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LOCALIZATION OF DIFFERENTIAL OPERATORS AND THE UNIQUNESS OF THE CAUCHY PROBLEM

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1. Introduction

Let $P(x, \partial_x)$ be a differential operator of order m with analytic coefficients in an open set U in $\boldsymbol{R}^n$ and Ω be an open subset of U with C^1 boundary $\partial\Omega$. Then the uniqueness theorem of Holmgren which is extended for distribution solutions ([3]) states that a distribution solution $u(x)$ of the equation $Pu=0$ in U vanishing in Ω must vanish in a neiborhood of $\partial\Omega$ if $\partial\Omega$ is non-characteristic. The extension of this theorem to the case near a characteristic point has been made by many authors relating to the problem of deciding the P-convexity domains. Among others Hörmander [3] showed that when the principal part is real the uniquness theorem holds if $\partial\Omega \in C^2$ and the characteristic points are simple and some convexity conditions are satisfied at these points. The refinements of this Hörmander's result are made by Treves [8], Zachmanoglou [10], [11] and Hörmander [5]. Recently Bony [2] introduced the notion of strongly characteristic and proved the uniquness theorem for degenerate equations. Bony's result is extended by Hörmander [6]. In this note we deal with a differential operator which is highly degenerated at some point p on $\partial\Omega$ and obtain the suffcient conditions to get the uniquness theorem. Though the uniquness theorem is invariant under the analytic change of coordinates, we here employ the weighted local coordinates at p such that the normal direction x_1 of $\partial\Omega$ at p is assigned the weight 2, while the tangential directions $x_2, \cdots, x_n$ are each assigned the weight 1. The motivation of this employment is that the boundary $\partial\Omega$ can be approximated by the quadratic hypersurface of the form

$$x_1 = \sum_{i,j\geq 2} a_{ij}x_i x_j\,. \tag{1.1}$$

The transformations of the coordinates in this note are limited to the ones which preserve the weights $(2, 1, \cdots, 1)$ (see the section 2 for the precise definition). In the section 3, the basic theorem is proved under some fixed local coordinates. The idea of the proof is due to Hörmander [3] and extensively used by Treves [8], Zachmanoglou [10], [11] and others. That is to construct the family of surfaces which are non-characteristic with respect to $P(x, \partial_x)$ and cover a neighborhood of p. This basic theorem is a generalization of Hörmander's theorem [3] of the simple characteristic case. In the last section, §4, we study the geometric conditions on $P(x, \partial_x)$ and $\partial\Omega$ to insure the existence of the local coordinates in the third section. The assumptions are made in relation to the localization of $P(x, \partial_x)$ at (p, N), where N is the normal direction of $\partial\Omega$ at p. The localization of an operator is also due to Hörmander [4] to research the location of the singularities of the solutions of $Pu=0$. Our method in this note is also used to show the holomorphic continuation of the solutions of $P(z, \partial_z)u=f$ in the complex n dimensional space, which is to appear in [9].

2. Weighted coordinates

As in the introduction, we shall approximate $\partial\Omega$ by the quadratic hypersurface of the form (1.1). For this sake we here introduce the weighted coordinates. Weighted coordinates are also used by T. Bloom and I. Graham [1] to determine the type of the real submanifold in $\boldsymbol{C}^n$ which is firstly introduced by Kohn in relation to the boundary regularity for the $\bar{\partial}$-Neumann problem. In this note we use the simplest weighted coordinates.

Let $(x_1, \cdots, x_n)$ be a local coordinates in U of $\boldsymbol{R}^n$. Then we say that $(x_1, \cdots, x_n)$ is the weighted coordinates system of the weights $(2, 1, \cdots, 1)$ if the coordinate function x_1 has the weight 2 and x_j $(j=2, \cdots, n)$ has the weight 1. The weight of a monomial x^α is determined by $2\alpha_1+\alpha_2+\cdots+\alpha_n$. An analytic function $f(x)$ at 0 has the weight l if l is the lowest weight among the monomials in the Taylor expansion of $f(x)$ at 0. For convenience, the weight of $f=0$ is assigned $+\infty$. The weight of a differential operator is defined by the corresponding negative weight. For a differential monomial $(\partial/\partial x)^\alpha$, its weight is defined by $-2\alpha_1-\alpha_2-\cdots-\alpha_n$. The weight of $a(x)(\partial/\partial x)^\alpha$ is equal to $\text{weight}(a(x))+\text{weight}((\partial/\partial x)^\alpha)$ and the weight of a linear partial differential operator $P(x, \partial_x)=\sum a_\alpha(x)(\partial/\partial x)^\alpha$ is determined by $\min \text{weight}(a_\alpha(x)(\partial/\partial x)^\alpha)$.

