Okuyama, T. and Wajima, S.
Osaka J. Math.
17 (1980), 801-806

CHARACTER CORRESPONDENCE AND p-BLOCKS OF P-SOLVABLE GROUPS

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(Received August 6, 1979)

1. Introduction

Let G be a finite group and p a prime. The group characters considered are defined over the complex numbers. Let B be a p-block of G with a defect group D and b the p-block of $N_G(D)$ such that $b^G=B$.

Alperin [1] conjectured that the number of irreducible characters of height 0 in B equals the number of irreducible characters of height 0 in b. In this paper, we prove this conjecture for p-solvable groups. Originally, McKay [10] conjectured that the number of irreducible characters of G of degree not divisible by p equals the number of irreducible characters of $N_G(P)$ of degree not divisible by p, for any Sylow p-subgroup P of G. Several interesting works have been done by Glauberman [6] and Isaacs [8] relating to the original conjecture and recently Wolf [13] proved it for solvable groups.

After authors have finished a proof of the result, they were informed that Professor E.C. Dade also had this result and announced it in Santa Cruz conference on finite group theory in 1979.

The authors would like to thank to Professor E.C. Dade who pointed out some errors in the original manuscript of this article.

2. Notations and preliminary results

Let K be a normal subgroup of G and $\theta\in\mathrm{Irr}(K)$. The inertia group $I_G(\theta)$ of θ in G is defined by

$$I_G(\theta)=\{x\in G;\ \theta^x=\theta\}\ .$$

If $I_G(\theta)=G$, then θ is called G-invariant. Let

$$\mathrm{Irr}(G\,|\,\theta)=\{\chi\in\mathrm{Irr}(G);\ (\chi_K,\theta)\neq 0\}\ .$$

Also, let $Ch(G\,|\,\theta)$ be the set of all sums of elements of $\mathrm{Irr}(G\,|\,\theta)$ and

$$\mathrm{Irr}_0(G\,|\,\theta)=\{\chi\in\mathrm{Irr}(G\,|\,\theta);\ p\not\mid\chi(1)\}\ .$$

Let B be a p-block of G. We regard B as a subset of $\mathrm{Irr}(G)$ and let

$$\mathrm{Irr}_0(B) = \{\chi \in B;\ \chi \text{ has height } 0\}\ ,$$
$$\mathrm{Irr}(B\,|\,\theta) = B \cap \mathrm{Irr}(G\,|\,\theta)$$
and
$$\mathrm{Irr}_0(B\,|\,\theta) = \mathrm{Irr}_0(B) \cap \mathrm{Irr}(G\,|\,\theta)\ .$$

The following theorem of Fong [5] is important for our study in this paper.

Theorem (Fong [5], Th. 2. D). *Let G be a finite group with a non-trivial normal p'-group K. Suppose $\theta \in \mathrm{Irr}(K)$ is G-invariant. Let $B \subseteq \mathrm{Irr}(G\,|\,\theta)$ be a p-block of G. Then there is a finite group $\hat{G}$ and a p-block $\hat{B}$ of $\hat{G}$ such that the following hold;*

(1) *B and $\hat{B}$ have isomorphic defect groups.*

(2) *There is a height preserving* 1–1 *correspondence between ordinary characters of B and $\hat{B}$.*

(3) *The group $\hat{G}$ has the following structure: there exists a cyclic normal p'-subgroup $\hat{A}$ in the center of G such that $\hat{G}/\hat{A} \cong G/K$.*

Clearly the same conclusion holds for any subgroup H of G containing K and a p-block of H. The following theorem shows the connection between the above correspondence and Brauer's block correspondence.

