Z(\langle t\rangle S)\cap C_S(t)=Z(C_S(t))\simeq E_{\sqrt{q}}$. Thus we have (i).

Suppose $S_1\neq S$. Then $\langle t\rangle S=S_1S\trianglerighteq S_1$ and $[\langle t\rangle S: S]=[S_1: S_1\cap S]=2$. If $Z(S_1)\not\leq S$, we have $S_1=\langle z\rangle\times(S_1\cap S)$ for an involution z in $Z(S_1)-S$. By (2.5) (ii), $z\in\Phi(S_1)$ and so $S_1=\langle z, S_1\cap S\rangle=S_1\cap S$, a contradiction. Hence $Z(S_1)\leq S$.

If $Z(S_1)=Z(S)$, $E_q\simeq Z(S)\leq Z(S_1S)=Z(\langle t\rangle S)\simeq E_{\sqrt{q}}$ by (i), which is a contradiction. Hence $Z(S_1)\neq Z(S)$.

Let z be an involution in $Z(S_1)-Z(S)$. Then $C_S(z)\simeq E_{q^2}$ by (2.5) (iii). On the other hand, $S_1\leq C_{\langle t\rangle S}(z)$ and $[C_{\langle t\rangle S}(z): C_S(z)]=1$ or 2. From this S_1 has an elementary abelian subgroup of index 2. Hence $q=2$, a contradiction. Thus we have (ii).

3. Proof of the theorem

Throughout the rest of the paper, G^Ω always denote a doubly transitive permutation group satisfying the hypotheses of the theorem.

Since $G_\alpha\trianglerighteq N^\alpha$, $|\beta^{N^\alpha}|=|\gamma^{N^\alpha}|$ for $\beta,\gamma\in\Omega-\{\alpha\}$ and so $|\Omega|=1+r|\beta^{N^\alpha}|$, where r is the number of N^α-orbits on $\Omega-\{\alpha\}$. Hence r is odd and N^α_β is a proper subgroup of N^α of odd index for $\alpha\neq\beta\in\Omega$. Therefore, by Lemma 2.8 $N^\alpha_\beta\trianglerighteq A$ for some elementary abelian subgroup A of order q^2. Let S be a Sylow 2-subgroup of N^α_β. Then, by (2.5) (iii) there exists a unique elementary abelian 2-subgroup B of S such that $A\simeq B\simeq E_{q^2}$ and $A\neq B$. Set $M_1=A^{N^\alpha}$, $M_2=B^{N^\alpha}$ and $K=G_\alpha(M_1)=G_\alpha(M_2)$. By (2.5) (iv), $M_1\cup M_2$ is the set of all elementary abelian 2-subgroup of N^α of order q^2 and G_α acts on $\{M_1, M_2\}$, so that $G_\alpha/K\leq Z_2$. Hence K is transitive on $\Omega-\{\alpha\}$.

(3.1) *Let $E=A$ or B. Then $N_{G_\alpha}(E)$ is transitive on $F(E)-\{\alpha\}$.*

Proof. If $E^h \leq K_\beta$ for some $h \in K$, $E^h \leq N^\alpha \cap K_\beta = N^\alpha_\beta$. Since $E^{N^\alpha} = E^K$ and $A^K \neq B^K$, E^h is conjugate to E in N^α_β. By a Witt's theorem $N_K(E)$ is transitive on $F(E)-\{\alpha\}$. Thus $N_{G_\alpha}(E)$ is transitive on $F(E)-\{\alpha\}$.

(3.2) *If* $q=2$, G^Ω *is of type* (i) *of the theorem.*

Proof. Assume $q=2$. We note that $PSL(3,2)$ is isomorphic to $PSL(2,7)$. It follows from [3] that G has a regular normal subgroup R.

Since K is transitive on $\Omega-\{\alpha\}$, by Lemmas 2.3 and 2.4

$$|F(A)| = 1+\frac{|N^\alpha \cap N(A)|}{|N^\alpha_\beta|}r = \frac{24r}{|N^\alpha_\beta|}+1 \quad \text{and}$$

$$|F(B)| = 1+\frac{|N^\alpha \cap N(B)|\,|N^\alpha_\beta : N^\alpha_\beta \cap N(B)|}{|N^\alpha_\beta|}r = \frac{24r}{|N^\alpha_\beta \cap N(B)|}+1\,.$$

Let $E=A$ or B. As $N_R(E) \neq 1$, $N_G(E)^{F(E)}$ is doubly transitive by (3.1). Hence $E \leq N^\beta$ and $|F(A)|=2^a$, $|F(B)|=2^b$ for some integers a, b. From this $S=\langle A,B\rangle \leq N^\alpha \cap N^\beta$ and $|N^\alpha_\beta : N^\alpha \cap N^\beta|$ is odd. Hence, if $S^g \leq G_{\alpha\beta}$, $S^g \leq N^\gamma_\beta \cap N^\gamma_\beta$, where $\gamma=\alpha^g$ and so $S^g \leq N^\alpha \cap N^\beta$. Since S and S^g are Sylow 2-subgroups of $N^\alpha \cap N^\beta$, S^g is conjugate to S in $N^\alpha \cap N^\beta$. By a Witt's theorem $N_G(S)^{F(S)}$ is a doubly transitive permutation group with a regular normal subgroup $N_R(S)$. Hence $|F(S)|=2^c$ for an integer c. By Lemmas 2.3 and 2.4,

$$|F(S)| = 1+\frac{8\times|N^\alpha_\beta : S|}{|N^\alpha_\beta|}r = r+1 = 2^c\,.$$

