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ISOTROPIC SUBMANIFOLDS WITH PARALLEL SECOND FUNDAMENTAL FORMS IN SYMMETRIC SPACES

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Introduction. In the study of submanifolds of a riemannian manifold, the notion of an isotropic submanifold has been introduced by B. O'Neill [13] as a generalization of a totally geodesic submanifold. On the other hand, as another generalization of a totally geodesic submanifold, there is the notion of a submanifold with parallel second fundamental form. It is interesting to study submanifolds that belong to both classes, in particular, those which are not totally geodesic, that is, to study nonzero isotropic submanifolds with parallel second fundamental forms. These submanifolds have the property that every geodesic in the submanifold is a circle in the ambient riemannian manifold (K. Nomizu [11]).

Now, as typical examples of such submanifolds, we have the following two; an extrinsic sphere and a nonzero isotropic Kähler submanifold with parallel second fundamental form. The former submanifold is totally umbilical and the latter is minimal.

When the ambient riemannian manifold is a symmetric space, extrinsic spheres have been studied by B.Y. Chen ([2],[3],[4]). Moreover when the ambient riemannian manifold is a complex projective space with Fubini-Study metric, nonzero isotropic Kähler submanifolds with parallel second fundamental forms have been studied by K. Nomizu [11] and T. Itoh [8].

In this paper, we shall show the following two results:

I) If the ambient riemannian manifold is a symmetric space, a complete extrinsic sphere of dimension$\geqq 2$ is isometric to a simply connected real space form (Theorem 8).

II) If the ambient riemannian manifold is a Hermitian symmetric space, a complete nonzero isotropic Kähler submanifold with parallel second fundamental form is the Veronese submanifold of degree 2 in some totally geodesic complex projective space in the Hermitian symmetric space (Theorem 25).

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1. Preliminares

Let $\bar{M}^m$ be an m-dimensional riemannian manifold furnished with a riemannian metric $\langle\ ,\ \rangle$ and M^n be an n-dimensional riemannian submanifold in $\bar{M}^m$. Denote by $\bar{\nabla}$ (resp. ∇) the riemannian connection on $\bar{M}$ (resp. M) and by $\bar{R}$ (resp. R) the riemannian curvature tensor for $\bar{\nabla}$ (resp. ∇). Moreover we denote by σ the second fundamental form of M, by D the normal connection on the normal bundle $N(M)$ of M and by $R^{\perp}$ the curvature tensor for D. For a point p in M, the tangent space $T_p(\bar{M})$ is orthogonally decomposed into the direct sum of the tangent space $T_p(M)$ and the normal space $N_p(M)$. For a vector $X\in T_p(\bar{M})$, the normal component of X will be denoted by $X^{\perp}$. Put

$$N_p^1(M)=\{\sigma(X,Y)\in N_p(M);\ X,Y\in T_p(M)\}_R\,,$$

where $\{*\}_R$ means the R-span of $*$. It is called the *first normal space* of M. Then we have the orthogonal decomposition

$$N_p(M)=N_p^1(M)+(N_p^1(M))^{\perp}\,,$$

where $(N_p^1(M))^{\perp}$ is the orthogonal complement of $N_p^1(M)$ in $N_p(M)$. Note that for a vector ζ in $(N_p^1(M))^{\perp}$, the shape operator A_ζ for ζ vanishes on $T_p(M)$. (Recall here that the shape operator A_ζ for $\zeta\in N_p(M)$ is a symmetric endomorphism of $T_p(M)$ satisfying $\langle A_\zeta(X),Y\rangle=\langle\sigma(X,Y),\zeta\rangle$ for all $X,Y\in T_p(M)$. It is also characterized by that

$$\bar{\nabla}_X\zeta=-A_\zeta X+D_X\zeta$$

for a vector field X of M and a normal vector field ζ.)

Now we recall the following fundamental equations, called the equations of Gauss, Codazzi-Mainardi, and Ricci respectively.

$$\begin{aligned}(1.1)\qquad \langle\bar{R}(X,Y)Z,W\rangle&=\langle R(X,Y)Z,W\rangle+\langle\sigma(X,Z),\sigma(Y,W)\rangle\\&\quad-\langle\sigma(X,W),\sigma(Y,Z)\rangle\,,\end{aligned}$$

$$(1.2)\qquad \{\bar{R}(X,Y)Z\}^{\perp}=(\nabla_X^*\sigma)(Y,Z)-(\nabla_Y^*\sigma)(X,Z)\,,$$

$$(1.3)\qquad \langle\bar{R}(X,Y)\zeta,\eta\rangle=\langle R^{\perp}(X,Y)\zeta,\eta\rangle-\langle[A_\zeta,A_\eta](X),Y\rangle$$

for $X,Y,Z,W\in T_p(M)$ and $\zeta,\eta\in N_p(M)$. Here ∇^* is the covariant derivation associated to the submanifold $M\subset\bar{M}$, defined by

$$(\nabla_X^*\sigma)(Y,Z)=D_X\sigma(Y,Z)-\sigma(\nabla_XY,Z)-\sigma(Y,\nabla_XZ)$$

for vector fields X,Y,Z of M. The second fundamental form σ is said to be *parallel* if $\nabla^*\sigma=0$. If σ is parallel, we have

$$D_X(\sigma(Y,Z)) = \sigma(\nabla_X Y,Z)+\sigma(Y,\nabla_X Z)\,, \tag{1.4}$$

$$\{\bar{R}(X,Y)Z\}^{\perp} = 0\,. \tag{1.5}$$

Moreover we have the following

Lemma 1. *If σ is parallel,*

$$R^{\perp}(T,S)\sigma(X,Y) = \sigma(R(T,S)X,Y)+\sigma(X,R(T,S)Y)$$

for vector fields, T,S,X,Y of M.