Theorem 1. *Let G be a finite group with a normal p'-subgroup K. Suppose $\theta \in \mathrm{Irr}(K)$ is G-invariant. Let H be a subgroup of G containing K and $b \subseteq \mathrm{Irr}(H\,|\,\theta)$ be a p-block of H with a defect group D such that $C_G(D) \subseteq H$. Then $b^G = B$ is defined and the following hold.*

(1) *$B \subseteq \mathrm{Irr}(G\,|\,\theta)$.*

(2) *Let $\hat{G}$ ($\hat{H}$ resp.) and $\hat{B}$ ($\hat{b}$ resp.) be the group and its p-block determined by Fong's theorem. Then $\hat{b}^{\hat{G}}$ is defined and $\hat{b}^{\hat{G}} = \hat{B}$.*

Proof. (1) is trivial. Let D_1 be a defect group of b. From the proof of Lemma (2.C) [5] D_1 can be chosen a Sylow p-subgroup of the inverse image of D in $\hat{G}$. So $C_G(D) \subseteq H$ implies $D_1 C_{\hat{G}}(D_1) \subseteq \hat{H}$ by the construction of $\hat{G}$ in Lemma (2.C) [5] since K is of p'-order. Thus $\hat{b}^{\hat{G}}$ is defined. Now it suffices to show $\hat{b}^{\hat{G}}$ is in fact $\hat{B}$. It is helpful to notice that Fong's correspondence of characters coincides with the isomorphism of Isaacs's character triples (see Isaacs [9], §11) which has the following properties:

There exists a $\hat{G}$-invariant character $\hat{\theta} \in \mathrm{Irr}(\hat{A})$. Let τ: $G/K \to \hat{G}/\hat{A}$ be an isomorphism. For $K \subseteq H \subseteq G$, let $\hat{H}$ deenote the inverse image in $\hat{G}$ of $\tau(H/K)$ For every such H. there exists a map σ_H: $Ch(H\,|\,\theta) \to Ch(\hat{H}\,|\,\hat{\theta})$ such that the following conditions hold for any χ, $\psi \in Ch(H\,|\,\theta)$:

(a). $\sigma_H(\chi+\psi) = \sigma_H(\chi) + \sigma_H(\psi)$

(b). $(\chi, \psi) = (\sigma_H(\chi), \sigma_H(\psi))$

(c). $\sigma_G(\psi^G) = (\sigma_H(\psi))^{\hat{G}}$.

This observation shows that $\sigma_G(B) = \hat{B}$ and $\sigma_H(b) = \hat{b}$. Let $\zeta \in b$ and

$\zeta^G=\sum a_\chi \chi$. Let n_p denote the p-part of an integer n. By Brauer's lemma ((3.A) [2]) we have

$$(\zeta^G(1))_p = (\sum_{\chi\in B} a_\chi \chi(1))_p$$

$$(\zeta^G(1))_p < (\sum_{\chi\in B_1} a_\chi \chi(1))_p \qquad \text{if } B_1 \neq b^G \text{ is a } p\text{-block of } G.$$

By the properties of σ_H, we have $\sigma_H(\zeta)^{\hat{G}}=\sum a_\chi \sigma_G(\chi)$. It follows that

$$(\sigma_H(\zeta)^G(1))_p = (\sum_{\chi\in B} a_\chi \sigma_G(\chi)(1))_p$$

$$(\sigma_H(\zeta)^G(1))_p < (\sum_{\chi\in B} a_\chi \sigma_G(\chi)(1))_p \qquad \text{if } B_1 \neq b^G.$$

Thus again by Brauer's lemma ((3.A) [2]) $\hat{b}^{\hat{G}}=\hat{B}$ as required.

The next theorem is a character theoretical version of Proposition 4.9 of [11] obtained by the first author.