Let $(x_1, \cdots, x_n)$ and $(u_1, \cdots, u_n)$ be two local coordinates with the same origin. We say that these coordinates are equivalent as the weighted coordinates if u_j has the same weight as x_j as an analytic function of x_j, and the converse is also true. In this note the weights are always equal to $(2, 1, \cdots, 1)$. Therefore $(x_1, \cdots, x_n)$ and $(u_1, \cdots, u_n)$ are equivalent if and only if

$$\frac{\partial(u_1, \cdots, u_n)}{\partial(x_1, \cdots, x_n)}(0) = \begin{vmatrix} c & 0 & \cdots & 0 \\ a_2 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ a_n & c_{n2} & \cdots & c_{nn} \end{vmatrix} \neq 0 .$$

It is easily derived that the weights of functions or differential operators are invariant under the equivalent transformation of the weighted coordinates.

We also remark that if the weights of covectors $(\xi_1, \cdots, \xi_n)$ are each assigned the $(-2, -1, \cdots, -1)$, then the weight of $P_m(x, \xi)$, the principal part of P, is invariant.

3. The basic theorem

The differential operator studied in this section is the following one:

$$P(x, \partial_x) = \left(\frac{\partial}{\partial x_1}\right)^{m-l}\left(\frac{\partial}{\partial x_2}\right)^{l} + \sum a_\alpha(x)\left(\frac{\partial}{\partial x}\right)^{\alpha} \tag{3.1}$$

where $a_\alpha(x)$ are analytic in some neighborhood U of 0 and the summation is taken over the multi-indices α such that $|\alpha| \leqq m$. The domain Ω is given by

$$\Omega = \{x \in U \mid \rho(x) < 0\} \tag{3.2}$$

where ρ is a real-valued C^2 function such that

$$\rho(0) = 0\,, \quad \frac{\partial \rho}{\partial x_1}(0) = 1\,, \quad \frac{\partial \rho}{\partial x_j}(0) = 0 \qquad j = 2, \cdots, n\,. \tag{Ω.1}$$

We consider this local coordinates as the weighted coordinates with the weights $(2, 1, \cdots, 1)$. Then we make the following conditions on the principal part $P_m(x, \partial_x)$ of the operator (3.1).

(P.1) *Every weight of $a_\alpha(x)(\partial/\partial x)^\alpha$ in $P_m(x, \partial_x)$ is larger than or equal to $l-2m$=the weight of $(\partial/\partial x_1)^{m-1}(\partial/\partial x_2)^l$.*

(P.2) *For the term in P_m with the weight $l-2m$, its coefficient does not vanish at* 0, *that is*

$$\text{weight}[(a_\alpha(x)-a_\alpha(0))(\partial/\partial x)^\alpha] \geqq l-2m+1\,,$$

when $|\alpha|=m$ and especially $a_\alpha(0)=0$ if $\alpha=(m-l, l, 0, \cdots, 0)$ in the second terms of the right hand side of (3.1).

(P.3) *There exists an integer μ $(2 \leqq \mu \leqq n)$ such that the term in P_m with the weight $l-2m$ is generated only by $\partial/\partial x_1, \cdots, \partial/\partial x_\mu$.*

REMARK 3.1 If P is simple characteristic at $(0, N)$ with $N=(1, 0, \cdots, 0)$, then it is possible to choose the local coordinates such that P is in the form (3.1) with $l=1$ and $\alpha_1 < m-1$ in the sum of the second terms. In this case all conditions (P.1,2,3) with $\mu=2$ are automatically fulfilled.

