# VANISHING THEOREMS FOR TYPE $(0, q)$ COHOMOLOGY
# OF LOCALLY SYMMETRIC SPACES

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## 1. Introduction

Let $G$ be a connected non-compact semisimple Lie group which admits a finite-dimensional faithful representation. Let $K$ be a maximal compact subgroup of $G$. We assume that the quotient $G/K$ admits a $G$-invariant complex structure. We also assume that the complexification $G^C$ of $G$ is simply-connected. Choose a Cartan subgroup $T$ of $G$ such that $T \subset K$. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ denote the complexifications of the (real) Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{h}_0$ of $G$, $K$, $T$, respectively. Let

$$(1.1) \qquad\qquad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

be a Cartan decomposition of $\mathfrak{g}_0$, let $\mathfrak{p}$ be the complexification of $\mathfrak{p}_0$, let $\Delta$ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$, and let $\Delta_n$, $\Delta_k$ denote the set of non-compact, compact roots, respectively. That is $\alpha \in \Delta$ is in $\Delta_n$(or $\Delta_k$) if and only if the corresponding (one-dimensional) root space $\mathfrak{g}_\alpha \subset \mathfrak{p}$ (or $\mathfrak{g}_\alpha \subset \mathfrak{k}$). Choose a system $\Delta^+$ of positive roots compatible with the complex structure on $G/K$. That is if

$$(1.2) \qquad\qquad \mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$$

is a splitting of the complex tangent space at the origin in $G/K$ into holomorphic and anti-holomorphic tangent vector $\mathfrak{p}^+$, $\mathfrak{p}^-$ respectively, then

$$(1.3) \qquad\qquad \mathfrak{p}^\pm = \sum_{\alpha \in \Delta^+ \cap \Delta_n} \mathfrak{g}_{\pm\alpha}$$

We now fix a discrete subgroup $\Gamma$ of $G$ such that $\Gamma$ acts freely on $G/K$ and such that the quotient $X = \Gamma \backslash G/K$ is compact. Thus $X$ is a compact locally symmetric Hermitian domain. Given any finite-dimensional irreducible representation $\tau$ of $K$ on a complex vector space, there is associated to $\tau$ a sheaf $\theta_\tau \to X$ over $X$ in the following way. Let $E_\tau \to G/K$ be the induced homogeneous $C^\infty$ vector bundle over $G/K$ associated to the principal $C^\infty$ fibration $K \to G \to G/K$. Then, as is well-known, $E_\tau$ has a holomorphic structure. We obtain a presheaf by assigning to each open set $U$ in $X$ the abelian group of $\Gamma$-invariant holomorphic sections of $E_\tau$ on the inverse image $\tilde{U}$ of $U$ in $G/K$. $\theta_\tau$ is the sheaf generated by this presheaf. Let $H^q(X, \theta_\tau)$ be the $q^{\text{th}}$ cohomology space of $X$ with coefficients

in $\theta_\tau$. *The main result of this paper is Theorem* 2.3, a general result governing the vanishing of the spaces $H^q(X, \theta_\tau)$ for $\tau$ whose highest weight $\Lambda$ relative to $\Delta_k^+ = \Delta_k \cap \Delta^+$ belongs to the set

(1.4)        $\mathscr{F}_0' = \{\Lambda \in \mathfrak{h}^* \mid \Lambda$  is integral,

$\qquad\qquad (\Lambda + \delta, \alpha) \neq 0$ for all $\alpha \in \Delta, (\Lambda + \delta, \alpha) > 0$ for all $\alpha \in \Delta_k^+\}$

and for which the system of positive roots defined by the regular element $\Lambda + \delta$,

$\delta = \dfrac{1}{2} \sum\limits_{\alpha \in \Delta^+} \alpha$, is also compatible with a $G$-invariant complex structure on $G/K$. Here $( , )$ denotes the Killing form and the integrality of $\Lambda$ means that $\dfrac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$ in an integer for every $\alpha$ in $\Delta$. Other results and applications are given in section 3.

The key point in the proof of Theorem 2.3 is the application of Parthasarathy's new unitarizability criteria for highest weight modules [15]. Theorem 2.3 extends, and implies in particular, results of [3], [4], [5], [6], [7], [9], [10], [11], [15].

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## 2.  Statement and proof of the main result

Let $\Lambda \in \mathscr{F}_0'$ in (1.4) and let $\tau_\Lambda$ be the finite-dimensional irreducible representation of $K$ on a complex vector space $V$ with highest weight $\Lambda$, relative to $\Delta_k^+$.  Since $\Lambda + \delta$ is regular

(2.1)                    $P^{(\Lambda)} = \{\alpha \in \Lambda \mid (\Lambda + \delta, \alpha) > 0\}$

is a system of positive roots.  We shall assume that *every non-compact root $\alpha$ in $P^{(\Lambda)}$ is totally positive*; i.e. $\alpha + \beta \in P_n^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_n$ for every $\beta \in \Delta_k$ such that $\alpha + \beta \in \Delta$.  It then follows, as is well-known, that there exists a $G$-invariant complex structure on $G/K$ such that the space of holomorphic tangent vectors at the origin is given by $\sum_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha$; cf. (1.3).  In general, if $Q \subset \Delta$ we shall write $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$.  Put $P_k^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_k$.  Let

(2.2)            $Q_\Lambda = \{\alpha \in \Delta_n^+ = \Delta^+ \cap \Delta_n \mid (\Lambda + \delta, \alpha) > 0\}$,

$\qquad\qquad Q_\Lambda' = \Delta_n^+ - Q_\Lambda, \qquad 2\delta^{(\Lambda)} = \langle P^{(\Lambda)} \rangle$,

$\qquad\qquad 2\delta_n^{(\Lambda)} = \langle P_n^{(\Lambda)} \rangle, \qquad 2\delta_k^{(\Lambda)} = \langle P_k^{(\Lambda)} \rangle$.