Proof. By (1.4),

$$\begin{aligned} R^{\perp}(T,S)\sigma(X,Y) &= D_T D_S\sigma(X,Y)-D_S D_T\sigma(X,Y)-D_{[T,S]}\sigma(X,Y) \\ &= \sigma(\nabla_T\nabla_S X,Y)+\sigma(\nabla_S X,\nabla_T Y)+\sigma(\nabla_T X,\nabla_S Y) \\ &\quad +\sigma(X,\nabla_T\nabla_S Y)-\sigma(\nabla_S\nabla_T X,Y)-\sigma(\nabla_T X,\nabla_S Y) \\ &\quad -\sigma(\nabla_S X,\nabla_T Y)-\sigma(X,\nabla_S\nabla_T Y)-\sigma(\nabla_{[T,S]}X,Y) \\ &\quad -\sigma(X,\nabla_{[T,S]}Y) \\ &= \sigma(R(T,S)X,Y)+\sigma(X,R(T,S)Y)\,. \end{aligned}$$

q.e.d.

For a given $\lambda\geqq 0$, a riemannian submanifold M in a riemannian manifold $\bar{M}$ is called a *λ-isotropic submanifold* if $|\sigma_p(X,X)|=\lambda$ for any point p in M and any unit tangent vector X in $T_p(M)$. In particular, a 0-isotropic submanifold is totally geodesic.

Now we study nonzero isotropic submanifold with parallel second fundamental form. At first, we recall the notion of circles in a riemannian manifold $\bar{M}$.

A curve x_t of $\bar{M}$ parameterized by arc length is called a *circle*, if there exists a field of unit vectors Y_t along the curve which satisfies, together with the unit tangent vector $X_t=\dot{x}_t$, the differential equations

$$\bar{\nabla}_t X_t = kY_t \text{ and } \bar{\nabla}_t Y_t = -kX_t\,,$$

where k is a positive constant, which is called the *curvature* of the circle x_t. Let p be an arbitrary point of $\bar{M}$. For a pair of orthonormal vectors X and Y in $T_p(\bar{M})$ and for a given constant $k>0$, there exists a unique circle x_t, defined for t near 0, such that

$$x_0 = p,\ X_0 = X, \text{ and } (\bar{\nabla}_t X_t)_{t=0} = kY\,.$$

If $\bar{M}$ is complete, x_t is defined for $-\infty<t<+\infty$. Moreover, it is known that a circle is characterized as a 1-dimensional submanifold immersed in $\bar{M}$ which has nonzero parallel mean curvature vector (See [12]).

Now a nonzero isotropic submanifold with parallel second fundamental form has the property as follows.

Lemma 2 (K. Nomizu [11]). *If M is a $\lambda(>0)$-isotropic submanifold in $\bar{M}$ with parallel second fundamental form, every geodesic in M is a circle with curvature λ in $\bar{M}$.*

2. Extrinstic spheres in a symmetric space

A nonzero isotropic submanifold M^n with parallel second fundamental form is called an *extrinsic sphere* in $\bar{M}^m$ if it is a totally umbilical submanifold in $\bar{M}^m$. If moreover $n+1=m$, it is called an *extrinsic hypersphere* in $\bar{M}^m$.

In this section, we study a complete extrinsic sphere $M^n(n\geqq 2)$ in a symmetric space $\bar{M}^m$. Since σ is parallel and $\bar{M}^m$ is a locally symmetric space, M is a locally symmetric space. Moreover B.Y. Chen [2] has shown the following

Proposition 3. *There exists a unique $(n+1)$-dimensional complete totally geodesic submanifold N^{n+1} of constant sectional curvature in $\bar{M}^m$ such that M^n is an extrinsic hypersphere in N^{n+1}.*

We see that N in the proposition is a symmetric submanifold of $\bar{M}$ since N is complete totally geodesic in $\bar{M}$. Now we know the following (cf. [6])

Lemma 4. *If N is a symmetric space of constant sectional curvature, it is isometric to one of the followings: a sphere, a real projective space, a real hyperbolic space, and an abelian group with a bi-invariant metric.*

We study an extrinsic hypersphere in each N of Lemma 4. The complete totally umbilical hypersurfaces of simply connected real space forms are well known (See [14], for example), and they are all extrinsic hyperspheres. Thus we have the following

Proposition 5. *If N is a euclidean space R^{n+1}, M^n is isometric to a sphere S^n. If N is a sphere S^{n+1}, M^n is isometric to a sphere S^n. If N is a real hyperbolic space $H^{n+1}(R)$, M^n is isometric to one of S^n, R^n, or $H^n(R)$.*

Proposition 6. *If N is a real projective space $P^{n+1}(R)$, M^n is isometric to S^n.*

Proof. Let π: $S^{n+1}\rightarrow P^{n+1}(R)$ be the canonical covering map and choose a point o in M. Since M is an extrinsic sphere in $P^{n+1}(R)$, each geodesic in M starting from o is a circle in $P^{n+1}(R)$ with the mean curvature vector H_o at o. Take a point $p\in S^{n+1}$ so that $\pi(p)=o$, and consider the subset $\tilde{M}=\bigcup_{x_t}\{Im\ x_t\}$ of S^{n+1}, where x_t runs over all circles in S^{n+1} starting from p with initial tangent vectors in $\pi_{*p}^{-1}(T_o(M))$ and with the initial mean curvature vector $\pi_{*p}^{-1}(H_o)$. Then we see that $\tilde{M}$ is a small sphere in S^{n+1} and that π maps $\tilde{M}$ onto M. Now we show that $\pi|\tilde{M}$ is injective. If $\pi(q)=\pi(r)$ for two distinct points q, r in $\tilde{M}$, q and r are antipodal in S^{n+1}. Let x_t be a geodesic in $\tilde{M}$ joining q and r.

Then x_t is a circle in S^{n+1}, since $\tilde{M}$ is a small sphere in S^{n+1}. This contradicts the fact that q and r are antipodal in S^{n+1}. Hence $\pi|\tilde{M}$ is injective, and M is a sphere. q.e.d.