REMARK 3.2 In P_m the condition (P.1) is only restrictive on the terms of the order larger than $m-l$ with respect to $\partial/\partial x_1$. Because by (P.1),

$$\text{weight}(a_\alpha(x)) \geqq \max\{0, l-2m+2\alpha_1+\alpha_2+\cdots+\alpha_n = l-m+\alpha_1\}\,.$$

REMARK 3.3 The conditions (P.1) and (P.2) imply that the term of the weight $l-2m$ in P_m is essentially of the form $a_\alpha(0)(\partial/\partial x)^\alpha$ with $\alpha_1=m-l$.

Concerning the boundary function $\rho(x)$ of $\partial\Omega$, we set

$$H=\left(\frac{\partial^2\rho}{\partial x_i\partial x_j}(0)\right) \qquad (2\leqq i, j\leqq n),$$

which is the tangential Hessian of ρ at 0. Then the following conditions are made in addition to $(\Omega.1)$.

$(\Omega.2)$ *H can be written as*

$$H=\begin{array}{c} \\ \mu-1\left\{\begin{array}{l}\lambda\{\\ \\ \end{array}\right. \\ \\ \end{array}\overset{\overbrace{\hphantom{A\ \ 0\ \ }}^{\mu-1}}{\overset{\overbrace{\hphantom{A}}^{\lambda}}{}}\left(\begin{array}{c|c|c} A & 0 & * \\ \hline 0 & 0 & 0 \\ \hline * & 0 & * \end{array}\right)$$

where A is strictly negative definite $(0<\lambda\leqq\mu-1)$.

We remark that if $\mu=2$ in (P.3), then $(\Omega.2)$ means only that $\partial^2\rho/\partial x_2^2(0)<0$. Such a case is happened when P is simple characteristic at $(0, N)$.

REMARK 3.4 It is easy to show that this condition $(\Omega.2)$ is independent of the choice of the defining function $\rho(x)$.

Now the basic theorem is as follows:

Theorem 3.1 *Let $P(x, \partial_x)$ be a differential operator of the form* (3.1) *which satisfies the conditions* (P. 1, 2, 3), *and Ω be an open set given by* (3.2) *with the conditions* $(\Omega.1, 2)$. *If $u(x)$ is a distribution solution of $Pu=0$ in U vanishing in Ω, then $u(x)$ must vanish in a neiborhood of* 0.

For the rest of this section, we devote ourselves to prove this theorem.

Lemma 3.1 *Let $\rho(x)$ be an defining function of Ω with the conditions* $(\Omega.1, 2)$. *Then by changing the defining function $\rho(x)$ if necessarily we may assume that*

$$(\Omega.3) \qquad \frac{\partial^2\rho}{\partial x_1\partial x_j}(0)=0 \qquad j=1, 2, \cdots, n\,.$$

in addition to $(\Omega.1, 2)$.

Proof. If we expand $\rho(x)$ to the second order, we have

$$\rho(x) = x_1 + (\sum_{j=1}^{n} a_j x_j) x_1 + \sum_{i,j\geqq 2} a_{ij} x_i x_j + o(|x|^2)$$

where $(a_{ij}) = \frac{1}{2} H$. Then $r(x) = \rho(x) \exp[-\sum_{j=1}^{n} a_j x_j]$ becomes the desired boundary function, which completes the proof.

Lemma 3.2 *If a real-valued C^2 function ρ satisfies the conditions* (Ω.1, 2 *and* 3), *then there exist positive constants α and M such that for any $\varepsilon > 0$ the following inequality holds in a sufficiently small neiborhood V of* 0.

$$\rho(x) \leqq x_1 - \alpha x_2^2 + \varepsilon(x_1^2 + x_3^2 + \cdots + x_\mu^2) + M(x_{\mu+1}^2 + \cdots + x_n^2). \tag{3.3}$$

Proof. We expand ρ in the Taylor series up to the second order. Then by (Ω.1 and 3),

$$\rho(x) = x_1 + \sum_{i,j\geqq 2} a_{ij} x_i x_j + o(|x|^2)$$

where $(a_{ij}) = \frac{1}{2} H$ satisfies (Ω.2). From (Ω.2), it is easy to derive the inequality (3.3). The details are omitted.