Theorem 2. *Let G be a group with a Sylow p-subgrpup P and a normal p'-subgroup K such that $G=KN_G(P)$. Let $\theta\in \mathrm{Irr}(K)$ be G-invariant. Put $N=N_G(P)$ and $L=N\cap K$. The following hold:*

(1). *There is a unique $\phi\in\mathrm{Irr}(L)$ such that $(\theta_L, \phi)\not\equiv 0$ (mod p). This ϕ is N-invariant.*

(2). *There is a unique extension $\theta_0\in\mathrm{Irr}(PK)$ of θ such that p does not divide $|\det(\theta_0)|$. This θ_0 is G-invariant. Also there is a unique extension $\phi\in\mathrm{Irr}_0(PL)$ of ϕ such that p does not divide $|\det(\phi_0)|$. This ϕ_0 is N-invariant.*

(3). *Assume G/PK is abelian. Then θ is extendible to G if and only if ϕ is extendible to N. Also θ_0 is extendible to G if and only if ϕ_0 is extendible to N.*

(4). $|\mathrm{Irr}(G|\theta_0)|=|\mathrm{Irr}(N|\phi_0)|$.

(5). $\mathrm{Irr}_0(G|\theta)|=|\mathrm{Irr}_0(N|\phi)|$.

Proof. (1) follows from the result of Glauberman (see Theorem 13.1, [9]).

(2) follows from Corollary 6.28 [9].

(3). First we claim that if θ is extendible to G then so is θ_0. Let $\hat{\theta}$ be an extension of θ to G and $\det(\hat{\theta})=\lambda\times\mu$ where $|\lambda|=p^a$ and p does not divide $|\mu|$. As p does not divide $\theta(1)$, there is an integer m such that $m\theta(1)\equiv -1$ (mod p^a). If we set $\theta_1=\hat{\theta}\lambda^m$, then θ_1 is an extension of θ and $\det(\theta_1)=\det(\hat{\theta})\times\lambda^{m\theta(1)}=\lambda^{1+m\theta(1)}\times\mu=\mu$. Thus by (2) θ_1 is an extension of θ_0 as required. Also we have that ϕ is extendible to N then so is ϕ_0. Hence to prove (3) it suffices to show that the first statement in (3) holds. It is proved by induction on $|G|$. Let M be a p-complement in N. As M/L is abelian, there is a subgroup U with $L\subseteq U\subseteq M$ such that M/U is cyclic and $C_{P/P'}(U)\neq 1$ (see Theorem 2.2, Chap. 3 [7]). Assume $C_{P/P'}(U)=P/P'$. Then $C_P(U)=P$ and every character in $\mathrm{Irr}(UK|\theta)$ or $\mathrm{Irr}(U|\phi)$ is P-invariant by Theorem 13.28 [9]. Furthermore by Theorem 13.1 and 13.29 [9] there is a 1–1 correspondence between $\mathrm{Irr}(UK\,|\,\theta)$ and $\mathrm{Irr}(U\,|\,\phi)$ such that $\xi\in\mathrm{Irr}(UK\,|\,\theta)$ corresponds to $\eta\in\mathrm{Irr}(U\,|\,\phi)$ if and only if $(\xi_U, \eta)\not\equiv 0 \pmod p$. If ϕ is extendible to N, then ϕ is extendible to M. Let $\hat{\phi}$ be an extension of ϕ to M and let $\eta=\hat{\phi}_U$. Since ϕ extends to $\hat{\phi}$ and U/L is abelian we have $|\mathrm{Irr}(U\,|\,\phi)|=|U/K|$ (see Corollary 6.17 [9]). So $|\mathrm{Irr}(UK\,|\,\theta)|=|\mathrm{Irr}(U\,|\,\phi)|=|U/L|=|UK/K|$. It follows that each character in $\mathrm{Irr}(UK\,|\,\theta)$ is an extension of θ to UK. In particular, the $\xi\in\mathrm{Irr}(UK\,|\,\theta)$ such that $(\xi_U, \eta)\not\equiv 0 \pmod p$ is such an extension. We have $I_{MK}(\xi)=MK$ since $I_M(\eta)=M$. As MK/UK is cyclic, ξ is extendible to MK and therefore θ is extendible to MK. Then θ is extendible to G (see Corollary 11.31 [9]). Conversely if θ is extendible to G, then by a similar argument it follows that ϕ is extendible to N. Thus we have proved (3) in case $C_{P/P'}(U)=P/P'$. Next assume $C_{P/P'}(U)=Q/P'\neq P/P'$. As U is normal in M, Q and QK are normal in N and G respectively. Let $H=QMK$ and $J=C_K(Q)$. There is a unique $\psi\in\mathrm{Irr}(J)$ such that $(\theta_J, \psi)\not\equiv 0 \pmod p$. This ψ is $N_G(Q)$-invariant and $(\psi_L, \phi)\not\equiv 0 \pmod p$ by Theorem 13.1 [9]. Considering the group $N_G(Q)/Q$ we have by induction that ϕ is extendible to N if and only if ψ is extendible to $N_G(Q)$. Also by induction we have that ψ is extendible to $N_H(Q)$ if and only if θ is extendible to H. As $|G\colon H|$ and $|N_G(Q)\colon N_H(Q)|$ are powers of p, we can conclude from Corollary 11.31 [9] that ϕ is extendible to N if and only if θ is extendible to G. Thus (3) is proved.

