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ON p-RADICAL DESCENT OF HIGHER EXPONENT

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0. Introduction

In the paper [8], P. Samuel has developed the theory of p-radical descent of exponent one by making use of logarithmic derivatives. In this article we shall give a generalization of his theory to the case of p-radical descent of higher exponent with the aid of a finite set of higher derivations of finite rank.

In the first section some preparatory results are collected. Let A be a Krull domain of characteristic $p>0$ and K be its quotient field. Let $\boldsymbol{D}=(\underline{D}^{(1)}, \cdots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations $\underline{D}^{(i)}$'s of rank m_i on K which leave A invariant. For simplicity we shall abuse the notation $\underline{D}^{(i)}$ to denote the ring homomorphism of K into a truncated polynomial ring of order m_i over K, i.e., $K[t_i: m_i]: =K[T_i]/T_i^{m_i+1}$ associated to the higher derivation $\underline{D}^{(i)}$. Let K' be the intersection of the fields of $\underline{D}^{(i)}$-constants $(1\leq i\leq r)$ and let $A': = A\cap K'$. Let $\boldsymbol{T}=(T_1, \cdots, T_r)$ be an r-ruple of indeterminates and let t_i be the residue class of T_i modulo $T_i^{m_i+1}$ in $K[T_i]/T_i^{m_i+1}$. We shall set $\boldsymbol{t}: =(t_1, \cdots, t_r)$ and $\boldsymbol{m}: =(m_1, \cdots, m_r)$. We shall denote $\prod_{i=1}^{r} K[t_i: m_i]$ by $K[\boldsymbol{t}: \boldsymbol{m}]$. Similarly we denote $\prod_{i=1}^{r} A[t_i: m_i]$ by $A[\boldsymbol{t}: \boldsymbol{m}]$ where $A[t_i: m_i]$ is a truncated polynomial ring of order m_i over A. Furthermore we shall define a ring homomorphism $\boldsymbol{D}$ of K into $K[\boldsymbol{t}: \boldsymbol{m}]$ by $\boldsymbol{D}(z)=(D^{(1)}(z), \cdots, \underline{D}^{(r)}(z))$ $(z\in K)$. Let $\mathcal{L}_A$ and $\mathcal{L}'_A$ be the sets of elements defined respectively by

$$\mathcal{L}_A = \{\boldsymbol{D}(z)/z\in K[\boldsymbol{t}: \boldsymbol{m}]\,|\, z\in K^*,\ \boldsymbol{D}(z)/z\in A[\boldsymbol{t}: \boldsymbol{m}]\}\ ,$$
$$\mathcal{L}'_A = \{\boldsymbol{D}(u)/u\,|\,u\in A^*\}\ .$$

Let $\boldsymbol{j}$: $\mathrm{Div}(A') \rightarrow \mathrm{Div}(A)$ be the homomorphism defined by $\boldsymbol{j}(\mathcal{G})=e(\mathcal{P})\mathcal{P}$ where, $\mathcal{G}$ is a prime ideal of height one in A', $\mathcal{P}$ is the unique prime ideal of height one in A with $\mathcal{P}\cap A'=\mathcal{G}$ and $e(\mathcal{P})$ is the ramification index of $\mathcal{P}$ over $\mathcal{G}$. Then we can define the homomorphism $\bar{\boldsymbol{j}}$: $\mathrm{Cl}(A')\rightarrow \mathrm{Cl}(A)$ induced by $\boldsymbol{j}$ (cf. [8]). Let $\mathcal{D}$ be the subgroup of $\mathrm{Div}(A')$ consisting of divisors E's such that $\boldsymbol{j}(E)$ is principal and let Φ_0: $\mathcal{D}\rightarrow\mathcal{L}_A/\mathcal{L}'_A$ be the homomorphism defined by $\Phi_0(E)=\boldsymbol{D}(x)/x$ modulo $\mathcal{L}'_A$, where $E\in\mathcal{D}$ and $\boldsymbol{j}(E)=\mathrm{div}_A(x)$. Let Φ: $\mathrm{Ker}(\bar{\boldsymbol{j}})=\mathcal{D}/F(A')\rightarrow\mathcal{L}_A/\mathcal{L}'_A$ be the homomorphism induced by Φ_0 where $F(A')$ denotes the subgroup of $\mathrm{Div}(A')$

generated by principal divisors. Furthermore we put $\mu_i = \min\{j \mid D_j^{(i)} \neq 0, 1 \leq j \leq m_i\}$ and, $n_i = \min\{n \mid m_i < \mu_i p^n\}$ where $\underline{D}^{(i)} = \{D_j^{(i)} \mid 0 \leq j \leq m_i\}$ $(1 \leq j \leq r)$. We denote the Jacobian $\det(D_{\mu_i}^{(i)}(\alpha_k))_{s \leq i, k \leq r}$ by $J(\boldsymbol{D}: \boldsymbol{a}; s, r)$ for $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_r) \in A^r$ and $1 \leq s \leq r$. We shall use the notation $J(\boldsymbol{D}: \boldsymbol{a})$ instead of $J(\boldsymbol{D}: \boldsymbol{a}; 1, r)$. Our main result in §1 is the following:

Theorem (cf. 1.6). *Assume that the following two conditions hold*:

(1) $[K: K'] = p^{n_1 + \cdots + n_r}$.

(2) *For each prime ideal $\mathcal{P}$ of height one in A, there exists $\boldsymbol{a}$ in A^r such that the Jacobian $J(\boldsymbol{D}: \boldsymbol{a})$ is not contained in $\mathcal{P}$.*

Then the homomorphism $\Phi: Ker(\bar{\boldsymbol{j}}) \to \mathcal{L}_A / \mathcal{L}'_A$ is an isomorphism.

The property (2) in the above theorem will be referred to as "the height one property". When the height one property is not satisfied, Φ is not necessarily surjective. Even if Φ is not surjective, we can determine, in some cases, the cokernel of Φ (§2). As a byproduct we get the following:

Theorem (cf. 2.7). *Assume that A is a unique factorization domain with $J(\boldsymbol{D}: A) := \{J(\boldsymbol{D}: \boldsymbol{a}) \mid \boldsymbol{a} \in A^r\} \neq \{0\}$ and $[K: K'] = p^{n_1 + \cdots + n_r}$. Let $\mathcal{P} = cA$ be a principal prime ideal of height one in A and let $s^{(i)}(\mathcal{P}) := \min\{s \in \boldsymbol{N} \mid (\underline{D}^{(i)}(c)/c)^s \in A[t_i: m_i]\}$ for $1 \leq i \leq r$, and $s(\mathcal{P}) := \max\{s^{(i)}(\mathcal{P}) \mid 1 \leq i \leq r\}$. Then the followings are equivalent to each other*:

(i) *$\Phi: Ker(\bar{\boldsymbol{j}}) \to \mathcal{L}_A / \mathcal{L}'_A$ is an isomorphism.*

(ii) *For each prime ideal $\mathcal{P}$ of height one in A, either $J(\boldsymbol{D}: A) \not\subset \mathcal{P}$ or $e(\mathcal{P}) = s(\mathcal{P})$ occurs.*

If A is a unique factorization domain, it turns out that $\mathrm{Ker}(\bar{\boldsymbol{j}})$ is isomorphic to $\mathrm{Cl}(A')$. Therefore, in order to determine $\mathrm{Cl}(A')$, it suffices to know $\mathrm{Ker}(\bar{\boldsymbol{j}})$. In the final section some examples of rings are presented whose divisor class groups are calculated by applying Theorem 1.6.

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Each ring appeared in this paper is commutative with identity. Our terminology and notation are as follows:

Let A be a Krull domain.

$P(A)$: the set of prime ideals of height one in A.

$\mathrm{Div}(A)$: the free abelian group generated by elements of $P(A)$. An element of $\mathrm{Div}(A)$ is called a divisor.

We shall define the divisor $\mathrm{div}_A(a)$ $(a \in A - \{0\})$ by $\mathrm{div}_A(a) = \sum v_{\mathcal{P}}(a)\mathcal{P}$ where the sum is taken over all prime ideals $\mathcal{P}$'s in $P(A)$ and $v_{\mathcal{P}}$ is the normalized valuation associated to the prime ideal $\mathcal{P}$. Let K be the quotient field of A and x be an element of K^*. We define $\mathrm{div}_A(x) := \mathrm{div}_A(a) - \mathrm{div}_A(b)$ where $x = a/b$ $(a, b \in A, b \neq 0)$.

$F(A)$: the subgroup of $\mathrm{Div}(A)$ generated by $\{\mathrm{div}_A(x) \mid x \in K^*\}$. We call an element of $F(A)$ a principal divisor.

$\mathrm{Cl}(A) := \mathrm{Div}(A)/F(A)$: the divisor class group of A.

$\mathrm{cl}(E)$: the divisor class of a divisor E.

$\mathrm{Supp}(E)$: the support of a divisor E, i.e., the set of all prime ideals $\mathcal{P}$'s in $P(A)$ such that $E = \sum n_{\mathcal{P}} \mathcal{P}$ and $n_{\mathcal{P}} \neq 0$.

1. Fundamental theorem

Let A and B be commutative rings with common identity such that $A \subset B$. A higher derivation $\underline{D} = \{D_j \mid 0 \leq j \leq m\}$ of rank m of A into B is a collection of additive homomorphisms of A into B satisfying the following conditions:

(1) $D_0(a) = a$ for all a in A.

(2) $D_n(ab) = \sum_{j=0}^{n} D_j(a) D_{n-j}(b)$

for $0 \leq n \leq m$ and $a, b \in A$ (cf. [5], [6]).

Let $B[t:m]$ be a truncated polynomial ring of order m over B, i.e., $B[t:m] = B[T]/T^{m+1}$. We can define the ring homomorphism $\phi_{\underline{D}}$ of A into $B[t:m]$ associated to a higher derivation $\underline{D}$ by the following manner:

$$\phi_{\underline{D}}(a) = \sum_{j=0}^{m} D_j(a) t^j \qquad \text{for} \quad a \in A .$$

For simplicity we shall abuse the notation $\underline{D}$ to denote the ring homomorphism $\phi_{\underline{D}}$ when there is no fear of confusion. If $\underline{D}(a) = a$, a is called a $\underline{D}$-constant. We say that $\underline{D}$ is non-trivial if there exists an element in A which is not a $\underline{D}$-constant. For a non-trivial higher derivation $\underline{D}$, the smallest integer among those j such that $D_j \neq 0$ for $1 \leq j \leq m$ is denoted by $\mu(\underline{D})$. Let C be a subset of A. We say that $\underline{D}$ leaves C invariant if $D_j(C) \subset C$ for $1 \leq j \leq m$. Let $\underline{D}^{(i)}$ be a higher derivation of rank m_i of A into B for $1 \leq i \leq r$. Let $\boldsymbol{T} = (T_1, \cdots, T_r)$ be an r-tuple of indeterminates $T_1, \cdots, T_r$ and let $\boldsymbol{t} := (t_1, \cdots, t_r)$ where t_i is the residue class of T_i modulo $T_i^{m_i+1}$ in $B[T_i]/T_i^{m_i+1}$. We shall denote $\prod_{i=1}^{r} B[t_i : m_i]$ by $B[\boldsymbol{t} : \boldsymbol{m}]$ where $\boldsymbol{m} := (m_1, \cdots, m_r)$. Then $B[\boldsymbol{t} : \boldsymbol{m}]$ is a B-algebra in the usual way. Let $\boldsymbol{D} = (\underline{D}^{(1)}, \cdots, \underline{D}^{(r)})$ be an r-tuple of higher derivations of rank $\boldsymbol{m}$ of A into B. A ring homomorphism $\boldsymbol{D}$ of A into $B[\boldsymbol{t} : \boldsymbol{m}]$ is defined by $\boldsymbol{D}(a) = (\underline{D}^{(1)}(a), \cdots, \underline{D}^{(r)}(a))$ $(a \in A)$. The intersection of $\underline{D}^{(i)}$-constants for $1 \leq i \leq r$ is called the ring of $\boldsymbol{D}$-constants. First we shall prove two lemmas:

Lemma 1.1. *Let $A \subset B$ be integral domains of characteristic $p > 0$ and let $\underline{D} = \{D_j \mid 0 \leq j \leq m\}$ be a non-trivial higher derivation of rank m of A into B. Set $\mu := \mu(\underline{D})$ and $d_i := D_{\mu p^i}$. Then $d_s(\alpha^{p^k}) = 0$ if $s < k$ and $d_s(\alpha^{p^k}) = d_{s-k}(\alpha)^{p^k}$ if $s \geq k$ $(\alpha \in A,\ \mu p^s \leq m)$.*

Proof. The proof is easy, hence we omit it. Q.E.D.

