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# ON THE SPECTRUM OF A RIEMANNIAN MANIFOLD OF POSITIVE CONSTANT CURVATURE II

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**Intoroduction.** A complete connected riemannian manifold of positive constant curvature 1 is called a Clifford-Klein spherical space form. In this paper, we call such a riemannian manifold simply a *spherical space form*. An $n$-dimensional spherical space form is obtained as $S^n/G$, the standard unit sphere $S^n$ modulo a finite group of fixed point free isometries $G$. In the previous paper [4], we raised the problem,

"*Does the spectrum of a spherical space form determine the spherical space form among all spherical space forms?*"

The above problem was solved affirmatively first for a 3-dimensional lens space $M$ by Tanaka [6] in case where the order of fundamental group of $M$, $|\pi_1(M)|$ is prime or 2-times prime, by the author and Yamamoto in case where $|\pi_1(M)|$ is prime power or 2-times prime power, and by Yamamoto [11] in case where $|\pi_1(M)|$ is any composite number. For any other 3-dimensional spherical space forms and homogeneous space forms, it was also affirmatively solved recently by the author [5].

In this paper, we shall attack the above problem for the spherical space form with dimension of the form $4k+1$ ($k\geqq 1$). Main tool in this paper is also the generating function associated to the spectrum of a spherical space form $S^{4k+1}/G$ constructed in [4] and [5]. The relations between the finite group $G$ and the generating function were studied in [5]. Here, we investigate the relations more details for $(4k+1)$-dimensional spherical space forms using the complete classification of the manifolds due to Vincent [9] (see also [10]).

Our main results are the followings.

**Theorem 3.1.** *Let $d$ be an odd prime. Let $M$, $N$ be $(2d-1)$-dimensional spherical space forms. Suppose $M$ is isospectral to $N$. Then their fundamental groups are isomorphic.*

**Theorem 3.9.** *Let $M$ be a 5-dimensional spherical space form with non-*

*cyclic fundamental group. Let $N$ be a compact connected riemannian manifold. Suppose $N$ is isospectral to $M$. Then they are isometric.*

## 1. Spherical space forms and their generating functions

In this section, we recall some properties of the generating function of a spherical space form obtained in the previous paper [5]. We state their properties without proofs. The proofs should be referred to [5].

Let $S^n$ $(n\geqq 2)$ be the unit sphere centered at the origin in $R^{n+1}$ the $(n+1)$-dimensional Euclidean space. We denote by $O(n+1)$ (resp. $SO(n+1)$) the orthogonal (resp. the special orthogonal) group acting on $R^{n+1}$. A finite subgroup $G$ of $O(n+1)$ is said to be *fixed point free* if for any $g\in G$ with $g\neq 1_{n+1}$ (the unit matrix in $O(n+1)$) 1 is not an eigenvalue of $g$. A fixed point free finite subgroup $G$ of $O(n+1)$ acts on $S^n$ fixed point freely, so that the quotient manifold $S^n/G$ becomes a riemannian manifold of positive constant curvature 1 in a natural way. Conversely, any compact connected riemannian manifold of positive constant curvature 1 is obtained in this way. We call a compact riemannian manifold of positive constant curvature 1 *a spherical space form.*

Fundamental properties for spherical space forms are

**Proposition 1.1** (see [5]). 1. *Even dimensional spherical space forms are only the standard spheres and the real projective spaces.*
2. *A finite fixed point free subgroup $G$ of $O(2n)$ is contained in $SO(2n)$.*
3. *Let $S^n/G$ and $S^n/G'$ be spherical space forms. Then they are isometric if and only if $G$ is conjugate to $G'$ in $O(n+1)$.*