(4) follows from (3) and the result of Gallagher (see Exercise 11.10, [9]).

(5). As $\mathrm{Irr}_0(PK\,|\,\theta)=\{\theta_0\lambda\,|\,\lambda\in\mathrm{Irr}(PK/K)=\mathrm{Irr}(P), \lambda(1)=1\}$ and $\mathrm{Irr}_0(PL\,|\,\phi)=\{\phi_0\mu\,|\,\mu\in\mathrm{Irr}(PL/L)=\mathrm{Irr}(P), \mu(1)=1\}$ by Corollary 6.17 [9], to prove (5) it suffices to show that $|\mathrm{Irr}(G\,|\,\theta_0\lambda)|=|\mathrm{Irr}(N\,|\,\phi_0\lambda)|$ for any linear character λ of P. We may assume λ is G-invariant. Since G/PK is a p'-group, λ has an extension λ' to G and $\mathrm{Irr}(G\,|\,\theta_0\lambda)=\{\lambda'\chi;\ \chi\in\mathrm{Irr}(G\,|\,\theta_0)\}$ and $\mathrm{Irr}(N\,|\,\phi_0\lambda)=\{\lambda'_N\chi';\ \chi'\in\mathrm{Irr}(N\,|\,\phi_0)\}$. Then the result follows from (4). Thus the theorem is proved.

3. Proof of the main theorem

Theorem. *Let G be a p-solvable group. Let B be a p-block of G with a defect group D and b the p-block of $N=N_G(D)$ such that $b^G=B$. Then* $|\mathrm{Irr}_0(B)|=|\mathrm{Irr}_0(b)|$.

Proof. The result is proved by induction on the index $|G\colon O_{p'}(G)|$. First we consider the case that the subgroup $H=NO_{p'}(G)$ is properly contained in G. Let b_1 be the p-block of H such that $b_1{}^G=B$ and $b^H=b_1$. There exists an irreducible character θ of $O_{p'}(G)$ such that $b_1\subseteq\mathrm{Irr}(H\,|\,\theta)$ and $D\subseteq I_H(\theta)$. For any $\phi\in b_1$, there exists $\chi\in B$ such that $(\phi, \chi_H)\neq 0$. So we have $(\phi_{0_{p'}(G)}, \chi_{0_{p'}(G)})\neq 0$ and it follows that $\mathrm{Irr}(G\,|\,\theta)$. Therefore $B\subseteq\mathrm{Irr}(G\,|\,\theta)$. By induction $|\mathrm{Irr}_0(b)|=\mathrm{Irr}_0(b_1)|$. Assume $T=I_G(\theta)$ is a proper subgroup. Let $\tilde{b}$

