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Osaka J. Math.
17 (1980), 763-767

ON A THEOREM OF A. SAKAI

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(Received May 7, 1979)

Let K be a compact subset of $\boldsymbol{C}^n$. Let $R_1, \cdots, R_s$, $0<s<n$, be continuous, complex-valued functions on K, each of which can be extended to a neighborhood U of K so as to be holomorphic in $z_{s+1}, \cdots, z_n$. Let A denote the algebra of continuous functions on K which can be approximated uniformly on K by polynomials in $z_1, \cdots, z_n$, $\bar{z}_1+R_1, \cdots, \bar{z}_s+R_s$. Clearly A is a subalgebra of the algebra B of continuous functions on K which can be approximated uniformly on K by functions which can be extended to some neighborhood of K so as to be holomorphic in $z_{s+1}, \cdots, z_n$. The goal of this paper is the following theorem which gives sufficient conditions for the equality of these two algebras.

Theorem. *Assume that $R_1, \cdots, R_s$ satisfy*

$$(*) \qquad \sum_{j=1}^{s} |R_j(z+w)-R_j(z)|^2 \leq k \sum_{j=1}^{s} |w_j|^2$$

for all $a \in U$ and all w such that $z+w \in U$ and $w_{s+1}=\cdots=w_n=0$, with $0 \leq k<1$. Assume further that for each $z' \in \boldsymbol{C}^s$ the set $K_{z'}=\{z'' \in \boldsymbol{C}^{n-s}: (z', z'') \in K\}$ is polynomially convex. Then $A=B$.

This theorem was formulated and proved by A. Sakai [4] undei the further assumption that $R_1, \cdots, R_s \in C^\infty(U)$. The special case when $s=n-1$ and $R_1, \cdots, R_{n-1}$ vanish identically was established much earlier by W. Rudin [3].

Sakai's proof is based on the method used by L. Hörmander and J. Wermer [2] who considered the case $s=n$ (where, of course, the sets $K_{z'}$ play no role) under the assumption that the functions $R_1, \cdots, R_n$ are differentiable of sufficiently high order. Our proof depends instead on the Cauchy-Fantappiè integral techniques used by the author [5, 6] to prove the Hörmander-Wermer theorem with minimal smoothness hypotheses, and also on Rudin's argument for the special case cited in the previous paragraph. More specifically, we use an argument due to Rudin to reduce the proof of the theorem to the assertion that if $h \in C_0^1(\boldsymbol{C}^s)$, and $\tilde{h}(z', z'')=h(z')$ then $\tilde{h}|K \in A$. The assertion is then proved using the Cauchy-Fantappiè formula as in the Appendix of [6].

1) This research was partialy supported by a grant from the National Science Foundation.

We begin with the following lemma, which is implicit in Rudin [3, theorem 4].

(If E is a set of functions on K and $z' \in \boldsymbol{C}^s$ we will use $E_{z'}$ to denote the set $\{f(z', \cdot) : f \in E\}$ of functions on $K_{z'}$.)

Lemma 1. *Let K be a compact subset of $\boldsymbol{C}^n$ and let s be an integer, $0<s<n$. Let A and B be closed subalgebras of $C(K)$ such that*

(i) $A \subset B$;

(ii) *A contains a subalgebra $\tilde{A}$ of functions which depend only on $z_{s+1}, \cdots, z_n$ such that, for each $z' \in \boldsymbol{C}^s$, $\tilde{A}_{z'}$ is dense in $V_{z'}$;*

(iii) *if $h \in C_0^1(\boldsymbol{C}^s)$ and $\tilde{h}(z', z'')=h(z')$ then $\tilde{h}|K \in A$.*

Then $A=B$.

