A NOTE ON HEIGHT OF EXCEPTIONAL CHARACTER DEGREES

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1. Introduction

We assume

(*) \( G \) is a finite group with a Sylow \( p \)-group \( P \) satisfying

\[
C_G(x) = C_G(P), \quad \text{all } x \in P^t.
\]

Hypothesis (*) implies that \( P \) is an abelian trivial intersection subgroup and \( C_G(P) = P \times V \), for some \( p' \)-group \( V \). In fact, \( S = (P \times V) - V \) is a T.I. set. Furthermore, \( N_G(P)/V \) is a Frobenius group with Frobenius kernel \( PV/V \). We set \( |P| = q, s = |N_G(P); C_G(P)| \) and \( st = q - 1 \), where \( t \) is the number of \( p \)-classes of \( G \). We set \( N = N_G(P) \).

Under this hypothesis (*) R. Brauer and H.S. Leonard, Jr. [1,3] have shown the following results.

(a) There is a one to one correspondence from the \( p \)-blocks of \( G \) of full defect onto the \( N \)-classes of irreducible characters of \( V \). (See (1D) [3] for details.)

(b) A \( p \)-block \( B \) of \( G \) of full defect associated with an \( N \)-class \( \Phi \) of irreducible characters of \( V \) contains a family of exceptional characters \( \{ \Lambda_i \} (1 \leq i \leq |\Phi|) \), if \( |\Phi| > 1 \).

(c) Let \( \varphi \in \Phi \), and let \( W(\varphi) \) be the inertia group of \( \varphi \) in \( N \). If \( f = |\Phi| \) and \( |W(\varphi)| = e |C_G(P)| \), then \( ef = s \). Let \( \Lambda \) be any member of \( \{ \Lambda_i \} \). Then

\[
\Lambda_{ij} = \delta \lambda + e \sum_{\varphi \in \Phi} 1_{\varphi},
\]

\[
\Lambda(1) = \delta \varphi(1) + cf \varphi(1) \equiv (\delta e + c)f \varphi(1) \pmod{q},
\]

where \( \delta = \pm 1, \varphi \in \Phi, c \in \mathbb{Z} \), and \( \lambda \) is an appropriately chosen exceptional character of \( N \). In particular, the \( p \)-block \( b \) of \( \lambda \) in \( N \) lifts to the \( p \)-block \( B \) of \( \Lambda \) in \( G \). In addition \( \lambda = (\mu \varphi)^N \), where \( \mu \) is some nonprincipal irreducible character of \( P \).

(d) Moreover it follows from (3.10) in [1] that

\[
\Lambda_{11} \equiv \delta \lambda_1 + e \sum_{\varphi \in \Phi} \varphi \equiv (\delta e + c) \sum_{\varphi \in \Phi} \varphi \pmod{q},
\]
in the ring of all algebraic integers.

Using (a),(b) and (c), D.A. Sibley has proved the following two theorems.

**Theorem 1** (Sibley [4]). *Suppose (*) holds and G has at least three classes of p-elements. Then c=0. In particular Λ(1)≡δλ(1) (mod q).*

**Theorem 2** (Sibley [5]). *Suppose

(♯) G is a finite group with a Sylow p-group P satisfying

\[ C_G(x) = P, \quad \text{all } x \in P^\dagger. \]

Under this hypothesis (♯), \( p \not| \Lambda(1) \) if G has at least two classes of p-elements.

We remark that instead of hypothesis (♯), Sibley has proved Theorem 2 under the following hypothesis:

(♯') A Sylow p-group P of G is an abelian T.I. set, and \( N_G(P) \) is a Frobenius group with Frobenius kernel P.

It is easily seen that hypothesis (♯) is equivalent to hypothesis (♯').

In this paper we shall prove the following theorem, which has been conjectured by Sibley [5].

