ON ONE-SIDED QF-2 RINGS I

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We first consider a right artinian ring. Then every projective module \( P \) is a direct sum of indecomposable submodules; \( P = \sum \oplus P_\alpha \). Furthermore for any simple module \( A \) in \( P/J(P) \) there exists a direct summand \( P_\alpha \) of \( P \) such that \( (P_\alpha + J(P))/J(P) = A \), where \( J(P) \) is the Jacobson radical of \( P \). It is clear that \( P + \sum \oplus P_\beta \) for any proper subset \( \beta \) of \( \alpha \).

In this paper we shall study those properties on injectives \( E \) with the condition (**) in [3] and [4], e.g. QF-2 algebra [11] (see §1). If \( E = \sum \oplus E'_\alpha \) and \( E + \sum \oplus E'_\beta \) for \( J \subseteq I \), we say \( \sum \oplus E'_\alpha \) be an irredundant sum. We shall give structure theorems of artinian rings over which every irredundant sum of injective in \( E \) is injective and every simple module in \( E/J(E) \) is lifted to an indecomposable submodule of \( E \). We have studied perfect rings satisfying (*) (see §1) in [4]. We shall show that they satisfy the above properties and they are right artinian from these facts.

We shall extend those ideas to more general modules in [5] and study the dual properties on projectives in [6].

1. Preliminaries

Throughout we consider a ring \( R \) with identity and every module is a unitary right \( R \)-module. Let \( M \) be an \( R \)-module. We shall denote the Jacobson radical and an injective envelope of \( M \) by \( J(M) \) and \( E(M) \), respectively. If \( M \) is a small submodule in \( E(M) \), \( M \) is called a small module [7] and [9] and otherwise we call \( M \) a non-small module [3]. If \( M \) contains a non-zero injective submodule, \( M \) is clearly non-small. We consider the converse case, namely

\[ (*) \quad \text{Every non-small module contains a non-zero injective submodule} \quad [4]. \]

In [4] we have studied perfect rings with \((*)\). We shall show that such rings are right artinian in §4. Furthermore, we shall give some weaker conditions than \((*)\) and show that rings satisfying new conditions give us new classes of rings.
Let $M$ be a module and $\{M_a\}_I$, a set of submodules of $M$. If $\sum I M_a \supseteq \sum J M_a$ for any proper subset $I'$ of $I$, then we say the sum $\sum I M_a$ be irredundant. It is clear that every direct sum is irredundant.

From now on, we assume $R$ is a left and right perfect ring [1]. Let $\{g_i\}_I$ be a complete set of mutually orthogonal primitive idempotents with $1=\sum g_i$. If $g_iR$ is a small (resp. non-small) module, we call $g_i$ a small (resp. non-small) idempotent. Then we obtain a partition $\{g_i\}_I = \{e_i\}_I \cup \{f_j\}_I^*$, where the $e_j$ is non-small and the $f_j$ is small [3]. We have the following lemma from [4], Theorem 2.3.

**Lemma 1.** If $R$ is a left and right perfect ring with (*), then every indecomposable injective module is of a form $e_i R[e_i](J^*)$, where $J=J(R)$, $e_i$ is non-small and $l(J^*)=\{x \in R | xJ^*=0\}$.

In this paper we always consider injective modules which are related to the above form. We note the injective $e_i R[e_i](J^*)$ contains a unique maximal submodule $e_i J/e_i l(J^*)$ and every epimorphism of $e_i R[e_i](J^*)$ onto itself is isomorphic, since $\text{End}_R(e_i R[e_i](J^*))$ is a homomorphic image of the local ring $e_i Re_i$ and $l(J^*)$ is a two-sided ideal. Hence, we quote here a condition in [3] and [4];

\( (** ) \) Every indecomposable and injective module contains a unique maximal submodule, i.e. a cyclic hollow module.

Furthermore, we consider a new condition;

\( (E-1) \) Every epimorphism of an $R$-module onto itself is isomorphic (cf. [5]).