Proof. Let M denote the projection of K on $\boldsymbol{C}^s$. Fix $f \in B$ and $\varepsilon>0$. For each $z' \in M$ there exists, by (ii), a function $g_{z'} \in \tilde{A}$ such that, for all $z'' \in K_{z'}$,

$$|g_{z'}(z'')-f(z', z'')|<\varepsilon .$$

Since K and M are compact, and f is uniformly continuous on K, there exist finitely many functions $g_1, \cdots, g_r \in \tilde{A}$ and open sets $N_1, \cdots, N_r$ which cover M such that, for all $z' \in N_i \cap M$, and all $z'' \in K_{z'}$

$$|g_i(z'')-f(z', z'')|<\varepsilon .$$

Let $\{h_i\}$ be a C^1 partition of unity subordinate to the covering $\{N_i\}$ of M Then

$$h_i(z')\,|g_i(z'')-f(z', z'')| \leq \varepsilon h_i(z')$$

for all i and all $(z', z'') \in K$. Since $\sum h_i=1$ on M we have

$$\left|\sum h_i(z')g_i(z'')-f(z', z'')\right| \leq \varepsilon$$

for all $(z', z'') \in K$. By (iii), $\sum h_i(z')g_i(z'')$ belongs to A. Hence $f \in A$, since A is closed and ε was arbitrary.

REMARK. A more abstract version of the proof of Lemma 1 can be obtained by using Bishop's generalized Stone-Weierstrass theorem [1] since each set of antisymmetry lies in one of the sets $K_{z'}$.

If A and B now denote the algebras of the Theorem, then by the Oka-Weil theorem we have (ii) of the Lemma if we take for $\tilde{A}$ the polynomials in $z_{s+1}, \cdots, z_n$. Thus we have reduced the Theorem to the following proposition.

Proposition. *If $h \in C_0^1(\boldsymbol{C}^s)$ and $\tilde{h}(w', w'')=h(w')$ then $\tilde{h}|K \in A$.*

Proof. By the Hahn-Banach theorem it suffices to show that if μ is a complex Borel measure on K which is orthogonal to the algebra A then

$$\int h(w')d\mu(w', w'') = 0$$

for all $h \in C_0^1(W)$, where W is the projection onto $\boldsymbol{C}^s$ of some neighborhood of K.

By convolution with an appropriate approximate identity we can obtain, for each j, $1 \leq j \leq s$, a sequence $\{R_j^\nu\}$ of C^1 functions defined on a neighborhood V of K such that

(a) $R_j^\nu(z) \rightarrow R_j(z)$ for each $z \in V$

(b) the functions $R_1^\nu, \cdots, R_s^\nu$ satisfy (*) with the same constant $k<1$, (independent of ν,)

(c) each R_j^ν is holomorphic in $z_{s+1}, \cdots, z_n$.

Let

$$G^\nu(z, w) = \sum_1^s (z_j - w_j)(\bar{z}_j + R_i^\nu(z) - \bar{w}_j - R_j^\nu(w))$$

$$\Omega_j^\nu(z, w) = G(z, w)^{-s}(\bar{z}_j + R_j^\nu(z) - \bar{w}_j - R_j^\nu(w))$$

$$Q_j^\nu(z) = (s-1)!(2\pi i)^{-s} \bigwedge_{\substack{k \neq j \\ 1 \leq k \leq s}} d\bar{z}_k + \bar{\partial} R_k^\nu$$

$$d'z = dz_1 \wedge \cdots \wedge dz_s\,.$$

Define $\Omega^\nu(z, w)$ by

$$\Omega^\nu(z, w) = \sum (-1)^{j-1} \Omega_j^\nu(z, w) Q_j^\nu(z) \wedge d'z$$

It follows easily from (*) that for each ν,

(d) $|G^\nu(z, w)| \geq (1-k)|z' - w'|^2$

(e) $Re\, G^\nu(z, w) > 0$ if $z' \neq w'$

(f) $|\Omega_j^\nu(z, w)| \leq (1+k)(1-k)^{-s}|z' - w'|^{1-2s}$

If we fix z'', $w'' \in \boldsymbol{C}^{n-s}$ it follows from the Corollary to Lemma 4 of [5] that

$$h(u') = \int_W \Omega^\nu(z', z'', w', w'') \wedge \bar{\partial} h(z')$$

where W is the projection of V onto $\boldsymbol{C}^s$ and the support of h lies in W. Consequently,

$$\int h(w')d\mu(w) = \int_W \sum (-1)^{j-1} \Big[\int_K \Omega_j^\nu(z', z'', w', w'')d\mu(w)\Big] Q_j^\nu \wedge d'z \wedge \bar{\partial} h$$