Set $\psi(x)$ as

$$\psi(x) = x_1 - \alpha x_2^2 + \varepsilon(x_1^2 + x_3^2 + \cdots + x_\mu^2) + M(x_{\mu+1}^2 + \cdots + x_n^2). \tag{3.4}$$

Then the above lemma showes that in some neiborhood V of 0, the open set $\{\psi(x) < 0\}$ is contained in Ω. Thus it is sufficient for the proof of the theorem 3.1 to obtain the uniqueness theorem across the surface $\psi(x) = 0$. For this purpose we construct the family of surfaces. Define $\phi(x)$ as

$$\phi(x) = x_1 - \frac{1}{2} \alpha r x_2 + 2\varepsilon(x_1^2 + x_3^2 + \cdots + x_\mu^2) + 2M(x_{\mu+1}^2 + \cdots + x_n^2), \tag{3.5}$$

where $r > 0$ is a parameter and determined later.

Lemma 3.3 *If s is real and $s \leqq \alpha r^2$, then the set $\{\psi(x) \geqq 0\} \cap \{\phi(x) \leqq s\}$ is compact and contained in $U(r)$, where*

$$\begin{aligned} U(r) = \{x \mid |x_1| < 2\alpha r^2,\ |x_2| < 2r, \\ |x_j| < (2\alpha/\varepsilon)^{1/2} r \qquad j = 3, \cdots, \mu \\ |x_k| < (2\alpha/M)^{1/2} r \qquad k = \mu+1, \cdots, n\} \end{aligned}$$

Proof. Set $R_\mu^2 = x_1^2 + x_3^2 + \cdots + x_\mu^2$ and $R_n^2 = x_{\mu+1}^2 + \cdots + x_n^2$. For any $x \in \{\psi(x) \geqq 0\} \cap \{\phi(x) \leqq s\}$ we have

$$2\alpha x_2^2 - 2x_1 \leqq 2\varepsilon R_\mu^2 + 2MR_n^2 \leqq s + \frac{1}{2} \alpha r x_2 - x_1,$$

which imply the next two inequalities:

$$x_1 \geqq 2\alpha x_2^2 - \frac{1}{2}\alpha r x_2 - s$$

$$s + \frac{1}{2}\alpha r x_2 - x_1 \geqq 0 .$$

Then it easily derived that $|x_1| < 2\alpha r^2$ and $|x_2| < 2r$ provided that $s \leqq \alpha r^2$. Using these estimates we have

$$0 \leqq \varepsilon R_\mu^2 + M R_n^2 < 2\alpha r^2 .$$

Thus the lemma is proved.

Now we determine the parameters ε and r so that the surface $\phi(x)=s$ is non-characteristic with respect to $P(x, \partial_x)$ in some neiborhood of 0. Let $Q(\partial_x)$ be the sum of the terms in P_m with the weight exactly $l-2m$. By the remark 3.3 and the condition (P.3), $Q(\partial_x)$ is expressed as followes:

$$Q(\partial_x) = \left(\frac{\partial}{\partial x_1}\right)^{m-l} \left(\frac{\partial}{\partial x_2}\right)^{l} + \sum a_\alpha \left(\frac{\partial}{\partial x}\right)^{\alpha} \tag{3.6}$$

where the summation is taken over the multi-indices α such that $\alpha_1 = m-l$, $\alpha_2 + \cdots \alpha_\mu = l$, $\alpha_2 < l$ and $\alpha_{\mu+1} = \cdots = \alpha_n = 0$. In (3.6) every a_α is a constant. If we set $\xi_j = \partial\phi/\partial x_j$ $(j=1, \cdots, n)$, then we have

$$\begin{aligned} \xi_1 &= 1 + 4\varepsilon x_1 \\ \xi_2 &= -\frac{1}{2}\alpha r \\ \xi_j &= 4\varepsilon x_j \qquad j = 3, \cdots, \mu \\ \xi_k &= 4M x_k \qquad k = \mu+1, \cdots, n . \end{aligned}$$