Let z be an involution of $Z(S)$ and assume $z^g \in G_\alpha$ for some $g \in G$. Then $z^g \in N^\gamma_\alpha$, where $\gamma=\alpha^g$. Since $|N^\gamma_\alpha : N^\gamma \cap N^\alpha|$ is odd, z^g is contained in N^α. By (2.5) (v), z^g is conjugate to z in N^α. Hence $C_G(z)$ is transitive on $F(z)$ and by Lemmas 2.3 and 2.4,

$$|F(z)| = 1+\frac{8\times|I(N^\alpha_\beta)|}{|N^\alpha_\beta|}r\,.$$

Suppose $N^\alpha_\beta=S$. Then $|F(A)|=3r+1=2^a=2^c+2r$ and $|F(z)|=5r+1$. Hence $r=1$. Since $N_R(A)=C_R(A) \leq C_G(z)$ and $N_R(A) \simeq E_4$, $|F(z)|$ is divisible by 4. But $|F(z)|=5r+1=6$. This is a contradiction.

Suppose $N^\alpha_\beta \neq S$. Then $N^\alpha_\beta=N_{N^\alpha}(A)$ as $N_{N^\alpha}(A) \simeq S_4$. From this, $|F(B)| =2^b=2^c+2r$ and so $r=1$. Hence $|\Omega|=1+|N^\alpha : N^\alpha_\beta|=8$. Thus $|R|=8$ and $G_\alpha \simeq GL(3,2)$, hence $G \simeq AL(2,3)$.

By (3.2), it suffices to consider the case $q>2$ to prove the theorem. From now on we assume the following.

Hypothesis (*): $q=2^n \geq 4$

(3.3) *The following hold.*

(i) $|N^{\alpha}_{\beta}/N^{\alpha}\cap N^{\beta}|$ *is odd.*

(ii) *Let* $\gamma\in\Omega$ *and* S_0 *a 2-subgroup of* N^{γ}. *Then* $F(S_0)=\{\delta\in\Omega\,|\,S_0\leq N^{\delta}\}$.

Proof. Suppose false and let T be a Sylow 2-subgroup of $N^{\beta}_{\alpha}N^{\alpha}_{\beta}$ such that $T\geq S$. Then $T\neq S$. Set $S_1=T\cap N^{\alpha}_{\beta}$ and $S_2=T\cap N^{\alpha}\cap N^{\beta}$. Then S_1 is a Syow 2-subgroup of N^{β}_{α}, $S_1\neq S$ and S_1, S_2 and S are normal subgroups of T. By Lemma 2.2, S_1N^{α}/N^{α} is isomorphic to a subgroup of the outer automorphism group of N^{α}. It follows from Lemma 2.6 that S_1N^{α}/N^{α} is abelian of 2-rank at most 2. Since $S_1N^{\alpha}/N^{\alpha}\simeq S_1/S_2$ and $S_1\simeq S$, we have $S_1/S_2\leq E_4$ by (2.5) (ii).

Let A_1, B_1 be the subgroups of S_1 such that $A_1\simeq B_1\simeq E_{q^2}$ and $A_1\cap S_2\leq A$, $B_1\cap S_2\leq B$. Since $A_1/A_1\cap S_2\simeq A_1S_2/S_2\leq S_1/S_1\leq E_4$ and by the hypothesis $(*)$, $q\geq 4$, we have $|A_1\cap S_2|\geq q^2/4$. Therefore, if $A_1\cap S_2\leq Z(S)$, then $q=4$, $A_1\cap S_2=Z(S)$ and $T=A_1S$ and so $Z(S)\leq Z(T)$, contrary to Lemma 2.9. Hence $A_1\cap S_2\not\leq Z(S)$. Similarly $B_1\cap S_2\not\leq Z(S)$.

Let $x\in A_1\cap S_2-Z(S)$. Then $x\in A^{y}\leq S$ for each $y\in A_1$ and so A_1 normalizes A. Hence A_1 normalizes B. Similarly B_1 normalizes A and B. From this $T=\langle A_1, B_1\rangle S\trianglerighteq A, B$ and so $S_1N^{\alpha}\leq K$. Hence $S_1N^{\alpha}/N^{\alpha}\simeq S_1/S_2\simeq Z_2$, so that there exists a field automorphism t of order 2 such that $T=\langle t\rangle S\triangleright S$. Since $S_1\leq T$ and $S_1\simeq S$, we have $S_1=S$ by Lemma 2.9, a contradiction. Thus (i) holds.