Lemma 1.2. *Let $M=(a_{ij})_{1\leq i,j\leq r}$ be a non-singular matrix. Then after a suitable change of columns we can bring M into the one such that every $M^{(k)}$ $(1\leq k\leq r)$ is a non-singular matrix where*

$$M^{(k)}=\begin{pmatrix} a_{kk}\cdots a_{kr} \\ \cdots \\ a_{rk}\cdots a_{rr}\end{pmatrix}.$$

Proof. Let α_{ij} be the cofactor of a_{ij}. Then $\det M=a_{11}\alpha_{11}+a_{12}\alpha_{12}+\cdots+a_{1r}\alpha_{1r}$. Since $\det M$ does not vanish, $\alpha_{1j'}\neq 0$ for some j'. Interchanging the first column with the j'-th column, we may assume $\alpha_{11}\neq 0$, i.e., $\det M^{(2)}\neq 0$. Continuing this process we will arrive at the desired situation. Q.E.D.

Let $\boldsymbol{D}=(\underline{D}^{(1)},\cdots,\underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations of rank $\boldsymbol{m}=(m_1,\cdots,m_r)$. We shall set $\mu_i;=\mu(\underline{D}^{(i)})$ and $n_i:=\min\{n\in\boldsymbol{N}\,|\,m_i<\mu_i p^n\}$ where $\boldsymbol{N}$ denotes the set of positive integers. Furthermore we shall set $n(\boldsymbol{D})=n_1+\cdots+n_r$. Then $D^{(i)}_{\mu_i}$ is a derivation. We denote the Jacobian $\det(D^{(i)}_{\mu_i}(\alpha_k))$ by $J(\boldsymbol{D}\colon\boldsymbol{\alpha})$ for $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_r)\in A^r$. Let $\boldsymbol{T}=(T_1,\cdots,T_r)$ be an r-tuple of indeterminates $T_1,\cdots,T_r$. We shall denote $(T_1^{\mu_1 p^j},\cdots,T_r^{\mu_r p^j})$ by $\boldsymbol{T}^{p^j\boldsymbol{\mu}}$ where $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_r)\in\boldsymbol{Z}^r$.

Proposition 1.3. *Let $L\subset F$ be fields of characteristic $p>0$ and let $\boldsymbol{D}=(\underline{D}^{(1)},\cdots,\underline{D}^{(r)})$ be an r-tuple of higher derivations of rank $\boldsymbol{m}=(m_1,\cdots,m_r)$ of L into F. Let L' be the field of $\boldsymbol{D}$-constants. Suppose that there exists an element $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_r)$ in L^r such that the Jacobian $J(\boldsymbol{D}\colon\boldsymbol{\alpha})$ does not vanish. Then we have $[L\colon L']\geq p^{n(\boldsymbol{D})}$. Furthermore if the equality holds, then $L=L'[\alpha_1,\cdots,\alpha_r]$.*

Proof. (I) First we shall prove the Proposition in the case $n:=n_1=\cdots=n_r$. Let L_j be a subfield of L defined by $\{z\in L\,|\,\boldsymbol{D}(z)=(z,\cdots,z)\bmod \boldsymbol{T}^{p^j\boldsymbol{\mu}}\}$ for $1\leq j\leq n$. Then we have $L_0\supset L_1\supset\cdots\supset L_n$ where we put $L_0:=L$ and $L_n:=L'$. It suffices to show that $[L_{j-1}\colon L_j]\geq p^r$ for $1\leq j\leq n$. For simplicity we shall set $d_j^{(i)}:=D^{(i)}_{\mu_i p^j}$. From the definition of L_{j-1}, the restriction of $d^{(i)}_{j-1}$ to L_{j-1} is a derivation of L_{j-1} for $1\leq i\leq r$. Let $\tilde{L}_{j-1}$ be the intersection of the kernels of these derivations. Then we have $L_{j-1}\supset\tilde{L}_{j-1}\supset L_j$. By Lemma 1.1 we know $J(\boldsymbol{D}|L_{j-1}\colon\boldsymbol{\alpha}^{p^{j-1}})=J(\boldsymbol{D}\colon\boldsymbol{\alpha})^{p^{j-1}}\neq 0$ and $\boldsymbol{\alpha}^{p^{j-1}}\in L^r_{j-1}$. Hence these derivations are linearly independent over F. This implies that $[L_{j-1}\colon\tilde{L}_{j-1}]\geq p^r$, hence $[L_{j-1}\colon L_j]\geq p^r$. From our argument we get the following sequence:

$$L_{j-1}\supset L_j^{\sharp}:=L_j[\alpha_1^{p^{j-1}},\cdots,\alpha_r^{p^{j-1}}]\supset L_j$$

for $1\leq j\leq n$. To prove the latter half, assume that $[L\colon L']=p^{nr}$. Then we have $[L_{j-1}\colon L_j]=p^r$. Since $d^{(i)}_{j-1}|L_j^{\sharp}$ $(1\leq i\leq r)$ are linearly independent over F, $[L_j^{\sharp}\colon L_j]\geq p^r$. Therefore we see that $L_{j-1}=L_j^{\sharp}$ for $1\leq j\leq n$, hence $L=$

$L'[\alpha_1, \cdots, \alpha_r]$.

(II) Next we shall prove the general case. Without loss of generality we may assume that $n_1 \leq n_2 \leq \cdots \leq n_r$. Moreover by Lemma 1.2 we may assume that $J(\boldsymbol{D}: \boldsymbol{\alpha}; k, r) \neq 0$ for $1 \leq k \leq r$. This implies that for every k there exists an integer k' such that $d_0^{(k)}(\alpha_{k'}) \neq 0$ and $k \leq k' \leq r$. Let $\bar{n}_1 < \cdots < \bar{n}_\rho$ be integers with the property $\{n_1, \cdots, n_r\} = \{\bar{n}_1, \cdots, \bar{n}_\rho\}$ and let $r_\lambda := \# \{i \mid n_i = \bar{n}_\lambda, 1 \leq i \leq r\}$ for $1 \leq \lambda \leq \rho$. Then we know

$$r_1 + r_2 + \cdots + r_\rho = r\,,$$
$$r_1\bar{n}_1 + r_2\bar{n}_2 + \cdots + r_\rho\bar{n}_\rho = n_1 + n_2 + \cdots + n_r\,.$$

For convenience sake we put $r_0 := 0$, $\bar{n}_0 := 0$ and $\delta_\lambda := r_0 + \cdots + r_\lambda$. Let K_λ be the subfield of L defined by

$$\begin{aligned}\{z \in L \mid \underline{D}^{(h)}(z) &\equiv z \mod T_h^{m_h+1} \qquad (1 \leq h \leq \delta_\lambda)\,,\\ \underline{D}^{(l)}(z) &\equiv z \mod T_l^{w_l} \qquad (w_l = \mu_l p^{\bar{n}_\lambda}\,,\ \delta_\lambda < l \leq r)\}\end{aligned}$$

for $1 \leq \lambda \leq \rho - 1$ (note that $n_l \geq \bar{n}_{\lambda+1} > \bar{n}_\lambda$). Then we have

$$K_0 := L \supset K_1 \supset \cdots \supset K_{\rho-1} \supset K_\rho := L'\,.$$

We shall claim the following inequality for $1 \leq \lambda \leq \rho$:

$$[K_{\lambda-1} : K_\lambda] \geq p^{\varepsilon_\lambda}$$

where $\varepsilon_\lambda := (r - \delta_{\lambda-1})(\bar{n}_\lambda - \bar{n}_{\lambda-1})$. Let $\underline{\Delta}^{(i)}$ be the restriction of $\underline{D}^{(i)}$ to $K_{\lambda-1}$. Then for $1 \leq \lambda < \rho$, $\underline{\Delta}^{(i)}$ is a higher derivation of $K_{\lambda-1}$ into F of rank m_i for $\delta_{\lambda-1} < i \leq \delta_\lambda$ and of rank $w_i - 1$ for $\delta_\lambda < i \leq r$ respectively. For $\lambda = \rho$, $\underline{\Delta}^{(i)}$ is a higher derivation of $K_{\lambda-1}$ into F of rank m_i for $\delta_{\rho-1} < i \leq r$. The following five assertions are easily verified:

(1) $K_\lambda = \bigcap_{i=\delta_{\lambda-1}+1}^{r}$ (the field of $\underline{\Delta}^{(i)}$-constants).

(2) $\mu(\underline{\Delta}^{(i)}) = \mu_i p^{\bar{n}_{\lambda-1}} \qquad (\delta_{\lambda-1} < i \leq r)$.

(3) For $1 \leq \lambda \leq \rho$,

$$\min \{s \in \boldsymbol{N} \mid m_u < \mu_u p^{\bar{n}_{\lambda-1}+s}\} = \bar{n}_\lambda - \bar{n}_{\lambda-1} \qquad (\delta_{\lambda-1} < u \leq \delta_\lambda)\,.$$

For $1 \leq \lambda < \rho$,

$$\min \{s \in \boldsymbol{N} \mid \mu_v p^{\bar{n}_\lambda} \leq \mu_v p^{\bar{n}_{\lambda-1}+s}\} = \bar{n}_\lambda - \bar{n}_{\lambda-1} \qquad (\delta_\lambda < v \leq r)$$

where $\boldsymbol{N}$ denotes the set of positive integers.

(4) $\alpha_i^q \in K_{\lambda-1}$ where $q := p^{\bar{n}_{\lambda-1}}$ $(\delta_{\lambda-1} < i \leq r)$.

(5) $J(\Delta : \boldsymbol{\alpha}^q; \delta_{\lambda-1}+1, r) = J(\boldsymbol{D}: \boldsymbol{\alpha}; \delta_{\lambda-1}+1, r)^q \neq 0$ where $\Delta = (\underline{\Delta}^{(1)}, \cdots, \underline{\Delta}^{(r)})$.

Therefore we get $[K_{\lambda-1} : K] \geq p^{\varepsilon_\lambda}$. Furthermore $\sum_{\lambda=1}^{\rho} \varepsilon_\lambda = n_1 + \cdots + n_r = n(\boldsymbol{D})$.

Hence we have $[L: L'] \geq p^{n(D)}$. In order to prove the latter half, it suffices to prove the following: $K_{\lambda-1}=\tilde{K}_\lambda$ where $\tilde{K}_\lambda := K_\lambda[\alpha_i^p; \delta_{\lambda-1}<i\leq r]$ for $1\leq\lambda\leq\rho$. Since $[L: L']=p^{n(D)}$, we have $[K_{\lambda-1}: K_\lambda]=p^{\varepsilon_\lambda}$. Applying the step (I) to $\tilde{K}_\lambda$ and $\underline{\Delta}^{(i)}|\tilde{K}_\lambda$ $(\delta_{\lambda-1}<i\leq r)$, it is seen that $[\tilde{K}_\lambda: K_\lambda]\geq p^{\varepsilon_\lambda}$. Since $K_{\lambda-1}\supset\tilde{K}_\lambda\supset K_\lambda$, we have $K_{\lambda-1}=\tilde{K}_\lambda$. Q.E.D.