Let $|S|$ denote the cardinality of a set $S$.  Our main theorem is

**Theorem 2.3.**  *Let $\Lambda \in \mathscr{F}_0'$ as above and assume that every non-compact root in $P^{(\Lambda)}$ is totally positive.  Suppose $H^q(\Gamma \mid G/K, \theta_{\tau_\Lambda}) \neq 0$.  Then there exists a*

*parabolic subalgebra* $\theta = \mathfrak{u} + \mathfrak{m}$ *of* $\mathfrak{g}$ *containing the Borel subalgebra* $\mathfrak{h} + \sum\limits_{\alpha \in P^{(\Lambda)}} \mathfrak{g}_\alpha$ *with* $\mathfrak{u} =$ *the unipotent radical of* $\theta$ *and* $\mathfrak{m} =$ *the reductive part of* $\theta$ *such that*

(i) *if* $\theta_{u,n}$ *is the set of non-compact roots in* $\mathfrak{u}$ *then* $q = 2|\theta_{u,n} \cap Q_\Lambda| + |Q'_\Lambda| - |\theta_{u,n}|$

(ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ *for every root* $\alpha$ *in* $\mathfrak{m}$. *In particular, if* $A_\Lambda = \{\alpha \in Q_\Lambda \cup -Q'_\Lambda | (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\}$, *then* $|A_\Lambda| \leqslant |\theta_{u,n}| = 2|\theta_{u,n} \cap Q_\Lambda| + |Q'_\Lambda| - q$.

REMARK: The conditions imposed on $\Lambda$ in Theorem 2.3 are the same as those formulated by Parthasarathy in Theorem 1 of [14].

We begin the proof of Theorem 2.3 by first proving

**Lemma 2.4.** *Let* $\Lambda$ *be as in the statement of Theorem 2.3 and suppose* $H^q(\Gamma \,|\, G/K, \theta_{\tau_\Lambda}) \neq 0$. *Then there exists an irreducible unitarizable highest weight* $\mathfrak{g}$-*module* $H_\pi$ *with respect to the system of positive roots* $\bar{P}^{(\Lambda)} = P_k^{(\Lambda)} \cup -P_n^{(\Lambda)} = \Delta_k^+ \cup -Q_\Lambda \cup Q'_\Lambda$ *with* $\Delta_k^+$ *highest weight* $\mu = \Lambda + \langle Q \rangle$ *where* $Q \subset \Delta_n^+$ *satisfies*

(i) $|Q| = q$,

(ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ *for every* $\alpha \in (Q_\Lambda \cap Q') \cup (Q \cap Q'_\Lambda)$, $Q' = \Delta_n^+ - Q$,

(iii) $|\delta_k - \delta_n| = |\delta_k - \delta_n + \langle Q \rangle|$ *where* $2\delta_k = \langle \Delta_k^+ \rangle$, $2\delta_n = \langle \Delta_n^+ \rangle$. *The representation* $\pi$ *satisfies* $\pi(\Omega) = (\Lambda, \Lambda + 2\delta)1$ *where* $\Omega$ *is the Casimir element of* $\mathfrak{g}$.

REMARK: In the special case when $\Lambda$ is actually $\Delta^+$ dominant, $P^{(\Lambda)} = \Delta^+$, $\bar{P}^{(\Lambda)} = \Delta_k^+ \cup -\Delta_n^+$, $Q_\Lambda = \Delta_n^+$, $Q'_\Lambda = \phi$, $\delta = \delta^{(\Lambda)}$, and Lemma 2.4 reduces to Lemma 2 of Hotta-Murakami [4]. We now prove Lemma 2.4 by abstracting parts of Parthasarathy's argument in his proof of Theorem 1 of [14]. Let $W_G$ be the subgroup of the Weyl group $W$ of $(\mathfrak{g}, \mathfrak{h})$ generated by reflections with respect to compact roots. One has

$$(2.5) \qquad P^{(\Lambda)} = \Delta_k^+ \cup Q_\Lambda \cup -Q'_\Lambda$$

so that

$$(2.6) \qquad \begin{aligned} P_k^{(\Lambda)} &= \Delta_k^+\,, \qquad P_n^{(\Lambda)} = Q_\Lambda \cup -Q'_\Lambda \\ Q_\Lambda &= \Delta_n^+ \cap P_n^{(\Lambda)}\,, \qquad \delta_n + \delta^{(\Lambda)} = \delta_k + \langle Q_\Lambda \rangle\,. \end{aligned}$$

Note

$$(2.7) \qquad \delta - \delta^{(\Lambda)} = \delta_n + \delta_k - (\delta_k^{(\Lambda)} + \delta_n^{(\Lambda)}) = \delta_n - \delta_n^{(\Lambda)}\,.$$

Since every non-compact root in $P^{(\Lambda)}$ is totally positive, by assumption, $\sigma P_n^{(\Lambda)} = P_n^{(\Lambda)}$ for every $\sigma$ in $W_G$. Already $\sigma \Delta_n^+ = \Delta_n^+$ for $\sigma$ in $W_G$. In particular, if $\kappa$ is the unique element of $W_G$ such that $\kappa \Delta_k^+ = -\Delta_k^+$, then (2.5), (2.6) imply

$$(2.8) \qquad -\kappa P^{(\Lambda)} = \Delta_k^+ \cup -Q_\Lambda \cup Q'_\Lambda \overset{\text{def.}}{=} \bar{P}^{(\Lambda)}$$

Now suppose $H^q(\Gamma \,|\, G/K, \theta_{\tau_\Lambda}) \neq 0$. Then by Theorem 3 of [12] or more explicitly formula (4.2) of [4], there exist an irreducible unitary representation $\pi$ of

$G$ on a Hilbert space $H_\pi$ such that $\pi(\Omega)=(\Lambda,\,\Lambda+2\delta)1$ and an irreducible $K$ module $V_\mu$ with $\Delta_k^+$ highest weight $\mu\in h^*$ such that $V_\mu$ is contained in both $\pi\,|_K$ and $V_\Lambda\otimes\bigwedge^q\mathfrak{p}^+$. $\mu$ has the form

(2.9) $$\mu=\Lambda+\langle Q\rangle,\quad Q\subset\Delta_n^+,\quad |Q|=q\,.$$

Parthasarathy's Theorem 1 of [14] is a statement about the vanishing of *square-integrable* cohomology and in his proof the role of $H_\pi$ is played by a so-called *discrete series* representation of $G$. However since we have $\pi(\Omega)=(\Lambda,\,\Lambda+2\delta)1$ his arguments on pages 608–682 are applicable in our present context and one may conclude that for a certain unit vector $\psi_\mu\in H_\pi$ and basis $\{E_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ (see pages 681, 682 of [14])