**Main Theorem.** *Suppose (*) holds and G has at least two classes of p-elements. Then \( p \not| \Lambda(1) \).*

**Example.** Let \( G=SL(2,q) \), where \( q \) is a power of an odd prime \( p \) and \( \frac{q-1}{2} \) is odd. Then \( G \) satisfies (*) with \(|V|=2\) and has two classes of p-elements. \( N_G(P) \) has two families of exceptional characters and both degrees are \( \frac{q-1}{2} \).

On the other hand \( G \) has two families of exceptional characters and their degrees are \( \frac{q-1}{2} \) and \( \frac{q+1}{2} \), which are prime to \( p \). Moreover if we choose \( \delta \) appropriately, we can take \( c=0 \) in both families.

### 2. Proof of Main Theorem

The main theorem can be proved by similar way to Theorem 2 with the addition of block calculations as the proof of Theorem 1.

Suppose by way of contradiction that \( p \not| \Lambda(1). \) Then first of all we claim also \( p \not| \Lambda(x) \) for any p-regular element \( x \) which is not conjugate to an element of \( V \) in \( G \). Let \( g_i \in P^\dagger \) and \( K_i \) be the class of \( G \) containing \( g_i \) and \( K \) be the class of \( G \) containing \( x \). We define a class function \( \theta_{ix} \) by

\[
\theta_{ix}(a) = |\{(g'_i, x')| g'_i \in K_i, x' \in K, g'_i x' = a\}|.
\]
We have the well-known formula
\[ \theta_{is} = \frac{|G|}{|C_G(g_i)| |C_G(x)|} \sum_{\chi} \chi(g_i) \chi(x), \]
where the sum is over all irreducible characters \( \chi \) of \( G \). We now define another class function \( \theta_{is}' \) by
\[ \theta_{is}'(a) = \begin{cases} \theta_{is}(a), & \text{if } a \text{ is } p\text{-singular,} \\ 0, & \text{otherwise.} \end{cases} \]
We may write
\[ \theta_{is}' = \sum_{a} \frac{\theta_{is}(a)}{|C_G(a)|} \sum_{\chi} \chi(a) \chi, \]
where the sum is over a complete set of representatives \( a \) of the \( p\)-singular classes \( a^G \) of \( G \). By \( \theta_{is}', \) we mean
\[ \theta_{is}', \text{ with the sum over } a \text{ for } \theta_{is}, \]
and by \( \theta_{is} \) we mean
\[ \theta_{is} = \frac{|G|}{|C_G(g_i)| |C_G(x)|} \sum_{\chi \in B} \chi(g_i) \chi(x). \]

**Lemma 1.** \( \theta_{is}(g_kv) \equiv 0 \pmod{q} \) for \( g_k \in P^4 \) and \( v \in V \).

**Proof.** The lemma follows easily, because \( P \) acts by conjugation fixed-point-free on the set of the pairs \((g', x')\), where \( g', x' \in K, x' \in K \) and \( g'x' = g_kv \).

Let \( m \) be \( \{ \frac{z}{y} | y \text{ is a rational integer which is prime to } p, \text{ and } z \text{ is an algebraic integer} \} \).

**Lemma 2.** \( \theta_{is}B(g_k) \) is in \( m \) and \( \theta_{is}'B(g_k) \equiv 0 \pmod{qm} \).

**Proof.** Since \( \theta_{is} - \theta_{is}' \) vanishes on \( p\)-singular elements, the "Truncation of Relations" theorem (see [2] (IV.6.3)) shows that \( \theta_{is}B - \theta_{is}'B \) vanishes on \( p\)-singular elements. In particular
\[ \theta_{is}B(g_k) = \theta_{is}'B(g_k) = \sum_{a} \frac{\theta_{is}(a)}{|C_G(a)|} \sum_{\chi \in B} \chi(a) \chi(g_k). \]
We can calculate \( \sum_{\chi \in B} \chi(a) \chi(g_k) \) by (5)[4] and it becomes

\[ \sum_{\chi \in B} \chi(a) \chi(g_k) \]
\[ \theta_{ix'}(g_k) = \sum_{v} \theta_{ix}(v) \sum_{v \in \mathcal{S}} \varphi(v) \varphi(1), \]

where the sum is over a complete set of representatives of \( p \)-singular classes in which \( g_k \) can be chosen as \( p \)-part. Since \( \theta_{ix}(g_k v) \equiv 0 \pmod{q} \) by Lemma 1, the result follows.