REMARK 1.4. The converse of the latter half of the Proposition 1.3 does not hold. Let k be a field of characteristic $p>0$. Let x, y be indeterminates over k and let $L:=k(x, y)$. Let $\underline{D}^{(i)}$ $(i=1, 2)$ be higher derivations on L over k of rank $p-1$ and p^2-1 defined respectively by

$$\underline{D}^{(1)}(x) = x(1+t_1)\,, \quad \underline{D}^{(1)}(y) = y+t_1\,,$$
$$\underline{D}^{(2)}(x) = x+t_2\,, \quad \underline{D}^{(2)}(y) = y(1+t_2)\,.$$

Then $n_1=1$, $n_2=2$ and $J(\boldsymbol{D}: (x, y))=xy-1\neq 0$. By a simple calculation we see that $L'=k(x^{p^2}, y^{p^2})$. Therefore $L=L'[x, y]$, while $[L: L']=p^4>p^{n_1+n_2}$.

(1.5) Let A be a Krull domain of characteristic $p>0$ with the quotient field K. Let $\boldsymbol{D}=(\underline{D}^{(1)}, \cdots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations of rank $\boldsymbol{m}=(m_1, \cdots, m_r)$ on K which leave A invariant. Let K' be the field of $\boldsymbol{D}$-constants and $A':=A\cap K'$. Then A' is also a Krull domain. Since any element of K is of the form a/b with $a\in A$, $b\in A'$, K' is the quotient field of A'. For any prime ideal $\mathcal{Q}$ in $P(A')$, there exists only one prime ideal $\mathcal{P}$ in $P(A)$ such that $\mathcal{P}\cap A'=\mathcal{Q}$. From this fact we define the homomorphism $\boldsymbol{j}: \mathrm{Div}(A')\to \mathrm{Div}(A)$ by $\boldsymbol{j}(\mathcal{Q})=e(\mathcal{P})\mathcal{P}$ where $e(\mathcal{P})$ stands for the ramification index of $\mathcal{P}$ over $\mathcal{Q}$. Since A is integral over A', we can define the canonical homomorphism $\bar{\boldsymbol{j}}: \mathrm{Cl}(A')\to\mathrm{Cl}(A)$ induced by the homomorphism $\boldsymbol{j}$ (cf. [8]).

Let $\mathcal{L}_A$ and $\mathcal{L}'_A$ be sets of elements defined respectively by

$$\mathcal{L}_A:= \{\boldsymbol{D}(z)/z\in K[\boldsymbol{t}: \boldsymbol{m}]\,|\,z\in K^*,\ \boldsymbol{D}(z)/z\in A[\boldsymbol{t}: \boldsymbol{m}]\}\,,$$
$$\mathcal{L}'_A:= \{\boldsymbol{D}(u)/u\,|\,u\in A^*\}$$

where * denotes the set of invertible elements. Since we have

$$(\boldsymbol{D}(z_1)/z_1)(\boldsymbol{D}(z_2)/z_2) = \boldsymbol{D}(z_1z_2)/z_1z_2$$

and

$$(\boldsymbol{D}(z)/z)^{-1} = \boldsymbol{D}(z^{-1})/z^{-1} \qquad (z\neq 0)\,,$$

$\mathcal{L}_A$ is an abelian group and $\mathcal{L}'_A$ is its subgroup.

Let $\mathcal{D}$ be the subgroup of $\mathrm{Div}(A')$ consisting of divisors E's such that $\boldsymbol{j}(E)$ is principal. Then we get $\mathrm{Ker}(\bar{\boldsymbol{j}})=\mathcal{D}/F(A')$. We shall define the homomorphism Φ_0 of $\mathcal{D}$ into $\mathcal{L}_A/\mathcal{L}'_A$ by the following manner: Let E be a divisor of $\mathcal{D}$ and x be an element of K^* satisfying $\boldsymbol{j}(E)=\mathrm{div}_A(x)$. Then we set $\Phi_0(E):=\boldsymbol{D}(x)/x$ modulo $\mathcal{L}'_A$. It is easily seen that Φ_0 is well-defined. Moreover if x'

is in K', $\Phi_0(\mathrm{div}_{A'}(x'))=\boldsymbol{D}(x')/x'=\mathbf{1}$ where $\mathbf{1}=(1, \cdots, 1)\in A^r$, hence the homomorphism Φ of $\mathrm{Ker}(\bar{\boldsymbol{j}})$ into $\mathcal{L}_A/\mathcal{L}'_A$ induced by the homomorphism Φ_0 is also well-defined. On the other hand, the relation $\boldsymbol{D}(x)/x=\boldsymbol{D}(u)/u$ $(x\in K^*, u\in A^*)$ implies $\boldsymbol{D}(xu^{-1})/xu^{-1}=\mathbf{1}$, i.e., $xu^{-1}\in K'$ and $E=\mathrm{div}_{A'}(xu^{-1})$. This implies that Φ is injective (cf. [8], p. 86). Set $\boldsymbol{\mu}:=(\mu_1, \cdots, \mu_r)$ and $\boldsymbol{n}(\boldsymbol{D}):=\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r$ where $\mu_i:=\mu(\underline{D}^{(i)})$ and $\boldsymbol{n}_i:=\min\{n\in \boldsymbol{N}\,|\,m_i<\mu_i p^n\}$ $(1\leq i\leq r)$.

Theorem 1.6. *Let A, K, K', $\boldsymbol{D}$ and $n(\boldsymbol{D})$ have the same meaning as in* 1.5. *Assume the following two conditions hold:*

(1) $[K: K']=p^{n(\boldsymbol{D})}$.

(2) *For each prime ideal $\mathcal{P}$ in $P(A)$, there exists an element $\boldsymbol{\alpha}$ in A^r such that the Jacobian $J(\boldsymbol{D}: \boldsymbol{\alpha})$ is not contained in $\mathcal{P}$.*

Then the homomorphism Φ: $Ker(\bar{\boldsymbol{j}})\to\mathcal{L}_A/\mathcal{L}'_A$ is an isomorphism.

Proof. Since Φ is injective, it suffices to prove the following: If $\boldsymbol{D}(x)/x$ is in $\mathcal{L}_A$ $(x\in K^*)$, then there exists a divisor E in $\mathcal{D}$ such that $\boldsymbol{j}(E)=\mathrm{div}_A(x)$. Set $n:=\max\{n_1, \cdots, n_r\}$. Note that for each prime ideal $\mathcal{Q}$ in $P(A')$ there exists a unique prime ideal in $P(A)$ which contracts to $\mathcal{Q}$ because $A^{p^n}\subset A'$. Therefore the surjectivity of Φ is seen by showing that if $\boldsymbol{D}(x)/x$ is in $\mathcal{L}_A$ $(x\in K^*)$, then $e(\mathcal{P})$ divides $v_{\mathcal{P}}(x)$ for every prime ideal $\mathcal{P}$ in $P(A)$ where $v_{\mathcal{P}}(x)$ denotes the normalized valuation of K associated to the prime ideal $\mathcal{P}$. Hence by localizing, we may assume that A is a discrete valuation ring with the maximal ideal $\mathcal{P}$. Thus we have only to show the following:

Proposition 1.7. *Let A be a discrete valuation ring with the maximal ideal $\mathcal{P}$ and let K, K', $\boldsymbol{D}$ and $n(\boldsymbol{D})$ have the same meaning as in* 1.5. *Assume that the following two conditions hold*:

(1) $[K: K']=p^{n(\boldsymbol{D})}$.

(2) *There exists an element $\boldsymbol{\alpha}$ in A^r such that the Jacobian $J(\boldsymbol{D}: \boldsymbol{\alpha})$ is not contained in $\mathcal{P}$.*

If $\boldsymbol{D}(x)/x$ is in $\mathcal{L}_A$ $(x\in K^*)$, then e divides $v(x)$ where we put $e:=e(\mathcal{P})$ and v is the normalized valuation of K associated to A.

Proof. Our proof consists of several steps:

(I) First we shall consider the case $m_i=1$ (hence $\mu_i=n_i=1$) for $1\leq i\leq r$. We shall set $\underline{D}^{(i)}=\{id, D^{(i)}\}$. Then $D^{(i)}$'s are derivations. We shall define the higher derivation $\underline{\Delta}^{(i)}=\{id, \Delta^{(i)}\}$ of rank 1 on K in the following way:

$$\Delta^{(i)}(z)=J^{-1}\det\begin{pmatrix} D^{(1)}(\alpha_1), & \cdots, & \overset{i}{\smile}{D^{(1)}(z)}, & \cdots, & D^{(1)}(\alpha_r) \\ & \cdots & & \cdots & \\ D^{(r)}(\alpha_1), & \cdots, & D^{(r)}(z), & \cdots, & D^{(r)}(\alpha_r) \end{pmatrix}$$

for $z\in K$ $(1\leq i\leq r)$ where $J:=J(\boldsymbol{D}:\boldsymbol{\alpha})$. Then we have $\Delta^{(i)}(\alpha_k)=\delta_{ik}$ where δ_{ik} denotes the Kronecker's delta $(1\leq i, k\leq r)$. Since J is not in $\mathcal{P}$, J is a unit of A, hence $\Delta^{(i)}(A)\subset A$ for $1\leq i\leq r$. Set $\Delta:=(\underline{\Delta}^{(1)}, \cdots, \underline{\Delta}^{(r)})$. Since $\Delta^{(i)}$ is an A-linear combination of $D^{(k)}$'s and $D^{(k)}$ is also an A-linear combination of $\Delta^{(k)}$'s, we have the following three assertions:

(1) K' is the field of Δ-constants.

(2) $J(\Delta:\boldsymbol{\alpha})=1$.

(3) $\boldsymbol{\Delta}(x)/x\in\mathcal{L}_A$.

Hence it suffices to prove the Proposition with respect to Δ instead of $\boldsymbol{D}$. We shall prove that e divides $v(x)$ by induction on r. As is well known e takes no other value than some power of p. Hence in the case $r=1$, it suffices to prove the following: If p does not divide $v(x)$, then $e=1$.

Let π be a uniformisant of the discrete valuation ring A. Then we can write $x=u\pi^{v(x)}$ for some $u\in A^*$. Since

$$\Delta^{(1)}(u)/u+v(x)\Delta^{(1)}(\pi)/\pi = \Delta^{(1)}(x)/x\in A$$

and since p does not divide $v(x)$, we have $\Delta^{(1)}(\pi)/\pi\in A$. This implies that we can define the derivation $\tilde{\Delta}^{(1)}$ of $A/\mathcal{P}$ induced by $\Delta^{(1)}$. Set $\mathcal{K}:=A/\mathcal{P}$ and $\mathcal{K}':=A'/\mathcal{G}$ where $\mathcal{G}:=\mathcal{P}\cap A'$. Since $\Delta^{(1)}(\alpha_1)=1$ implies $\tilde{\Delta}^{(1)}\neq 0$, we have $[\mathcal{K}:\mathcal{K}']>1$. Therefore from the inequality $e[\mathcal{K}:\mathcal{K}']\leq[K:K']=p$, it follows that $e=1$.