(2.10) $$-2\sum_{\alpha\in\bar{P}_n^{(\Lambda)}}\|\pi(E_\alpha)\psi_\mu\|^2=-2(\Lambda+\delta-\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle)$$
$$-(\gamma-\langle Q_\Lambda\rangle+2\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle)$$

where $\gamma=\langle Q\rangle$, and both terms on the r.h.s. of (2.10) are non-negative. Here one needs that $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant. $\psi_\mu$ is chosen moreover to satisfy $\pi(E_\alpha)\psi_\mu=0$ for any $\alpha\in\Delta_k^+$ and $\pi(H)\psi_\mu=\mu(H)\psi_\mu$ for every $H$ in $\mathfrak{h}$. Thus one has $\pi(E_\alpha)\psi_\mu=0$ for any $\alpha\in\bar{P}^{(\Lambda)}$so that $H_\pi$ is a highest weight module relative to the system of positive roots $\bar{P}^{(\Lambda)}$with $\Delta_k^+$ highest weight $\mu$. Also by (2.10)

(2.11) $$(\Lambda+\delta-\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle)=0\,,$$
$$(\gamma-\langle Q_\Lambda\rangle+2\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle)=0\,,$$

with $\gamma-\langle Q_\Lambda\rangle=\langle Q\cap Q_\Lambda'\rangle-\langle Q_\Lambda\cap Q'\rangle$ (since $Q\cup(Q_\Lambda\cap Q')=(Q\cap Q_\Lambda')\cup Q_\Lambda$).

Therefore, $-(\Lambda+\delta-\delta^{(\Lambda)},\,\gamma-\langle Q\rangle)=0$ implies

$$0=\sum_{\alpha\in-(Q\cap Q_\Lambda')}(\Lambda+\delta-\delta^{(\Lambda)},\,\alpha)+\sum_{\beta\in Q_\Lambda\cap Q'}(\Lambda+\delta-\delta^{(\Lambda)},\,\beta)$$

and since $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant, (2.5) implies

(2.12) $$(\Lambda+\delta-\delta^{(\Lambda)},\,\alpha)=0\quad\text{for}\quad \alpha\in-(Q\cap Q_\Lambda')\cup(Q_\Lambda\cap Q')$$

which proves statement (ii) of Lemma 2.4. By page 682 of [14]

(2.13) $$-(\gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)})+(\delta^{(\Lambda)},\,\delta^{(\Lambda)})$$
$$=-(\gamma-\langle Q_\Lambda\rangle+2\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle)\,.$$

Hence by (2.11)

(2.14) $$(\gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)},\,\gamma-\langle Q_\Lambda\rangle+\delta^{(\Lambda)})=(\delta^{(\Lambda)},\,\delta^{(\Lambda)})\,.$$

Now let

(2.15) $$\Delta_+'=\Delta_k^+\cup-\Delta_n^+\,,\quad 2\delta'=\langle\Delta_+'\rangle\quad\text{so that}\quad \delta'=\delta_k-\delta_n\,.$$

One knows that $\Delta'_+$ is a system of positive roots. Hence $(\delta^{(\Lambda)}, \delta^{(\Lambda)}) = (\delta', \delta')$. Since $\gamma = \langle Q \rangle$ and since $-\langle Q_\Lambda \rangle + \delta^{(\Lambda)} = \delta_k - \delta_n$ by (2.6), equation (2.14) is statement (iii) of Lemma 2.4. Thus the proof of Lemma 2.4 is completed.

By (iii) of Lemma 2.4, $|\delta'| = |\delta' - \langle -Q \rangle|$ (see (2.15)) so by a lemma of Kostant [8] there exists $\sigma \in W$ such that

$$(2.16) \qquad \sigma\delta' = \delta' - \langle -Q \rangle, \quad l(\sigma) = q, \quad \sigma(-\Delta'_+) \cap \Delta'_+ = -Q.$$

One knows in fact that

$$(2.17) \qquad \Delta^+_k \subset \sigma\Delta'_+.$$

In (2.16), $l(\sigma)$ denotes the *length* of $\sigma$. It is easy to check that (2.16) implies

**Proposition 2.18.** $Q = \{\alpha \in \Delta^+_n \mid (\sigma\delta', \alpha) > 0\}$.

**Lemma 2.19.** $(\mu + \delta^{(\Lambda)}_k - \delta^{(\Lambda)}_n, \alpha) \neq 0$ *for every $\alpha$ in $\Delta$ and for $\mu$ in Lemma 2.4; (see 2.2).*

Proof. $\delta^{(\Lambda)}_k = \delta_k$ and $\delta' = \delta_k - \delta_n$ so

$$(2.20) \qquad \mu + \delta^{(\Lambda)}_k - \delta^{(\Lambda)}_n = \Lambda + \langle Q \rangle + \delta_k - \delta^{(\Lambda)}_n$$
$$= \Lambda + \sigma\delta' + \delta_n - \delta^{(\Lambda)}_n = \Lambda + \sigma\delta' + \delta - \delta^{(\Lambda)}$$

by (2.16) and (2.7). It suffices to check Lemma 2.19 for $\alpha \in P^{(\Lambda)} = \Delta^+_k \cup Q_\Lambda \cup -Q'_\Lambda$ (see (2.5)). Again we use that $\Lambda + \delta - \delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant. For $\alpha \in \Delta^+_k \subset P^{(\Lambda)}$ in particular, $\alpha = \sigma\alpha_1$, $\alpha_1 \in \Delta'_+$ by (2.17) so that $(\sigma\delta', \alpha) = (\delta', \sigma^{-1}\alpha) = (\delta', \alpha_1) > 0$. Hence $(\Lambda + \delta - \delta^{(\Lambda)} + \sigma\delta', \alpha) > 0$. Also $Q_\Lambda \cap Q \subset Q_\Lambda \subset P^{(\Lambda)}$ and $(\sigma\delta', \alpha) > 0$ for $\alpha \in Q$ by Proposition 2.18 so by the same argument

$$(2.21) \qquad (\Lambda + \delta - \delta^{(\Lambda)} + \sigma\delta', \alpha) > 0 \quad \text{for} \quad \alpha \in Q_\Lambda \cap Q.$$

Suppose $\alpha \in (Q'_\Lambda \cap Q) \cup (Q_\Lambda \cap Q')$. Then by (ii) of Lemma 2.4 $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ and $(\Lambda + \delta - \delta^{(\Lambda)} + \sigma\delta', \alpha) = (\sigma\delta', \alpha) \neq 0$ with $(\sigma\delta', \alpha) > 0$ for $\alpha \in Q'_\Lambda \cap Q$ by Proposition 2.18. Then by (2.21)

$$(2.22) \qquad (\Lambda + \delta - \delta^{(\Lambda)} + \sigma\delta', \alpha) > 0 \quad \text{for} \quad \alpha \in Q = (Q_\Lambda \cap Q) \cup (Q'_\Lambda \cap Q).$$