Therefore we have the next estimates on $U(r)$:

$$\begin{cases} \dfrac{1}{2} \leqq |\xi_1| \leqq 2 & \text{if } \varepsilon\alpha r^2 \leqq 4^{-2} \\ |\xi_2| = \dfrac{1}{2}\alpha r & \\ |\xi_j| \leqq 4(2\alpha\varepsilon)^{1/2} r & j = 3, \cdots, \mu \\ |\xi_k| \leqq 4(2\alpha M)^{1/2} r & k = \mu+1, \cdots, n . \end{cases} \tag{3.7}$$

Lemma 3.4 *If we take ε sufficiently small ($\varepsilon\alpha r^2 \leqq 4^{-2}$), then $Q(\xi)$ does not vanish on $U(r)$.*

Proof. We use the notation $C(\alpha)$ which is a different constant in each position depending only on α. By (3.7),

$$|\xi_1^{m-l}\xi_2^l| \geqq \left(\frac{1}{2}\right)^m \alpha^l r^l$$

$$|a_\alpha \xi^\alpha| \leqq C(\alpha)\varepsilon^{(1/2)(\alpha_3+\cdots+\alpha_\mu)} r^l ,$$

for $|\alpha|=m$, $\alpha_1=m-l$ and $\alpha_{\mu+1}=\cdots=\alpha_n=0$. If we put these estimates into the corresponding terms in (3,6), we have that

$$|Q(\xi)| \geqq \left\{\left(\frac{1}{2}\right)^m \alpha^l - C(\alpha)\varepsilon^{(1/2)(\alpha_3+\cdots+\alpha_\mu)}\right\} r^l .$$

Since $\alpha_3+\cdots+\alpha_\mu \neq 0$, $|Q(\xi)| \geqq C(\alpha)r^l$ with $C(\alpha)>0$ for a sufficiently small ε. This proves the lemma.

From now on, the constant ε is taken as in this lemma and always fixed. For the determination of the parameter r, we have the next lemma.

Lemma 3.5 *If we take r sufficiently small, then $P_m(x, \xi)$ does not vanish on $U(r)$.*

Proof. If the weight of an analytic function $a(x)$ is equal to k, then the inequality

$$\sup_{U(r)} |a(x)| \leqq \text{const. } r^k$$

holds for a sufficiently small r. Thus for a term $a(x)(\partial/\partial x)^\alpha$ in P_m with the weight larger than $l-2m$, the inequality

$$\begin{aligned}\text{weight } a(x) &\geqq l-2m+1+2\alpha_1+\alpha_2+\cdots+\alpha_n \\ &= l-m+\alpha_1+1\end{aligned}$$

implies

$$\begin{aligned}|a(x)\xi^\alpha| &\leqq \text{const. } r^{l-m+\alpha_1+1} \text{const. } r^{\alpha_2+\cdots+\alpha_n} \\ &= \text{const. } r^{l+1}\end{aligned}$$

While $|Q(\xi)| \geqq C(\alpha)r^l$ on $U(r)$. Since P_m is the sum of Q and the terms of the weight larger than $l-2m$, we can choose r sufficiently small so that P_m does not vanish on $U(r)$. This proves the lemma.

Under these preparations we now prove the basic theorem. The key lemma of this proof is the following one which is due to Hörmander [3].

Lemma 3.6 *Suppose that there exist a real-valued C^1 function $\phi(x)$ and constants s_0, s_1 such that in some neighborhood V of 0,*

(i) $P_m(x, \text{grad } \phi(x)) \neq 0$

(ii) $s_0 < \phi(0) < s_1$

(iii) $\{x \in V \mid \phi(x) \leqq s_1\} \cap \overline{\Omega^c}$ *is compact,*

(iv) $\{x \in V \mid \phi(x) \leqq s_0\} \cap \overline{\Omega^c}$ *is empty,*

(v) $\{x \in V \mid \phi(x) \leqq s_0\}$ *is not empty.*

Then every distribution solution $u(x)$ in V of the equation $Pu=0$ vanishing in Ω must vanish in $\{x\in V \mid \phi(x)<s_1\}$.