In the followings, we shall consider only odd dimensional spherical space forms of dimension greater than 3. Let $M=S^{2n-1}/G$ $(n\geqq 2)$ be a $(2n-1)$-dimensional spherical space form and $\Delta$ the Laplacian acting on the space of smooth functions on $M$. Then each eigenvalue of $\Delta$ on $M$ is of the form $k(k+2n-2)$ $(k=0, 1, 2, \cdots)$ and the eigenspace $E_{k(k+2n-2)}$ for eigenvalue $k(k+2n-2)$ is isomorphic to the space of $G$-invariant elements in the eigenspace $H_k$ of the Laplacian on $S^n$ for eigenvalue $k(k+2n-2)$. Then the generating function associated to the spectrum of the Laplacian on $M$ is defined by

$$F_G(z)=\sum_{k=0}^{\infty}(\dim E_{k(k+2n-2)})z^k.$$

By the definition, we have

**Proposition 1.2.** *Let $S^{2n-1}/G$ and $S^{2n-1}/G'$ be spherical space forms. Then $S^{2n-1}/G$ is isospectral to $S^{2n-1}/G'$ if and only if $F_G(z)=F_{G'}(z)$.*

An important formula for the generating function $F_G(z)$ of spherical space form $S^{2n-1}/G$ is given by

**Theorem 1.3.** *We have*

$$F_G(z) = \frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\det(1_{2n}-gz)} = \frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\prod_{\gamma\in E(g)}(z-\gamma)},$$

*where $|G|$ is the order of $G$ and $E(g)$ is the set of eigenvalues of $g$, with multiplicity counted.*

By the above theorem, we see the generating function $F_G(z)$ has a unique meromorphic extension to the whole complex plane $\boldsymbol{C}$. Moreover, $F_G(z)$ is a rational function on $\boldsymbol{C}$, and has a zero at infinity.

DEFINITIONS. Let $G$ be a finite group. The subset $G_k$ of the finite group $G$ consists of *all elements of order $k$ in $G$*, so that $G=\bigcup_k G_k$ (disjoint union). $\sigma(G)$ is *the set of positive integers consisting of orders of elements in $G$*.

For a spherical space form $S^{2n-1}/G$ and for $k\in\sigma(G)$,

$$F_G^k(z) = \sum_{g\in G_k}\frac{1-z^2}{\det(1_{2n}-gz)}.$$

Then we have the following two Propositions.

**Proposition 1.4.** *$F_G^k(z)$ has a pole at any primitive $k$-th root of* 1, *but no poles eleswhere.*

**Proposition 1.5.** *Let $S^{2n-1}/G$ and $S^{2n-1}/G'$ be spherical space forms. Suppose they are isospectral. Then*

1. $|G|=|G'|$,
2. $\sigma(G)=\sigma(G')$, *particularly* max. $\sigma(G)=$max. $\sigma(G')$,
3. $F_G^k(z)=F_{G'}^k(z)$ *for any* $k\in\sigma(G)$.

Proof. Only the proof of 3 is not given in [5]. By the assumption, for any $k\in\sigma(G)$ we have

$$F_G^k(z)-F_{G'}^k(z) = \sum_{\substack{l\in\sigma(G)\\ l\neq k}}(F_{G'}^l(z)-F_G^l(z)).$$

Applying Proposition 1.4 to the identity, we can see easily that the rational function $F_G^k(z)-F_{G'}^k(z)$ is holomorphic on the whole complex plane $\boldsymbol{C}$. Clearly this function has a zero at infinity, so that the function is identically zero. Thus we have proved 3. q.e.d.

## 2. Vincent's results for spherical space forms

The complete classification of 3-dimensional spherical space forms was obtained by Seifert and Threlfall [8]. In this section we shall state Vincent's results [9] on the classification of spherical space forms for higher dimensional cases, mainly according to Wolf [10].

Definitions. A finite dimensional orthogonal representation of a finite group is *fixed point free if it is faithful and its image is a fixed point free subgroup of the orthogonal group*. A finite group is called *a fixed point free group if it has a fixed point free representation.*

Remarks. A fixed point free representation is decomposed into the sum of irreducible representations, all of which are fixed point free. The sum of fixed point free representations is fixed point free. For a fixed point free subgroup $G\subset SO(2n)$, $G$ is considered as the image of the natural representation $i_G$ of the abstract group $G$, so that the abstract group $G$ is fixed point free.