Let $\delta\in F(S_0)-\{\gamma\}$. Then $S_0\leq N^{\gamma}_{\delta}$. Since $N^{\gamma}_{\delta}\trianglerighteq N^{\gamma}\cap N^{\delta}$ and $|N^{\gamma}_{\delta}/N^{\gamma}\cap N^{\delta}|$ is odd by (i), $S_0\leq N^{\gamma}\cap N^{\delta}\leq N^{\delta}$. Hence $F(S_0)\subseteq\{\delta\in\Omega\,|\,S_0\leq N^{\delta}\}$. The converse implication is clear. Thus (ii) holds.

(3.4) *The following hold.*

(i) $N_G(B)^{F(B)}$ *is doubly transitive.*

(ii) *If* $F(A)\neq\{\alpha,\beta\}$, $N_G(A)^{F(A)}$ *is doubly transitive.*

Proof. Let $E=A$ or B. By (3.3) (i), S is a Sylow 2-subgroup of N^{α}_{β}. Therefore, by a similar argument as in (3.1), $N_{G_\beta}(E)$ is transitive on $F(E)-\{\beta\}$. Suppose $N_G(E)^{F(E)}$ is not doubly transitive. Then, $F(E)=\{\alpha,\beta\}$ by (3.1) and (3.3). Since $N_{N^{\alpha}}(E)$ acts on $F(E)$ and fixes $\{\alpha\}$, we have $N_{N^{\alpha}}(E)\leq N^{\alpha}_{\beta}$. On the other hand $N_{N^{\alpha}}(E)$ is a maximal subgroup of N^{α} by (2.5) (vi). Hence $N_{N^{\alpha}}(E)=N^{\alpha}_{\beta}$. If $E=B$, then $N^{\alpha}_{\beta}\not\trianglerighteq A$, a contradiction. Thus $E=A$ and (3.4) follows.

(3.5) *The following hold.*

(i) *Put* $M=(N_{N^{\alpha}}(A))'$. *Then* $F(M)=F(A)$.

(ii) $N^{\alpha}_{\beta}=N^{\alpha}_{\gamma}$ *for each* $\gamma\in F(A)-\{\alpha\}$.

Proof. Suppose $F(M)\neq F(A)$. Then $M\not\leq N_G(A)_{F(A)}$. It follows from (3.4) that $F(A)\neq\{\alpha,\beta\}$ and $N_G(A)^{F(A)}$ is doubly transitive. Moreover by (2.5) (vii) $N_{G_\alpha}(A)^{F(A)}\trianglerighteq M^{F(A)}\simeq PSL(2,q)$ as $q>2$. By Lemma 2.1, $r=1$ and either (1) $q=$

4 and $N_G(A)^{F(A)}=A_6$ or S_6 or (2) $|F(A)|=q^2$.

If (1) holds, $|F(A)|=1+|N_{N^\alpha}(A): N^\alpha_\beta|=1+2^6\cdot 3\cdot 5/|N^\alpha_\beta|=6$ and so $|N^\alpha_\beta|=2^6 3$. Hence $|\Omega|=1+|N^\alpha: N^\alpha_\beta|=1+2^6\cdot 3^2\cdot 5\cdot 7/2^6\cdot 3=2\cdot 53$. Let z be an involution of $N^\alpha\cap N^\beta$. Then, by (2.5) (v) and (3.3), $z^G\cap G_\alpha=z^{G_\alpha}$, so that $C_G(z)^{F(z)}$ is transitive by a Witt's theorem. On the other hand $|F(z)|=1+\dfrac{|C_{N^\alpha}(z)|\times|I(N^\alpha_\beta)|}{|N^\alpha_\beta|}=1+2^6\cdot 3^3/2^6\cdot 3=10$. In particular $|C_G(z)|$ is divisible by 5. Let R be a Sylow 5-subgroup of $C_G(z)$. Then $|\Omega|$, $|G_\alpha: N^\alpha|$ and $|N^\alpha_\beta|$ are not divisible by 5 and so $F(R)=\{\gamma\}$ and $R\le N^\gamma$ for some $\gamma\in\Omega$. Therefore $\langle z\rangle\times R\le N^\gamma$ by (3.3) (ii). But $|C_{N^\gamma}(z)|=2^6$ by (2.5) (v). This is a contradiction.

If (2) holds, $q^2=|F(A)|=1+|N_{N^\alpha}(A): N^\alpha_\beta|$, hence $|N^\alpha_\beta|=(q-1)q^3/(3,q-1)$. From this $|\Omega|=1+|N^\alpha: N^\alpha_\beta|=1+(q-1)(q+1)(q^2+q+1)=q(q^3+q^2-1)$. Hence $|G|_2=|\Omega|_2\times|G_\alpha|_2=q\times|G_\alpha: K|\times|K|_2$. On the other hand $|N_G(A)|_2=|F(A)|\times|N_{G_\alpha}(A)|_2=q^2|K|_2$ because $K=N_{G_\alpha}(A)N^\alpha$. Therefore $q^2|K|_2=|N_G(A)|_2\le|G|_2=q\times|G_\alpha: K|\times|K|_2\le 2q|K|_2$ and we obtain $q=2$, contrary to the hypothesis (*). Thus we have (i).