If $R$ is a finite dimensional algebra over a field $K$, then we can consider the duality. The above condition (***) is dual to

\( (** )^* \) Every indecomposable and projective module contains a unique minimal submodule.

If $R$ further satisfies (**) for every left projective module, we call $R$ a *QF-2 ring* following Thrall [11]. Hence, in general, we shall call a ring satisfying (**) a right *QF-2* ring in this note. We shall study a right QF-2 ring (satisfying (***)*) in [6].

From now on, we always assume (**). Then if $E_a$ is indecomposable and injective, $J(E_a)=E_a J$ is a unique maximal submodule and $E_a/E_a J$ is simple, since $R$ is perfect. Let $x$ be any element in $E_a-E_a J$, then $xR=E_a$ and so $E_a \approx eR[eA]$, where $e$ is a non-small idempotent and $A$ is a right ideal of $R$. We denote $E_a$ by $E(S_a)$, where $S_a$ is a simple submodule. Let $M$ be an $R$-
module. We denote $M/MJ$ and the natural epimorphism of $M$ onto $M/MJ$ by $\bar{M}$ and $\varphi_M$, respectively. If $M_1$ is a submodule of $M$, $\varphi_M|_{M_1} = \rho \varphi_{M_1}$; $\rho: M_1/M_1J \to M/MJ$. If there are no confusions, we denote $\varphi_M(M_1)$ by $\bar{M}_1$ (actually $\bar{M}_1 = M_1/M_1J$). If $M_1$ is a direct summand, $\varphi_M|_{M_1} = \varphi_{M_1}$.

**Lemma 2.** We assume an $R$-module $M$ is equal to a sum of injective submodules $\{E(S_a)\}_1$. Then $\sum_1 E(S_a)$ is irredundant if and only if $\varphi_M(M) = \sum_1 \varphi_M(E(S_a))$.

Proof. It is clear that $\bar{M} = \sum_1 \bar{E}(S_a)$ and the lemma is trivial since $MJ$ is a small submodule in $M$ [1].

**Lemma 3.** Let $R$ be a right QF-2* and artinian ring. Then for every non-small primitive idempotent $e$ there exists a right ideal $A$ such that $eR/eA$ is injective.

Proof. Let $E(eR) = \sum_{i=1}^{l} \oplus e_i'R/e_i'A_i$, where $e_i'$ is non-small and the $e_i'A_i$ is a right ideal. Since $eR$ is non-small and $e_i'R/e_i'A_i$ is hollow, $\pi_i(eR) = e_i'R/e_i'A_i$ for some $i$, where $\pi_i: E(eR) \to e_i'R/e_i'A_i$ is the projection. Hence, $eR \cong e_i'R$.

2. Right artinian rings with (*)

We shall show in §4 that perfect rings with (*) are right artinian. Hence we shall first add here a characterization of such rings (cf. [4], Theorem 2.3).

Let $R$ be a right artinian. Then we have a standard decomposition of $R$:

$$R = \sum_{i=1}^{g} \sum_{j=1}^{r_i} g_{ij} R,$$

where $\{g_{ij}\}$ a set of mutually orthogonal primitive idempotents such that $g_{ij}R \cong g_{ij}'R$ and $g_{ij}R \not\cong g_{i'}j' R$ if $i \neq k$. As in §1 we denote non-small idempotent by $e_{ij}$ and put $E_i = \sum_{j=1}^{r_i} e_{ij}$.

Now it is clear that $R$ satisfies (*) if and only if every module $M$ is a direct sum of an injective submodule and a small submodule. We can restate [4], Theorems 2.3 and 2.4 as follows:

**Theorem 1.** Let $R$ be right artinian. Then the following conditions are equivalent.

1) $R$ satisfies (*).
2) There exists $n_i > 0$ for each non-small idempotent $e_i = e_{i1}$ such that $e_iR/e_iJ^{n_i}$ is injective for all $n_i \leq k$, and $e_iR/e_iJ^{n_i-1}$ is small.
3) $R/[R(J)]J^k$ is a direct sum of an injective module and a small projective
module for all $k>0$ as $R$-modules.