**Proposition 2.1.** *Let $G_1$, $G_2$ be fixed point free subgroups of $SO(2n)$. Then $G_1$ is conjugate to $G_2$ in $O(2n)$ if and only if $G_1$ is isomorphic to $G_2$ and there is an automorphism $\alpha$ of $G_1$ such that $i_{G_1}\circ\alpha$ is equivalent to $i_{G_2}$, identifying $G_1$ with $G_2$.*

Under these considerations, Vincent reduced the classification problem to the followings (for details see [9], [10]);

1. To determine finite groups which admit fixed point free representations.
2. To determine irreducible fixed point free representations of a finite fixed point free group.
3. To determine all the automorphisms of a fixed point free finite group $G$ and their actions on irreducible fixed point free representations of $G$.

Fixed point free groups are divided into following two classes;
First type: Every Sylow subgroup of $G$ is cyclic.
Second type: Every $p$-Sylow subgroup ($p\neq 2$) of $G$ is cyclic and the 2-Sylow subgroups of $G$ are generalized quoternion groups $Q2^a$ ($a>2$), where $Q2^a$ is the finite group of order $2^a$ generated by two elements $A$ and $B$ with relations;

$$A^{2^{a-1}}=1,\ A^{2^{a-2}}=B^2 \quad\text{and}\quad BAB^{-1}=A^{-1}.$$

In [9], Vincent determined finite fixed point free groups of First type and their fixed point free representations.

For any non-zero integer $m$, $K_m$ denotes the multiplicative group of residue classes modulo $m$ of integers prime to $m$. The order of $K_m$ is denoted by $\phi(m)$. Let $m$, $n$ be positive integers and $r$ be an integer with $((r-1)n, m)=1$, $r^n\equiv 1 \pmod m$. We consider the finite group of order $N=mn$ generated by two elements $A$ and $B$ with defining relations

$$A^m = B^n = 1, \quad BAB^{-1} = A^r,$$

Let $d$ be the order of the residue class of $r$ in $K_m$ and $n=n'd$. Assume $n'$ is divisible by every prime divivor of $d$. Then this finite group is denoted by $\Gamma_d(m, n, r)$. Note that the following four conditions are equivalent for the $\Gamma_d(m, n, r)$: (i) $\Gamma_d(m, n, r)$ is cyclic, (ii) $A=1$, (iii) $r=1$, and (iv) $d=1$. We shall determine the automorphisms of $\Gamma_d(m, n, r)$. Whenever $s$, $t$ and $u$ are integers with $(s, m)=1=(t, n)$ and $t\equiv 1 \pmod d$, we put

$$\psi_{s,t,u}(A) = A^s \quad \text{and} \quad \psi_{s,t,u}(B) = B^t A^u .$$

Then we can see easily $\psi_{s,t,u}$ defines an autormorphism of $\Gamma_d(m, n, r)$ onto itself.

**Proposition 2.2** (cf. Wolf [10]). *The automorphisms of $\Gamma_d(m, n, r)$ are just the $\psi_{s,t,u}$'s.*

Proof. Let $\alpha$ be an automorphism of $\Gamma_d(m, n, r)$. Since the commutator subgroup $\Gamma'$ of $\Gamma_d(m, n, r)$ is the cyclic subgroup $\{A\}$ generated by $A$, $\alpha(A)=A^s$ for some integer $s$ with $(s, m)=1$. Let $\alpha(B)=B^tA^u$. Since $B\Gamma'$ generates the quotient group $\Gamma_d(m, n, r)/\Gamma'$, we have $(t, n)=1$. Then $\alpha$ is an automorphism if and only if $\alpha(A^r)=\alpha(BAB^{-1})=\alpha(B)\alpha(A)\alpha(B)^{-1}$. On the other hand, $\alpha(B)\alpha(A)\alpha(B)^{-1}=B^tA^uA^sA^{-u}B^{-t}=A^{sr^t}$. Hence, $sr\equiv sr^t \pmod m$. This means $t\equiv 1 \pmod d$. q.e.d.