Proposition 7. *If N is an abelian group with a bi-invariant metric, M^n is isometric to S^n.*

Proof. Let $\pi\colon R^{n+1}\to N$ be the riemannian covering map. Then the covering transformations are parallel translations of R^{n+1}. As in the proof of Proposition 6, we construct a sphere $\tilde{M}$ in R^{n+1}. Then π maps $\tilde{M}$ onto M. Now we show that $\pi|\tilde{M}$ is injective. If $\pi(q)=\pi(r)$ for two distinct points q, r in $\tilde{M}$, there exists a covering transformation τ such that $\tau(q)=r$. Since $\pi(\tilde{M})=M$, there exist a neighbourhood U of q in R^{n+1} and a neighbourhood V of r in R^{n+1} such that $\tau(U\cap\tilde{M})=V\cap\tilde{M}$. Since τ is a nontrivial parallel translation of R^{n+1} and M is a sphere in R^{n+1}, this does not occur. Hence $\pi|\tilde{M}$ is injective and thus M is a sphere. q.e.d.

Summing up our results, we have the following

Theorem 8. *If $\bar{M}^m$ is a symmetric space and $M^n(n\geqq 2)$ is a complete extrinsic sphere in $\bar{M}^m$, then M^n is isometric to a simply connected real space form.*

Remark 9. In [2] and [7], B.Y. Chen and C.S. Houh have shown that if $M^n(n\geqq 2)$ is a complete simply connected extrinsic sphere with flat normal connection in a Hermitian symmetric space $\bar{M}^m$, then dim $M<$rank $\bar{M}$. Theroem 8 shows that the assumption of simply-connectedness of M may be omitted.

Remark 10. On the classification of complete extrinsic spheres in a symmetric space, the classification of totally geodesic submanifolds N in Proposition 3 has been still left, but the maximum of dimensions of such N in an irreducible symmetric space has been studied by B.Y. Chen and T. Nagano [5].

3. Isotropic Kähler submanifolds

Let $\bar{M}$ be a Kähler manifold furnished with an almost complex structure J and a Kähler metric $\langle\, ,\rangle$ and M be a Kähler submanifold in $\bar{M}$. In this case, both the tangent space $T_p(M)$ and the normal space $N_p(M)$ are invariant under the action of J. Since

$$\sigma(JX,Y)=\sigma(X,JY)=J(\sigma(X,Y))$$

for $X,Y\in T_p(M)$ (cf. [10]), the first normal space $N_p^1(M)$ is also a J-invariant subspace in $N_p(M)$. Recall (cf. [10]) that

$$R(JX,JY)=R(X,Y) \text{ and } JR(X,Y)=R(X,Y)J.$$

The same holds also for $\bar{R}$.

Let $\bar{H}$ (resp. H) denote the holomorphic curvature of $\bar{M}$ (resp. of M). Since a Kähler imbedding preserves holomorphic planes, we can define the *holomorphic difference*

$$\Delta(X)=\bar{H}(X)-H(X)$$

for a unit vector X in $T_p(M)$. If M is a λ-isotropic Kähler submanifold in $\bar{M}$, we have

$$\Delta(X)=2\lambda^2 \tag{3.1}$$

for any unit vector X in $T_p(M)$, which is an easy consequence of the equation of Gauss.

From now on we assume that M is a λ-isotropic Kähler submanifold with $\lambda>0$. Put

$$A(X,Y;Z,W)=\langle\sigma(X,Z),\ \sigma(Y,W)\rangle-\langle\sigma(X,W),\ \sigma(Y,Z)\rangle$$

for tangent vectors X,Y,Z,W of M. Then A is a curvaturelike tensor on M with the holomorphic sectional curvature $2\lambda^2$ such that

$$A(JX,JY;Z,W)=A(X,Y;Z,W)\,.$$

By the theorem of F. Shur, we have

$$\begin{aligned}A(X,Y;Z,W)=\frac{\lambda^2}{2}\{&\langle Y,Z\rangle\langle X,W\rangle+\langle JY,Z\rangle\langle JX,W\rangle-\langle X,Z\rangle\langle Y,W\rangle\\&-\langle JX,Z\rangle\langle JY,W\rangle+2\langle X,JY\rangle\langle JZ,W\rangle\}\end{aligned} \tag{3.2}$$

for tangent vectors X,Y,Z,W of M (cf. [10]). Now we have the following

Lemma 11. *For tangent vectors X,Y,Z,W of M,*

$$\begin{aligned}\langle\sigma(X,Z),\ \sigma(Y,W)\rangle=\frac{\lambda^2}{2}\ \{&\langle Y,Z\rangle\langle X,W\rangle+\langle X,Y\rangle\langle Z,W\rangle\\&+\langle X,JY\rangle\langle JZ,W\rangle+\langle JY,Z\rangle\langle JX,W\rangle\}\,.\end{aligned}$$

Proof. By the J-invariance property of σ, we have

$$A(JX,Y;JZ,W)=-\langle\sigma(X,Z),\ \sigma(Y,W)\rangle-\langle\sigma(X,W),\ \sigma(Y,Z)\rangle\,,$$

and thus

$$\langle\sigma(X,Z),\ \sigma(Y,W)\rangle=\frac{1}{2}\ \{A(X,Y;\,Z,W)-A(JX,Y;JZ,W)\}\,.$$

Now the required equality follows from (3.2). q.e.d.

Lemma 12. *For tangent vectors X,Z,W of M the shape operator $A_{\sigma(Z,W)}$*

is given by

$$A_{\sigma(Z,W)}(X)=\frac{\lambda^2}{2}\{\langle W,X\rangle Z+\langle Z,X\rangle W+\langle Z,JX\rangle JW+\langle JX,W\rangle JZ\}\,.$$

Proof. For tangent vectors X,Y,Z,W of M, we have

$$\langle A_{\sigma(Z,W)}(X),Y\rangle=\langle\sigma(Z,W),\sigma(X,Y)\rangle\,.$$

Now the formula follows from Lemma 11. q.e.d.

4. Isotropic Kähler submanifolds in a Hermitian symmetric space

Throughout in this section, let $\bar{M}^m$ be a complex m-dimensional Hermitian symmetric space and M^n be a complex n-dimensional $\lambda(>0)$-isotropic complete Kähler submanifold in $\bar{M}^m$ with the parallel second fundamental form σ.