Let $\gamma\in F(A)-\{\alpha\}$. By (i) and (3.4) (ii), $N^\alpha_\gamma\trianglerighteq A$ and $M\le N^\alpha_\gamma$. Since $N_{N^\alpha}(A)/M\simeq Z_k$, where $k=(q-1)/(3,q-1)$ and $|N^\alpha_\beta/M|=|N^\alpha_\gamma/M|$, we have $N^\alpha_\beta=N^\alpha_\gamma$. Thus (ii) holds.

(3.6) $B\notin A^G$ *and* $G_\alpha=K$.

Proof If $B\in A^G$, by (3.4) (i), there is an element $g\in G_{\alpha\beta}$ such that $B=A^g$. Hence $N^\alpha_\beta=g^{-1}N^\alpha_\beta g\trianglerighteq g^{-1}Ag=B$ and so M normalizes $\langle A,B\rangle=S$, a contradiction.

(3.7) *Set* $L=(N_{N^\alpha}(B))'$. *Then* $r=1$, $L_{F(B)}=B$, $L^{F(B)}=L/B\simeq PSL(2,q)$, $L_\beta=S$ *and one of the following holds.*

(i) $C_G(N^\alpha)=1$, $|F(B)|=6$, $q=4$ *and* $N_G(B)^{F(B)}=A_6$ *or* S_6.

(ii) $C_G(N^\alpha)\le Z_{q-1}$, $|F(B)|=q^2$ *and* $N_G(B)^{F(B)}$ *has a regular normal subgroup.*

Proof. By (3.4) (i), $N_G(B)^{F(B)}$ is doubly transitive. If $L\le G_{\alpha\beta}$, then $L\le N^\alpha_\beta$ and so $B\trianglelefteq L=L'\le(N^\alpha_\beta)'=M$. Therefore $L=M$ and $M\trianglerighteq\langle A, B\rangle=S$, a contradiction. Hence $L\not\le G_{\alpha\beta}$. From this $N_{G_\alpha}(B)^{F(B)}\trianglerighteq L^{F(B)}\simeq PSL(2,q)$ and (3.7) follows from Lemmas 2.1 and 2.2.

(3.8) *If* (i) *of* (3.7) *occurs, then we have* (ii) *or* (iii) *of the theorem.*

Proof. Since $|F(B)|=1+|N_{N^\alpha}(B): N_{N^\alpha_\beta}(B)|=6$ and $|N^\alpha_\beta: N_{N^\alpha_\beta}(B)|=|N^\alpha_\beta: N_{N^\alpha_\beta}(S)|=5$, we have $|N^\alpha_\beta|=2^6\cdot 3\cdot 5$. Hence $N^\alpha_\beta=N_{N^\alpha}(A)$ and so $|\Omega-\{\alpha\}|=|N^\alpha: N^\alpha_\beta|=21$. By (3.6), $PSL(3,4)\le(G_\alpha)^{\Omega-\{\alpha\}}\le P\Gamma L(3,4)$ in their natural doubly transitive permutation representation and hence (3.8) follows from Satz 7 of [7].

In the rest of this paper, we consider the case (ii) of (3.7). From now on we assume the following.

Hypothesis (**): $r=1$, $q=2^n>2$, $|F(B)|=q^2$ and $N_G(B)^{F(B)}$ is a doubly transitive permutation group with a regular normal subgroup.

(3.9) *The following hold.*

(i) $N^\alpha_\beta=N^\alpha\cap N^\beta=M$ *and* $|N^\alpha_\beta|=(q-1)(q+1)q^3$.

(ii) n *is odd.*

(iii) $|F(A)|=q$.

Proof. Since $q^2=|F(B)|=1+|N_{N^\alpha}(B): N_{N^\alpha_\beta}(B)|$ by (3.7), we have $|N_{N^\alpha_\beta}(B)|=|N_{N^\alpha}(B)|/(q^2-1)=(q-1)q^3/(3,q-1)$. As $N^\alpha_\beta\trianglerighteq A$, $N_{N^\alpha_\beta}(B)=N_{N^\alpha_\beta}(\langle A,B\rangle)=N_{N^\alpha_\beta}(S)$. On the other hand, from (2.5) (vi) $|N_{N^\alpha_\beta}(S)|=|N^\alpha_\beta: M|\times|N_M(S)|=|N^\alpha_\beta: M|\times(q-1)q^3$. Therefore $(3,q-1)=1$ and $|N^\alpha_\beta: M|=1$. Hence $N^\alpha_\beta=M$ and n is odd. By (3.3) (i) and (2.5) (vii), $N^\alpha_\beta=N^\alpha\cap N^\beta$. Hence $F(A)=1+|N_{N^\alpha}(A)|/|N^\alpha|=q$. Thus we have (3.9).

(3.10) *Put* $m=|G_\alpha: N^\alpha|$. *Then the following hold.*

(i) m *is odd and* S *is a Sylow* 2*-subgroup of* G_α.

(ii) $|\Omega|=q^3$ *and* $|G|=q^6(q-1)^2(q+1)(q^2+q+1)m$.