Since $Q'_\Lambda = (Q'_\Lambda \cap Q) \cup (Q'_\Lambda \cap Q')$ the final case to check is $\alpha \in Q'_\Lambda \cap Q'$. By Proposition 2.18, $(\sigma\delta', \alpha) < 0$ for $\alpha \in Q'$; i.e. $(\sigma\delta', -\alpha) > 0$. Also $-\alpha \in -Q'_\Lambda$ for $\alpha \in Q'_\Lambda$ and $(\Lambda + \delta - \delta^{(\Lambda)}, -\alpha) \geq 0$ since $-Q'_\Lambda \subset P^{(\Lambda)}$ and since $\Lambda + \delta - \delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant. That is

$$(2.23) \qquad (\Lambda + \delta - \delta^{(\Lambda)} + \sigma\delta', -\alpha) > 0 \quad \text{for} \quad \alpha \in Q'_\Lambda \cap Q'.$$

This completes the proof of Lemma 2.19.

REMARK: One can observe directly that $(\mu + \delta^{(\Lambda)}_k - \delta^{(\Lambda)}_n, \alpha) \neq 0$ for $\alpha \in \Delta^+_k$.

Hence equation (2.17) is not needed for the proof of Lemma 2.19, nor for the proof of Theorem 2.3.

We now state Parthasarathy's necessary conditions for the unitarizability of highest weight modules. This is Theorem A of [15]. For sufficient conditions see Theorem B of [15].

**Theorem 2.24** (Parthasarathy). *Suppose that $P$ is a system of positive roots compatible with a $G$-invariant complex structure on $G/K$ (here $G$ is linear, as we have assumed in section 1). Let $2\delta_{P,n}=\langle\Delta_n\cap P\rangle$, $2\delta_{P,k}=\langle\Delta_k\cap P\rangle$. Suppose $\mu\in h^*$ is integral and $\Delta_k\cap P$ dominant and suppose that $H_\mu$ is an irreducible highest weight $\mathfrak{g}$-module (or $G$-module) with respect to the positive system $(\Delta_k\cap P)\cup -(\Delta_n\cap P)$, with $\Delta_k\cap P$ highest weight $\mu$. Suppose that*

$$(2.25) \qquad (\mu+\delta_{P,k}-\delta_{P,n},\ \alpha)\neq 0 \quad \text{for every}\quad \alpha\in\Delta.$$

*Then if $H_\mu$ is unitarizable there exists a parabolic subalgebra $\theta$ of $\mathfrak{g}$, $\theta$ containing the Borel subalgebra $\mathfrak{h}+\sum_{\alpha\in P}\mathfrak{g}_\alpha$ of $\mathfrak{g}$, such that $\mu=\Lambda_0+2\delta_{\theta,n}$ where*

( i ) *$2\delta_{\theta,n}=$the sum of non-compact roots in the unipotent radical of $\theta$.*

( ii ) *$\Lambda_0$ is $P$ dominant integral, and*

(iii) *$(\Lambda_0,\alpha)=0$ for every root $\alpha$ in the reductive part of $\theta$.*

As indicated in the Introduction the proof of Theorem 2.3 will be based upon Theorem 2.24. Suppose $\Lambda\in\mathcal{F}_0'$ satisfies the hypothesis of Theorem 2.3 and suppose $H^q(\Gamma\,|\,G/K,\theta_{\tau_\Lambda})\neq 0$. By the remarks following (2.1) we may take $P$ of Theorem 2.24 to be $P^{(\Lambda)}$. Then $\Delta_k\cap P=P_k^{(\Lambda)}=\Delta_k^+$and $-(\Delta_n\cap P)=-P_n^{(\Lambda)}$. By Lemma 2.19 condition (2.25) is satisfied for $\mu$ in Lemma 2.4. Thus by Lemma 2.4 and Theorem 2.24 we can conclude the following: There exists a parabolic subalgebra $\theta=\mathfrak{u}+\mathfrak{m}$ of $\mathfrak{g}$ with unipotent radical $\mathfrak{u}$ and reductive part $\mathfrak{m}$ such that $\theta\supset\mathfrak{h}+\sum_{\alpha\in P^{(\Lambda)}}\mathfrak{g}_\alpha$, $\mu=\Lambda_0+2\delta_{\theta,n}$ where if $\theta_{u,n}$ is the set of non-compact roots in $\mathfrak{u}$, $2\delta_{\theta,n}=\langle\theta_{u,n}\rangle$. Also $\Lambda_0$ is $P^{(\Lambda)}$ dominant integral and $(\Lambda_0,\alpha)=0$ for every root $\alpha$ of $\mathfrak{m}$. Now we also know that $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ dominant integral. We will show that in fact $\Lambda+\delta-\delta^{(\Lambda)}=\Lambda_0$. For this we need the following remark: The subalgebra $\theta=\mathfrak{u}+\mathfrak{m}$ in Theorem 2.24 can be chosen so that if $P'=\{\alpha\in\Delta\,|\,(\mu+\delta_{P,k}-\delta_{P,n},\ \alpha)>0\}$ is the positive system defined by the regular element $\mu+\delta_{P,k}-\delta_{P,n}$ (see (2.25)) then $\theta_{u,n}=(\Delta_n\cap P)\cap(\Delta_n\cap P')$. This follows by (3.49) of [15]. Thus we have by (2.20)

$$(2.26) \qquad\qquad \theta_{u,n}=P'\cap P_n^{(\Lambda)}$$

where $\qquad\qquad P'=\{\alpha\in\Delta\,|\,(\Lambda+\delta-\delta^{(\Lambda)}+\sigma\delta,\ \alpha)>0\}$ .

If $\Delta(\mathfrak{m})$ denotes the set of roots of $\mathfrak{m}$, then

$$(2.27) \qquad \mathfrak{m} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{m})} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha \in P^{(\Lambda)} - \Delta(\mathfrak{m})} \mathfrak{g}_\alpha$$

and hence $\theta_{\mathfrak{u},n} = P_n^{(\Lambda)} - \Delta(\mathfrak{m})$.