4) $R/A$ is a direct sum of an injective module and a small module for every right ideal $A$ contained in $r(J)$.

In this case $A=\bigoplus A_i$ and the $A_i$ is a right ideal in $E_i r(J)$ where $J=\oplus (R)$ and $r(J)=\{x\in R | Jx=0\}$.

Proof. Let $R=\bigoplus e_i R \oplus \bigoplus f_j R$ as in §1 and $D=r(J)$. Since the $f_j R$ is small, $f_j D=0$ by [9], Proposition 4.8. Hence, $D=\bigoplus e_i D$ and $DJ^s=\bigoplus e_i DJ^s$.

1)\(\Rightarrow\)2). We assume that $e_i=e$ and $eR$ is injective and $eJ^s\ni0$, $eJ^s=0$. Then $eJ^s\ni$ is a unique minimal submodule in $eR$. Hence, $eJ^s=0J$. Similarly, we obtain $eJ^s=0J$ if $eR/eJ^s$ is injective, $eD<eJ^s$ and hence, $eD=0J$, since $eJ^s/eJ^s$ is unique minimal. Hence, we have proved 1)\(\Rightarrow\)2) by [4], Theorem 2.3.

3)\(\Rightarrow\)1). We always have $DJ^s=\bigoplus e_i D$. Hence, $R/DJ^s=\bigoplus e_i R \oplus \bigoplus f_j R/eJ^s$. Therefore, the $e_i R/e_i DJ^s$ is injective for any $k>0$ by Krull-

Remak-Schmidt theorem, since $e_i R$ is non-small. If $e_i DJ^s=0$ and $e_i DJ^s=0J$, then $e_i R$ is injective and $e_i DJ^s=0J$. Repeating those arguments as in the proof of [4], Theorem 2.3, there exist an integer $n_i$ and a unique series of submodules $e_i l(J^t)$ of $e_i R$ such that $e_i R/e_i l(J^t)$ is injective for $t<n_i$ and $e_i D=0 l(J^s)$. Therefore, $R$ satisfies (*) by [4], Theorem 2.3.

1)\(\Rightarrow\)4). It is clear from the fact mentioned before the theorem.

4)\(\Rightarrow\)1). Since $DJ^s\subseteq D$ for $k>0$, $e_i R/e_i DJ^s$ is not small by [9], Proposition 4.8. Hence, we can use the same method in 3)\(\Rightarrow\)1).

Finally we assume (*) and $e_i=e_i$. Let $e_i D \ni e_i A_i$ be right ideals. Then $e_i B_i = e_i J^t$ for some $t_i$ and $e_i R/e_i B_i$ is injective by [4], Theorem 2.3. If $e_i A_i/e_i B_i \simeq e_j A_j/e_j B_j$, we can extend $\varphi$ to an isomorphism of $e_i R/e_i B_i$ to $e_j R/e_j B_j$, since $e_p R/e_p B_p$ is indecomposable and injective for $p=i, j$. Hence $i=j$. Since $e_i D \ni e_i D$, the set of simple factor modules of composition series of $E_i D$ coincides with one of $e_i D$. Therefore, $A=\bigoplus (E_i D \cap A)$.

Example 1. There exists a commutative ring $R$ with $R/J$ artinian, which satisfies the condition in [4], Theorem 2.3. We quote the example in [7], p. 378. Let $R=Z(p) \oplus Z(p)$, where $Z(p)=\operatorname{End}_Z(Z(p))$. Then $R$ has a ring structure as usual. $J(R)=pZ(p) \oplus Z(p)$ and $r(J)=\{(0, 1/p)\}$. Furthermore, $0 \rightarrow \{(0, 1/p)\} \rightarrow
\( \rightarrow R \rightarrow pZ(p)\oplus Z_{p^*}(\approx J(R)) \rightarrow 0 \) is exact. Hence, \( R \) is self-injective and \( R/r(J) \) is a small module. However \( R \) does not satisfy (*) by [4], Lemma 2.1, since \( Q(p) \) does not contain any cyclic injective modules.