In the same way as the above, we have

**Proposition 2.3.** *The group $\Gamma_d(m, n, r_1)$ is isomorphic to the group $\Gamma_d(m, n, r_2)$ if and only if there exists an integer $c$ such that $r_1\equiv r_2^c \pmod m$.*

**Theorem 2.4** (see [9], [10]). *A finite fixed point free group of First type is isomorphic to some $\Gamma_d(m, n, r)$.*

**Theorem 2.5** (see [9], [10]). *Let $G=\Gamma_d(m, n, r)$, and let $R(\theta)$ denotes the rotational matrix on the plane:*

$$R(\theta)=\begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

*Given integers $k$ and $l$ with $(k, m)=1=(l, n)$, let $\pi_{k,l}$ be the real representation of degree $2d$ of $G$ defined by*

$$\hat{\pi}_{k,l}(A)=\begin{pmatrix} R(k/m) & & & \\ & R(kr/m) & & 0 \\ & & \ddots & \\ & 0 & & R(kr^{d-1}/m) \end{pmatrix}$$

*and*

$$\hat{\pi}_{k,l}(B)=\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ R(l/n') & 0 & & 0 \end{pmatrix}.$$

*Then $\hat{\pi}_{k,l}$ is irreducible and a real representation of $G$ is fixed point free if and only if it is equivalent to a sum of these representations $\hat{\pi}_{k,l}$. $\hat{\pi}_{k,l}$ is equivalent to $\hat{\pi}_{k',l'}$ if and only if there exist numbers $e=\pm 1$ and $c=0, 1, \cdots, d-1$ such that $k'\equiv ekr^c \pmod m$ and $l'\equiv el \pmod{n'}$. $\hat{\pi}_{k,l}\circ\psi_{s,t,u}$ is equivalent to $\hat{\pi}_{sk',tl'}$, where $\psi_{s,t,u}$ is the automorphism of $G$ defined before. There are just $\phi(N)/d^2$ or just $\phi(N)/2d^2$ inequivalent $\hat{\pi}_{k,l}$ according to whether $n'=2$ or $n'\neq 2$.*

REMARK. Any irreducible fixed point free representation of $\Gamma_d(m,n,r)$ has *the same degree* $2d$.

**Theorem 2.6** (see [10]). *The fundamental group of every spherical space form with dimension of the form $4k+1$ is of First type.*

From these theorems we have

**Corollary 2.7.** *Let $d$ be an odd prime. Then every non-cyclic finite fixed point free subgroup of $SO(2d)$ is conjugate to the image of an irreducible fixed point free representation of some $\Gamma_d(m,n,r)$.*

**Corollary 2.8.** *Let $G$ and $G'$ be finite fixed point free subgroups of $SO(6)$. Assume $G$ is isomorphic to $G'$ and is not of cyclic. Then $G$ is conjugate to $G'$ in $O(6)$.*

Proof. By Corollary 2.7, $G$, $G'$ are isomorphic to a $\Gamma_3(m,n,r)$. Hence, it suffices to show that for any two irreducible finite fixed point free representations $\hat{\pi}_{k,l}$, $\hat{\pi}_{k',l'}$ of $\Gamma_3(m,n,r)$, there exists an automorphism $\alpha$ of $\Gamma_3(m,n,r)$ such that $\hat{\pi}_{k,l}\circ\alpha$ is equivalent to $\hat{\pi}_{k',l'}$. We fix integers $k^*$, $l^*$ such that $k^*k\equiv 1 \pmod m$ and $l^*l\equiv 1 \pmod{n/3}$. Take the automorphism $\alpha=\psi_{k^*k',l^*l',0}$ or $\psi_{-k^*k',-l^*l',0}$ according to whether $l^*l'\equiv 1 \pmod 3$ or $l^*l'\equiv -1 \pmod 3$, respectively. By Theorem 2.5, $\hat{\pi}_{k,l}\circ\alpha$ is equivalent to $\hat{\pi}_{k',l'}$ or $\hat{\pi}_{-k',-l'}$, respectively. It follows easily from Theorem 2.5 that $\hat{\pi}_{k',l'}$ is equivalent to $\hat{\pi}_{-k',-l'}$. Hence the representation $\hat{\pi}_{k,l}\circ\alpha$ is equivalent to $\hat{\pi}_{k',l'}$. This completes the proof of the corollary. q.e.d.