Suppose $r>1$ and the assertion holds for $r-1$. Set $\bar{K}:=$the field of $\underline{\Delta}^{(1)}$-constants and $\bar{A}:=A\cap\bar{K}$. Since $[K:K']=p^r$ and $J(\Delta|\bar{K}:\boldsymbol{\alpha};2,r)=1$, Propositoin 1.3 implies that $[K:\bar{K}]=p$ and $[\bar{K}:K']=p^{r-1}$. Furthermore we have $K=\bar{K}[\alpha_1]$ and $\bar{K}=K'[\alpha_2, \cdots, \alpha_r]$. Then the restriction of $\Delta^{(i)}$ to $\bar{K}$ is a derivation on $\bar{K}$ such that $\Delta^{(i)}(\bar{A})\subset\bar{A}$ for $2\leq i\leq r$. Let e_1 be the ramification index of $\mathcal{P}$ over $\mathcal{P}\cap\bar{A}$. Since $[K:\bar{K}]=p$ and $\Delta^{(1)}(\alpha_1)=1$, e_1 divides $v(x)$ from the argument in the case $r=1$. Therefore we can write $x=uy$ for some u in A^* and y in $\bar{K}^*$. It follows from $\Delta(x)/x=(\Delta(u)/u)(\Delta(y)/y)$ that $\Delta(y)/y\in(A\cap\bar{K})\times[\boldsymbol{t}:\boldsymbol{m}]=\bar{A}[\boldsymbol{t}:\boldsymbol{m}]$. Furthermore $J(\Delta|\bar{K}:\boldsymbol{\alpha};2,r)=1\in\bar{A}^*$ and $\alpha_2, \cdots, \alpha_r\in\bar{A}$. Let e_2 be the ramification index of $\bar{\mathcal{G}}:=\mathcal{P}\cap A$ over $\mathcal{G}':=\mathcal{P}\cap A'$ and $\bar{v}$ be the normalized valuation of K associated to the prime ideal $\bar{\mathcal{G}}$. Apply the induction assumption to $\Delta|\bar{K}$, then we see that e_2 divides $\bar{v}(y)$. On the other hand $v(x)=v(y)=e_1\bar{v}(y)$ and $e=e_1e_2$. Hence e divides $v(x)$

(II) Suppose that $n:=n_1=\cdots=n_r$. We shall prove the Proposition by induction on n. For the case $n=1$, let $\tilde{K}=\{z\in K\,|\,\boldsymbol{D}(z)\equiv(z, \cdots, z) \bmod \boldsymbol{T}^{\mu+1}\}$. Then $K\supset\tilde{K}\subset K'$ and Proposition 1.3 implies that $[K:\tilde{K}]\geq p^r$. Since $[K:K']=p^r$, we get $\tilde{K}=K'$ and e divides $v(x)$ by the previous argument. Suppose that $n>1$ and the Proposition is proved for $n-1$. Let $L_1=\{z\in K\,|\,\boldsymbol{D}(z)\equiv(z, \cdots, z) \bmod \boldsymbol{T}^{p\mu}\}$ and $A_1':=A\cap L_1$. It is easily seen that

(1) $\mu(\underline{D}^{(i)}|L_1)=\mu_i p$.

(2) $\min\{s\in\boldsymbol{N} \mid m_i<\mu_i p^{1+s}\}=n_i-1=n-1$ $(1\leq i\leq r)$.

(3) $J(\boldsymbol{D}|L_1\colon \boldsymbol{\alpha}^p)=J(\boldsymbol{D}\colon \boldsymbol{\alpha})^p\notin \mathcal{Q}_1\colon =\mathcal{P}\cap A'_1$.

(4) $\boldsymbol{\alpha}^p\in A_1^r$.

Hence Proposition 1.3 implies that $[K\colon L_1]=p^r$ and $[L_1\colon K']=p^{(n-1)r}$ because $[K\colon K']=p^{nr}$. We shall prove that the restriction of $\boldsymbol{D}$ to L_1 is an r-tuple of non-trivial higher derivations of rank $\boldsymbol{m}$ on L_1 which leave A'_1 invariant. We know $L_1=K'[\alpha_1^p, \cdots, \alpha_r^p]$ by Proposition 1.3. For any element z in L_1, z is of the form

$$z=\sum_{i_1,\cdots,i_r\in \boldsymbol{Z}_+} c_{i_1\cdots i_r}(\alpha_1^p)^{i_1}\cdots(\alpha_r^p)^{i_r} \ (c_{i_1\cdots i_r}\in K')$$

where $\boldsymbol{Z}_+$ denotes the set of non-negative integers. Therefore we get

$$\boldsymbol{D}(z)=\sum c_{i_1\cdots i_r}\boldsymbol{D}(\alpha_1^p)^{i_1}\cdots \boldsymbol{D}(\alpha_r^p)^{i_r}\,.$$

From Lemma 1.1 and the definition of L_1, it follows that $\boldsymbol{D}(\alpha_k^p)\in L_1[\boldsymbol{t}\colon \boldsymbol{m}]$. This implies that $\boldsymbol{D}(L_1)\subset L_1[\boldsymbol{t}\colon \boldsymbol{m}]$. Since $A'_1=A\cap L_1$, $\boldsymbol{D}|L_1$ becomes an r-tuple of non-trivial higher derivations of rank $\boldsymbol{m}$ on L_1 with the desired property. Let e_1 be the ramification index of $\mathcal{P}$ over $\mathcal{Q}_1$. Let $\tilde{K}$ be a subfield of K defined by $\{z\in K \mid \boldsymbol{D}(z)\equiv(z, \cdots, z) \bmod \boldsymbol{T}^{\mu+1}\}$ where $\boldsymbol{1}=(1, \cdots, 1)$. Then we have $K\supset \tilde{K}\supset L_1$ and Proposition 1.3 implies $[K\colon \tilde{K}]\geq p^r$. Since $[K\colon L_1]=p^r$, we get $\tilde{K}=L_1$ and e_1 divides $v(x)$ by the argument in (I). Hence we can write $x=uy$ for some u in A^* and y in L_1^*. Therefore $\boldsymbol{D}(y)/y\in A_1[\boldsymbol{t}\colon \boldsymbol{m}]$. Let e_2 be the ramification index of $\mathcal{Q}_1$ over $\mathcal{P}\cap A'$ and v' be the normalized valuation of L_1 associated to the prime idea $\mathcal{Q}_1$. By induction hypothesis, we know that e_2 divides $v'(y)$ and therefore e divides $v(x)$.

(III) We shall prove the general case. Without loss of generality we may assume the following:

(1) $n_1\leq n_2\leq\cdots\leq n_r$.

(2) $J(\boldsymbol{D}\colon \boldsymbol{\alpha}; k, r)\notin\mathcal{P}$ for $1\leq k\leq r$.

Let $\bar{n}_1, \cdots, \bar{n}_\rho$ and K_λ have the same meaning as in the step (II) of the proof of Proposition 1.3. We shall use the induction on ρ. The case $\rho=1$ is treated in (II). Suppose that $\rho>1$ and the Proposition is proved for $\rho-1$. Proposition 1.3 and its proof shows $[K_{\lambda-1}\colon K_\lambda]\geq p^{e_\lambda}$. Since $[K\colon K']=p^{n(\boldsymbol{D})}$, we have $[K\colon K_1]=p^{r\bar{n}_1}$ and $[K_1\colon K']=p^{n(\boldsymbol{D})-r\bar{n}_1}$. Let $A_1\colon =A\cap K_1$ and e_1 be the ramification index of $\mathcal{P}$ over $\mathcal{Q}_1\colon =\mathcal{P}\cap A_1$. Then the step (II) implies that e_1 divides $v(x)$. Hence we can write $x=uy$ for some u in A^* and y in K_1^*. Then $\boldsymbol{D}(y)/y\in A_1[\boldsymbol{t}\colon \boldsymbol{m}]$. For $r_1<i\leq r$, we have the followings:

(1) $\mu(\underline{D}^{(i)}|K_1)=\mu_i p^{\bar{n}_1}$.

(2) $\min\{s\in\boldsymbol{N} \mid m_i<\mu_i p^{\bar{n}_1+s}\}=n_i-\bar{n}_1$.

(3) $J(\boldsymbol{D}|K_1\colon \boldsymbol{\alpha}^q; r_1+1, r)=J(\boldsymbol{D}\colon \boldsymbol{\alpha}; r_1+1, r)^q\in A_1^*$ where $q\colon =p^{\bar{n}_1}$.

(4) $\#\{n_i-\bar{n}_1 \mid r_1<i\leq r\}<\rho$.

Let e_2 be the ramification index of $\mathcal{G}_1$ over $\mathcal{P}\cap A'$ and v' be the normalized valuation of K_1 associated to the prime ideal $\mathcal{G}_1$. Then induction hypothesis implies that e_2 divides $v'(y)$, hence e divides $v(x)$. Q.E.D.

2. Cokernel of Φ

We shall retain the same notations and assumptions used in §1, (1.5).

Proposition 2.1. *Let S be a multiplicatively closed subset of A' consisting of prime elements in A. Let H be the subgroup of $Div(A')$ generated by $\mathcal{G}\in P(A')$ such that $\mathcal{G}\cap S\neq\phi$, and L be the subgroup of $\mathcal{L}_A$ generated by the set $\{\boldsymbol{D}(a)/a\in\mathcal{L}_A \mid a\in A\cap A^*_S\}$. Let $L\vee\mathcal{L}'_A$ denote the subgroup of $\mathcal{L}_A$ generated by L and $\mathcal{L}'_A$. Let f be the restriction of Φ to $(H+F(A')/F(A'))\cap Ker(\bar{\boldsymbol{j}})$. Let the homomorphisms $\bar{\boldsymbol{j}}_S$: $Cl(A'_S)\to Cl(A_S)$, Φ_S: $Ker(\bar{\boldsymbol{j}}_S)\to\mathcal{L}_{A_S}/\mathcal{L}'_{A_S}$ be defined in a similar way as $\bar{\boldsymbol{j}}$ and Φ respectively. Then we have the following commutative diagram of exact rows and columns:*

$$\begin{tikzcd}
& 0 & 0 & 0 \\
0 \arrow[r] & \mathrm{Coker}(f) \arrow[u] \arrow[r] & \mathrm{Coker}(\Phi) \arrow[u] \arrow[r] & \mathrm{Coker}(\Phi_S) \arrow[u] \\
0 \arrow[r] & L\vee\mathcal{L}'_A/\mathcal{L}'_A \arrow[u] \arrow[r] & \mathcal{L}_A/\mathcal{L}'_A \arrow[u] \arrow[r] & \mathcal{L}_{A_S}/\mathcal{L}'_{A_S} \arrow[u] \\
0 \arrow[r] & (H+F(A')/F(A'))\cap Ker(\bar{\boldsymbol{j}}) \arrow[u, "f"] \arrow[r] & Ker(\bar{\boldsymbol{j}}) \arrow[u, "\Phi"] \arrow[r] & Ker(\bar{\boldsymbol{j}}_S) \arrow[u, "\Phi_S"] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where $\mathcal{L}_A/\mathcal{L}'_A\to\mathcal{L}_{A_S}/\mathcal{L}'_{A_S}$ is the homomorphism induced by the inclusion $\mathcal{L}_A\to\mathcal{L}_{A_S}$ and $Ker(\bar{\boldsymbol{j}})\to Ker(\bar{\boldsymbol{j}}_S)$ is the natural homomorphism $Cl(A')\to Cl(A'_S)$.

Proof. The homomorphism $\mathrm{Ker}(\bar{\boldsymbol{j}})\to\mathrm{Ker}(\bar{\boldsymbol{j}}_S)$ is well-defined since we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Cl}(A) \arrow[r] & \mathrm{Cl}(A_S) \\
\mathrm{Cl}(A') \arrow[u, "\bar{\boldsymbol{j}}"] \arrow[r] & \mathrm{Cl}(A'_S) \arrow[u, "\bar{\boldsymbol{j}}_S"]
\end{tikzcd}.$$

The middle sequence forms evidently a complex. For any element $\boldsymbol{D}(x)/x\in\mathcal{L}_A\cap\mathcal{L}'_{A_S}$ $(x\in K^*)$, we can write

$$\boldsymbol{D}(x)/x=\boldsymbol{D}(a/s)/(a/s)=\boldsymbol{D}(a)/a$$

for some $a/s\in A_S^*$ ($a\in A$, $s\in S$). Since a/s is a unit of A_S, a is in A_S^*. Hence $\boldsymbol{D}(a)/a$ is in $L\vee\mathcal{L}'_A$ and the middle row is exact. The exactness of the third row is seen as follows:

$$\begin{tikzcd}
0 \arrow[r] & H+F(A')/F(A') \arrow[r] & \mathrm{Cl}(A') \arrow[r] & \mathrm{Cl}(A'_S) \arrow[r] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[u] & \mathrm{Ker}(\bar{\boldsymbol{j}}) \arrow[r] \arrow[u] & \mathrm{Ker}(\bar{\boldsymbol{j}}_S) \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

is commutative where $G=(H+F(A')/F(A'))\cap\mathrm{Ker}(\bar{\boldsymbol{j}})$. Since S is generated by prime elements of A, we have $\mathrm{Cl}(A)\cong\mathrm{Cl}(A_S)$ ([4], Cor. 7.3, [7]). Therefore $\mathrm{Ker}(\bar{\boldsymbol{j}})\to\mathrm{Ker}(\bar{\boldsymbol{j}}_S)$ is surjective. Furthermore $\mathrm{Im}(f)\subset L\vee\mathcal{L}'_A/\mathcal{L}'_A$. The rest is immediate from the Snake lemma ([2], Chap. 1, §1. Prop. 2). Q.E.D.