**Lemma 2.28.** $\theta_{\mathfrak{u}\, n} = (Q \cap Q_\Lambda) \cup (-Q_\Lambda' \cap -Q')$ and $P_n^{(\Lambda)} \cap \Delta(\mathfrak{m}) = P_n^{(\Lambda)} - P'$
$= (Q' \cap Q_\Lambda) \cup (-Q_\Lambda' \cap -Q)$.

Proof. By (2.6) and (2.26), $\theta_{\mathfrak{u},n} = P' \cap P_n^{(\Lambda)} = P' \cap (Q_\Lambda \cup -Q_\Lambda')$. By (2.22) and (2.23) $Q, -Q_\Lambda' \cap -Q' \subset P'$ so that $Q \cap Q_\Lambda, -Q_\Lambda' \cap -Q' \subset P' \cap (Q_\Lambda \cup -Q_\Lambda')$. Conversely let $\alpha \in P' \cap (Q_\Lambda \cap -Q_\Lambda')$. We consider two cases:

(i) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ and (ii) $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) \neq 0$.

If (i) holds then $\alpha \in P'$ implies $(\sigma \delta', \alpha) > 0$. Hence if $\alpha \in Q_\Lambda \subset \Delta_n^+$ then $\alpha \in Q_\Lambda \cap Q$ by Proposition 2.18. If $\alpha \in -Q_\Lambda'$ then $-\alpha \in Q_\Lambda' \subset \Delta_n^+$ such that $(\sigma \delta', -\alpha) < 0$. By Proposition 2.18, $-\alpha \in Q'$ so $\alpha \in -Q'$ implies $\alpha \in -Q_\Lambda' \cap -Q'$. Suppose (ii) holds. Then by (ii) of Lemma 2.4 we have $\alpha \notin Q_\Lambda \cap Q'$, $-\alpha \notin Q \cap Q_\Lambda'$. Since $\alpha \in Q_\Lambda \cup -Q_\Lambda'$ we have $\alpha \in Q_\Lambda \cap Q$ if $\alpha \in Q_\Lambda$, and if $\alpha \in Q_\Lambda'$ then $-\alpha \in Q'$ so $\alpha \in -Q_\Lambda' \cap -Q'$. Since $P_n = Q_\Lambda \cup (-Q_\Lambda')$, it follows that $P_n^{(\Lambda)} - P' = (Q' \cap Q_\Lambda) \cup (-Q_\Lambda' \cap -Q)$. Then $P_n^{(\Lambda)} = (P_n^{(\Lambda)} \cap \Delta(\mathfrak{m})) \cup (P_n^{(\Lambda)} - \Delta(\mathfrak{m}))$ and $P_n^{(\Lambda)} - \Delta(\mathfrak{m}) = P' \cap P_n^{\Lambda)}$ implies that $P_n^{(\Lambda)} \cap \Delta(\mathfrak{m}) = P_n^{(\Lambda)} - P'$, which completes the proof of Lemma 2.28.

By Lemma 2.28 we have

$$(2.29) \qquad Q \cap Q_\Lambda = \theta_{\mathfrak{u},n} \cap Q_\Lambda$$

and $|\theta_{\mathfrak{u},n}| = |Q \cap Q_\Lambda| + |Q_\Lambda' \cap Q'| = |Q \cap Q_\Lambda| + |Q_\Lambda'| - |Q_\Lambda' \cap Q| = 2|Q \cap Q_\Lambda| - |Q| + |Q_\Lambda'|$ so that by (2.29) and (i) of Lemma 2.4, $q = 2|\theta_{\mathfrak{u},n} \cap Q_\Lambda| - |\theta_{\mathfrak{u},n}| + |Q_\Lambda'|$, which proves statement (i) of Theorem 2.3. Recalling that $2\delta_{\theta,n} \overset{\text{def.}}{=} \langle \theta_{\mathfrak{u},n} \rangle$ we also have by Lemma 2.28 that $2\delta_{\theta,n} = \langle Q \cap Q_\Lambda \rangle - \langle Q_\Lambda' \cap Q' \rangle = \langle Q \cap Q_\Lambda \rangle + \langle Q_\Lambda' \cap Q \rangle - \langle Q_\Lambda' \rangle = \langle Q \rangle + \langle Q_\Lambda \rangle - 2\delta_n = \langle Q \rangle - \delta_n + \delta^{(\Lambda)} - \delta_k$ (by (2.6)) $= \langle Q \rangle + \delta^{(\Lambda)} - \delta$. That is we have $\Lambda + \langle Q \rangle = \mu = \Lambda_0 + 2\delta_{\theta,n} = \Lambda_0 + \langle Q \rangle + \delta^{(\Lambda)} - \delta$ implies $\Lambda + \delta - \delta^{(\Lambda)} = \Lambda_0$. Hence $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ for every $\alpha \in \Delta(\mathfrak{m})$, which proves the first statement in (ii) of Theorem 2.3. Since $(\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0$ for every $\alpha$ in $\Delta(\mathfrak{m}) \cap P_n^{(\Lambda)}$ in particular, we must have the set

$$A_\Lambda = \{\alpha \in P_n^{(\Lambda)} | (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\} \subset P_n^{(\Lambda)} - \Delta(\mathfrak{m}) = \theta_{\mathfrak{u},n}.$$

Hence $|A_\Lambda| \leqslant |\theta_{\mathfrak{u},n}|$ and we have completed the proof of Theorem 2.3.

REMARK: One may check that statement (i) of Theorem 2.3 is equivalent to

the statement

$$(2.30) \qquad n-q = 2\,|\,(P_n^{(\Lambda)} \cap \Delta(\mathfrak{m})) \cap Q_\Lambda\,| - |\,P_n^{(\Lambda)} \cap \Delta(\mathfrak{m})\,| + |\,Q_\Lambda'\,|$$

where $n = |\,\Delta_n^+\,| = \dim_C G/K$.

For later computational purposes it is convenient to consider parabolic subalgebras of $\mathfrak{g}$ which contain the Borel subglgebra $\mathfrak{h} + \sum\limits_{\phi \in \overline{P}^{(\Lambda)}} \mathfrak{g}_\alpha$. Thus we give the following equivalent formulation of Theorem 2.3.