## 3. Isospectral problem for $(2d-1)$-dimensional spherical space forms, where $d$ is an odd prime

Our main goal in this section is to prove

**Theorem 3.1.** *Let $d$ be an odd prime, and let $M$, $N$ be $(2d-1)$-dimensional spherical space forms. Suppose $M$ is isospectral to $N$. Then their fundamental groups are isomorphic.*

Let $m$ be a positive integer. We denote by $Z_m$ *the ring of residue classes modulus $m$ of integers.* For any real number $x$, we denote by $[x]$ *the largest integer which does not exceed $x$.*

**Lemma 3.2.** *Let $p$ be a prime number and $k$ a positive integer. Let $q$ be an integer with $q\equiv\pm 1 \pmod p$. Then the order of the residue class of $q$ in $K_{p^k}$ is $p^t$ or $2p^t$ for some integer $t\geqq 0$.*

Proof. If $p=2$, then the lemma is clear. Suppose $p$ is an odd prime. Let $q=lp\pm 1$, where $l=(q\mp 1)/p$. It suffices to show that $q^{p^{k-1}}\equiv\pm 1 \pmod{p^k}$. We have

$$(lp\pm 1)^{p^{k-1}}=(lp)^{p^{k-1}}+\sum_{t=1}^{p^{k-1}-1}\binom{p^{k-1}}{t}(\pm lp)^t\pm 1\,,$$

where $$\binom{p^{k-1}}{t}=\frac{p^{k-1}(p^{k-1}-1)\cdots(p^{k-1}-t+1)}{t(t-1)(t-2)\cdots 2\cdot 1}\,.$$

It is easy to see $p^{k-1}\geqq k$. We shall show $\binom{p^{k-1}}{t}$ is divisible by $p^{k-1}$. Let $\alpha$ be the largest integer such that $p^\alpha$ is a divisor of $t(t-1)\cdots 2\cdot 1$. Then we have

$$\begin{aligned}\alpha&=[t/p]+[t/p^2]+\cdots\\ &\leqq[t/p+t/p^2+\cdots]\\ &\leqq[t/(p-1)]\\ &\leqq t-1\,.\end{aligned}$$

Hence, $\binom{p^{k-1}}{t}$ is divisible by $p^{k-t}$. Thus we have

$$q^{p^{k-1}}=(lp\pm 1)^{p^{k-1}}\equiv\pm 1 \pmod{p^k}\,,$$

which implies the lemma. q.e.d.

**Lemma 3.3.** *Let $m$, $r$ be positive integers with $(r(r-1), m)=1$ and $d$ the order of $r$ in $K_m$. Suppose $d$ is an odd prime with $(d, m)=1$. Then the integers $\pm r^j\pm 1$ $(j=1, 2, \cdots, d-1)$ are all prime to $m$.*

Proof. By the assumption, it is easy to see that $m$ is odd. Let $m=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, where $p_1, \cdots, p_k$ are odd primes with $p_1<p_2<\cdots<p_k$ and $\alpha_i \geqq 1$ $(i=1, \cdots, k)$. Then an isomorphism between the rings $Z_m$ and $Z_{p_1^{\alpha_1}} \times \cdots \times Z_{p_k^{\alpha_k}}$ is given by

$$x \mapsto ((x \bmod p_1^{\alpha_1}), \cdots, (x \bmod p_k^{\alpha_k})) .$$

This isomorphism induces an isomorphism of the group $K_m$ onto $K_{p_1^{\alpha_1}} \times \cdots \times K_{p_k^{\alpha_k}}$. Since $r-1$ is prime to $m$, $r-1$ is also prime to any prime divisor of $m$. Hence, $r^t$ is also of order $d$ in $K_{p_i^{\alpha_i}}$ $(i=1, \cdots, k)$. Thus, for any integer $t$ prime to $d$, $r^t$ is of order $d$ in $K_{p_i^{\alpha_i}}$ $(i=1, \cdots, k)$. Together this fact with Lemma 3.2, we see that $\pm(r^j \pm 1)$ $(j=1, \cdots, d-1)$ are all prime to $p_i$ $(i=1, \cdots, k)$, because $d$ is prime to $p_i$ $(i=1, \cdots, k)$. q.e.d.