3. **Lifting property on injectives**

In this section we assume \( R \) is a right artinian ring. Let \( M \) be an \( R \)-module. If for any simple submodule \( A_\alpha \) of \( M/MJ \) there exists a direct summand \( M_\alpha \) of \( M \) such that \( \phi(M_\alpha) = A_\alpha \), then we say \( M \) have the *lifting property of simple module*.

Now we shall study injective modules over right \( \text{QF-2}^* \) ring. We define two weaker conditions than (*).

*(1)* Every non-small module which is a homomorphic image of an injective module contains a non-zero injective module.

And

*(2)* For every non-small module \( M \) there exists an indecomposable direct summand \( E_1 \) of \( E(M) \) such that \( E(M) \supset M \supset E_1 \).

We know from (**) that every indecomposable and injective module is of a form \( eR/eA \), where \( e \) is a non-small primitive idempotent and \( eA \) is a right ideal of \( R \).

**Theorem 2.** Let \( R \) be a right \( \text{QF-2}^* \) and artinian ring. Then the following conditions are equivalent.

1) \( R \) satisfies (*1).
2) Every irredundant sum of direct summand \( E_\alpha \) in an injective module \( E \) such that \( \phi(\sum E_\alpha) = \sum \phi(E_\alpha) \) is injective.
3) For each non-small idempotent \( e \), there exist right ideals \( eA_1 \subset eA_2 \subset \cdots \subset eA_{i-1} \subset eA_i \) such that
   i) \( eA_i/eA_i \) is a uni-serial module such that \( eA_i/eA_{i-1} \) is the socle of \( eR/eA_i \).
   ii) \( eR/eB \) is small for all right ideals \( eB \supset eA_i \).
   iii) The set of those \( eR/eA_i \) is the representative set of indecomposable injectives. (It is possible that \( eR \) contains more than one such increasing series).

Proof. 1)\( \rightarrow \) 2). Let \( E \) be injective and \( \sum E_\alpha \) an irredundant sum in \( E \) with \( E_\alpha \) injective and indecomposable as in 2). Since \( R \) is artinian (hence noetherian) we may show that \( \sum E_\alpha \) is injective for every finite subset \( K \) of \( I \).

Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \) be a finite subset of \( I \) and \( E(p) = \sum_{i=1}^p E_{\alpha_i} \). We shall show the above fact by induction on \( p \). We assume \( E(p-1) \) is injective and \( E = E(p-1) \oplus K' \). Let \( \pi: E \rightarrow K' \) be the projection. Then \( \pi(E_{\alpha_p}) \) is not a small
submodule in \( K' \), since \( \varphi_E(\sum E_a) = \sum \varphi_E(E_a) \). Hence, \( \pi(E_{ap}) \) is injective by \((1)\) for \( E_{ap} \) is hollow. Therefore, \( E(p) = E(p-1) \oplus \pi(E_{ap}) \) is injective.

2)\( \rightarrow \)3). We shall show that if \( eR/eA \) is injective and \( eR/eB \) is non-small for \( eB \supset eA \), then \( eR/eB \) is injective for \( e = e_i \). Put \( E = E(eR/eB) \) and \( F = eR/eA \oplus E \). Let \( f: eR/eA \rightarrow eR/eB \) be the natural epimorphism and \( G = \{ x+f(x) \mid x \in eR/eA \} \subset F \). Then \( G \approx eR/eA \) and \( \varphi_f(eR/eA+G) = \varphi_f(eR/eA) \oplus \varphi_f(G) \) since \( eR/eB \) is non-small and hence \( f(x) \neq 0 \) for some \( x \). Hence, \( eR/eA+G = eR/eA \oplus f(eR/eA) \) is injective. Therefore, \( eR/eB = f(eR/eA) \) is injective. From Lemma 3 we have injective \( eR/eC \) for each \( e \). We may assume that \( eA_i \) is a minimal one among \( eC_i \). Let \( eA_2/eA_1 \) be the socle of \( eR/eA_1 \). If \( eR/eA_2 \) is non-small, \( eR/eA_2 \) is injective by the above. Repeating those arguments, we get a series of right ideals \( eA_1 \supset eA_2 \supset \cdots \supset eA_i \) such that \( eR/eA_j \) is injective and \( eR/eB \) is small for all \( eB \supsetneq eA_j \) by [4], Lemma 1.1. Hence, we have proved 2)\( \rightarrow \)3) by \((**))