We can write $Q_j^\nu(z) \wedge d'z \wedge \bar{\partial} h$ as $\sigma_j^\nu(z')dm_s$ where m_s denotes Lebsegue measure on $\boldsymbol{C}^s$ and σ_j^ν is a function. Moreover, the functions σ_j^ν are uniformly bounded because (b) implies that $\partial R_j^\nu / \partial \bar{z}_k$ are uniformly bounded.

Define G and Ω_j using $R_1, \cdots, R_s$ in the same way that G^ν and Ω_j^ν were defined using $R_1^\nu, \cdots, R_s^\nu$.

(g) For almost all $z\in V$, and each j, $1\leq j\leq s$,

$$\int|\Omega_j(z, w)|d|\mu|(w)<\infty.$$

Furthermore, there exists $L\in L^1(dm_s(z'))$, independent of ν such that

$$|\int \Omega_j^\nu(z, w)d\mu(w)|\leq L(z').$$

Indeed, let $L(z')$ be defined by

$$L(z')=(1+k)^{-1}(1-k)^s\int\frac{d|\mu|(w)}{|z'-w'|^{2s-1}}.$$

Since, for any compact set $A\subset \boldsymbol{C}^s$,

$$\sup_{w'\in A}\int_A\frac{dm_s(z')}{|z'-w'|^{2s-1}}<\infty,$$

(g) follows from (f) together with Fubini's theorem.

Now $\Omega_j^\nu\rightarrow\Omega_j$ pointwise. By (g), for almost all $z\in V$,

$$\int_K \Omega_j^\nu(z, w)d\mu(w)\rightarrow\int\Omega_j(z, w)d\mu(w).$$

Consequently,

$$\begin{aligned}|\int h(w')d\mu(w)|&=|\int\sum(-1)^{j-1}[\int\Omega_j^\nu(z, w)d\mu(w)]\sigma_j^\nu(z)dm_s(z')|\\&\leq\sum\int|\int\Omega_i^\nu(z, w)d\mu(w)|\,|\sigma_j^\nu|(z)dm_s(z')\\&\leq C\sum\int|\int\Omega_j^\nu(z, w)d\mu(w)|dm_s(z')\end{aligned}$$

Letting $\nu\rightarrow\infty$,

$$|\int h(w')d\mu(w)|\leq C\sum\int|\int\Omega_j(z, w)d\mu(w)|dm_s(z')$$

By Lemma 1 of [7] there exist holomorphic functions P_ν on $\{Re\,\lambda\geq 0\}$ such that

$$P_\nu(\lambda)\rightarrow\frac{1}{\lambda}\qquad\text{if }\lambda\neq 0 \tag{1}$$

$$|P_\nu(\lambda)|\leq\frac{2}{|\lambda|} \tag{2}$$

By Runge's theorem, each P_ν is the uniform limit on compact subsets of $\{Re\,\lambda\geq 0\}$ of a sequence of polynomials in λ.

Since $Re\, G(z, w)>0$ if $z'\neq w'$ and since $G(z, \cdot)\in A$ for each $z\in V$, there exist $Q_\nu\in A$ such that

$$Q_\nu(w) \to G(z, w)^{-1} \quad \text{on} \quad \{w\in K: w'\neq z'\} \tag{3}$$

$$|Q_\nu(w)| \leq \frac{2}{|G(z, w)|}. \tag{4}$$

But (3) and (4) together with (g) imply that each $\Omega_j(z, \cdot)$ is the pointwise limit *a.e.* $-d|\mu|$ of a sequence of elements of A, and that for all z except perhaps a set of Lebesgue measure zero, this convergence is dominated with respect to $d|\mu|$. By the dominated convergence theorem, then, $\int \Omega_j(z, w)d\mu(w)=0$ each j. Hence $\int h(w')d\mu(w)=0$ which is what we set out to prove.

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