REMARK. In the proof of the above lemma, we see also that for any prime divisor $p$ of $m$, $p-1$ is divisible by $d$, because $\phi(p^\alpha)=p^{\alpha-1}(p-1)$.

The following lemma due to W. Franz [3].

**Lemma 3.4** (cf. Franz [3]). *Let $m$ be a positive integer, $\{a_n\}_{n\in K_m}$ a set of integers indexed by $K_m$, and let $\gamma$ be a primitive $m$-th root of* 1. *Suppose*

(i) $a_n=a_{-n}$ *for any* $n \in K_m$,

(ii) $\sum_{n\in K_m} a_n=0$,

(iii) $\prod_{n\in K_m} (1-\gamma^{kn})^{a_n}=1$ *for any integer* $k \not\equiv 0 \pmod m$.

*Then we have* $a_n=0$ *for all* $n \in K_m$.

Recall that $\Gamma_d(m, n, r)$ is the finite group generated by two elements $A$, $B$ with relations:

$$A^m = B^n = 1, \quad BAB^{-1} = A^r$$
$$(n(r-1),m) = 1, \quad r^n \equiv 1 \pmod m$$
$$d \text{ is the order of } r \text{ in } K_m \text{ and } n=n'd .$$

In the followings, we assume $d$ is an odd prime.

Let $\hat{\pi}_{k,l}$ be the fixed point free irreducible real representation of $\Gamma_d(m, n, r)$ as in Theorem 2.5. Let $G=\hat{\pi}_{k,l}(\Gamma_d(m, n, r))$ and $G_0$ the cyclic subgroup of $G$ generated by $\hat{\pi}_{k,l}(AB^d)$.

**Lemma 3.5.** *The order of any element in $G-G_0$ is a divisor of $n$. Particularly, it is prime to $m$.*

Proof. Any element in $G-G_0$ is represented as $\hat{\pi}_{k,l}(A^s B^{td+j})$ where $t$, $j$, $s$ are integers with $0<j<d$. we have

$$
\begin{aligned}
(A^s B^{td+j})^n &= A^{s(1+r^j+r^{2j}+\cdots+r^{(n-1)j})} B^{n(td+j)} \\
&= A^{s(1+r^j+r^{2j}+\cdots+r^{(n-1)j})}.
\end{aligned}
$$

Since $(d, j)=1=(r-1,m)$, we have

$$
\begin{aligned}
(1+r^j+r^{2j}+\cdots+r^{(n-1)j}) &\equiv n'(1+r+r^2+\cdots+r^{d-1}) \qquad (\mathrm{mod}\ m) \\
&\equiv 0 \qquad (\mathrm{mod}\ m)\,.
\end{aligned}
$$

Thus, we have $\hat{\pi}_{k,l}((A^s B^{td+j})^n)=1_{2d}$, which implies tbe lemma. q.e.d.

**Corollary 3.6.** $\sigma(G)=\{k$: *k is a divisor of* $mn'$ *or* $n\}$.

Let $m$ be a positive integer, and let $r_1$, $r_2$ be integers prime to $m$. Suppose that $r_1$, $r_2$ are of the same order $d$ in $K_m$. We consider the cyclic groups $G_i$ $(i=1, 2)$ generated by

$$
(3.1) \qquad g_i = \begin{pmatrix} R(1/m) & & & 0 \\ & R(r_i/m) & & \\ & & \ddots & \\ 0 & & & R(r_i^{d-1}/m) \end{pmatrix} \qquad (i = 1, 2);
$$

$$
G_i = \{g_i^k\}_{k=1}^m \qquad (i = 1, 2)\,.
$$

Then $G_1$, $G_2$ are fixed point free finite subgroups of $SO(2d)$ and these yield the lens spaces $L(m$: $1, r_i, \cdots, r_i^{d-1})$ $(i=1, 2)$ (see [5]).