Now for a point p in M, set

$$O_p^1(M)=T_p(M)+N_p^1(M)\,,$$

which is called the *first osculating space* at p. Note that dimensions of $N_p^1(M)$ and $O_p^1(M)$ are constant on M, and hence $N^1(M)=\bigcup_p N_p^1(M)$ and $O^1(M)=\bigcup_{p\in M} O_p^1(M)$ are subbundles of $T(\bar{M})|M$, the restriction to M of the tangent bundle $T(\bar{M})$ of $\bar{M}$.

Now fix a point p in M. Let $\bar{G}$ be the identity component of the group of isometries of $\bar{M}$, and set

$$\bar{K}=\{g\in\bar{G};\,g(p)=p\}\,.$$

Let $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{k}}$ be the Lie algebras of $\bar{G}$ and $\bar{K}$, and let

$$\bar{\mathfrak{g}}=\bar{\mathfrak{k}}+\bar{\mathfrak{m}}$$

be the associated Cartan decomposition. Then the tangent space $T_p(\bar{M})$ is identified with $\bar{\mathfrak{m}}$, and hence $O_p^1(M)$ is identified with a subspace of $\bar{\mathfrak{m}}$. Then we have the following

Lemma 13. *The first osculating space $O_p^1(M)$ at p is a Lie triple system in $\bar{\mathfrak{m}}$, that is,* $[[O_p^1(M),O_p^1(M)],O_p^1(M)]\subset O_p^1(M)$.

Proof. Since $\bar{R}_p(X,Y)Z=-ad[X,Y]Z$ for $X,Y,Z\in\bar{\mathfrak{m}}$ (cf. [6]), it is sufficient to show the followings:

(1) $\bar{R}_p(T_p(M),T_p(M))T_p(M)\subset T_p(M)\subset O_p^1(M)\,,$

(2) $\bar{R}_p(T_p(M),T_p(M))N_p^1(M)\subset N_p^1(M)\subset O_p^1(M)\,,$

(3) $\bar{R}_p(T_p(M),N_p^1(M))T_p(M)\subset N_p(M)\subset O_p^1(M)\,,$

(4) $\bar{R}_p(T_p(M), N_p^1(M))N_p^1(M) \subset O_p^1(M)$,
(5) $\bar{R}_p(N_p^1(M), N_p^1(M))T_p(M) \subset O_p^1(M)$,
(6) $\bar{R}_p(N_p^1(M), N_p^1(M))N_p^1(M) \subset O_p^1(M)$,

for each point p of M.

(1) is seen by the equation (1.5) of Codazzi-Mainardi.

Now we shall prove (2). By the equation (1.3), we have

$$\langle \bar{R}(X,Y)H,\eta\rangle = \langle R^{\perp}(X,Y)H,\eta\rangle - \langle [A_H,A_\eta]X,Y\rangle$$

for $X,Y \in T_p(M)$, $H \in N_p^1(M)$, and $\eta \in (N_p^1(M))^{\perp}$. The equation (1.4) shows that $R^{\perp}(X,Y)H \in N_p^1(M)$. Since $A_\eta = 0$, we get

$$\langle \bar{R}(X,Y)H,\eta\rangle = 0\,,$$

and thus $\bar{R}(X,Y)H \in O_p^1(M)$. But $\bar{R}(X,Y)H$ is orthogonal to $T_p(M)$ by (1), thus we have $\bar{R}(X,Y)H \in N_p^1(M)$.

Next we shall prove (3). Since $\bar{M}$ is a locally symmetric space, we have $(\bar{\nabla}_T\bar{R})(X,Y)Z=0$ for tangent vector fields X,Y,Z,W on M, and thus

$$\bar{\nabla}_T(\bar{R}(X,Y)Z) = \bar{R}(\bar{\nabla}_TX,Y)Z + \bar{R}(X,\bar{\nabla}_TY)Z + \bar{R}(X,Y)\bar{\nabla}_TZ\,. \tag{4.1}$$

Since $\bar{R}(X,Y)Z$ is a vector field on M by (1), by the equation of Gauss,

$$\bar{\nabla}_T\bar{R}(X,Y)Z = \nabla_T(\bar{R}(X,Y)Z) + \sigma(T,\bar{R}(X,Y)Z)\,.$$

Similarly, we have

$$\begin{aligned}\bar{R}(\bar{\nabla}_TX,Y)Z &= \bar{R}(\nabla_TX,Y)Z + \bar{R}(\sigma(T,X),Y)Z\,,\\ \bar{R}(X,\bar{\nabla}_TY)Z &= \bar{R}(X,\nabla_TY)Z + \bar{R}(X,\sigma(T,Y))Z\,,\\ \bar{R}(X,Y)\bar{\nabla}_TZ &= \bar{R}(X,Y)\nabla_TZ + \bar{R}(X,Y)\sigma(T,Y)\,.\end{aligned}$$

Here $\nabla_T(\bar{R}(X,Y)Z)$, $\bar{R}(\nabla_TX,Y)Z$, $\bar{R}(X,\nabla_TY)Z$, and $\bar{R}(X,Y)\nabla_TZ$ are tangent vector fields on M by (1), and $\bar{R}(X,Y)\sigma(T,Y)$ is an $N^1(M)$-valued vector field along M by (2), and moreover $\bar{R}(\sigma(T,X),Y)Z$ and $\bar{R}(X,\sigma(T,Y))Z$ are normal vector fields on M by (1) together with the symmetry property of the curvature tensor $\bar{R}$. Thus we have

$$\nabla_T(\bar{R}(X,Y)Z) = \bar{R}(\nabla_TX,Y)Z + \bar{R}(X,\nabla_TY) + \bar{R}(X,Y)\nabla_TZ\,, \tag{4.2}$$

$$\sigma(T,\bar{R}(X,Y)Z) = \bar{R}(\sigma(T,X),Y)Z + \bar{R}(X,\sigma(T,Y))Z + \bar{R}(X,Y)\sigma(T,Z)\,. \tag{4.3}$$

In particular, by (4.3) and (2)

$$\bar{R}(\sigma(T,X),Y)Z + \bar{R}(X,\sigma(T,Y))Z \in N^1(M)\,. \tag{4.4}$$

Since M is a Kähler submanifold in $\bar{M}$, substituting JT (resp. JY) for T (resp.