Proof. Set $C^\alpha=C_G(N^\alpha)$. By (3.6), (3.9) (ii) and Lemma 2.6, $|G_\alpha/C^\alpha N^\alpha|$ is odd. Since $C^\alpha\cap N^\alpha=1$, $m=|G_\alpha/C^\alpha N^\alpha|\cdot|C^\alpha|$ and so m is odd by Lemma 2.2. Therefore S is a Sylow 2-subgroup of G_α and so (i) holds.

Since $|\Omega|=1+|N^\alpha: N^\alpha_\beta|$, $|\Omega|=q^3$ by (3.9). From this $|G|=|\Omega|\times|G_\alpha|=q^3m|N^\alpha|=q^6(q-1)^2(q+1)(q^2+q+1)m$. Thus (ii) holds.

(3.11) *Let z be an involution of* G_α. *Then* $|F(z)|=q^2$. *In particular* B *is semi-regular on* $\Omega-F(B)$.

Proof. By (3.10) (ii), z is contained in N^α. By (2.5) (vii) and (3.9) (ii), $|I(N^\alpha_\beta)|=|N^\alpha_\beta: N_{N^\alpha_\beta}(S)|\times(q^2-q)+q^2-1=(q+1)(q^2-q)+q^2-1=(q-1)(q+1)^2$, hence $|F(z)|=1+q^3(q-1)\times(q-1)(q+1)^2/q^3(q-1)(q+1)=q^2$ by Lemma 2.3. As $|F(B)|=q^2$, B is semi-regular on $\Omega-F(B)$.

(3.12) *Set* $\Delta=F(B)$. *Then the following hold.*

(i) $G_\Delta\trianglerighteq B$ *and* B *is a Sylow* 2*-subgroup of* G_Δ.

(ii) $G(\Delta)=N_G(B)$ *and* $|N_G(B)|=q^5(q-1)^2(q+1)m$.

Proof. Since $N_{N^\alpha}(B)\leq N^\alpha(\Delta)\neq N^\alpha$ and $N_{N^\alpha}(B)$ is a maximal subgroup of N^α, we have $N_{N^\alpha}(B)=N^\alpha(\Delta)$. By (3.7), B is a normal Sylow 2-subgroup of $(N^\alpha)_\Delta$ and (i) follows immediately from (3.10) (i).

Since $G(\Delta)\trianglerighteq G_\Delta$ and B is a characteristic subgroup of G_Δ by (i), we have $G(\Delta)\leq N_G(B)$. The converse implication is clear. Thus $G(\Delta)=N_G(B)$. By (3.6), $G_\alpha=N_{G_\alpha}(B)N^\alpha$ and so $|N_{G_\alpha}(B): N_{N^\alpha}(B)|=|G_\alpha:N^\alpha|=m$. Hence $|N_G(B)|$

$=|F(B)|\times|N_{G_\alpha}(B)|=q^2m\times|N_{N^\alpha}(B)|=q^5(q-1)^2(q+1)m$. Thus we have (ii).

(3.13) *Let T_1 be a Sylow 2-subgroup of $N_G(B)$ and T_2 a Sylow 2-subgroup of $N_G(T_1)$. Then $T_1 \neq T_2$. Let x be an element of T_2-T_1 and set $U=BB^x$. Then $U\simeq E_{q^4}$ and for each $\gamma\in\Omega$, $U_\gamma\simeq E_{q^2}$, $U_\gamma\in B^G$, $\gamma^U=F(U_\gamma)$ and $|\gamma^U|=q^2$. Moreover $U_\gamma=U_\delta$ for all $\delta\in\gamma^U$.*

Proof. If $B\cap B^x\neq 1$, by (3.11) and (3.12) (i), we have $B=B^x$ and so $x\in T_1$, contrary to the choice of x. Hence $B\cap B^x=1$. As $T_1 \trianglerighteq B$ and $T_1=T_1^x \trianglerighteq B^x$, $U=B\times B^x$ and $U\simeq E_{q^4}$.

Let $\gamma\in\Omega$ and put $D=U_\gamma$. Then $F(D)\supseteq\gamma^U$ as U is abelian. Therefore $|U: D|=|\gamma^U|\leq q^2$ by (3.11), while $|D|\leq q^2$ because D is an elementary abelian subgroup of N^γ. Hence $D\simeq E_{q^2}$ and $|F(D)|=|\gamma^U|=q^2$. By (3.6) and (3.9) (iii), $D\in B^G$. Since $U_\gamma\leq U_\delta\simeq E_{q^2}$ for each $\delta\in\gamma^U$, we have $U_\gamma=U_\delta$.

(3.14) *Let U be as in (3.13). Let $\Gamma=\{X_i \mid 1\leq i\leq s\}$ be the set of U-orbits on Ω and set $B_i=U_\gamma$ for $\gamma\in X_i$ with $1\leq i\leq s$. Then the following hold.*

(i) $s=q, \Omega=\bigcup_{i=1}^{q} X_i$ and $|X_i|=q^2$.

(ii) *B_i is semi-regular on $\Omega-X_i$ and $B_i\cap B_j=1$ for each i, j with $i\neq j$.*

Proof. By (3.10) (ii) and (3.13), $|X_i|=q^2$ and $|\Omega|=q^3$, hence $s=q$. Cleary $\Omega=\bigcup_{i=1}^{q} X_i$. Thus we have (i).