**Theorem 3.7.** *Let* $G_1$, $G_2$ *be as in the above. Suppose the spherical space form* $S^{2d-1}/G_1$ *is isospectral to* $S^{2d-1}/G_2$. *Then they are isometric.*

Proof. Let $F_{G_1}(z)$, $F_{G_2}(z)$ be the generating functions of $S^{2d-1}/G_1$ and $S^{2d-1}/G_2$, respectively. Then

$$
F_{G_i}(z) = \frac{1}{m}\sum_{k=1}^{m} \frac{1-z^2}{(z-\gamma^k)(z-\gamma^{-k})(z-\gamma^{kr_i})\cdots(z-\gamma^{kr_i^{d-1}})(z-\gamma^{-kr_i^{d-1}})}\,,
$$

where $\gamma$ is a primitive $m$-th root of 1.
By Lemma 3.3, $\pm(r_i^t\pm 1)$ $(0<t<f,\ i=1, 2)$ are all prime to $m$. Hence, the order of pole of $F_{G_i}(z)$ at any $m$-th root $z$ of 1 $(z\neq 1)$ is 1. We compute the residue of $F_{G_i}(z)$ $(i=1, 2)$ at $z=\gamma^k$ $(k\not\equiv 0\ (\mathrm{mod}\ m))$,

$$
\lim_{z\to\gamma^k}(z-\gamma^k)F_{G_i}(z) = \frac{1}{m}\,\frac{2d(1-\gamma^{2k})}{(\gamma^k-\gamma^{-k})\prod_{t=1}^{d-1}(\gamma^k-\gamma^{kr_i^t})(\gamma^k-\gamma^{-kr_i^t})}\,.
$$

Hence, by the assumption, we see

$$
(3.2) \qquad \prod_{t=1}^{d-1}(1-(\gamma^k)^{r_1^t-1})(1-(\gamma^k)^{-r_1^t-1}) = \prod_{t=1}^{d-1}(1-(\gamma^k)^{r_2^t-1})(1-(\gamma^k)^{-r_2^t-1})
$$

for any $k=1, 2, 3, \cdots, m-1$.
Multiplying each side of (3.2) by its conjugate, we obtain

$$
\begin{aligned}
(3.3)\qquad & \prod_{t=1}^{d-1}(1-(\gamma^k)^{r_1^t-1})(1-(\gamma^k)^{-r_1^t+1})(1-\gamma^k)^{r_1^t+1})(1-(\gamma^k)^{-r_1^t-1}) \\
& (1-(\gamma^k)^{r_2^t-1})^{-1}(1-(\gamma^k)^{-r_2^t+1})^{-1}(1-(\gamma^k)^{r_2^t+1})^{-1}(1-(\gamma^k)^{-r_2^t-1})^{-1}=1\,.
\end{aligned}
$$

Now, we apply Lemma 3.4 to (3.3). Suppose for any $s$ $(0<s<d)$, there exist $s'$ and $s''$ $(0<s', s''<d)$ such that

$$r_1^s-1\equiv\pm r_2^{s'}+1 \qquad (\mathrm{mod}\ m)$$

and

$$-r_1^s-1\equiv\pm r_2^{s''}+1 \qquad (\mathrm{mod}\ m)\,.$$

Then by Lemma 3.4

$$\sum_{s=1}^{d-1}(r_1^s-1)+\sum_{s=1}^{d-1}(-r_1^s-1)\equiv\sum_{s=1}^{d-1}(r_2^s+1)+\sum_{s=1}^{d-1}(-r_2^s+1) \qquad (\mathrm{mod}\ m).$$