Y) in (4.4), we have

$$\bar{R}(\sigma(T,X),Y)Z-\bar{R}(X,\sigma(T,Y))Z\in N^1(M)\,. \tag{4.5}$$

By (4.4) and (4.5), we have $\bar{R}(\sigma(T,X),Y)Z\in N^1(M)$, and thus $\bar{R}(T_p(M),N_p^1(M))T_p(M)\subset N_p^1(M)$ for each $p\in M$.

We shall prove (4). Since $\bar{M}$ is a locally symmetric space, we have $(\bar{\nabla}_T\bar{R})(X,Y)H=0$ for tangent vector fields X,Y,T on M and $N^1(M)$-valued vector field H along M, and thus

$$\bar{\nabla}_T(\bar{R}(X,Y)H)=\bar{R}(\bar{\nabla}_TX,Y)H+\bar{R}(X,\bar{\nabla}_TY)H+\bar{R}(X,Y)\bar{\nabla}_TH\,. \tag{4.6}$$

Since $\bar{R}(X,Y)H$ is an $N^1(M)$-valued vector field along M by (2),

$$\bar{\nabla}_T(\bar{R}(X,Y)H)=-A_{\bar{R}(X,Y)H}(T)+D_T(\bar{R}(X,Y)H)\,.$$

Similarly, we have

$$\begin{aligned}\bar{R}(\bar{\nabla}_TX,Y)H&=\bar{R}(\nabla_TX,Y)H+\bar{R}(\sigma(T,X),Y)H\,,\\ \bar{R}(X,\bar{\nabla}_TY)H&=\bar{R}(X,\nabla_TY)H+\bar{R}(X,\sigma(T,Y))H\,,\\ \bar{R}(X,Y)\bar{\nabla}_TH&=-\bar{R}(X,Y)A_H(T)+\bar{R}(X,Y)D_TH\,.\end{aligned}$$

Here $\bar{R}(\nabla_TX,Y)H$, $\bar{R}(X,\nabla_TY)H$, $D_T(\bar{R}(X,Y)H)$, and $\bar{R}(X,Y)D_TH$ are $N^1(M)$-valued vector field along M by (2) and (1.4), and $\bar{R}(X,Y)A_H(T)$ is a tangent vector on M by (1). Thus we have

$$-A_{\bar{R}(X,Y)H}(T)=-\bar{R}(X,Y)A_H(T)+\{\bar{R}(\sigma(T,X),Y)H+\bar{R}(X,\sigma(T,Y)H\}^T\,, \tag{4.7}$$

$$D_T(\bar{R}(X,Y)H)=\bar{R}(\nabla_TX,Y)H+\bar{R}(X,\nabla_TY)H+\bar{R}(X,Y)D_TH+\{\bar{R}(\sigma(T,X),Y)H+\bar{R}(X,\sigma(T,Y))H\}^{\perp}\,, \tag{4.8}$$

where $\{*\}^T$ is the tangent component of $*$. In particular, by (4.8),

$$\bar{R}(\sigma(T,X),Y)H+\bar{R}(X,\sigma(T,Y))H\in O^1(M)\,. \tag{4.9}$$

Since M is a Kähler submanifold in $\bar{M}$, substituting JT (resp. JY) for T (resp. Y) in (4.9), we have

$$\bar{R}(\sigma(T,X),Y)H-\bar{R}(X,\sigma(T,Y))H\in O^1(M)\,. \tag{4.10}$$

By (4.9) and (4.10), we have $\bar{R}(\sigma(T,X),Y)H\in O^1(M)$, and thus $\bar{R}(T_p(M),N_p^1(M))N_p^1(M)\subset O_p^1(M)$ for each $p\in M$.

Now we shall prove (5). Since $\bar{M}$ is a locally symmetric space, we have $(\bar{\nabla}_T\bar{R})(X,H)Y=0$ for tangent vector fields X,Y,T on M and an $N^1(M)$-valued vector field H along M, and thus, as in the proof of (3) and (4),

$$(4.11)\quad \nabla_T(\bar{R}(X,H)Y) = \bar{R}(\nabla_T X,H)Y+\bar{R}(\sigma(T,X),H)Y-\bar{R}(X,A_H(T))Y \\ +\bar{R}(X,D_T H)Y+\bar{R}(X,H)\nabla_T Y+\bar{R}(X,H)\sigma(T,Y)\,.$$

By (1), (3), (4) and (1.4), we have $\bar{R}(\sigma(T,X),H)Y\in O^1(M)$, and thus $\bar{R}(N_p^1(M), N_p^1(M))T_p(M)\subset O_p^1(M)$ for each $p\in M$.

At last we shall prove (6). Since $\bar{M}$ is a locally symmetric space, we have $(\nabla_T\bar{R})\,(X,H)H'=0$ for tangent vector fields T,X on M and $N^1(M)$-valued vector fields H,H' along M, and thus

$$(4.12)\quad \nabla_T(\bar{R}(X,H)H') = \bar{R}(\nabla_T X,H)H'+\bar{R}(\sigma(T,X),H)H'-\bar{R}(X,A_H(T))H' \\ +\bar{R}(X,D_T H)H'-\bar{R}(X,H)A_{H'}(T)+\bar{R}(X,H)D_T H'\,.$$

By (2), (3), (4) and (1.4), we have $\bar{R}(\sigma(T,X),H)H'\in O^1(M)$, and thus $\bar{R}(N_p^1(M), N_p^1(M))N_p^1(M)\subset O_p^1(M)$ for each $p\in M$. q.e.d.

REMARK 14. Lemma 13 holds for any Kähler submanifold M with parallel second fundamental form in a locally Hermitian symmetric space $\bar{M}$.