(q.e.d.)

**Lemma 3.** If \( p \mid \Lambda(1) \), then \( p \mid \Lambda(x) \) in \( m \) for any \( p \)-regular element \( x \in V^c \).

Proof. We can compute the difference between \( \theta_{ix'}(g_k) \) and \( \theta_{ix}(g_k) \) for \( g_i, g_j \in P^x \), as Sibley did in [5]:

\[ \theta_{ix}(g_k) - \theta_{ix'}(g_k) = \frac{|G|}{|PV||C_{G}(x)|} \sum_{\Lambda} \left\{ \overline{\Lambda}(g_i) \Lambda(x) \Lambda(g_k) - \overline{\Lambda}(g_j) \Lambda(x) \Lambda(g_k) \right\} \frac{\Lambda(1)}{\Lambda(1)} \]

\[ = \frac{|G|}{q |V||C_{G}(x)|} \sum_{\Lambda} \left\{ \overline{\Lambda}(g_i) \Lambda(g_k) - \overline{\Lambda}(g_j) \Lambda(g_k) \right\}, \]

where \( \{\Lambda\} \) are the exceptional characters in \( B \). (These equalities follow from the facts that \( X(g_i) = X(g_j) \) for any nonexceptional character \( X \) in \( B \)((1D) \) [3]), and that \( \Lambda(x) \) and \( \Lambda(1) \) are independent of the choice of \( \Lambda((2B) \) [1]).

On the other hand,

\[ \sum_{\Lambda} \left\{ \overline{\Lambda}(g_i) \Lambda(g_k) - \overline{\Lambda}(g_j) \Lambda(g_k) \right\} = \sum_{x \in B} \left\{ \overline{\mathbb{R}}(g_i) \mathbb{R}(g_k) - \overline{\mathbb{R}}(g_j) \mathbb{R}(g_k) \right\} \]

\[ = qf \varphi(1)^2 (\delta_{g_i g_k} - \delta_{g_j g_k}), \]

where \( \delta_{g h} \) is defined for \( g, h \in P^x \) by

\[ \delta_{g h} = \begin{cases} 1 & g \sim h, \\ 0 & \text{otherwise}. \end{cases} \]

The last equality holds by (5) [4]. As \( G \) has at least two classes of \( p \)-elements, we can choose \( g_i = g_k \) and \( g_j \not\sim g_k \). Then by Lemma 2

\[ 0 \equiv \theta_{ix}(g_k) - \theta_{ix'}(g_k) = \frac{|G| \overline{\Lambda}(x) f \varphi(1)^2}{|V||C_{G}(x)||\Lambda(1)|} \pmod{q m}. \]

Then \( p \mid \Lambda(x) \).

(q.e.d.)

We now calculate \( ||\Lambda||^2 \). This gives

\[ 1 = ||\Lambda||^2 = \frac{1}{|G|} \sum_{g \in \mathbb{G}} \frac{|G| \Lambda(g) \overline{\Lambda}(g)}{|C_{G}(g)|} + \frac{1}{|G|} \sum_{v \in \mathbb{V}} \frac{|G| \Lambda(v) \overline{\Lambda}(v)}{|C_{G}(v)|} \]

\[ + \frac{1}{|G|} \sum_{x \in \mathbb{W}} \frac{|G| \Lambda(x) \overline{\Lambda}(x)}{|C_{G}(x)|}, \]

(4)

where the first and the second sums are over complete sets of representatives of
G-conjugacy classes and the third sum is over that of G-conjugacy classes of \( p \)-regular elements which are not in \( V^c \). Then by Lemma 3 we may write the third sum as \( p^3R \) where \( R = \frac{\pi}{y} \) for some algebraic integer \( \pi \) and some rational integer \( y \) which is prime to \( p \).