3)\( \rightarrow \)1). Let \( M \) be a non-small module which is a homomorphic image of injective \( E \). Let \( E = \sum \oplus E(S_a) \). Then there exists \( E(S_a)(= eR/eA_i) \) whose image is a non-small submodule \( M_0 \) of \( M \). Hence, \( M_0 \approx eR/eC, \quad eC \supset eA_i \). On the other hand, either \( eC = eA_j \supset eA_i \) for some \( j \) or \( eC \supsetneq eA_i \) from 3) i). Hence, \( eC = eA_j \) by 3) i).

**Corollary.** Let \( R \) be right artinian. Then \( R \) satisfies \((*)\) if and only if \( R \) is a right QF-2* and QF-3 ring [11] satisfying \((1)\), (see Example 2).

**Proof.** It is clear from Theorems 1 and 2 and [4], Theorem 1.3.

We have considered special irredundant sums in an injective module in Theorem 2. If we drop the assumption in 2) of Theorem 2, we have the well known theorem:

**Let \( R \) be a right noetherian ring.** Then the following conditions are equivalent.
1) \( R \) is right hereditary.
2) Every irredundant sum of indecomposable injective submodules in an injective module is injective.

We shall give a proof for the sake of completeness.

1)\( \rightarrow \)2). It is clear.

2)\( \rightarrow \)1). Let \( E \) be injective and let \( E_1, E_2 \) be injective submodules. We shall show \( E_1+E_2 \) is injective. We have decompositions \( E_i = \sum \oplus E_{ia} (i=1, 2) \) where the \( E_{ia} \) is indecomposable. Since \( R \) is right noetherian, we may show \( \sum E_{ia} + \sum E_{i\beta} \) is injective for finite subsets \( K_i \). Hence, since we may assume that its sum is irredundant, \( E_1+E_2 \) is injective. Let \( A \) be a submodule of \( E \). Then we can show by the same argument as in the proof 2)\( \rightarrow \)3) of Theorem 2.
that $E/A$ is injective. Therefore, $R$ is right hereditary.

Using the above and Theorem 2 we have

**Theorem 2'.** Let $R$ be a right artinian QF-2* ring. Then the following conditions are equivalent.

1) Every homomorphic image of an injective module is injective ($R$ is hereditary).

2) Every irredundant sum of indecomposable injective submodules in an injective modules is injective.

3) For each non-small idempotent $e$, there exists a right ideal $eA$ such that $eR/eA$ is a uni-serial module and $eR/eB$ is injective for all $eR^eB^eA$. The set $\{eR/eB\}_{e,B}$ is the representative set of indecomposable injective modules (cf. Example 3).

Proof. 2)$\Rightarrow$3). It is clear from Theorem 2 and the above.

3)$\Rightarrow$2). Since every homomorphic image of $eR/eB$ is injective, we can prove the implication by the same argument as in Theorem 2.

We assume that a module $M$ has the lifting property of simple module. Then for any decomposition $M=\bigoplus A_a$ with $A_a$ simple, there exists a set of direct summands $M_a$ of $M$ such that $M_a=A_a$ and $M=\bigoplus M_a$ is an irredundant sum (cf. Lemma 2).

**Theorem 3.** Let $R$ be a right (QF-2* and) artinian ring. Then the following conditions are equivalent.

1) $R$ satisfies $(*2)$.

2) Every injective module has the lifting property of simple module.