Proposition 2.2. *Let $\underline{D}=\{D_j\,|\,0\leq j\leq m\}$ be a higher derivation of rank m on A and let $\mathcal{P}$ be a principal prime ideal in $P(A)$, say, $\mathcal{P}=cA$. Let*

$$s_0:=\min\{s\in\boldsymbol{N}\,|\,(\underline{D}(c)/c)^s\in A[t:m]\}$$

and

$$r_0:=\min\{\gamma\in\boldsymbol{N}\,|\,D_\gamma(c)\notin\mathcal{P}\}$$

(if $D_\gamma(c)\in\mathcal{P}$ for all $1\leq\gamma\leq m$, we put $r_0:=m+1$).

Then the following three assertions hold:

(1) *s_0 is a power of p.*

(2) *Write $s_0=p^{\alpha_0}$, then $\alpha_0=\min\{\alpha\in\boldsymbol{Z}_+\,|\,r_0p^\alpha\geq m+1\}$ where $\boldsymbol{Z}_+$ denotes the set of non-negative integers.*

(3) *$(\underline{D}(c)/c)^h\in A[t:m]$ if and only if s_0 divides h.*

Proof. (1) Write $s_0=s'p^\alpha$, $p\nmid s'$. Then it suffices to prove that $s'=1$. In the relation

$$(\underline{D}(c)/c)^{s_0}=(1+\cdots+(D_{r_0}(c)/c)^{p^\alpha}t^{r_0p^\alpha}+\cdots)^{s'},$$

the coefficient of $t^{r_0p^\alpha}$ is of the form $s'(D_{r_0}(c)/c)^{p^\alpha}+a$ $(a\in A)$. If $r_0p^\alpha>m$, then $(\underline{D}(c)/c)^{p^\alpha}\in A[t:m]$, i.e., $s'=1$ because of the minimality of s_0. Hence if $s'>1$, we must have $r_0p^\alpha\leq m$. Then the coefficient of $t^{r_0p^\alpha}$ is in A and $D_{r_0}(c)^{p^\alpha}$ is in $c^{p^\alpha}A$. This implies that $D_{r_0}(c)$ is in $cA=\mathcal{P}$, which contradicts to the definition of r_0 (note that $r_0\leq m$).

(2) Set $\alpha':=\min\{\alpha\in\boldsymbol{Z}_+\,|\,r_0p^\alpha\geq m+1\}$. Then we have $(\underline{D}(c)/c)^{p^{\alpha'}}\in A[t:m]$, hence by the minimality of s_0 we have $s_0\leq p^{\alpha'}$. On the other hand $r_0p^{\alpha_0}\geq m+1$ because otherwise $(\underline{D}(c)/c)^{p^{\alpha_0}}\notin A[t:m]$. Hence $\alpha_0\geq\alpha'$. Combin-

ing these, $\alpha_0=\alpha'$.

(3) It suffices to prove the "only if" part. Write $h=s_0q+h'$, $0\leq h'<s_0$. Suppose that $(\underline{D}(c)/c)^h\in A[t:m]$. Since $(\underline{D}(c)/c)^{s_0}\in A[t:m]$ and $(\underline{D}(c)/c)^{s_0}$ is a unit of $A[t:m]$, we see that $(\underline{D}(c)/c)^{-s_0q}\in A[t:m]$. Hence $(\underline{D}(c)/c)^{h'}\in A[t:m]$ and $h'=0$ by the minimality of s_0. Q.E.D.

Corollary 2.3. *In the above notations, s_0 divides e where $e:=e(\mathcal{P})$.*

Proof. Notice that e is a power of p because $\mathcal{P}^{p^n}\subset\mathcal{P}\cap A'$ for some n. Hence it remains only to prove that $(\underline{D}(c)/c)^e\in A[t:m]$. For every prime ideal Q in $P(A)$, we can write $c^e=ux$ for some $u\in A_Q^*$ and $x\in K'$. Then we know that $(\underline{D}(c)/c)^e=\underline{D}(u)/u\in A_Q[t:m]$. Since $A=\bigcap_Q A_Q$, we have $(\underline{D}(c)/c)^e\in A[t:m]$.
Q.E.D.

Lemma 2.4. *Let A be a Krull domain and let $a_1,\cdots,a_\nu$ $(\nu\geq 2)$ be elements of A such that $Supp(div_A(a_k))\cap Supp(div_A(a_l))=\phi$ for $1\leq k$, $l\leq\nu$, $k\neq l$. Let $f_k(X)$ $(1\leq k\leq\nu)$ be polynomials in one variable X over the quotient field of A defined by*

$$f_k(X)=1+(\alpha_1^{(k)}X+\cdots+\alpha_m^{(k)}X^m)/a_k$$

with $\alpha_1^{(k)},\cdots,\alpha_m^{(k)}\in A$. If the product $f_1(t)\cdots f_\nu(t)$ is in $A[t:m]$, then all $f_k(t)$'s are in $A[t:m]$ $(1\leq k\leq\nu)$.

Proof. We shall use the induction on ν. Let γ_k be the smallest integer among those j such that $\alpha_j^{(k)}/a_k\notin A$ (if $\alpha_j^{(k)}/a_k\in A$ for all $1\leq j\leq m$, we put $\gamma_k=m+1$). In the case $\nu=2$, we may assume that $\gamma_1\leq\gamma_2$. If $\gamma_1=m+1$, then $\gamma_2=m+1$ and $f_1(t)$, $f_2(t)$ are already in $A[t:m]$, hence the Lemma is proved. Suppose that $\gamma_1\leq m$. The coefficient of $t^{\gamma_1}$of $f_1(t)f_2(t)$ is

$$(\alpha_{\gamma_1}^{(1)}/a_1)+(\alpha_{\gamma_1-1}^{(1)}/a_1)(\alpha_1^{(2)}/a_2)+\cdots+(\alpha_{\gamma_1}^{(2)}/a_2)\,.$$

Hence $(\alpha_{\gamma_1}^{(1)}/a_1)+(\alpha_{\gamma_1}^{(2)}/a_2)$ is in A. This means that $a_2\alpha_{\gamma_1}^{(1)}+a_1\alpha_{\gamma_1}^{(2)}$ is in a_1a_2A, hence $a_2\alpha_{\gamma_1}^{(1)}$ is in a_1A. Since $\mathrm{Supp}(\mathrm{div}_A(a_1))\cap\mathrm{Supp}(\mathrm{div}_A(a_2))=\phi$, $\alpha_{\gamma_1}^{(1)}$ is in a_1A. This is absurd. Suppose that $\nu>2$ and the assertion holds for $\nu-1$. Notice that $\mathrm{Supp}(\mathrm{div}_A(a_1))\cap\mathrm{Supp}(\mathrm{div}_A(a_2\cdots a_\nu))=\phi$. By our argument in the case $\nu=2$, $f_1(t)$ is in $A[t:m]$ and $f_2(t)\cdots f_\nu(t)$ is in $A[t:m]$. From the induction hypothesis, it follows that $f_2(t),\cdots,f_\nu(t)$ is in $A[t:m]$. Q.E.D.

Proposition 2.5. *Let $\underline{D}$ be a higher derivation of rank m on A and let $a=uc_1^{j_1}\cdots c_\nu^{j_\nu}$ $(u\in A^*$, $j_1,\cdots,j_\nu\in\boldsymbol{Z}$ and $c_1,\cdots,c_\nu$ are distinct prime elements of $A)$. Let*

$$s_k:=\min\{s\in\boldsymbol{N}\,|\,(\underline{D}(c_k)/c_k)^s\in A[t:m]\}\,.$$

Then $\underline{D}(a)/a\in A[t:m]$ if and only if s_k divides j_k for $1\leq k\leq\nu$.

Proof. The "if" part of the Proposition is obvious. We shall prove the "only if" part. Assume that $\underline{D}(a)/a$ is in $A[t: m]$. Then we have $(\underline{D}(c_1)/c_1)^{j_1}\cdots(\underline{D}(c_\nu)/c_\nu)^{j_\nu}$ is in $A[t: m]$. Since $c_1, \cdots, c_\nu$ are distinct prime elements of A, the assumptions of Lemma 2.4 are satisfied. Hence by Lemma 2.4, $(\underline{D}(c_k)/c_k)^{j_k}$ is in $A[t: m]$ for $1\leq k\leq \nu$. Therefore Proposition 2.2, (3) implies that s_k divides j_k for $1\leq k\leq \nu$ Q.E.D.

Let $\boldsymbol{D}=(\underline{D}^{(1)}, \cdots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations of rank $\mathbf{m}=(m_1, \cdots, m_r)$ on A. Let c be a prime element of A. Set

$$s^{(i)}: = \min \{s\in \boldsymbol{N} \,|\, (\underline{D}^{(i)}(c)/c)^s\in A[t_i: m_i]\} \qquad (1\leq i\leq r)$$

and

$$s_0: = \max \{s^{(i)} \,|\, 1\leq i\leq r\} .$$

Then s_0 is a power of p by Proposition 2.2, (1) and s_0 divides the ramification index of cA over $cA\cap A'$ by Corollary 2.3.

Let $J(\boldsymbol{D}: A): = \{J(\boldsymbol{D}: \boldsymbol{\alpha}) \,|\, \boldsymbol{\alpha}=(\alpha_1, \cdots, \alpha_r)\in A'^r\}$. If $J(\boldsymbol{D}: A)\neq \{0\}$, $\{\mathcal{P}\in P(A) \,|\, J(\boldsymbol{D}: A)\subset \mathcal{P}\}$ is a finite set because A is a Krull domain.