Now let N be the complete totally goedesic submanifold in $\bar{M}$ through p with $T_p(N)=O_p^1(M)$ (cf. [6]). Since $O_p^1(M)$ is a J-invariant subspace in $T_p(\bar{M})$, N is a Kähler sumbanifold in $\bar{M}$. Now we have the following

Proposition 15. *If M is a complete nonzero isotropic Kähler submanifold in $\bar{M}$ with parallel second fundamental form, there exists a unique complete totally geodesic Kähler submanifold N in $\bar{M}$ such that*

(*a*) *M is a Kähler submanifold in N,*

and

(*b*) *$O_q^1(M)=T_q(N)$ for any point q in M.*

Proof. Let N be the totally geodesic Kähler submanifold in $\bar{M}$ defined as above. Since M is complete, for any point q in M there exists a geodesic γ_t in M such that $\gamma_0=p$ and $\gamma_1=q$. By Lemma 2, γ_t is a circle in $\bar{M}$. Since $\dot{\gamma}_0$ and $(\nabla_t\dot{\gamma}_t)_{t=0}$ are vectors in $O_p^1(M)=T_p(N)$, there exists a unique circle $\tilde{\gamma}_t$ in N such that $\tilde{\gamma}_0=p$, $\dot{\tilde{\gamma}}_0=\dot{\gamma}_0$, and $(\nabla_t\dot{\tilde{\gamma}}_t)_{t=0}=(\nabla_t\dot{\gamma}_t)_{t=0}$. Since N is totally geodesic in $\bar{M}$, $\tilde{\gamma}$ is a circle in $\bar{M}$ with the same initial conditions. Thus we have $\tilde{\gamma}_t=\gamma_t$ by the uniqueness of circle, and consequently $q\in N$. This shows the assertion (a). The assertion (b) follows from that both $O^1(M)$ and $T(N)|M$ are invariant under the parallel translation of $\bar{M}$ along a curve in M. q.e.d.

Now we shall calculate the curvature of the Kähler manifold M.

Lemma 16. *For tangent vectors X,Y,Z,T of M,*

$$\bar{R}(X,J\sigma(T,Y))JZ+\bar{R}(X,\sigma(T,Y))Z = 0\,.$$

Proof. Substituting JT (resp. JZ) for T (resp. Z) in the equation (4.3), we have

$$(4.13)\qquad -\sigma(T,\bar{R}(X,Y)Z)=\bar{R}(J\sigma(T,X),Y)JZ+\bar{R}(X,J\sigma(T,Y)JZ\\ -\bar{R}(X,Y)\sigma(T,Z)\,.$$

By (4.3) and (4.13), we have

$$(4.14)\qquad \bar{R}(J\sigma(T,X),Y)JZ+\bar{R}(X,J\sigma(T,Y))JZ+\bar{R}(\sigma(T,X),Y)Z\\ +\bar{R}(X,\sigma(T,Y))Z=0\,.$$

Substituting JT (resp. JX) for T (resp. X) in (4.14), we get the lemma. q.e.d.

Let $\bar{K}(A,B)$ denote the sectional curvature for the plane spanned by orthogonal vectors A,B of $\bar{M}$.

Lemma 17. *For a unit vector Z of M,*

$$\bar{H}(Z)=4\bar{K}(Z,\sigma(Z,Z))\,.$$

Proof Substituting JT (resp JX) for T (resp. X) in (4.3), we have

$$(4.15)\qquad \sigma(JT,\bar{R}(JX,Y)Z)=-\bar{R}(\sigma(T,X),Y)Z+\bar{R}(X,\sigma(T,Y))Z\\ +\bar{R}(JX,Y)\sigma(JT,Z)\,.$$

By (4.3) and (4.15), we have

$$(4.16)\qquad \sigma(T,\bar{R}(X,Y)Z)+\sigma(JT,\bar{R}(JX,Y)Z)\\ =2\bar{R}(X,\sigma(T,Y))Z+\bar{R}(X,Y)\sigma(T,Z)+\bar{R}(JX,Y)\sigma(JT,Z)\,.$$

Setting $T=X=Y=Z$, we have

$$(4.17)\qquad \sigma(JZ,\bar{R}(JZ,Z)Z)=2\bar{R}(Z,\sigma(Z,Z))Z+\bar{R}(JZ,Z)\sigma(JZ,Z)\,.$$

When Z is a unit vector of M, by Lemma 11,

$$(4.18)\qquad \langle\sigma(JZ,\bar{R}(JZ,Z)Z),\sigma(Z,Z)\rangle=\lambda^2\langle\bar{R}(JZ,Z)JZ,Z\rangle\\ =-\lambda^2\bar{H}(Z)\,.$$

By the Bianchi identity and Lemma 16,

$$(4.19)\qquad \langle\bar{R}(JZ,Z)\sigma(JZ,Z),\sigma(Z,Z)\rangle=-\langle\bar{R}(Z,J\sigma(Z,Z))JZ,\sigma(Z,Z)\rangle\\ -\langle\bar{R}(\sigma(Z,Z),Z)Z,\sigma(Z,Z)\rangle\\ =2\langle\bar{R}(Z,\sigma(Z,Z))Z,\sigma(Z,Z)\rangle\,.$$

Thus, by (4.17), (4.18), and (4.19), we have

$$-\lambda^2\bar{H}(Z)=4\langle\bar{R}(Z,\sigma(Z,Z))Z,\sigma(Z,Z)\rangle\,.$$

Since $|\sigma(Z,Z)|=\lambda$, $\bar{H}(Z)=4\bar{K}(Z,\sigma(Z,Z))$. q.e.d.

Lemma 18. *For a unit vector Z of M,*

$$2\bar{K}(Z,\sigma(Z,Z)) = -\bar{H}(Z)+6\lambda^2 .$$

Proof. Substituting JX (resp. JT) for X (resp. T) in the equation (4.7), we have

$$\begin{aligned}(4.20)\qquad -A_{\bar{R}(JX,Y)H}(JT) &= -\bar{R}(JX,Y)A_H(JT)\\ &\quad +\{-\bar{R}(\sigma(T,X),Y)H+\bar{R}(X,\sigma(T,Y))H\}^T .\end{aligned}$$

By (4.7) and (4.20), we have

$$\begin{aligned}&-A_{\bar{R}(X,Y)H}(T)-A_{\bar{R}(JX,Y)H}(T)\\ &= -\bar{R}(X,Y)A_H(T)-\bar{R}(JX,Y)A_H(JT)+2\{\bar{R}(X,\sigma(T,Y))H\}^T .\end{aligned}$$