**Theorem 2.3'.** *Let $\Lambda \in \mathscr{F}_0'$ such that every non-compact root in $P^{(\Lambda)}$ is totally positive. Suppose that $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda}) \neq 0$. Then there exists a parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h} + \sum\limits_{\alpha \in \overline{P}^{(\Lambda)}} \mathfrak{g}_\alpha$ with $\mathfrak{u} = $ the unipotent radical of $\theta$ and $\mathfrak{m} = $ the reductive part of $\theta$ such that*
  (i)  *$n-q = 2\,|\,\theta_{u,m} \cap Q_\Lambda'\,| + |\,Q_\Lambda\,| - |\,\theta_{u,n}\,|$*
  (ii) *$(\Lambda + \delta - \delta^{(\Lambda)}, \kappa\alpha) = 0$ for every root $\alpha$ of $\mathfrak{m}$.*
*Here $n = |\,\Delta_n^+\,| = \dim_C G/K$, $\theta_{u,n}$ is the set of non-compact roots in $\mathfrak{u}$, and $\kappa$ denotes the unique element of $W_G$ such that $\kappa\Delta_\kappa^+ = -\Delta_\kappa^+$.*

Proof: One has $\kappa\Delta_n^+ = \Delta_n^+$ and $\kappa^2 = 1$. Given $\Delta \in \mathscr{F}_0'$ we have $-\kappa\Lambda - 2\delta_n \in \mathscr{F}_0'$ since $-\kappa\Lambda - 2\delta_n + \delta = -\kappa\Lambda - \kappa\delta = -\kappa(\Lambda + \delta)$. The latter equation also shows that $P^{(-\kappa\Lambda - 2\delta_n)} = -\kappa P^{(\Lambda)} = \overline{P}^{(\Lambda)}$; see (2.8). Moreover one may check that every non-compact root in the positive system $\overline{P}^{\Lambda)}$ is totally positive. By Serre duality, $H^q(\Gamma|G/K, \theta_{\tau_\Lambda}) \cong H^{n-q}(\Gamma|G/K, \theta_{\tau_{-\kappa\Lambda-2\delta_n}})$. Hence if $H^q(\Gamma|G/K, \theta_{\tau_\Lambda}) \neq 0$. Theorem 2.3 says there exists a parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ containing $\mathfrak{h} + \sum\limits_{\alpha \in \overline{P}^{(\Lambda)}} \mathfrak{g}_\alpha$ such that (i) $n - q = 2\,|\,\theta_{u,n} \cap Q_{-\kappa\Lambda-2\delta_n}| + Q_{-\kappa\Lambda-2\delta_n}'| - |\,\theta_{u,n}|$, (ii) $(-\kappa\Lambda - 2\delta_n + \delta - \delta^{(\Lambda)}, \alpha) = 0$ for every $\alpha$ in $\Delta(\mathfrak{m})$. Now $Q_{-\kappa\Lambda-2\delta_n} = \kappa Q_\Lambda'$ so that $Q_{-\kappa\Lambda-2\delta_n}' = \kappa Q_\Lambda$. Also $-\kappa\delta^{(\Lambda)} = \overline{\delta}^{(\Lambda)}$ by (2.8) so that

$$-\kappa\Lambda - 2\delta_n + \delta - \overline{\delta}^{(\Lambda)} = -\kappa(\Lambda + \delta) + \kappa\delta^{(\Lambda)} = -\kappa(\Lambda + \delta - \delta^{(\Lambda)})$$

implies $(\Lambda + \delta - \delta^{(\Lambda)}, \kappa\alpha) = 0$ for every $\alpha$ in $\Delta(\mathfrak{m})$. Thus Theorem 2.3' follows.

## 3.  Some applications

In the present discussion we shall see how Theorem 2.3 incorporates and extends some of the classical results on the vanishing of $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda})$. Here we consider the two extreme cases of $\Delta \in \mathscr{F}_0'$:
  (i)  $(\Lambda + \delta, \alpha) > 0$ for every in $\Delta_n^+$ (i.e. $\Lambda$ is $\Delta^+$ dominant) and
  (ii) $(\Lambda + \delta, \alpha) < 0$ for every $\alpha$ in $\Delta_n^+$.
In case (i), $Q_\Lambda = \Delta_n^+$ so that $\delta^{(\Lambda)} = \delta$ and $P^{(\Lambda)} = \Delta^+$ by (2.5). In case (ii) $Q_\Lambda = \phi$, $P^{(\Lambda)} = \Delta_k^+ \cup -\Delta_n^+$ (by (2.5)) and $\delta^{(\Lambda)} = \delta' = \delta_k - \delta_n$. Thus in both cases every non-compact root in $P^{(\Lambda)}$ is totally positive and therefore Theorem 2.3 is applicable. For case (i) we get the following:

**Theorem 3.1.** *Suppose $\Lambda$ is $\Delta^+$ dominant integral. Suppose $H^q(\Gamma \mid G/K, \theta_{\tau_\Lambda})$ $\neq 0$. Then there exists a parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ such that*

    (i)   $q = |\theta_{\mathfrak{u},n}| =$ *the no. of non-compact roots in the unipotent radical $\mathfrak{u}$ of $\theta$*

    (ii)  $(\Lambda, \alpha) = 0$ *for every root in the reductive part $\mathfrak{m}$ of $\theta$.*

*In particular, if $n_\Lambda = |\{\alpha \in \Delta_n^+ \mid (\Lambda, \alpha) > 0\}$ then we have $H^q(\Gamma \mid G/K, \theta_{\tau_\Lambda}) = 0$ for $q < n_\Lambda$.*

The last statement follows since the set $A_\Lambda$ in Theorem 2.3 is the set $\{\alpha \in \Delta_n^+ \mid (\Lambda, \alpha) > 0\}$ and $|\theta_{\mathfrak{u},n}| = q$. This statement moreover was first proved by Y. Matsushima and S. Murakami; see [10], [11] [12],. Let $r$ be the real rank of $G$. Suppose $G$ is simple so that the Hermitian symmetric space $G/K$ is irreducible. Then in [15] it is shown that there exists no parabolic subalgebra $\theta = \mathfrak{u} + \mathfrak{m}$ of $\mathfrak{g}$ such that $q = |\theta_{\mathfrak{u},n}|$ for $1 \leqslant q < r$, $\theta \supset \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Hence, in particular, Theorem 3.1 implies

**Corollary 3.2.** $H^q(\Gamma \mid G/K, \theta_{\tau_\Lambda}) = 0$ *for $1 \leqslant q < r$, $G$ simple and $\Lambda$ $\Delta^+$ dominant integral.*

Corollary 3.2 was also proved by R. Hotta and S. Murakami in [4]. However we shall see that statement (i) of Theorem 3.1 is generally sharper than the statement of Corollay 3.2. If we taken $\Lambda = 0$ in Theorem 3.1, then $\dim H^q(\Gamma \mid G/K, \theta_{\tau_\Lambda})$ is just the $(0, q)$ Betti number of the locally symmetric space $\Gamma \mid G/K$. Then Corollary 3.2 is a result of Y. Matsushima [18] and R. Hotta - N. Wallach [6] (also see A. Borel - N. Wallach [2]). Moreover Theorem 3.1 for $\Lambda = 0$ coincides with the sharper results of [15] obtained by R. Parthasarathy for the vanishing of $(0, q)$ Betti numbers.