By (3.13) (ii), B_i is conjugate to B for each i. Hence B_i is semi-regular on $\Omega-X_i$ by (3.11). Therefore, if $B_i\cap B_j\neq 1$, then $X_i=F(B_i)=F(B_j)=X_j$, so that $i=j$. Thus we have (ii).

(3.15) *Set $Y=\{B_i \mid 1\leq i\leq q\}$ and let $D\in Y$. Then $N_G(D)\leq N_G(U)$ and U is a unique Sylow 2-subgroup of $C_G(D)$.*

Proof. Suppose $N_G(D)\nleq N_G(U)$. Since $[N_G(D), U]\nleq U$, there exist $g\in N_G(D)$ and $B_i\in Y-\{D\}$ such that $(B_i)^g\nleq U$. Set $D_1=(B_i)^g$. Since $[D_1, D]=[B_i, D]^g=1$, it follows from (3.10) (i) that $F(D_1)\cap F(D)=\phi$ and so D is regular on $F(D_1)$ by (3.11). Hence $F(D_1)=\gamma^D=\gamma^U$ for $\gamma\in F(D_1)$. By (3.14), $F(D_1)=F(B_j)$ for some $B_j\in Y$. By (3.12) (i), $D_1=B_j$, so that $D_1\leq U$, a contradiction. Thus we have $N_G(D)\leq N_G(U)$. Hence $U\leq 0_2(C_G(D))$. Since $U\leq C_G(B)$, $C_G(B)$ is transitive on $F(B)$. Hence $|C_G(B)|_2=|F(B)|\times|C_{G_\alpha}(B)|_2=q^4$ by (3.10) (i). Therefore $|C_G(D)|_2=q^4$ as $D\in B^G$ and so U is a unique Sylow 2-subgroup of $C_G(D)$.

(3.16) $|N_G(U)|=q^6(q-1)^2(q+1)m$.

Proof. Let S_1 be a Sylow 2-subgroup of $N_G(U)$ and S_2 be a Sylow 2-subgroup of $N_G(S_1)$. Suppose $S_1\neq S_2$ and let w be an element of S_2-S_1.

Set $\gamma=\alpha^{w^{-1}}$. Then $(U_\gamma)^w\in B^G$ by (3.13) and $(U_\gamma)^w\leq(G_\gamma)^w=G_\alpha$. Since U and U^w are normal subgroups of S_1, $\langle B, (U_\gamma)^w\rangle$ is 2-subgroup of $G_\alpha\cap S_1=S$. Hence $B=(U_\gamma)^w$ by (2.5) (iii) and (3.6). Therefore U, $U^w\leq C_G(B)$, so that $U=U^w$ by (3.15) and $w\in S_2\cap N_G(U)=S_1$, contrary to the choice of w. Hence $S_1=S_2$ and S_1 is a Sylow 2-subgroup of G. It follows from (3.10) that $|S_1|=q^6$.

We now consider the action of $N_G(U)$ on $\Gamma=\{X_i\,|\,1\leq i\leq q\}$. Set $\Delta=F(B)$. By (3.12), $S_1(\Delta)\leq G(\Delta)=N_G(B)$ and $|N_G(B)|_2=q^5$ and so $|S_1\colon S_1(\Delta)|$ is divisible by q. Hence S_1 is transitive on Γ and so $N_G(U)$ is transitive on Γ. Therefore $|N_G(U)|=q\times|N_G(U)\cap N_G(B)|=q\times|N_G(B)|=q^6(q-1)^2(q+1)m$ by (3.12) (ii) and (3.15).

(3.17) *Let R be a cyclic subgroup of N^α_β of order $q+1$. Then $|F(R)|=q$ and R is semi-regular on $\Omega-F(R)$.*

Proof. Since $N^\alpha_\beta/A\simeq PSL(2,q)$, there exists a cyclic subgroup R of N^α_β of order $q+1$. Let $Q\neq 1$ be a subgroup of R. Then, by Lemma 2.7 $|F(Q)|$
$$=1+\frac{|N_{N^\alpha}(Q)|\times|N^\alpha_\beta\colon N_{N^\alpha_\beta}(Q)|}{|N^\alpha_\beta|}=1+\frac{2(q-1)(q+1)}{2(q+1)}=q.$$ Thus (3.17) holds.