3) i) For each non-small idempotent $e$ there exists a chain of right ideals $A_i$ such that $eR/eA_i$ is injective and $eA_i\subseteq eA_{i+1}$ for all $eR=eB=_{eA}\neq eR/eA_i$. Each element in $\text{End}_R(eR/eJ)$ is induced from some element in $\text{Hom}_R(eR/eA_i, eR/eA_j)$ for any $i\geq j$.

ii) The set of $\{e_iR/e_iA_{i,j}\}$ is the representative set of indecomposable injectives.

Proof. 1)$\Rightarrow$2). Let $E$ be injective and $E/EJ=\bigoplus A_a$; the $A_a$ is simple. We take an element $x$ in $E$ such that $Rx=A_a$. Then $Rx$ is non-small by the definition. Hence, there exists an indecomposable direct summand $E_a$ of $E$ such that $A_a=\varphi_R(xR)\supseteq \varphi_R(E_a)$ by $(*2)$. Therefore, $E_a=A_a$.

2)$\Rightarrow$3). We know from $(*2)$ and Lemma 3 that the representative set of indecomposable injectives is $\{e_iR/e_iB_{i,j}\}$. We put $E(i)=\bigoplus_{j=1}^{\infty} e_iR/e_iB_{i,j}$. Then there exists an epimorphism either $f_{im}: eR/eB_m\rightarrow eR/eB_i$ or $f_{mi}: eR/eB_i\rightarrow eR/eB_m$ for any pair $i$ and $m$ by 2) and [5], Corollary to
Theorem 2. We denote this situation by $eR/eB_m \geq \text{resp.} \leq eR/eB_j$. Then the relation $\geq$ is linear. We take the maximal one among $eR/eB_j$, say $eB_j (= eA_j)$ with respect to the relation $\geq$. Let $eR/eB_j$ be the second one. Since there exists an epimorphism $f_{21}$, there exists a right ideal $A_2$ such that $eA_2 \supset eA_1$ and $eR/eA_2 \approx eR/eB_2$. Repeating those arguments, we have a chain of right ideals $A_i$ satisfying i) by [5], Theorem 2.

3) → 2). It is clear from 3) and [5], Theorem 2.
2) → 1). It is clear from the definition.

4. Perfect rings with $(*)$

We shall show, in this section, that a left and right perfect ring satisfying $(*)$ is right artinian.

**Theorem 4.** Let $R$ be a perfect and right QF-2* ring. We assume every indecomposable injective module satisfies $(E-I)$. Further if every injective module $E$ has the lifting property of simple module, then $R$ is right artinian.

Proof. Let $E$ be an indecomposable injective module, say $E = E(S); S$ is simple. We shall show that $E$ is $\Sigma$-injective [2]. Put $T = \sum E_a; E_a = E$ and $Q = E(T)$. It is clear that $\sum T \oplus E_a$ is a direct summand in $Q = Q/QJ$. Now we can express $Q = \sum T \oplus E_a + \sum E(S_\beta)$ by the assumption and Lemma 2 and $E(S_\beta) = A_\beta$. We shall show $L = \phi$. Otherwise, there would exist $A_\beta \neq 0$. $\sum T \oplus S$ is the socle of $T$ and hence of $Q$. Hence, $S_\beta \approx S$ and $E(S_\beta) \approx E$. Let $S_\beta \subseteq \sum K \oplus S$ for some finite subset $K$ of $I$ and $E_K = \sum \oplus E_\gamma$. Then $Q = E_K \oplus F$. We denote the projection of $Q$ onto $F$ by $\pi$. ker $\pi | E(S_\beta) \supset S_\beta$ and $\pi(E(S_\beta)) \subseteq FJ$ since $Q = \sum \oplus E_a + \sum E(S_\gamma)$. It is clear $\pi(E(S_\beta)) \subseteq FJ$. Hence, $F/FJ = C/CJ \oplus B_2 \oplus \cdots$, where $C = \pi(E(S_\beta))$ and the $B_i$ is simple. Then we have an irredundant sum $\sum E_{\beta_i}'$ of indecomposable injective submodules of $F$ such that $\varphi_F(E_{\beta_i}') = C/CJ$. It is clear as above that $E_{\beta_i}' \approx E$. Put $F = E_{\beta_i}' \oplus L$ and $\pi_1$ the projection of $F$ onto $E_{\beta_i}'$. Since $C + FJ = E_{\beta_i}' + FJ$, $\pi_1 \pi: E(S_\beta) \to E_{\beta_i}'$ is an epimorphism. Hence, $\pi_1 \pi | E(S_\beta)$ is isomorphic by the assumption on $E$. However, ker $\pi_1 \pi | E(S_\beta) \supset S_\beta$, which is a contradiction. Thus, $E$ is $\Sigma$-injective. Let $V$ be any injective module and $\sum \oplus S_{i\alpha}$ the socle of $V$, where $S_\alpha \approx S_\beta$ and $S_\alpha \neq S_\beta$ if $i \neq j$. Then $V = E(\sum \oplus S_{i\alpha}) = \sum \oplus E(\sum \oplus S_{i\alpha}) = \sum \oplus \sum_i E(S_{i\alpha})$ by the above. Hence, $R$ is right artinian by [10], Theorem 4.5 in p. 85.
Theorem 5. Let $R$ be a left and right perfect ring. If $R$ satisfies (*), $R$ is right artinian.