Theorem 2.6. *Let A, A', K, K', $\boldsymbol{D}$ and $n(\boldsymbol{D})$ be as before. Assume that $J(\boldsymbol{D}: A)\neq \{0\}$ and let $\mathcal{P}_1, \cdots, \mathcal{P}_\nu$ be all of $\mathcal{P}$'s in $P(A)$ such that $J(\boldsymbol{D}: A)\subset \mathcal{P}$. Furthermore assume that $[K: K']=p^{n(\boldsymbol{D})}$ and $\mathcal{P}_k$'s $(1\leq k\leq \nu)$ are principal. Set $\mathcal{P}_k=c_kA$,*

$$s_k^{(i)}: = \min \{s\in \boldsymbol{N} \,|\, (\underline{D}^{(i)}(c_k)/c_k)^s\in A[t_i: m_i]\} \qquad (1\leq i\leq r)$$

and,

$$s_k: = \max \{s_k^{(i)} \,|\, 1\leq i\leq r\} .$$

Let e_k be the ramification index of $\mathcal{P}_k$ over $\mathcal{P}_k\cap A'$ for $1\leq k\leq \nu$. Then we get the following exact sequence:

$$0 \to \operatorname{Ker}(\bar{\boldsymbol{j}}) \xrightarrow{\Phi} \mathcal{L}_A/\mathcal{L}'_A \to \prod_{k=1}^{\nu} \boldsymbol{Z}/(e_k/s_k)\boldsymbol{Z} \to 0 .$$

Proof. Let $n: =\max \{n_1, \cdots, n_r\}$ and S be the multiplicatively closed subset of A' generated by $c_1^{p^n}, \cdots, c_\nu^{p^n}$. Then we get an isomorphism $\Phi_S: \operatorname{Ker}(\bar{\boldsymbol{j}}_S)\to \mathcal{L}_{A_S}/\mathcal{L}'_{A_S}$ from Theorem 1.6. Therefore Proposition 2.1 implies that $\operatorname{Coker}(f)\cong \operatorname{Coker}(\Phi)$. Hence it suffices to prove $\operatorname{Coker}(f)\cong \prod_{k=1}^{\nu} \boldsymbol{Z}/(e_k/s_k)\boldsymbol{Z}$. Set $\mathcal{G}_k: = \mathcal{P}_k\cap A'$ $(1\leq k\leq \nu)$. Then $\mathcal{G}_1, \cdots, \mathcal{G}_\nu$ are all prime ideals in $P(A')$ with $\mathcal{G}_k\cap S\neq \phi$. For each k $(1\leq k\leq \nu)$, we have $\boldsymbol{j}(\mathcal{G}_k)=e_k\mathcal{P}_k=\operatorname{div}_A(c_k^{e_k})$ by the definition. Hence $f(\operatorname{cl}(\mathcal{G}_k))=(\boldsymbol{D}(c_k)/c_k)^{e_k}$ and

$$\operatorname{Im}(f) = \langle(\boldsymbol{D}(c_k)/c_k)^{e_k} \,|\, 1\leq k\leq \nu\rangle \vee \mathcal{L}'_A/\mathcal{L}'_A .$$

Next we shall prove the following:

$$L \vee \mathcal{L}'_A / \mathcal{L}'_A = \langle (\boldsymbol{D}(c_k)/c_k)^{s_k} \mid 1 \leq k \leq \nu \rangle \vee \mathcal{L}'_A / \mathcal{L}'_A \, .$$

Suppose that $\boldsymbol{D}(a)/a \in L$ $(a \in A \cap A_S^*)$, then it is seen that

$$\mathrm{Supp}\,(\mathrm{div}_A\,(a)) \subset \{\mathcal{P}_1, \cdots, \mathcal{P}_\nu\} \, .$$

Hence we can write $a = uc_1^{j_1} \cdots c_\nu^{j_\nu}$ for some $u \in A^*$ and $j_1, \cdots, j_\nu \in \boldsymbol{Z}$. Notice that $\underline{D}^{(i)}(a)/a \in A[t_i \colon m_i]$ for $1 \leq i \leq r$. Then Proposition 2.5 implies that $s_k^{(i)}$ divides j_k for $1 \leq i \leq r$ and $1 \leq k \leq \nu$. Therefore s_k divides j_k for $1 \leq k \leq \nu$. Conversely, it is easily seen that $(D(c_k)/c_k)^{s_k}$ is in L $(1 \leq k \leq \nu)$. So we have the required result. Consequently we know

$$\mathrm{Coker}\,(f) \cong \frac{\langle (\boldsymbol{D}(c_k)/c_k)^{s_k} \mid 1 \leq k \leq \nu \rangle \vee \mathcal{L}'_A}{\langle (\boldsymbol{D}(c_k)/(c_k)^{e_k} \mid 1 \leq k \leq \nu \rangle \vee \mathcal{L}'_A} \, .$$

We shall define the homomorphism θ by the following manner:

$$\theta \colon \prod_{k=1}^{\nu} \boldsymbol{Z}/(e_k/s_k)\boldsymbol{Z} \to \mathrm{Coker}\,(f) \, ,$$

$$\theta \text{ (the residue class of } (j_1, \cdots, j_\nu))$$

$$= \text{the residue class of } \prod_{k=1}^{\nu} (\boldsymbol{D}(c_k)/c_k)^{s_k j_k} \, .$$

Then it is easily seen that θ is well-defined and surjective. We shall show that θ is injective. Suppose that

$$\theta \text{ (the residue class of } (j_1, \cdots, j_\nu)) = \boldsymbol{1} \, .$$

Then there exist elements $i_1, \cdots, i_\nu \in \boldsymbol{Z}$ and $\alpha \in A^*$ such that

$$(\boldsymbol{D}(\alpha)/\alpha) \prod_{k=1}^{\nu} (\boldsymbol{D}(c_k)/c_k)^{s_k j_k} = \prod_{k=1}^{\nu} (\boldsymbol{D}(c_k)/c_k)^{e_k i_k} \, .$$

Put $x := \prod_{k=1}^{\nu} \alpha c_k^{d_k}$ where $d_k := s_k j_k - e_k i_k$. Then $\boldsymbol{D}(x)/x = \boldsymbol{1}$ and $x \in K'$. Let v_k be the normalized valuation of K associated to the prime ideal $\mathcal{P}_k$ and A'_k be the localization of A' with respect to $\mathcal{Q}_k$. Let u_k be a uniformisant of A'_k for $1 \leq k \leq \nu$. Since x is in K', there exist elements $\alpha_k \in A_k'^*$ and $f_k \in \boldsymbol{Z}$ such that $x = \alpha_k u_k^{f_k}$ for $1 \leq k \leq \nu$. Then we have $d_k = v_k(x) = v_k(\alpha_k u_k^{f_k}) = v_k(u_k^{f_k}) = f_k e_k$. Hence e_k divides $s_k j_k$, i.e., e_k/s_k divides j_k for $1 \leq k \leq \nu$. This implies that θ is injective. Q.E.D.

Let $\mathcal{P} = cA$ be a principal prime ideal in $P(A)$ and let $s^{(i)}(\mathcal{P}) := \min\,\{s \in \boldsymbol{N} \mid (\underline{D}^{(i)}(c)/c)^s \in A[t_i \colon m_i]\}$ $(1 \leq i \leq r)$, and $s(\mathcal{P}) := \max\,\{s^{(i)}(\mathcal{P}) \mid 1 \leq i \leq r\}$.

Theorem 2.7. *Assume that A is a unique factorization domain and let $\boldsymbol{D} = (\underline{D}^{(1)}, \cdots, \underline{D}^{(r)})$ be an r-tuple of non-trivial higher derivations on A satisfying the conditions $J(\boldsymbol{D} \colon A) \neq \{0\}$ and $[K \colon K'] = p^{n(\boldsymbol{D})}$. Then the followings are equi-*

valent to each other:

(i) $\Phi: Ker(\bar{j}) \to \mathcal{L}_A/\mathcal{L}'_A$ *is an isomorphism.*

(ii) *For each prime ideal* $\mathcal{P}$ *in* $P(A)$, *either* $J(\boldsymbol{D}: A) \not\subset \mathcal{P}$ *or* $e(\mathcal{P}) = s(\mathcal{P})$ *occurs where* $e(\mathcal{P})$ *stands for the ramification index of* $\mathcal{P}$ *over* $\mathcal{P} \cap A'$.

Proof. Immediate from Theorem 2.6. Q.E.D.

3. Calculus of divisor class groups

In this section we shall determine divisor class groups of certain rings as applications of the preceding results. As before k will be a field of characteristic $p>0$ unless otherwise specified.

Proposition 3.1. *Let* $A=k[x, y]$ *be a two-dimensional polynomial ring over* k *with the quotient field* K. *Let* α, β *be integers such that* $0<\alpha$, $\beta<p^n$. *Let* $\underline{D}$ *be the higher derivation of rank* p^n-1 *on* K *over* k *defined by*

$$\underline{D}(x) = x(1+t)^{\alpha}\,, \quad \underline{D}(y) = y(1+t)^{\beta}$$

and let K' *be the field of* $\underline{D}$*-constants. Let* p^{γ} *be the maximal* p*-th power which divides* $GCD(\alpha, \beta)$. *Set* $\alpha=\alpha' p^{\gamma}$, *and* $\beta=\beta' p^{\gamma}$. *Then we have the following assertions*:

(1) $[K: K']=p^{n-\gamma}$.

(2) $\mathcal{L}_A/\mathcal{L}'_A=\boldsymbol{Z}/p^{n-\gamma}\boldsymbol{Z}$.

(3) *Assume that* p *does not divide either* α *or* β. *Then* $Cl(A') \cong \boldsymbol{Z}/p^n\boldsymbol{Z}$ *where* $A':=A\cap K'$, *and* A' *is the normalization of* $k[x^{p^n}, y^{p^n}, x^{p^n-\beta'}y^{\alpha'}]$.

Proof. (1) We may assume that p does not divide α'. Set $F_s:=k(x^{p^s}, y^{p^s}, x^{-\beta'}y^{\alpha'})$ for $0\leq s\leq n$. Then we have

$$K = F_0 \supset F_1 \supset \cdots \supset F_{n-1} \supset F_n\,.$$

Hence $\mathrm{GCD}(\alpha', p^s)=1$ implies that $F_{s-1}=F_s(x^{p^{s-1}})$ and $x^{p^{s-1}} \in F_{s-1}-F_s$. Therefore $[F_{s-1}: F_s]=p$ for $1\leq s\leq n$. Set $s_0:=\min\{s \mid x^{p^s}\in K', 1\leq s\leq n\}$. We shall show that $s_0=n-\gamma$. From $\underline{D}(x^{p^{n-\gamma}})=x^{p^{n-\gamma}}$, it follows that $x^{p^{n-\gamma}}\in K'$ and $s_0\leq n-\gamma$. On the other hand $\underline{D}(x^{p^{n-\gamma-1}}) \neq x^{p^{n-\gamma-1}}$ because p does not divide α'. This implies that $s_0=n-\gamma$. Since $\mu(\underline{D})=p^{\gamma}$, we know that $[K: K']\geq p^{n-\gamma}$ by Proposition 1.3. Then we get $K'=F_{s_0}$ because $F_{s_0}\subset K'\subset K=F_0$ and $[F_0: F_{s_0}]=p^{s_0}=p^{n-\gamma}$. Hence $[K: K']=p^{s_0}=p^{n-\gamma}$.

(2) Since $A^*=k^*$, we have $\mathcal{L}'_A=\{1\}$. We shall show that $\mathcal{L}_A=\{(1+t)^{ds}\in k[t: m] \mid s\in\boldsymbol{Z}\}$ where $d:=\mathrm{GCD}(\alpha, \beta)$ and $m:=p^n-1$. Notice that

$$\mathcal{L}_A = \{\underline{D}(f)/f \in K[t: m] \mid f\in A-\{0\},\ \underline{D}(f)/f\in A[t: m]\}$$

because $\underline{D}(f_1/f_2)/(f_1/f_2)=\underline{D}(f_1 f_2^m)/f_1 f_2^m$ $(f_1, f_2(\neq 0)\in A)$. For every polynomial

$f \in A - \{0\}$, the total degree of the coefficient of t^j in $\underline{D}(f)$ is not more than that of f for $0 \leq j \leq m$ by the definition of $\underline{D}$. Hence $\underline{D}(f)/f \in A[t: m]$ implies that $\underline{D}(f)/f \in k[t: m]$. Set $f := \sum a_{ij} x^i y^j$ $(a_{ij} \in k^*)$ and $\underline{D}(f)/f = h(t)$ where $h(T) \in k[T]$. Then we see

$$\sum a_{ij} x^i y^j (1+t)^{i\alpha + j\beta} = \sum a_{ij} x^i y^j h(t) .$$

Since x, y and T are algebraically independent over k, we get $(1+t)^{i\alpha+j\beta} = h(t)$. Hence $i\alpha + j\beta$ is constant modulo p^n for any i, j with $a_{ij} \neq 0$. On the other hand $i\alpha + j\beta$ is a multiple of $d = \mathrm{GCD}(\alpha, \beta)$. Therefore we know $\underline{D}(f)/f = (1+t)^{ds'}$ where $s' = (i\alpha + j\beta)/d$. This means that $\mathcal{L}_A$ is contained in $\{(1+t)^{ds} \in k[t: m] \mid s \in \boldsymbol{Z}\}$. Since $\mathrm{GCD}(\alpha, \beta) = d$, there exist integers a, b such that $a\alpha + b\beta = d$. Then we have $\underline{D}(x^a y^b)/x^a y^b = (1+t)^d$. This implies that $(1+t)^d$ is in $\mathcal{L}_A$. Hence $\mathcal{L}_A = \{(1+t)^{ds} \in k[t: m] \mid s \in \boldsymbol{Z}\}$. Let θ: $\boldsymbol{Z}/p^{n-\gamma}\boldsymbol{Z} \to \mathcal{L}_A$ be the homomorphism defined by θ(the residue class of s) $= (1+t)^{ds}$. Then we see easily that θ is well-defined and surjective. We shall prove the injectivity of θ. Assume that θ(the residue class of s) $= 1$. Then $(1+t)^{ds} = 1$ in a truncated polynomial ring $k[t: m]$. Write $d = d' p^{\gamma}$ and $s = s' p^{\delta}$ ($p \nmid d'$ and $p \nmid s'$). Since $(1+t)^{ds} = (1+t^{p^{\gamma+\delta}})^{d's'}$ and $p \nmid d's'$, the coefficient of $t^{p^{\gamma+\delta}}$ does not vanish. Hence $p^{\gamma+\delta} \geq p^n$ and $\delta \geq n - \gamma$. This implies that $s \in p^{n-\gamma}\boldsymbol{Z}$ and θ is injective. Finally we have $\mathcal{L}_A / \mathcal{L}'_A \cong \mathcal{L}_A \cong \boldsymbol{Z}/p^{n-\gamma}\boldsymbol{Z}$.