(3.18) *Let $V\in U^G$. If $V\neq U$, then $|F(U_\gamma)\cap F(V_\gamma)|=1$ or q for $\gamma\in\Omega$.*

Proof. Suppose $\gamma\neq\delta\in F(U_\gamma)\cap F(V_\gamma)$. By (3.13), U_γ, $V_\gamma\in B^G$ and so by (3.3) (ii), U_γ, $V_\gamma\leq N^\gamma\cap N^\delta$. Set $H=0_2(N^\gamma_\delta)$. Then, by (3.6) and (3.9) (i), $U_\gamma H$ and $V_\gamma H$ are Sylow 2-subgroups of N^γ_δ. If $U_\gamma H=V_\gamma H$, then $U_\gamma=V_\gamma$ and U, $V\leq C_G(U_\gamma)$. By (3.15) we have $U=V$, a contradiction. Therefore $U_\gamma H\neq V_\gamma H$. Set $X=\langle U_\gamma, V_\gamma\rangle$. Then $XH=N^\gamma_\delta$ because $N^\gamma_\delta/H\simeq PSL(2,q)$, $q=2^n$, and $PSL(2,q)$ is generated by its two distinct Sylow 2-subgroups. Hence $N^\gamma_\delta\trianglerighteq X\cap H$. By (2.5) (iii), $E_q\simeq U_\gamma\cap H\leq X\cap H$. Since N^γ_δ acts irreducibly on H by (2.5) (vii), $X\cap H=H$ and hence $H\leq X$. From this $X=N^\gamma_\delta$. Thus, by (3.5)(i) and (3.9), $|F(U_\gamma)\cap F(V_\gamma)|=|F(X)|=|F(N^\gamma_\delta)|=q$.

(3.19) *Let Q be a cyclic subgroup of $N_{N^\alpha}(B)$ of order $q+1$, $V\in U^G$ and set $P=N_Q(V)$. Then the following hold.*

(i) *Q is semi-regular on $\Omega-F(Q)$ and $|F(Q)|=q$.*

(ii) *If $P\neq 1$ and $V\geq D\in B^G$, then P normalizes D and $|F(P)\cap F(D)|=1$.*

Proof. Since $N_{N^\alpha}(B)/B\simeq PSL(2,q)$, there exists a cyclic subgroup Q of $N_{N^\alpha}(B)$ of order $q+1$. Clearly Q is a cyclic Hall subgroup of N^α, hence Q is conjugate to R defined in (3.17). By (3.17), Q is semi-regular on $\Omega-F(Q)$ and $|F(Q)|=q$. Thus (i) holds.

Suppose $P\neq 1$ and let $\gamma\in F(P)$. Then, by (3.9) (i), $P\leq N^\gamma$ and hence P normalizes $N^\gamma\cap V$. By (3.10) (i) and (3.13), $N^\gamma\cap V=V_\gamma$ and $V_\gamma\in B^G$ and so $P\leq N_{N^\gamma}(V_\gamma)$ and $N_G(V_\gamma)^{F(V_\gamma)}\simeq N_G(B)^{F(B)}$. Hence we have $F(P)\cap F(V_\gamma)=\{\gamma\}$ by (3.7). As $|F(P)|=q$ by (i), (ii) holds.

(3.20) *Let $V\in U^G-\{U\}$ and let Q be a cyclic subgroup of $N_{N^\alpha}(B)$ of order $q+1$. Then $N_Q(V)=1$.*

Proof. Set $P=N_Q(V)$ and assume $P\neq 1$. Let $\gamma\in\Omega-F(Q)$ and set $B_1=U_\gamma$, $B_2=V_\gamma$. By (3.15), Q normalizes U and so by (3.19) Q normalizes B_1. Similarly P normalizes B_2. Therefore $F(B_1)\cap F(B_2)\geq\gamma^P\neq\{\gamma\}$ as $P\neq 1$ and P is semi-regular on $\Omega-F(Q)$. By (3.18), we have $|F(B_1)\cap F(B_2)|=q$. Since P acts on $F(B_1)\cap F(B_2)$ and $|P|$ divides $q+1$, P fixes at least two points of $F(B_1)\cap F(B_2)$, which contradicts to (3.19).

(3.21) *Let T be a Sylow 2-subgroup of $N_G(U)$. Then, for each $V\in U^G-\{U\}$, $|T: N_T(V)|$ is divisible by q.*

Proof. Suppose $|T: N_T(V)|<q$ and set $T_1=N_T(V)$. Then $|T_1|>q^5$ as $|T|=q^6$ by (3.16). Hence $q>|T_1V: T_1|=|V: V\cap T_1|$ and so $|V\cap T_1|>q^3$. Therefore, for each $B_1\in B^G$ such that $B_1\leq V$, $q>|B_1(V\cap T_1): V\cap T_1|=|B_1: B_1\cap T_1|=|B_1: B_1\cap T|$. Hence $|B_1\cap T|>q$. Let $\gamma\in F(B_1\cap T)$ and set $B_2=U_\gamma$. Then $\langle B_1\cap T, B_2\rangle\leq N^\gamma\cap T$. As $|B_1\cap T|>q$ by (2.5) (iii), $B_1\cap T\cap B_2\neq 1$. By (3.11), $\langle B_1\cap T, B_2\rangle\leq G_{F(B_2)}$. By (3.12) (i), we have $B_1\cap T\leq B_2$, so that $F(B_1)=F(B_1\cap T)=F(B_2)$. Again, by (3.12) (i), $B_1=B_2$ and so $U, V\leq C_G(B_2)$. Therefore $U=V$ by (3.15), a contradiction.