If we put $X=Y=T=Z$ and $H=\sigma(Z,Z)$, where Z is a unit vector of M, we have

$$\begin{aligned}(4.21)\qquad -A_{\bar{R}(JZ,Z)\sigma(Z,Z)}(JZ) &= -\bar{R}(JZ,Z)A_{\sigma(Z,Z)}(JZ)\\ &\quad +2\{\bar{R}(Z,\sigma(Z,Z))\sigma(Z,Z)\}^T .\end{aligned}$$

On the other hand, by Lemma 12,

$$(4.22)\qquad \begin{cases}A_{\sigma(Z,Z)}Z = \lambda^2 Z\,, \ A_{\sigma(Z,Z)}JZ = -\lambda^2 JZ\,,\\ A_{J\sigma(Z,Z)}Z = \lambda^2 JZ\,,\ A_{J\sigma(Z,Z)}JZ = \lambda^2 Z\,.\end{cases}$$

By (4.22), we have

$$(4.23)\qquad \langle\bar{R}(JZ,Z)A_{\sigma(Z,Z)}(JZ),Z\rangle = \lambda^2\bar{H}(Z)\,.$$

By (4.22), (1.3), Lemma 1, and Lemma 11, we have

$$\begin{aligned}(4.24)\qquad &\langle A_{\bar{R}(JZ,Z)\sigma(Z,Z)}(JZ),Z\rangle\\ &= \langle\bar{R}(JZ,Z)\sigma(Z,Z),\sigma(JZ,Z)\rangle\\ &= \langle R^{\perp}(JZ,Z)\sigma(Z,Z),\sigma(JZ,Z)\rangle-\langle[A_{\sigma(Z,Z)},A_{\sigma(JZ,Z)}]JZ,Z\rangle\\ &= 2\langle\sigma(R(JZ,Z)Z,Z),\sigma(JZ,Z)\rangle-2\lambda^4\\ &= \lambda^2\{\langle R(JZ,Z)Z,JZ\rangle-\langle R(JZ,Z)JZ,Z\rangle\}-2\lambda^4\\ &= 2\lambda^2 H(Z)-2\lambda^4 .\end{aligned}$$

Thus, by (4.21), (4.23), (4.24), and (3.1), we have

$$2\bar{K}(Z,\sigma(Z,Z)) = -\bar{H}(Z)+6\lambda^2 .$$ q.e.d.

Combining Lemma 17 with Lemma 18, we have

$$(4.25)\qquad \bar{H}(Z) = 4\lambda^2 \ \text{ and } \ H(Z) = 2\lambda^2$$

for any unit vector Z of M. Thus we have proved the following

Proposition 19. *If M is $\lambda(>0)$-isotropic Kähler submanifold in $\bar{M}$ with parallel second fundamental form, then M has constant holomorphic sectional curvatures $2\lambda^2$.*

Now we calculate the curvature of the totally geodesic Kähler submanifold N. The connection on N induced by $\bar{\nabla}$ will be denoted by the same $\bar{\nabla}$. We define a tensor field $\bar{r}$ on N by

$$(4.26)\quad \langle \bar{r}(A,B)C,D\rangle = \lambda^2\{\langle B,C\rangle\langle A,D\rangle+\langle JB,C\rangle\langle JA,D\rangle-\langle A,C\rangle\langle B,D\rangle -\langle JA,C\rangle\langle JB,D\rangle+2\langle A,JB\rangle\langle JC,D\rangle\}$$

for tangent vectors A,B,C,D of N. Then $\langle\bar{r}(\ ,\ )\ ,\ \rangle$ is a curvature-like tensor with the holomorphic sectional curvatures $4\lambda^2$ such that $\langle\bar{r}(JA,JB)C,D\rangle=\langle\bar{r}(A,B)C,D\rangle$ and $\bar{\nabla}\bar{r}=0$.

Lemma 20. *For tangent vectors X,Y,Z of M,*

$$\bar{R}(X,Y)Z=\bar{r}(X,Y)Z\,.$$

Proof. $\langle\bar{R}(X,Y)Z,W\rangle$ and $\langle\bar{r}(X,Y)Z,W\rangle$ are curvature-like tensors on M such that $\langle\bar{R}(JX,JY)Z,W\rangle=\langle\bar{R}(X,Y)Z,W\rangle$ and $\langle\bar{r}(JX,JY)Z,W\rangle=\langle\bar{r}(X,Y)Z,W\rangle$. Since they have the same holomorphic sectional curvatures $4\lambda^2$ by (4.25), the theorem of F. Shur shows that

$$\langle\bar{R}(X,Y)Z,W\rangle=\langle\bar{r}(X,Y)Z,W\rangle\,.$$

Since $\bar{R}(X,Y)Z$ and $\bar{r}(X,Y)Z$ are tangent vectors of M, we have $\bar{R}(X,Y)Z=\bar{r}(X,Y)Z$. q.e.d.

Lemma 21. *For tangent vectors T,S,X,Y of M,*

$$\bar{R}(T,S)\sigma(X,Y)=\bar{r}(T,S)\sigma(X,Y)\,.$$

Proof. By (4.25) and Shur's theorem,

$$(4.27)\quad R(T,S)X=\frac{\lambda^2}{2}\{\langle S,X\rangle T+\langle JS,X\rangle JT-\langle T,X\rangle S-\langle JT,X\rangle JS +2\langle T,JS\rangle JX\}\,.$$

By (1.3), Lemma 1, Lemma 11, Lemma 12, and (4.27), we have

$$\begin{aligned}(4.28)\quad &\langle\bar{R}(T,S)\sigma(X,Y),\sigma(Z,W)\rangle\\ &=\langle R^{\perp}(T,S)\sigma(X,Y),\sigma(Z,W)\rangle-\langle[A_{\sigma(X,Y)},A_{\sigma(Z,W)}]T,S\rangle\\ &=\langle\sigma(R(T,S)X,Y),\sigma(Z,W)\rangle+\langle\sigma(X,R(T,S)Y),\sigma(Z,W)\rangle\\ &\quad-\langle A_{\sigma(Z,W)}T,A_{\sigma(X,Y)}S\rangle+\langle A_{\sigma(X,Y)}T,A_{\sigma(Z,W)}S\rangle\end{aligned}$$

$$= 2\lambda^2\langle T,JS\rangle\{\langle Z,X\rangle\langle JY,W\rangle+\langle JY,Z\rangle\langle X,W\rangle+\langle Y,Z\rangle\langle JX,W\rangle + \langle Z,JX\rangle\langle Y,W\rangle\}\,.$$