(3) Since p does not divide either α or β, we see that the height one property for $\underline{D}$ is satisfied. It follows from (1) that $[K: K'] = p^n$ (note that $\gamma = 0$). Therefore Theorem 1.6 implies that $\mathrm{Ker}(\bar{\boldsymbol{j}}) \cong \mathcal{L}_A / \mathcal{L}'_A$. Since A is a unique factorization domain, we have $\mathrm{Cl}(A') = \mathrm{Ker}(\bar{\boldsymbol{j}})$, hence $\mathrm{Cl}(A') \cong \boldsymbol{Z}/p^n\boldsymbol{Z}$. The rest is obvious from the fact A' is normal and integral over $k[x^{p^n}, y^{p^n}, x^{p^n - \beta'} y^{\alpha'}]$ (note that $K' = F_n$). Q.E.D.

By making use of Proposition 3.1 we get the following:

Proposition 3.2. *The divisor class group of a surface S: $Z^{p^n} = XY$ is a cyclic group of order p^n.*

Proof. Let x, y be independent variables over k. Then the coordinate ring of the surface S is isomorphic to $A'_1 := k[x^{p^n}, y^{p^n}, xy]$. Set $\alpha := 1$ and $\beta := p^n - 1$ in Proposition 3.1, then we have $\mathrm{Cl}(A') \cong \boldsymbol{Z}/p^n\boldsymbol{Z}$ where $A' = A \cap K'$ is a Krull domain in Proposition 3.1. We shall show that $A'_1 = A'$. We see that A'_1 is normal because the surface S has only isolated singular point (cf. [4], Th. 4.1). Since A' is the normalization of $k[x^{p^n}, y^{p^n}, xy]$ by Proposition 3.1, (3), we get $A'_1 = A'$. Q.E.D.

REMARK 3.3. Let $\mathcal{Q}$ be a prime ideal in $P(A')$ generated by x^{p^n} and xy. Since $\boldsymbol{j}(\mathcal{Q}) = \mathrm{div}_A(x)$ and since $\Phi(\mathrm{cl}(\mathcal{Q})) = \underline{D}(x)/x$, $\mathrm{cl}(\mathcal{Q})$ generates $\mathrm{Cl}(A') \cong \boldsymbol{Z}/p^n\boldsymbol{Z}$.

In order to generalize Proposition 3.2, we shall prove $\mathrm{Cl}(R_1 \otimes_k \cdots \otimes_k R_r) \cong \prod_{i=1}^{r} \mathrm{Cl}(R_i)$ in a certain restricted case as an application of Theorem 1.6.

Proposition 3.4. *Let A_i be a polynomial ring in a finite set of variables over k and set $K_i := Q(A_i)$ $(1 \leq i \leq r)$. Let $\underline{D}^{(i)}$ be a non-trivial higher derivation of rank m_i on K_i over k leaving A_i invariant. Let K_i' be the field of $\underline{D}^{(i)}$-constants and set $A_i' := A_i \cap K_i'$ $(1 \leq i \leq r)$. Assume that the height one property holds for $\underline{D}^{(i)}$ and $[K_i : K_i'] = p^{n_i}$ where $n_i := n(\underline{D}^{(i)})$ for $1 \leq i \leq r$. Set $A := A_1 \otimes_k \cdots \otimes_k A_r$ and $A' := A_1' \otimes_k \cdots \otimes_k A_r'$ with $L := Q(A)$ and $L' := Q(A')$. Then we have $Cl(A') \cong \prod_{i=1}^{r} Cl(A_i')$.*

Proof. We have only to prove the Proposition in the case $r=2$ because we can get the general case by induction on r. Set $A_1 = k[x_1, \cdots, x_d]$ and $A_2 = k[y_1, \cdots, y_e]$ where $x_1, \cdots, x_d$ and $y_1, \cdots, y_e$ are independent variables over k. Then $A \cong k[x_1, \cdots, x_d, y_1, \cdots y_e]$. We shall extend $\underline{D}^{(1)}$ to L by the following way:

$$\underline{D}^{(1)}(y_1) = y_1, \cdots, \underline{D}^{(1)}(y_e) = y_e .$$

Similarly we shall extend $\underline{D}^{(2)}$ to L. Then $\boldsymbol{D} := (\underline{D}^{(1)}\ \underline{D}^{(2)})$ is a 2-tuple of non-trivial higher derivations of rank $\boldsymbol{m} := (m_1 .\ m_2)$ on L over k leaving A invariant.

We shall show that $A' = A \cap L'$. Since K_i $(i=1\ 2)$ are regular extensions of k, $K_i'(i=1\ 2)$ are also regular extensions of k. Besides, $A_i'(i=1, 2)$ are integrally closed integral domains. Therefore $A' = A_1' \otimes_k A_2'$ is an integrally closed integral domain ([2], Chap. 5, §1, Cor. of Prop. 19). Furthermore $A \cap L'$ is an integral extension of A' with the same quotient field $L' = Q(A \cap L') = Q(A')$ Hence we have $A' = A \cap L'$

Next we shall prove that L' is the field of $\boldsymbol{D}$-constants. It is easily seen that $A_1' \otimes_k A_2 = A_1'[y_1, \cdots, y_e]$ is the ring of $\underline{D}^{(1)}$-constants in A. Similarly $A_1 \otimes_k A_2'$ is the ring of $\underline{D}^{(2)}$-constants in A. We know that $A_1' \otimes_k A_2' = (A_1' \otimes_k A_2) \cap (A_1 \otimes_k A_2')$ ([2], Chapter 1, §2, Proposition 7). Therefore $A' = A_1' \otimes_k A_2'$ is the ring of $\boldsymbol{D}$-constants in A. It is clear that $L' = Q(A')$ is contained in the field of $\boldsymbol{D}$-constants. Since A is the integral closure of A' in L, any element of L is of the form a/b $(a \in A, b \in A')$. Suppose that $\boldsymbol{D}(a/b) = a/b$ $(a \in A, b \in A')$. Then we have $\boldsymbol{D}(a) = (\boldsymbol{D}(a/b)b) = \boldsymbol{D}(a/b)\boldsymbol{D}(b) = (a/b)b = a$, hence a is in A'. This implies that a/b is in L'. Finally L' is the field of $\boldsymbol{D}$-constants.

We shall show that the height one property holds for $\boldsymbol{D}$. Since A is A_i-flat, we know that $ht(\mathcal{P} \cap A_i) \leq 1$ $(i=1, 2)$ for all $\mathcal{P} \in P(A)$ ([4], Proposition 6.4). Set $\mathcal{P}_i := \mathcal{P} \cap A_i$. Then there exists an element α_i in A_i such that the Jacobian $J(\underline{D}^{(i)} : \alpha_i)$ is not contained in $\mathcal{P}_i$ because the height one property holds for $\underline{D}^{(i)}$.

On the other hand we have $J(\boldsymbol{D}\colon (\alpha_1, \alpha_2))=J(\underline{D}^{(1)}\colon \alpha_1)J(\underline{D}^{(2)}\colon \alpha_2)$. Suppose that $J(\boldsymbol{D}\colon (\alpha_1, \alpha_2))\in\mathcal{P}$, then either $J(\underline{D}^{(1)}\colon \alpha_1)$ or $J(\underline{D}^{(2)}\colon \alpha_2)$ is in $\mathcal{P}$, say, $J(\underline{D}^{(1)}\colon \alpha_1)\in\mathcal{P}$. This means that $J(\underline{D}^{(1)}\colon \alpha_1)\in\mathcal{P}\cap A_1=\mathcal{P}_1$, which contradicts to the height one property for $\underline{D}^{(1)}$.

We shall show that $[L\colon L']=p^{n(\boldsymbol{D})}$. Set $L_1=Q(A_1'\otimes_k A_2)$, then we have $L\supset L_1\supset L'$. We know that $[L\colon L']\geq p^{n(\boldsymbol{D})}$ because of Proposition 1.3. Since $[L\colon L']=[L\colon L_1][L_1\colon L']$, it suffices to prove that $[L\colon L_1]\leq p^{n_1}$ and $[L_1\colon L']\leq p^{n_2}$. We shall prove that $[L\colon L_1]\leq p^{n_1}$. It is easily verified that $L=Q(K_1\otimes_k K_2)$, $L_1=Q(K_1'\otimes_k K_2)$ and $K_1'\otimes_k K_2=L_1\cap(K_1\otimes_k K_2)$. Therefore any element of L is of the form α/β with $\alpha\in K_1\otimes_k K_2$ and $\beta\in K_1'\otimes_k K_2$. Let $a_1, \cdots, a_\nu$ $(\nu\colon=p^{n_1})$ be K_1'-basis of K_1. Then $K_1\otimes_k K_2$ is generated by $a_1\otimes 1, \cdots, a_\nu\otimes 1$ over $K_1'\otimes_k K_2$. Since any element of L is of the form α/β $(\alpha\in K_1\otimes_k K_2,\ \beta\in K_1'\otimes_k K_2)$, L is generated by $a_1\otimes 1, \cdots, a_\nu\otimes 1$ over L_1, hence $[L\colon L_1]\leq\nu=p^{n_1}$. Similarly we have $[L_1\colon L']\leq p^{n_2}$.