Proof. It is clear from Lemma 1 and Theorems 3 and 4.

Proposition 1. Let $R$ and $E$ be as above. We assume $E = \sum E(S_\alpha)$ is irredundant. Then there exists a set of epimorphisms $\psi_\alpha: E(S_\alpha) \to E(S'_\alpha)$ for all $\alpha \in I$ such that $E = \sum \oplus E(S'_\alpha)$.

Proof. We denote the injective $e_1R/\ker(f)$ by $E(e_1, t)$. We assume $E = \sum F(e_1, 0) + \sum F(e_1, 1) + \cdots + \sum F(e_1, s_1) + \cdots + \sum F(e_n, s_n)$ is an irredundant sum of $E$, where $F(e_i, j)_\alpha = E(e_i, j)$. Let $F_0$ be an indecomposable direct summand of $F(e_1, t)_\alpha$. $F(e_1, t)$ is injective by Theorem 2 and Lemma 2 and $F_{j,t} = F_0 \oplus E'$. Let $\pi$ be the projection of $F_{j,t}$ onto $F_0$. Then there exists some $F(e_j, t)$ such that $\pi(F(e_j, t)_\alpha) = F_0$ and so $F_0 \approx E(e_1, p)$ for $p \geq t$. Let $F_{j,t} = F_1, s_1 \oplus L$ and $F_{j,t} = \sum I(1, \alpha) \oplus E(S_\alpha)$. Then $E(S_\alpha) \approx E(e_1, s_1)$ by the above and $|\{1, s_1\}| = |\{1, s_1\}|$ by Lemma 2, where $|P|$ means the cardinal of $P$. Hence, there is a set of epimorphisms $\psi_\alpha: E(S_\alpha) \to E(S'_\alpha)$. Let $\pi': F_{1,s_1} + F_{1,s_1-1} \to L$ be the projection. Since $F_{1,s_1-1}$ is an irredundant sum, $\pi'(F(e_1, s_1-1)_\alpha)$ is injective by Theorem 1. Let $L = \sum I(1, \alpha) \oplus E(S'_\alpha)$. Then $E(S'_\alpha)$ is a homomorphic image of $\pi'(F(e_1, s_1-1))$ and hence of $F(e_1, s_1-1)_\alpha$. Further $|\{1, s_1-1\}| = |\{1, s_1-1\}|$. Hence, we obtain a set of epimorphisms $\psi_\alpha': F(e_1, s_1-1)_\alpha \to E(S'_\alpha)$. Repeating those arguments, we obtain the proposition.

Proposition 2. Let $R$ and $E$ be as above. We assume $E$ is a direct sum of $E(S_\alpha)$ whose proper homomorphic images are not injective. Then every irredundant sum of indecomposable injectives of $E$ is a direct sum.