**Lemma 4.** Let \( T_1 \) be the first term of (4). Then

\[
T_1 = 1 + \frac{|V| \cdot f}{|N|} \{- (\delta e + c)^2 + c^2 \}.
\]

**Proof.** By (1),

\[
T_1 = \frac{1}{|G|} \sum_{\ell \in s} \frac{|G|}{|C_\ell(g)|} (\delta \lambda(g) + c \sum_{\varphi \in \Phi} \varphi(p(g))) (\delta \bar{\lambda}(g) + c \sum_{\varphi \in \Phi} \varphi(p(g)));
\]

where the sum is over a complete set of representatives of G-conjugacy classes. Since \( S \) is a T.I. set, the representatives of G-conjugacy classes of \( S \) coincide with those of N-conjugacy classes of \( S \) and \( |C_\ell(g)| = |C_N(g)| \). Then

\[
T_1 = \frac{|G : N|}{|G|} \sum_{\ell \in s} \frac{|N|}{|C_N(g)|} (\delta \lambda(g) + c \sum_{\varphi \in \Phi} \varphi(p(g))) (\delta \bar{\lambda}(g) + c \sum_{\varphi \in \Phi} \varphi(p(g)));
\]

where the sum is over a complete set of representatives of N-conjugacy classes. Since \( \lambda \) is a character of \( N \) and \( \sum_{\varphi \in \Phi} \varphi(p(g)) \) is an N-invariant character of \( PV \),

\[
T_1 = \frac{1}{|N|} \sum_{\ell \in s} \lambda(g) \bar{\lambda}(g) + \delta c \sum_{\ell \in s} [\lambda(g) (\sum_{\varphi \in \Phi} \varphi(p(g))) + \bar{\lambda}(g) (\sum_{\varphi \in \Phi} \varphi(p(g)))]
\]

\[
+ c^2 \sum_{\ell \in s} (\sum_{\varphi \in \Phi} \varphi(p(g))) (\sum_{\varphi \in \Phi} \varphi(p(g)));
\]

where the sums \( \sum_{\ell \in s} \) are over all elements of \( S \). We can express \( \lambda_{1_{P \times V}} \) as follows:

\[
\lambda_{1_{P \times V}} = \sum_n (\mu_1 + \mu_2 + \cdots + \mu_e)^n \varphi^n,
\]

where \( n \) ranges over a cross section of \( W(\varphi) \) in \( N \), and \( \mu_1, \mu_2, \cdots, \mu_e \) are distinct irreducible nonprincipal characters of \( P \). Note that

\[
\sum_n \varphi^n = \sum_{\varphi \in \Phi} \varphi.
\]

From the orthogonality relations we get

\[
\sum_{\ell \in s} \lambda(g) \bar{\lambda}(g) = |N| - \sum_{\ell \in s} |\lambda(\ell)|^2
\]

\[
= |N| - \sum_{\ell \in s} |c \sum_{\varphi \in \Phi} \varphi(\ell)|^2
\]

\[
= |N| - e^2 |V| f,
\]
\[
\sum_{g \in G} \lambda(g) (\sum_{\varphi \in \Phi} 1 \rho(\varphi(g))) = \sum_{g \in G} \left( \sum_{*} (\mu_1 + \mu_2 + \cdots + \mu_s)^n \varphi^n(g) \right) (\sum_{\varphi \in \Phi} 1 \rho(\varphi(g))) \\
= -\sum_{v \in V} (\sum_{*} e \varphi^*(v)) (\sum_{\varphi \in \Phi} \varphi(v)) \\
= -e \sum_{v \in V} (\sum_{\varphi \in \Phi} \varphi(v)) (\sum_{\varphi \in \Phi} \varphi(v)) \\
= -e |V| f,
\]

and
\[
\sum_{g \in G} \overline{\lambda}(g) (\sum_{\varphi \in \Phi} 1 \rho(\varphi(g))) = -e |V| f,
\]

and
\[
\sum_{g \in G} | \sum_{\varphi \in \Phi} 1 \rho(\varphi(g))|^2 = (q-1) \sum_{\varphi \in \Phi} | \varphi(v) |^2 = (q-1) |V| f.
\]