be the p-block of $I_H(\theta)$ determined by Fong's result (Theorem (2.B) [5]) so that $\tilde{b}^H=b_1$. As $D\subseteq I_H(\theta)$ and $N\cap T\subseteq I_H(\theta)$, $\tilde{b}^T=\tilde{B}$ is defined and $\tilde{B}^G=B$. Then by induction $|\mathrm{Irr}_0(\tilde{b})|=|\mathrm{Irr}_0(\tilde{B})|$. Fong's result (Theorem (2.B) [5]) shows that $|\mathrm{Irr}_0(b_1)|=|\mathrm{Irr}_0(\tilde{b})|$ and $|\mathrm{Irr}_0(B)|=|\mathrm{Irr}_0(\tilde{B})|$. Thus $|\mathrm{Irr}_0(b)|=|\mathrm{Irr}_0(B)|$. If θ is G-invariant, then by Theorem 1 and Fong's remarks on p-solvable groups [5], we may assume $O_{p'}(G)$ is contained in the center of G and D is a Sylow p-subgroup of G. Notice that then $B=\mathrm{Irr}(G|\theta)$. Put $P=O_p(G)$ and $\bar{G}=G/P'$. The group $G/O_{p'p}(G)$ acts faithfully on P/P', since G is p-solvable. It follows that $O_{p'p}(\bar{G})$ is precisely the image of $O_{p'p}(G)=O_{p'}(G)\times P$, that $O_{p'}(\bar{G})\cong O_{p'}(G)$ and that the character $\bar{\theta}$ of $O_{p'}(\bar{G})$ corresponding to $\theta\in\mathrm{Irr}(O_{p'}(\bar{G}))$ is $\bar{G}$-invariant. So $\mathrm{Irr}(\bar{G}|\bar{\theta})$ is a p-block $\bar{B}$ of $\bar{G}$. Obviously $\bar{B}=\{\chi\in B;\ P'\subseteq\mathrm{Ker}\,\chi\}$ has the p-Sylow subgroup $\bar{D}$ as a defect group by Fong's Theorem. Thus we may assume P is abelian since every irreducible character of G of p'-degree contains P' in its kernel. Let $O_{pp'}(G)=PK$ where K is a Hall p'-subgroup of $O_{pp'}(G)$. If $K=O_{p'}(G)$, then $G=O_p(G)\times O_{p'}(G)$ and the result follows. Thus we may assume $N_G(K)$ is a proper subgroup of G. By the Frattini argument, $G=O_{pp'}(G)N_G(K)=PN_G(K)$. Since P is abelian, it follows that $P=[P,K]\times C_P(K)$ by Theorem 5.2, Chap. 3 [7]. We conclude that G is the semidirect product of $[P,K]$ by $N_G(K)$. Let $Q=[P,K]$ and $L=N_G(K)$. Q is not 1. Let Ω be $\{\mu\in\mathrm{Irr}(Q);\ \mu$ is D-invariant$\}$. Divide Ω into N-conjugate classes. We claim the above N-cnjugate classes coincide with G-conjugate classes. Suppose λ, μ are conjugate in G. Then there is $g\in G$ such that $\lambda=\mu^g$. Now D and D^g are Sylow p-subgroups of $I_G(\lambda)$. By Sylow's theorem, there is $h\in I_G(\lambda)$ such that $D=D^{gh}$, thus λ and μ are conjugate in N. By Clifford's theorem, it follows that $\mathrm{Irr}_0(B)=\cup\,\mathrm{Irr}_0(B|\mu)$, $\mathrm{Irr}_0(b)=\cup\,\mathrm{Irr}_0(b|\mu)$ where μ runs over a complete set of representatives of N-conjugate classes of Ω. To prove the theorem, it suffices to show that $\mathrm{Irr}_0(B|\mu)=\mathrm{Irr}_0(b|\mu)$ for each μ. We show the above equality by using the method of Wigner (Proposition 2.5 [12]). From the proof of Proposition 2.5 [12], there exists an extension $\tilde{\mu}$ ($\hat{\mu}$ resp.) of μ to $I_G(\mu)(I_N(\mu)$ resp.) and