Suppose now again that the Hermitian symmetric space $G/K$ is irreducible and $r = r(G)$ is the real rank of $G$. Then $G/K$ is one of the following spaces on E. Cartan's list:

|        |     |                                    |                                      |
| ------ | --- | ---------------------------------- | ------------------------------------ |
|        | I   | $SU(n, m)/S(U(n) \times U(m))$,    | $r = \min(n, m)$                     |
|        | II  | $Sp(n, R)/U(n)$,                   | $r = n$                              |
| (3.3)  | III | $SO_0(n, 2)/SO(n) \times SO(2)$,   | $n > 2, \; r = 2$                    |
|        | IV  | $SO^*(2n)/U(n)$,                   | $n > 3, \; r = \left[\frac{n}{2}\right]$ |
|        | V   | $G/K$,                             | $r = 2, \; G^c = E_6$                |
|        | VI  | $G/K$,                             | $r = 3, \; G^c = E_7$.               |

Parthasarathy [15] has computed all of the numbers $|\theta_{\mathfrak{u},n}|$ as $\theta = \mathfrak{u} + \mathfrak{m}$ varies over the parabolics containing $\mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. We present his list in the form of

the following

<div align="center">Table 3-4</div>

| $G$ | $\{|\theta_{u,m}|\,|\theta\supset\mathfrak{h}+\sum\limits_{\alpha\in\Delta_+}\mathfrak{g}_\alpha\}$ |
|---|---|
| $SU(n,\,m),\ n\geqslant m$ | $\{nm-n'm'\,|\,0\leqslant n'\leqslant n,\ 0\leqslant m'\leqslant m\}$ |
| $SP(n,\,R)$ | $\{0\}\cup\{n+(n-1)+\cdots+(n-j)\,|\,j=0,1,\cdots,n-1\}$ |
| $SO_0(n,\,2),\ n>2$ | $\{0\}\cup\left\{\left[\dfrac{n+1}{2}\right]\right\},\cdots,n\right\}$ |
| $SO^*(2n),\ n>3$ | $\left\{\dfrac{n(n-1)}{2}-\dfrac{j(j-1)}{2}\,\middle|\,j=3,\cdots,n\right\}$ <br> $\cup\left\{\dfrac{n(n-1)}{2}-i\,\middle|\,i=0,1,\cdots,n-1\right\}$ |
| real form of $E_6$ | $\{0,\ 8,\ 11,\ 12,\ 13,\ 14,\ 15,\ 16\}$ |
| real from of $E_7$ | $\{0\ ,17,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27\}$ |

Theorem 3.1 now implies

**Theorem 3.5.** *Suppose $G$ is simple as in Table 3.4 and suppose $\Lambda$ is $\Delta^+$ dominant integral. Then $H^q(\Gamma\,|\,G/K,\,\theta_{\tau_\Lambda})$ vanishes unless $q$ belongs to the set $\{|\theta_{u,n}|\,|\,\theta\supset\mathfrak{h}+\sum\limits_{\alpha\in\Delta^+}\mathfrak{g}_\alpha\}$ corresponding to $G$ in the Table 3.4.*

Consider the exceptional cases V, VI of (3.3) for example. For $G^c=E_6$, $r(G)=2$ and the classical result Corollary 3.2 predicts vanishing of $H^q$ for $q=1$. However by Theorem 3.5 we get $H^q=0$ for $1\leqslant q\leqslant 7$, $q=9$, $10$, which shows that Theorem 3.4 (or Theorem 3.1) is sharper than the main Theorem of [4] as we asserted earlier. For $G^c=E_7$ Corollary 3.2 gives $H^q=0$ for $q=1,2$ whereas Theorem 3.5 implies $H^q=0$ for $1\leq q\leq 16$ and $q=18$, $19$, $20$. We remark that $\dim_C G/K=16$, $27$ respectively in cases V, VI. In case IV of (3.3) our result gives $H^q=0$ for $1\leqslant q\leqslant n-2$ (and for certain other values of $q$) even though $r(G)=[n/2]$. In case III of (3.3), $H^q=0$ for $1\leqslant q<[(n+1)/2]$, even though the real rank is only 2. Similarly in the other cases Theorem 3.5 improves known results.

Next let $\gamma_1$ be the unique non-compact simple root of $\Delta_n^+$ and let $\beta_0\in\Delta^+$ be the largest root. Then $\beta_0$ is the highest $\Delta^+$ weight of the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ and $\beta_0$ is the highest $\Delta_k^+$ weight of the adjoint representation $\mathrm{ad}_+$ of $\mathfrak{k}$ on $\mathfrak{p}^+$; $\beta_0\in\Delta_n^+$. For the special representation $\tau=\mathrm{ad}_+$ of $K$ one has

$$(3.6)\qquad\qquad H^q(\Gamma\,|\,G/K,\,\theta_{\tau_{\beta_0}})\cong H^q(\Gamma\,|\,G/K,\,\Theta)$$

where $\Theta$ is the sheaf of germs of holomorphic vector fields on $\Gamma | G/K$. The number $n_\Lambda = n_{\beta_0}$ in the statement of Theorem 3.1 (for $G$ simple) is given by

$$(3.7) \qquad n_{\beta_0} = \frac{1}{(\gamma_1, \gamma_1)} - 1$$

; see [11] or [13]. Thus, following Matsushima and Murakami [11], we obtain from Theorem 3.1 the following classical theorem of E. Calabi and E. Vesentini [3]:

**Theorem 3.8.** $H_q(\Gamma | G/K, \Theta) = 0$ for $q < \dfrac{1}{(\gamma_1, \gamma_1)} - 1$, $G$ simple.

One knows also that

$$(3.9) \qquad n_{\beta_0} = m(G/K) + 1$$

where $\qquad m(G/K) = |\{\alpha \in \Delta_n^+ - \{\gamma_1\} \mid \alpha - \gamma_1 \in \Delta\}|$ .