Then
\[
T_1 = \frac{1}{|N|} \{ |N| - e^2 |V| f - 2\delta c e |V| f + c^2 (q-1) |V| f \} \\
= 1 + \frac{|V| f}{|N|} \{ -(\delta e + c)^2 + c^2 q \} .
\]

(q.c.d.)

Multiplying (4) by \( q |V| \) we get
\[
\frac{|V| f}{s} \{ -(\delta e + c)^2 + c^2 q \} + T_2 + p^2 q R |V| = 0 ,
\]
where
\[
T_2 = \frac{1}{|G : PV|} \sum_{v \in V} |G| \Lambda(v) \overline{\Lambda}(v).
\]

Then
\[
\frac{|V| f}{s} \{ -(\delta e + c)^2 + c^2 q \} + T_2 \equiv 0 \pmod{pqm} . \tag{5}
\]

Lemma 5.
\[
T_2 \equiv \frac{|V| f (\delta e + c)^2}{s} \pmod{pqm} .
\]

Proof. Let \( \{v_j\} (1 \leq j \leq u) \) be the representatives of \( G \)-conjugacy classes of \( V \). Then these are also the representatives of \( N \)-conjugacy classes, because \( N_G(P) \) controls fusion of \( C_G(P) \). Note that \( p | (\delta e + c) \) from (2), because we have assumed that \( p | \Lambda(1) \). By (3),
\[
T_2 = \frac{1}{|G : PV|} \sum_{j=1}^{u} \frac{|G|}{|C_G(v_j)|} \Lambda(v_j) \overline{\Lambda}(v_j) \\
= \frac{1}{|G : PV|} \sum_{j=1}^{u} \frac{|G|}{|C_G(v_j)|} (\delta e + c)^2 \{ (\sum_{\varphi \in \Phi} \varphi(v_j)) (\sum_{\varphi \in \Phi} \varphi(v_j)) \} \pmod{pqm} .
\]

We now set \( \zeta = \sum_{\varphi \in \Phi} \varphi \). Since \( \zeta \) is an \( N \)-invariant character of \( V \) and \( \{v_j\} \) are also the representatives of \( N \)-conjugacy classes,
Then

\[ T_2 \equiv \frac{(\delta e+c)^2}{s} \sum_{v \in P} |C(v)|^2 \equiv \frac{|G|}{|G:PV|} \sum_{i=1}^s \left\{ \frac{|G| |C_N(v_j)|}{|C_G(v_j)|} \sum_{v \in P} |\zeta(v)|^2 \right\} \equiv \frac{(\delta e+c)^2}{s} \sum_{v \in P} \left| \frac{|\zeta(v)|^2}{|C_G(v):C_N(v)|} \right| \equiv 0 \pmod{pqm}. \]

Since \( P \subseteq C_G(v) \) and \( P \) is a T.I. Sylow \( p \)-group of \( C_G(v) \),

\[ |C_G(v):C_N(v)| \equiv |C_G(v):N_G(P) \cap C_G(v)| \equiv 1 \pmod{q}. \]

Thus

\[ T_2 \equiv \frac{(\delta e+c)^2}{s} \sum_{v \in P} |\zeta(v)|^2 \equiv \frac{|V| f(\delta e+c)^2}{s} \equiv 0 \pmod{pqm}. \]

(q.e.d.)

Then by (5) we get the congruence

\[ \frac{|V| f^2 q}{s} \equiv 0 \pmod{pqm}. \]

Hence we get \( p | c^2 \). This contradicts \( p | \Lambda(1) \). This completes the proof of the main theorem.

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References