Proof of the Theorem 3.1 By the lemma 3.2 we may take Ω as the set $\{x\in V \mid \psi(x)<0\}$. Now take $U(r)$ in the lemma 3.5 as the neighborhood V of 0 in the lemma 3.6. Then the condition (i) is fulfilled. Set $s_0=-\alpha r^2$ and $s_1=\alpha r^2$. Then (ii) becomes trivial and (iii) is derived from the lemma 3.3. The other conditions (iv) and (v) are easily derived from the expression of $\phi(x)$ and $\psi(x)$, so we omitt their prooves. This ends the proof of the theorem 3.1.

4. Choice of the local coordinates in the basic theorem

Let $(x_1, \cdots, x_n)$ be the local coordinates such that the surface $x_1=0$ is tangent to $\partial\Omega$ at $x=0$. We consider this coordinates as the weighted coordinates with the weights $(2, 1, \cdots, 1)$. The other local coordinates with the same property become equivalent to this coordinates as the weighted coordinates.

Let $P(x, \partial_x)$ be a linear differential operator of order m with analytic coefficients which is characteristic at 0 in the cotangential direction $N=(1, 0, \cdots, 0)$. We set l the multiplicity of P at $(0, N)$. That is, for a cotangent vector $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_n)$,

$$P_m(0, N+t\zeta) = L(\zeta)t^l + \text{higher order terms of } t \tag{4.1}$$

where P_m is the principal part of P and $L(\zeta)$ is a non-zero polynomial of ζ. This polynomial $L(\zeta)$ is called the localization of P_m at $(0, N)$, which is originally introduced by Hörmander [4]. When $N=(1, 0, \cdots, 0)$, (4.1) means that in $P_m(0, \partial_x)$ there is none of the terms of order larger than $m-l$ with respect to $\partial/\partial x_1$ and the sum of the coefficients of $(\partial/\partial x_1)^{m-l}$ is equal to $L(\partial/\partial x_2, \cdots, \partial/\partial x_n)$. Therefore $L(\zeta)$ is a homogeneous polynomial of degree l in the variables $(\zeta_2, \cdots, \zeta_n)$. Since the weight of $L(\partial/\partial x_2, \cdots, \partial/\partial x_n)(\partial/\partial x_1)^{m-l}$ is equal to $l-2m$, we make the assumption:

(P.I) *the weight of $P_m(x, \xi)$ is equal to $l-2m$, if the weight of ξ are assigned by $(-2, -1, \cdots, -1)$.*

Relating to the localization $L(\zeta)$ of P_m, we introduce some linear spaces in the tangent space T_0 and the cotangent space T_0^* of the surface $\partial\Omega$ at 0. For the polynomial $L(\zeta)$, we set

$$\Lambda^*(L) = \{\eta\in T_0^* \mid L(\xi+\eta t) = L(\xi) \quad \text{for all } t \text{ and } \xi\}, \tag{4.2}$$

which is a linear subspace, and we introduce the annihilator

$$\Lambda(L) = \{v\in T_0 \mid \langle v, \eta\rangle = 0 \quad \text{for any } \eta\in\Lambda^*(L)\}, \tag{4.3}$$

where $\langle\ ,\ \rangle$ denotes the contraction between cotangent vectors and tangent vectors. $\Lambda(L)$ is the smallest subspace along which $L(\partial/\partial x)$ operates and is called the bicharacteristic space of P at $(0, N)$. These subspaces are introduced by Hörmander [4].

DEFINITION 4.1 An analytic function $\phi(x)$ with grad $\phi(0)=N$ is said to be a weighted characteristic function of $P(x, \partial_x)$ if it satisfies the following condition:

$$\text{weight } P_m(x, t \operatorname{grad}\phi(x)) \geqq l-2m+1\,, \tag{4.4}$$

where the parameter t is assigned the weight -2.

To find such a weighted characteristic function $\phi(x)$, it is sufficient that ϕ is in the form

$$\phi(x) = x_1 + \sum_{i,j\geqq 2} a_{ij}x_i x_j\,.$$

Assume that

(P.II) *there exists a weighted characteristic function* $\phi(x)$.