$$\mathrm{Irr}(I_G(\mu)|\mu)=\{\tilde{\mu}\zeta\,|\,\zeta\in\mathrm{Irr}(L_1)=\mathrm{Irr}(I_G(\mu)/Q)\}$$
$$\mathrm{Irr}(I_N(\mu)|\mu)=\{\hat{\mu}\xi\,|\,\xi\in\mathrm{Irr}(M_1)=\mathrm{Irr}(I_N(\mu)/Q)\}$$

where $L_1=L\cap I_G(\mu)$ and $M_1=L_1\cap N$. Since D is the semidirect product of Q by $D_1\cap LD$, it follows that $M_1=N_{L_1}(D_1)$. The theorem holds for L_1 by induction and the fact that L is a proper subgroup of G. Let $\hat{B}=\mathrm{Irr}(I_G(\mu)|\theta)$ and $\hat{b}=\mathrm{Irr}(I_N(\mu)|\theta)$. Then since $O_{p'}(I_G(\mu))=O_{p'}(G)$, $\hat{B}$ and $\hat{b}$ are p-blocks of $I_G(\mu)$ and $I_N(\mu)$ respectively and $\hat{B}^G=B$ and $\hat{b}^N=b$. Since an induction map of characters defines a 1–1 correspondence between $\mathrm{Irr}(I_G(\mu)|\mu)$ and $\mathrm{Irr}(G|\mu)$, $|\mathrm{Irr}_0(B|\mu)|=|\mathrm{Irr}_0(\hat{B}|\mu)|$. Also we have $\mathrm{Irr}_0(b|\mu)|=|\mathrm{Irr}_0(\hat{b}|\mu|)|$. Thus it suffices to show that $|\mathrm{Irr}_0(\hat{B}|\mu)=\mathrm{Irr}_0(\hat{b}|\mu)|$. Let B_i, $1\leqq i\leqq s$ be all p-blocks of $L_1=I_G(\mu)/Q$ which are included in $\hat{B}$ and have defect group D_1. Let b_j,

$1\leqq j\leqq t$ be all p-blocks of $M_1=I_N(\mu)/Q$ included in b. It follows easily that $s=t$ and after suitable renumbering $b_i^{L_1}=B_i$ for each i. By induction $|\mathrm{Irr}_0(B_i)|=|\mathrm{Irr}_0(b_i)|$ for each i. Since $\mathrm{Irr}_0(B|\mu)=\cup\,\mathrm{Irr}_0(B_i)$ and $\mathrm{Irr}_0(b|\mu)=\cup\,\mathrm{Irr}_0(b_i)$ the result follows. Thus the theorem is proved when H is a proper subgroup of G.

So we may assume $H=NO_{p'}(G)=G$. Set $K=O_{p'}(G)$ and $L=N\cap K$. there is an irreducible character θ of K such that $B\subseteq\mathrm{Irr}(G|\theta)$. Observing that $L=C_K(P)$, there is a unique irreducible character ϕ of L such that $(\theta_L, \phi)\not\equiv 0 \pmod p$ by Glauberman's theorem. By Mackey decomposition $b\subseteq\mathrm{Irr}(N|\phi)$. If θ is not G-invariant, then by Fong's result (Theorem (2.B) [5]) and induction it follows that $|\mathrm{Irr}_0(B)|=|\mathrm{Irr}_0(b)|$. Now assume θ is G-invariant. Then D is a Sylow p-subgroup of G by Fong's remark on p-solvable groups [5] and $B=\mathrm{Irr}(G|\theta)$, $b=\mathrm{Irr}(N|\theta)$. Thus it follows from Theorem 2 that $|\mathrm{Irr}_0(B)|=|\mathrm{Irr}_0(b)|$. The theorem is proved.

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