In [1] A. Borel shows that for $E_6$, $E_7$ respectively $m(G/K) = 10, 16$. Thus by (3.7), (3.9) the Calabi-Vesentini theorem gives $H^q = 0$ for $q < 11$ for $G$ = the real form of $E_6$, and $H^q = 0$ for $q < 17$ for $G$ = the real form of $E_7$. However we have already observed that $H^q = 0$ for $1 \leqslant q \leqslant 16$ and $q = 18, 19, 20$ for $G$ = the real form of $E_7$. Thus we have the following slight improvement of the Calabi-Vesentini theorem:

**Theorem 3.10.** Let $G$ be the unique real form of $E_7$ such that $G/K$ is Hermitian symmetric. Then $H^q(\Gamma | G/K, \Theta) = 0$ for $0 \leqslant q < 17$ and for $q = 18, 19, 20$.

For $G^c = E_6$ and for the cases III, IV in (3.3) our results give no improvement of the Calabi-Vesentini result. However in cases I, II we do obtain further improvements (even more so than in case VI of Theorem 3.10).

Indeed, for the irreducible Hermitian symmetric spaces $G/K$ in (3.3) the corresponding complex dimensions $n$ and the values $n_{\beta_0} = \dfrac{1}{(\gamma_1, \gamma_1)} - 1$ in Theorem 3.8 are given as follows:

| $G/K$ | $n$ | $n_{\beta_0}$ |
|---|---|---|
| I | $nm$ | $n+m-1$ |
| II | $\dfrac{n(n+1)}{2}$ | $n$ |
| III | $n$ | $n-1$ |
| IV | $\dfrac{n(n-1)}{2}$ | $2n-3$ |
| V | 16 | 11 |
| VI | 27 | 17 |

We turn now to the consideration of (ii) above: $(\Lambda+\delta, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$. Here, as we have seen, $P^{(\Lambda)}=\Delta_+'=\Delta_k^+ \cup -\Delta_n^+$, $\delta^{(\Lambda)}=\delta'=\delta_k-\delta_n$, $Q_\Lambda=\phi$; we assume that $\Lambda$ is integral and $\Delta_k^+$ dominant. Let

(3.11) $$B_\Lambda = \{\alpha\in\Delta_n^+ \,|\, (\Lambda+2\delta_n, \alpha)<0\} \,.$$

Then the set $A_\Lambda$ in the statement of Theorem 2.3 is given by $A=\{\alpha\in -\Lambda_n^+ \,|\, (\Lambda+2\delta_n, \alpha)>0\}=-B_\Lambda$. Hence by Theorem 2.3 and Theorem 2.3' we get

**Theorem 3.12.** *Suppose $\Lambda$ is integral, $\Delta_k^+$ dominant, and satisfies $(\Lambda+\delta, \alpha)$ $<0$ for every $\alpha\in\Delta_n^+$. Then*

(i) $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda})=0$ *for* $q>n-|B_\Lambda|$ *(see 3.11) where* $n=|\Delta_n^+|= \dim_C G/K$.

(ii) *If* $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda})\neq0$ *then a parabolic subalgebra* $\theta=\mathfrak{u}+\mathfrak{m}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{h}+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ *with* $\mathfrak{u}=$*the unipotent radical of* $\theta$ *and* $\mathfrak{m}=$*the reductive component of* $\theta$ *such that* (a) $n-q=|\theta_{u,n}|$, $\theta_{u,n}=$*set of non-compact roots in* $u$, *and* (b) $(\Lambda+2\delta_n, \kappa\alpha)=0$ *for every root* $\alpha$ *in* $\mathfrak{m}$; $\kappa=$*unique element of* $W_G$ *such that* $\kappa\Delta_k^+=-\Delta_k^+$.

REMARK. In Theorem 3.12 $G$ of course is *not* assumed to be simple.

For $\Lambda$ integral and $\Delta_k^+$ dominant, consider the following three assumptions:

(X)  $(\Lambda+2\delta_n, \alpha)\leqslant0$ for every $\alpha$ in $\Delta_n^+$

(Y)  $(\Lambda+2\delta_n, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$

(Z)  $(\Lambda+2\delta, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$.

One has that $(Z)\Rightarrow(Y)\Rightarrow(X)\Rightarrow(\Lambda+\delta, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$ (using that $\delta= 2\delta_n+\delta'$). In cases (Z) and (Y), $B_\Lambda=\Delta_n^+$ in (3.11). Hence by (i) of Theorem 3.12 we obtain the following result of Hotta-Parthasarathy [5] and M. Ise [7].

**Corollary 3.13.** *Suppose $\Lambda$ is integral and $\Delta_k^+$ dominant. If $\Lambda$ satisfies either (Y) or (Z), then $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda})=0$ for $q>0$; $(Z)\Rightarrow(Y)$.*

REMARK: In Corollary 2 page 231 of [5], Hotta and Parthasarathy assume that (Y) holds and that $\Lambda\in\mathcal{F}_0'$ such that $(\Lambda+\delta, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$. However as we have just observed, $(Y)\Rightarrow\Lambda\in\mathcal{F}_0'$ and that $(\Lambda+\delta, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$. Thus the latter two assumptions are superfluous. In particular the Hotta-Parthasarathy multiplicity formula of Corollary 2 for holomorphic discrete series representations is valid under assumption (Y) only.

In case (X), (i) of Theorem 3.12 implies that $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda})=0$ for $q>|\{\alpha\in\Delta_n^+ \,|\, (\Lambda+2\delta_n, \alpha)=0\}|$. From (ii) of Theorem 3.12 we obtain

**Theorem 3.14.** *Suppose $G$ is simple as in Table 3.4 and $\Lambda$ is integral, $\Delta_k^+$ dominant, and satisfies $(\Lambda+\delta, \alpha)<0$ for every $\alpha$ in $\Delta_n^+$. Then $H^q(\Gamma\,|\,G/K, \theta_{\tau_\Lambda})=0$*

unless $n-q$ belongs to the set $\{|\theta_{u,n}| \, | \, \theta \supset \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha\}$ corresponding to $G$ in the Table 3.4. Again $n = |\Delta_n^+| = \dim_C G/K$.

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