By the suitable equivalent change of the weighted coordinates we can assume that $\phi(x)=x_1$. Then the following proposition is easy to prove.

Proposition 4.1 *If* $\phi(x)=x_1$, *then* (4.4) *is equivalent to that there is none of the differential monomials of the weight* $l-2m$ *in* $P_m(x, \partial_x)$ *which is generated only by* $\partial/\partial x_1$.

We now fix some weighted characteristic function $\phi(x)$ and consider the local coordinates $(x_1, \cdots, x_n)$ as $\phi(x)=x_1$ (mod weight 3) and each x_j $(j=2, \cdots, n)$ has the weight 1. This means that the coordinates transformation considered from now on is in the following form:

$$\begin{cases} u_1 = x_1 + \text{an analytic function of the weight } \geqq 3\,, \\ u_j = \sum_{k=2}^{n} c_{jk}x_k + \text{an analytic function of the weight } \geqq 2 \\ \qquad\qquad\qquad\qquad j = 2, \ldots, n\,. \end{cases} \tag{4.5}$$

Then we make the last assumption on P_m such that

$$\text{weigh } [P_m(x, \xi) - P_m(0, \xi)] \geqq l-2m+1\,. \tag{P.III}$$

Proposition 4.2 *This assumption* (P.III) *is invariant under the change of variables of form* (4.5).

Proof. If we remark that by (4.5),

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial u_1} + \text{ terms of the weight larger than } -2$$

$$\frac{\partial}{\partial x_j}=\sum_{k=1}^{n} c_{kj}\frac{\partial}{\partial u_k}+\text{terms of the weight larger than } -1,$$
$$j=2,\cdots,n$$

the invariance of (P.III) is easy to prove.

Assumption (P.III) means that the terms with the lowest weight in P_m do not degenerate at 0.

Now we proceed to examine the conditions on Ω under the assumptions (P. I, II, III).

If Ω is given by $\{\rho(x)<0\}$ with a real-valued C^2 function ρ, we denote by H_ρ the tangential Hessian of ρ at 0. That is

$$H_\rho=\left(\frac{\partial^2\rho}{\partial x_i\partial x_j}(0)\right) \qquad 2\leqq i,j\leqq n\,.$$

Then it easily derived that the symmetric bilinear form $\sum_{i,j\geqq 2}\frac{\partial^2\rho}{\partial x_i\partial x_j}(0)\,dx_i\otimes dx_j$ on $T_0\times T_0$ is invariant under the transformation of the coordinates of the form (4.5). We set N_ρ as the kernel of the linear map H_ρ: $T_0\rightarrow T_0^*$ defined by

$$H_\rho(v)=\sum\frac{\partial^2\rho}{\partial x_i\partial x_j}(0)\langle dx_i,\,v\rangle dx_j\,.$$

Then H_ρ is derived to the bilinear form on the space $T_0/N_\rho\times T_0/N_\rho$. Similary if we set

$$\tilde{\Lambda}(L)=\Lambda(L)/N_\rho\cap\Lambda(L)\,,$$

where $\Lambda(L)$ is the bicharacteristic space of P, H_ρ is also derived to the bilinear form on $\tilde{\Lambda}(L)\times\tilde{\Lambda}(L)$.

The first assumption on $\partial\Omega$ is as follows:

(Ω.I) $\tilde{\Lambda}(L)\neq 0$ *and* H_ρ *is strictly negative definite on* $\tilde{\Lambda}(L)\times\tilde{\Lambda}(L)$.

This condition means that Ω is concave in the direction of the bicharacteristic space at 0.

Lastly we demand that $L(\zeta)$, the localization of P, is non-characteristic at some covector ξ_0 for which Ω is strictly concave at 0. For this sake we introduce N_ρ^* the annihilator of N_ρ,

$$N_\rho^*=\{\xi\in T_0^*\,|\,\langle\xi,v\rangle=0 \qquad \forall v\in N_\rho\}\,.$$

Then we assume

(Ω.II) *there exists a covector* ξ_0 *in* N_ρ^* *such that* $L(\xi_0)\neq 0$.