Proof. Let $E = \sum E_\alpha$ be an irredundant sum of indecomposable injectives. We put $E(n) = \sum E_{\alpha_i}$ for a finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. We show $E(n) = \sum \oplus E_{\alpha_i}$ by the induction on $n$. We assume $E(n-1) = \sum \oplus E_{\alpha_i}$ and $E(n) = E(n-1) \oplus K$. Let $\pi$ be the projection of $E(n)$ onto $K$. Then $\pi|E_{\alpha_i}$ is epimorphic. Since $K$ is injective by Theorem 2, $\pi|E_{\alpha_i}$ is isomorphic by the assumption. Hence, $0 = \ker \pi \cap E_{\alpha_i} = E(n-1) \cap E_{\alpha_i}$. Therefore, $E(n) = \sum \oplus E_{\alpha_i}$.

Proposition 3. Let $R$ be as above. Then $R$ is a QF ring if and only if every irredundant sum $\sum E(S_\alpha)$ of injective module is a direct sum.

Proof. If $R$ is a QF ring, each $e_1R$ has no proper homomorphic injective
images, since every injective module is projective. Hence, we obtain "only if" by Proposition 2. If $R$ is not a QF ring, then there exists a non-small idempotent $e$ such that $eR$ and $eR/eA$ are injective by Theorem 1, where $eA$ is a proper ideal of $eR$. Then $E = eR \oplus eR/eA$ has an irredundant sum $(eR, 0) \oplus \{(er, er) \mid r \in R\}$, where $er$ denotes the image of $er$ by the natural epimorphism: $eR \to eR/eA$. $\{(er, er) \mid r \in R\} \approx eR$ and $(eR, 0) \cap \{(er, er) \mid r \in R\} = (eA, 0) \neq 0$.

We shall give some examples which show that (*1) and (*2) are independent. It is clear that (*) implies (*1) and (*2).

**Example 2.** Let $K$ be a field and $C = K \oplus M$; $M = K$, the trivial extension and

$$R = \begin{pmatrix} C & C \\ 0 & C \end{pmatrix}.$$ 

Put $e_i = e_{ii}$ (matrix units). Then $e_i R = \text{Hom}_C (Re_i, C)$. Since $C$ is self-injective, \{ $e_i R, e_i R/(0, C)\}$ is the complete set of indecomposable injectives. Hence, $R$ is right QF-2* and QF-3. Furthermore, $R$ satisfies (*2) by Theorem 3 but not (*1) by 3, ii) in Theorem 2.

**Example 3.** Put

$$R = \begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}.$$ 

Then $e_i R/(0, K, K)$, $e_i R/(0, K, 0)$ and $e_i R/(0, 0, K)$ is the complete set of indecomposable injectives. Hence, $R$ is right QF-2* and satisfies (*1) by Theorem 2. Since $(0, K, 0) \cup (0, 0, K) = (0, K, K)$, $R$ does not satisfy (*2) by Theorem 3.

**Example 4.** Put

$$R = \begin{pmatrix} K & K & K & K \\ K & 0 & K & K \\ 0 & K & K \end{pmatrix}.$$ 

Then $e_i R, e_i R/e_1 J, e_i R/(0, 0, K, K)$ and $e_i R/(0, K, 0, K)$ is the complete set of indecomposable injectives. Hence $R$ is right QF-2* and furthermore, every indecomposable projective is uniform and so $R$ is QF-2. However, $R$ satisfies neither (*1) nor (*2).

**Example 5.** Put

$$R = \begin{pmatrix} K & K & 0 \\ K & K & K \\ 0 & K & 0 \end{pmatrix} (e_1 e_3 = 0).$$
Then $e_1 R, e_1 R/e_1 J, e_2 R/e_2 J^2$ and $e_3 R/(0, 0, 0, K)$ is the complete set of indecomposable injectives. Hence, $R$ is right QF-2* and satisfies (*1) and (*2). However, (*) is not satisfied. We note $R$ is not QF-3 (cf. Corollary to Theorem 2).

References


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