Hence,

$$4(d-1)\equiv 0 \qquad (\mathrm{mod}\ m)\,.$$

Since $m$ and $d$ are odd, we see

$$(d-1)/2\equiv 0 \qquad (\mathrm{mod}\ m)\,.$$

Thus

$$m\leqq(d-1)/2\,.$$

On the other hand,

$$d\leqq\phi(m)<m\,.$$

Hence we have a contradiction. Thus we can assume there exist integers $s$, $s'$ $(0<s, s'<d)$ such that

$$r_1^s-1\equiv -r_2^{s'}-1 \qquad (\mathrm{mod}\ m)$$

or

$$r_1^s-1\equiv r_2^{s'}-1 \qquad (\mathrm{mod}\ m)\,.$$

If $r_1^s\equiv -r_2^{s'}$ (mod $m$), then

$$1\equiv(r_1^s)^d\equiv(-r_1^{s'})^d\equiv-(r_2^{s'})^d\equiv-1 \qquad (\mathrm{mod}\ m)\,,$$

which is a contradiction, since $m\neq 2$.
Hence we get

$$r_1^s\equiv r_2^{s'} \qquad (\mathrm{mod}\ m)\,.$$

This means that $g_1^s$ is conjugate to $g_2^{s'}$ in $O(2d)$. Hence $G_1$ is conjugate to $G_2$ in $O(2d)$, which shows the theorem. q.e.d.

REMARK. In the proof of the above theorem, we obtained $r_1^s \equiv r_2^{s'}$ (mod $m$) only from the assumption $F_{G_1}(z)=F_{G_2}(z)$.

**Proof of Theorem 3.1.** By Proposition 1.5, $\sigma(G_1)=\sigma(G_2)$ and $|G_1|=|G_2|$. From these we see easily that if $G_1$ is cyclic then $G_2$ is also cyclic and isomorphic to $G_2$. Hence, by Corollary 2.7, we may assume that $G_1$, $G_2$ are isomorphic to some $\Gamma_d(m_1, n_1, r_1)$ and $\Gamma_d(m_2, n_2, r_2)$, respectively. Since $|G_1|=|G_2|$, $m_1n_1=m_1n_1'd=m_2n_2'd=m_2n_2$. By Corollary 2.7, we can assume that $G_i$ $(i=1, 2)$ is the image of some irreducible fixed point free representation $\hat{\pi}_{k_i,l_i}$ of $\Gamma_d(m_i, n_i, r_i)$. By Theorem 2.5, we can assume $k_i=1$ $(i=1, 2)$. Now, we shall show $n_1=n_2$. By Corollary 3.6, $n_2$ is a divisor of $n_1$ or $m_1n_1'=m_2n_2'$. Since $n_2$ is prime to $m_2$, $n_2$ is a divisor of $n_1$. Interchanging $n_1$ and $n_2$, we see $n_1$ is a divisor of $n_2$. Thus $n_1=n_2$. We put $m=m_1=m_2$ and $n=n_1=n_2=n'd$. Let $H_1$ (resp. $H_2$) be the cyclic subgroup generated by $g_1$ of (3.1) (resp. $g_2$ of (3.1)). Let $F_{g_i}(z)$ $(i=1, 2)$ be the generating function of $S^{2d-1}/G_i$ $(i=1, 2)$. Then

$$|G_i|F_{G_i}(z)=\sum_{t=1}^{m}\frac{1-z^2}{\det(z-g_i^t)}+\sum_{g\in G_i-H_i}\frac{1-z^2}{\det(z-g)}.$$

Since the order of any element of $G_i-H_i$ is not a divisor of $m$, we have

$$\sum_{t=1}^{m}\frac{1-z^2}{\det(z-g_1^t)}=\sum_{t=1}^{m}\frac{1-z^2}{\det(z-g_2^t)}.$$

By the remark of Theorem 3.7, we have $r_1\equiv r_2^c$ (mod $m$) for some $c=1, 2, \cdots, d-1$. By Proposition 2.3, this implies $G_1$ is isomorphic to $G_2$. q.e.d.

Together this with Corollary 2.8, we obtain

**Theorem 3.8.** *If two 5-dimensional spherical space forms with fundamental groups non-cyclic are isospectral, then they are isometric.*

From this with the result due to Tanno [7], we have

**Theorem 3.9.** *Let $M$ be a 5-dimensional spherical space form with non-cyclic fundamental group. Let $N$ be a compact connected riemannian manifold. Suppose $N$ is isospectral to $M$. Then they are isometric.*

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