Now we construct the local coordinates $(x_1,\cdots,x_n)$ so that the operator

$P(x, \partial_x)$ is reduced to the form (3.1) and all assumptions in the basic theorem are satisfied.

First we fix the weighted characteristic function $\phi(x)$ in (P.II) and set $\phi(x)=x_1$.

Secondly we choose the tangential coordinates $(x_2, \cdots, x_n)$ such that the vectors $\partial/\partial x_2, \cdots, \partial/\partial x_\mu$ span the bicharacteristic space $\Lambda(L)$ at 0. Since $L(\zeta)$ does not vanish identically, the dimension of $\Lambda(L)$ is $\mu-1$ which becomes positive (i.e. $\mu\geqq 2$).

Thirdly under the suitable linear change of variables $(x_2, \cdots, x_\mu)$, we may assume that for some λ $(2<\lambda\leqq\mu)$, $\partial/\partial x_{\lambda+1}, \cdots, \partial/\partial x_\mu$ span the subspace $N_\rho \cap \Lambda(L)$. At this moment, the condition (Ω.1) means that the matrix

$$\left(\frac{\partial^2 \rho}{\partial x_i \partial x_j}(0)\right) \qquad 2\leqq i, j\leqq \lambda$$

is strictly negative definite and

$$\frac{\partial^2 \rho}{\partial x_i \partial x_j}(0) = 0 \quad \text{if } \lambda+1\leqq i \text{ or } j\leqq \mu.$$

Lastly we shall prove that $L(\zeta)$ can be non-characteristic at dx_2 by the linear change of variables $(x_2, \cdots, x_\lambda)$.

Proposition 4.3 *By the suitable linear change of variables* $(x_2, \cdots, x_\lambda)$, $L(\zeta)$ *is non-characteristic at the covector* dx_2.

Proof. Let $\xi_0 \in T_0^*$ be the covector in the condition (Ω.II). Since $\langle \xi_0, \partial/\partial x_j\rangle=0$ $(j=\lambda+1, \cdots, \mu)$, ξ_0 is written as

$$\begin{aligned}\xi_0 &= c_2 dx_2+\cdots+c_\lambda dx_\lambda+c_{\mu+1}dx_{\mu+1}+\cdots+c_n dx_n\\ &= c_2 dx_2+\cdots+c_\lambda dx_\lambda+\xi_0'\end{aligned}$$

where $\xi_0' \in \Lambda^*(L)$ which is generated by $dx_{\mu+1}, \cdots, dx_n$. Then

$$\begin{aligned}0\neq L(\xi_0) &= L(c_2 dx_2+\cdots+c_\lambda dx_\lambda+\xi_0')\\ &= L(c_2 dx_2+\cdots+c_\lambda dx_\lambda)\,.\end{aligned}$$

Therefore this proposition is easily derived from this relation.

From this proposition, $L(\partial/\partial x)$ is written as

$$L(\partial_x) = a\left(\frac{\partial}{\partial x_2}\right)^l+\sum a_\alpha\left(\frac{\partial}{\partial x}\right)^\alpha$$

where $a\neq 0$ and the summation is taken over the multi-indices $|\alpha|=l$, $\alpha_2<l$, $\alpha_1=\alpha_{\mu+1}=\cdots=\alpha_n=0$.

Thus the operator $P(x, \partial_x)$ is expressed in the form (3.1) under this coordinates. Since (P.I) implies (P.1), (P.III) implies (P.2), (P.3) follows from the fact that $\partial/\partial x_2, \cdots, \partial/\partial x_\mu$ span $\Lambda(L)$, (Ω.1) is trivial from the choice of x_1 and (Ω.2) is derived from (Ω.I), all conditions in the theorem 3.1 are satisfied under this coordinates. Summing up these results we have the final theorem:

Theorem 4.1 *Let $P(x, \partial_x)$ be a differential operator of order m with analytic coefficients in a neighborhood V of p and Ω be an open set with C^2 boundary $\partial\Omega \in p$. We suppose that P and Ω satisfy the conditions (P. I, II, III) and (Ω. I, II). Then every distribution solution $u(x)$ of $Pu=0$ in V vanishing in Ω must vanish in a neighborhood of p.*

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