On the other hand, by (4.26) and Lemma 11, we have

$$\begin{aligned}(4.29)\qquad &\langle \bar{r}(T,S)\sigma(X,Y),\sigma(Z,W)\rangle \\ &= 2\lambda^2\langle T,JS\rangle\{\langle Z,X\rangle\langle JY,W\rangle+\langle JY,Z\rangle\langle X,W\rangle+\langle Y,Z\rangle\langle JX,W\rangle \\ &\quad + \langle Z,JX\rangle\langle Y,W\rangle\}\,.\end{aligned}$$

Thus we have

$$\langle \bar{R}(T,S)\sigma(X,Y),\sigma(Z,W)\rangle = \langle \bar{r}(T,S)\sigma(X,Y),\sigma(Z,W)\rangle\,.$$

Since $\bar{R}(T,S)\sigma(X,Y)$ and $\bar{r}(T,S)\sigma(X,Y)$ are normal vectors by Lemma 13, (2), and (4.26), we have $\bar{R}(T,S)\sigma(X,Y)=\bar{r}(T,S)\sigma(X,Y)$. q.e.d.

Lemma 22. *For tangent vectors X,Y,Z,T of M,*

$$\bar{R}(X,\sigma(T,Y))Z = \bar{r}(X,\sigma(T,Y))Z\,.$$

Proof. Note that the equation (4.16) was derived from only $\bar{\nabla}\bar{R}=0$. Thus, by $\bar{\nabla}\bar{r}=0$, we have

$$\begin{aligned}(4.30)\qquad &\sigma(T,\bar{r}(X,Y)Z)+\sigma(JT,\bar{r}(JX,Y)Z) \\ &= 2\bar{r}(X,\sigma(T,Y))Z+\bar{r}(X,Y)\sigma(T,Z)+\bar{r}(JX,Y)\sigma(JT,Z)\,.\end{aligned}$$

By (4.16), Lemma 20, and Lemma 21, we have

$$\begin{aligned}(4.31)\qquad &\sigma(T,\bar{r}(X,Y)Z)+\sigma(JT,\bar{r}(JX,Y)Z) \\ &= 2\bar{R}(X,\sigma(T,Y))Z+\bar{r}(X,Y)\sigma(T,Z)+\bar{r}(JX,Y)\sigma(JT,Z)\,.\end{aligned}$$

Thus we have $\bar{R}(X,\sigma(T,Y))Z=\bar{r}(X,\sigma(T,Y))Z$ by (4.30) and (4.31). q.e.d.

Similarly, by (4.6), (4.11), and (4.12), we have the following

Lemma 23. *For $X\in T_p(M)$ and $H_0,H_1,H_2\in N_p^1(M)$, with $p\in M$, we have*

$$\begin{aligned}\bar{R}(X,H_0)H_1 &= \bar{r}(X,H_0)H_1\,,\\ \bar{R}(H_0,H_1)X &= \bar{r}(H_0,H_1)X\,,\\ \bar{R}(H_0,H_1)H_2 &= \bar{r}(H_0,H_1)H_2\,.\end{aligned}$$

Summing up Lemma 20, Lemma 21, and Lemma 22, we have $\bar{R}=\bar{r}$ on N. Thus we have proved the following

Proposition 24. *The complete totally geodesic Kähler submanifold N in $\bar{M}$ is holomorphically isometric to a complex projective space with constant holomorphic sectional curvatures $4\lambda^2$.*

By Propositions 15, 19, and 24, M is a complete Kähler submanifold in the complex projective space N with constant holomorphic sectional curvatures $4\lambda^2$ such that

(1) M has constant holomorphic sectional curvatures $\frac{1}{2}(4\lambda^2)$;

(2) M is not contained in any proper complete totally geodesic Kähler submanifold of M.

It is known (E. Calabi [1]) that such a submanifold M is the Veronese submanifold of degree 2 in N, up to holomorphic isometries of N. Thus we have the following

Theorem 25. *Let $\bar{M}$ be a Hermitian symmetric space. If M is a complete $\lambda(>0)$-isotropic Kähler submanifold in $\bar{M}$ with parallel second fundamental form, then there exists a unique complete totally geodesic Kähler submanifold N in $\bar{M}$ such that*

(*a*) *N is holomorphically isometric to a complex projective space with constant holomorphic curvatures $4\lambda^2$,*

and

(*b*) *M is the Veronese submanifold of degree 2 in N.*

Corollary 26. *If $\bar{M}$ is a Hermitian symmetric space whose holomorpnic sectional curvature is non positive for any holomorphic plane, then there exists no nonzero isotropic Kähler submanifold in $\bar{M}$ with parallel second fundamental form,*

REMARK 27. On the classification of complete nonzero isotropic Kähler submanifolds with parallel second fundamental form in Hermitian symmetric space, the classification of totally geodesic Kähler submanifolds N in Theorem 25 has been still left, but the maximum complex dimension $n(\bar{M})$ of such N in an irreducible compact Hermitian symmetric space $\bar{M}$ is calculated easily by using the result of S. Ihara [9]. The value of $n(\bar{M})$ is; q for $\bar{M}=SU(p+q)/S(U(p)\times U(q))$ $(1\leqq p\leqq q)$; $p-1$ for $\bar{M}=SO(2p)/U(p)$ $(p\geqq 5)$ or $S_p(p)/U(p)$ $(p\geqq 2)$; $\left[\frac{p}{2}\right]$ for $\bar{M}=SO(p+2)/S(O(p)\times O(2))$ $(p\geqq 5)$; 5 for $\bar{M}=E_6/\mathrm{Spin}(10)\cdot T$; 6 for $\bar{M}=E_7/E_6\cdot T$.

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