(3.22) *Put $W=U^G$. Then $|W|=q^2+q+1$ and G^W is doubly transitive.*

Proof. Set $H=N_G(U)$. By (3.10) (ii) and (3.16), $|W|=|G: H|=q^2+q+1$. Let $V\in W-\{U\}$ and let Q be as defined in (3.20). By (3.15), $Q\leq H$ and by (3.20), Q acts semi-regularly on $W-\{U\}$. Hence $|V^H|$ is divisible by $q+1$. On the other hand, by (3.21), $|V^H|$ is divisible by q and so we have $|V^H|=q(q+1)$. Thus (3.22) holds.

(3.23) $G_W\cap U\neq 1$.

Proof. Suppose $G_W\cap U=1$. Since $G\trianglerighteq G_W$ and $H\trianglerighteq U$, $[G_W, U]\leq G_W\cap U=1$. Hence $G_W\leq C_G(U)$. By (3.15), U is a unique Sylow 2-subgroup of $C_G(U)$ and so $G_W\leq 0(G)$. On the other hand, as $|\Omega|$ is even and G is doubly transitive on Ω, we have $0(G)=1$. Therefore $G_W=1$ and hence G acts faithfully on W. Since U is not semi-regular on $W-\{U\}$, by [4], $PSL(n_1,q_1)\leq G\leq P\Gamma L(n_1 q_1)$ for some $n_1\geq 3$ and q_1 with q_1 even. As $|W|=q^2+q+1=q_1^{n_1-1}+\cdots+q_1+1$, $q(q+1)=q_1(q_1^{n_1-2}+\cdots+1)$ and so $q=q_1$ and $n_1=3$. Therefore $PSL(3,q)\leq G\leq P\Gamma L(3,q)$. But $|P\Gamma L(3,q)|_2=q^3$ by (3.9) (ii) and Lemma 2.6. Hence $q^3=q^6$ by (3.10) (ii). This is a contradiction. Thus $G_W\cap U\neq 1$.

(3.24) *G^Ω has a regular normal subgroup.*

Proof. Since $G_W\leq N_G(U)$, $G_W\cap U$ is a normal subgroup of G_W. As $G_W\cap$

$U \leq 0_2(G_W)$ and $G \trianglerighteq G_W$, $0_2(G_W)$ is a normal subgroup of G. Let E be a minimal normal subgroup of G contained in $0_2(G_W)$. Then E is an elementary abelian 2-subgroup of G and acts regularly on Ω.

(3.25) *If* (ii) *of* (3.7) *occurs, we have* (i) *of the theorem.*

Proof. By (3.9), (3.10) and (3.24), G has a regular normal subgroup E of order q^3, where $q=2^n$ and $n\equiv 1 \pmod 2$ and N^α is transitive on $\Omega-\{\alpha\}$. Moreover $G=G_\alpha E$ and G_α is isomorphic to a subgroup of $GL(E)$. As in the proof of Lemma 2.1, we may assume $\Omega=E$, $\alpha=0\in E$ and $GL(E)E\leq \mathrm{Sym}(\Omega)$. There exists a subgroup H of $GL(E)$ such that $H\simeq\Gamma L(3,q)$ and $HE\simeq A\Gamma L(3,q)$. Let L be a normal subgroup of H isomorphic to $SL(3,q)$. Since $q=2^n$ and $n\equiv 1 \pmod 2$, L is isomorphic to $PSL(3,q)$.

By (3.9) (i) and by the structure of $A\Gamma L(3,q)$, there exist an automorphism f from N^α to L and $g\in \mathrm{Sym}(\Omega)$ such that $\alpha^g=\alpha$ and $(\beta^x)^g=(\beta^g)^{f(x)}$ for each $\beta\in\Omega-\{\alpha\}$ and $x\in N^\alpha$. From this $(\beta^g)^{g^{-1}xg}=(\beta^x)=(\beta^g)^{f(x)}$ for each $\beta\in\Omega-\{\alpha\}$ and so $g^{-1}xg=f(x)$. Hence $g^{-1}N^\alpha g=L$.

Set $X=N(L)\cap \mathrm{Sym}(\Omega)$ and $D=C_X(L)$. Then D is semi-regular on $\Omega-\{\alpha\}$ as L is transitive on $\Omega-\{\alpha\}$. Put $T=f(A)$. Then $N_L(T)^{F(T)}\simeq Z_{q-1}$ and it is semi-regular on $F(T)-\{\alpha\}$ by (3.5) (i) and (3.9) (i), (iii). It follows that $D\leq Z_{q-1}$. Since X/DL is isomorphic to a subgroup of the outer automorphism group of $PSL(3,q)$ and $f(A)$ and $f(B)$ are not conjugate in $\mathrm{Sym}(\Omega)$ by the hypothesis (**) and (3.9) (ii), it follows from Lemma 2.6 (i) that $|X/DL|\leq n$. Hence $|X|\leq n(q-1)|L|=|\Gamma L(3,q)|$. On the other hand $\Gamma L(3,q)\simeq H<X$ and so $X=H$. Therefore $g^{-1}G_\alpha g\trianglerighteq g^{-1}N^\alpha g=L$ and $g^{-1}G_\alpha\leq X=H$. Thus we have (3.25).

The conclusion of the theorem now follows immediately from steps (3.2), (3.7), (3.8) and (3.25).

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