Let

$$\begin{aligned}\mathcal{L}_i &= \{\underline{D}^{(i)}(z_i)/z_i \mid z_i\in K_i^*,\ \underline{D}^{(i)}(z_i)/z_i\in A_i[t_i\colon m_i]\}\ ,\\ \mathcal{L}_i' &= \{\underline{D}^{(i)}(u_i)/u_i \mid u_i\in A_i^*\} \qquad \text{for} \quad i=1,2\,,\\ \mathcal{L} &= \{\boldsymbol{D}(z)/z \mid z\in L^*,\ \boldsymbol{D}(z)/z\in A[\boldsymbol{t}\colon \boldsymbol{m}]\}\end{aligned}$$

and,

$$\mathcal{L}' = \{\boldsymbol{D}(u)/u \mid u\in A^*\}$$

where $\boldsymbol{t}=(t_1, t_2)$. Since we know that $\mathrm{Cl}(A_i')\cong\mathcal{L}_i/\mathcal{L}_i'$ $(i=1,2)$, $\mathrm{Cl}(A')\cong\mathcal{L}/\mathcal{L}'$ and $\mathcal{L}_i'=\mathcal{L}'=\{1\}$, it remains only to prove that $\mathcal{L}_1\times\mathcal{L}_2\cong\mathcal{L}$. Let θ be the homomorphism of $\mathcal{L}_1\times\mathcal{L}_2$ into $\mathcal{L}$ defined by

$$(\underline{D}^{(1)}(a_1)/a_1,\ \underline{D}^{(2)}(a_2)/a_2) = \boldsymbol{D}(a_1a_2)/a_1a_2\,, \quad (a_i\in K_i^*)\,.$$

It is easily seen that θ is injective. We shall show that θ is surjective. Suppose that $\boldsymbol{D}(f)/f\in\mathcal{L}$ $(f\in A-\{0\})$. Then there exist polynomials $g_i(T_i)$ in $A[T_i]$ $(i=1,2)$ such that $\boldsymbol{D}(f)/f=(g_1(t_1), g_2(t_2))$. Comparing the total degree with respect to $y_1, \cdots, y_e$ of $\underline{D}^{(1)}(f)$ with that of $fg_1(t_1)$, we see that $g_1(t_1)$ is in $A_1[t_1\colon m_1]$. Write $f=\sum_\gamma a_\gamma b_\gamma$ $(a_\gamma\in A_1,\ b_\gamma\in A_2$ and $\{b_\gamma\}$ is linearly independent over $k)$, then we have

$$\sum_\gamma (\underline{D}^{(1)}(a_\gamma)-g_1(t_1)a_\gamma)b_\gamma = 0\,.$$

This implies that $\underline{D}^{(1)}(a_\gamma)=g_1(t_1)a_\gamma$ for all γ. Therefore $\underline{D}^{(1)}(a)/a=g_1(t_1)$ for some $a\in A_1$ Similarly $\underline{D}^{(2)}(b)/b=g_2(t_2)$ for some $b\in A_2$. Hence $\theta(\underline{D}^{(1)}(a)/a,\ \underline{D}^{(2)}(b)/b)=\boldsymbol{D}(f)/f$. Furthermore we know that $\mathcal{L}=\{\boldsymbol{D}(f)/f \mid f\in A-\{0\},\ \boldsymbol{D}(f)/f\in A[t\colon m]\}$. Therefore θ is surjective and we get the desired result. Q.E.D.

REMARK 3.5. By the similar method as the proof of Proposition 3.4, we can get the following fact using units theorem ([10], Corollary 1.8). But the proof is more complicated, so we omit it:

"*Let* $A_i := \bigoplus_{s\in Z_+} (A_i)_s$ $(1\leq i\leq r)$ *be graded unique factorization domains with* $(A_i)_0 = k$ *and let* K_i *be its quotient field. Assume that* K_i $(1\leq i\leq r)$ *are regular extensions of* k. *Let* $\underline{D}^{(i)}$ *be a non-trivial higher derivation of rank* m_i *on* K_i *over* k *leaving* A_i *invariant for* $1\leq i\leq r$. *Let* K_i' *be the field of* $\underline{D}^{(i)}$*-constants and set* $A_i' := A_i \cap K_i'$ $(1\leq i\leq r)$. *Assume that the height one property holds for* $\underline{D}^{(i)}$ *and* $[K_i : K_i'] = p^{n_i}$ *where* $n_i := n(\underline{D}^{(i)})$ *for* $1\leq i\leq r$. *Set* $A := A_1 \otimes_k \cdots \otimes_k A_r$ *and* $A' := A_1' \otimes_k \cdots \otimes_k A_r'$ *with* $L := Q(A)$ *and* $L' := Q(A')$. *Forthermore assume that* $A_1 \otimes_k \cdots \otimes_k A_i$ $(1\leq i\leq r)$ *are unique factorization domains. Then we have* $\mathrm{Cl}(A') \simeq \prod_{i=1}^{r} \mathrm{Cl}(A_i')$".

The following Proposition is immediate from Proposition 3.4.

Proposition 3.6. *The divisor class group of an affine variety in* $\boldsymbol{A}^{3r}$ *defined by the equations* $Z_i^{q_i} = X_i Y_i$ $(1\leq i\leq r)$ *is isomorphic to* $\prod_{i=1}^{r} \boldsymbol{Z}/q_i\boldsymbol{Z}$ *where* $q_i := p^{n_i}$.

REMARK 3.7. The coordinate ring of this variety is isomorphic to $A' := k[x_1^{q_1}, y_1^{q_1}, x_1y_1, \cdots, x_r^{q_r}, y_r^{q_r}, x_ry_r]$. And if we denote by $\mathcal{Q}_i$ a prime ideal in $P(A')$ generated by $x_i^{q_i}$, x_iy_i for $1\leq i\leq r$, then $\mathrm{cl}(\mathcal{Q}_i)$ $(1\leq i\leq r)$ generate $\mathrm{Cl}(A')$.

As another generalization of Proposition 3.2 we have the following:

Proposition 3.8. *The divisor class group of a hypersurface* S: $Z^{p^n} = X_1X_2\cdots X_r$ $(r\geq 2)$ *is isomorphic to* $(\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1}$. *The coordinate ring of this hypersurface* S *is isomorphic to* $A' := k[x_1^{p^n}, x_2^{p^n}, \cdots, x_r^{p^n}, x_1x_2\cdots x_r]$ *where* $x_1, x_2, \cdots, x_r$ *are independent variables over* k. *If we denote by* $\mathcal{Q}_i$ *a prime ideal in* $P(A')$ *generated by* $x_i^{p^n}$ *and* $x_1x_2\cdots x_r$ *for* $1\leq i\leq r-1$, *then* $cl(\mathcal{Q}_i)$ $(1\leq i\leq r-1)$ *generate* $Cl(A')$.

Proof. We see easily that A' is the coordinate ring of the hypersurface S. We shall set $A = k[x_1, x_2, \cdots, x_r]$ and $K := Q(A)$. Let $\underline{D}^{(i)}$ be the higher derivation of rank $p^n - 1$ on K over k satisfying

$$\begin{aligned} \underline{D}^{(i)}(x_i) &= x_i(1+t_i)\,, \\ \underline{D}^{(i)}(x_j) &= x_j \quad (1\leq j\leq r-1,\ j\neq i)\,, \\ \underline{D}^{(i)}(x_r) &= x_r(1+t_i)^{-1} \end{aligned}$$

for $1\leq i\leq r-1$. Then we have

$$J(\boldsymbol{D}\colon (x_1, \cdots, \hat{x}_s, \cdots, x_r)) = (-1)^{r+s} x_1\cdots\hat{x}_s\cdots x_r$$

for $1\leq s\leq r$ where $\boldsymbol{D} = \underline{D}(^{(1)}, \cdots, \underline{D}^{(r-1)})$ and the symbol $\wedge$ over a letter means that the letter is missing. Let K' be the field of $\boldsymbol{D}$-constants. Then Proposition 1.3 implies that $[K: K'] \geq p^{n(r-1)}$. We shall set

$$K_i := k(x_1^{p^n}, x_2^{p^n}, \cdots, x_i^{p^n}, x_{i+1}, \cdots, x_r, x_1 \cdots x_r)$$

for $1 \leq i \leq r-1$ and $K_r := k(x_1^{p^n}, \cdots, x_r^{p^n}, x_1 \cdots x_r)$. Then $K = K_1$, $K_i = K_{i+1}(x_{i+1})$ and $x_{i+1}^{p^n} \in K_{i+1}$ for $1 \leq i \leq r-1$. Besides, $K \supset K' \supset K_r$. This implies that $[K: K'] \leq p^{n(r-1)}$, hence $[K: K'] = p^{n(r-1)}$. Since the hypersurface S has no singularity of codimension one, we see that A' is normal. Then we get $A' = A \cap K'$. Therefore we have $\mathrm{Cl}(A') \cong \mathcal{L}_A / \mathcal{L}'_A$ by Theorem 1.6. Let θ be the homomorphism of $(\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1}$ into $\mathcal{L}_A$ defined by

$$\begin{aligned} &\theta(\text{the residue class of } (j_1, \cdots, j_{r-1})) \\ &= \boldsymbol{D}(a)/a \\ &= ((1+t_1)^{j_1}, \cdots, (1+t_{r-1})^{j_{r-1}}) \end{aligned}$$

where $a := x_1^{j_1} \cdots x_{r-1}^{j_{r-1}}$. Then θ is well-defined and bijective by the similar device to the proof of Proposition 3.1. Consequently $\mathrm{Cl}(A') \cong \mathcal{L}_A / \mathcal{L}'_A \cong \mathcal{L}_A \cong (\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1}$. Since $\boldsymbol{D}(x_i)/x_i$ $(1 \leq i \leq r-1)$ generate $\mathcal{L}_A$, $\mathrm{cl}(\mathcal{G}_i)$ $(1 \leq i \leq r-1)$ generate $\mathrm{Cl}(A')$. Q.E.D.

For future reference we shall recollect the known results concerning Galois descent and semigroup rings. Let G be a finite group of automorphisms of a Krull domain A and let A' be the invariant subring of A with respect to G. Since A is integral over A', we can define the homomorphism $\bar{\boldsymbol{j}}$: $\mathrm{Cl}(A') \to \mathrm{Cl}(A)$ by $\bar{\boldsymbol{j}}(\mathrm{cl}(\mathcal{G})) = \mathrm{cl}(\sum e(\mathcal{P})\mathcal{P})$ where the sum is taken over all prime ideal $\mathcal{P}$ in $P(A)$ such that $\mathcal{P} \cap A' = \mathcal{G}$. If every prime ideal $\mathcal{P}$ in $P(A)$ is unramified over $\mathcal{P} \cap A'$, A is called divisorially unramified over A'.

Lemma 3.9. *If A is divisorially unramified over A', there is an isomorphism $Ker(\bar{\boldsymbol{j}}) \cong H^1(G, A^*)$* (cf. [4], Theorem 16.1).

Lemma 3.10. *Let $\mathcal{D}(A/A')$ be the Dedekind different of A over A'. Then we have the following; a prime ideal $\mathcal{P}$ in $P(A)$ is unramified over $\mathcal{P} \cap A'$ if and only if $\mathcal{D}(A/A') \not\subset \mathcal{P}$* ([4], Proposition 16.3).

Let $f(X)$ be the minimal polynomial for a primitive element α of $Q(A)$ over $Q(A')$. Let $f'(X)$ denote the derivative of $f(X)$ with respect to X. Then we have $f'(\alpha) \in \mathcal{D}(A/A')$. Hence each prime ideal $\mathcal{P}$ in $P(A)$ such that $f'(\alpha) \notin \mathcal{P}$ is unramified over $\mathcal{P} \cap A'$ by Lemma 3.10.

Furthermore we need the following fact concerning semigroup rings.

Lemma 3.11. *Let $K_i[\Gamma]$ be a semigroup ring over a field K_i generated by a semigroup $\Gamma \subset \boldsymbol{Z}_+^n$ $(i=1, 2)$. Assume that $K_i[\Gamma]$ $(i=1, 2)$ are Krull domains. Then we have $Cl(K_1[\Gamma]) = Cl(K_2[\Gamma])$* (cf. [1], Proposition 7.3).

By making use of Proposition 3.8 and Galois descent we get the following:

Proposition 3.12. *Let k be a field of arbitrary characteristic. Then the divisor class group of a hypersurface $S\colon Z^d=X_1X_2\cdots X_r$ $(r\geq 2)$ over k is isomorphic to $(\boldsymbol{Z}/d\boldsymbol{Z})^{r-1}$.*

Proof. It is easily seen that the coordinate ring of the hypersurface S is isomorphic to $A'\colon=k[x_1^d, \cdots, x_r^d, x_1\cdots x_r]$ where $x_1, \cdots, x_r$ are independent variables over k. Since A' is generated by monomials, we may assume that k is algebraically closed by Lemma 3.11. Let p denote the characteristic of k. In the case $p=0$, we can conclude the result simply through Galois descent. So we omit the proof. Assume that $p>0$ and write $d=ap^n$, $p\nmid a$. We shall set $B=k[x_1^{p^n}, \cdots, x_r^{p^n}, x_1\cdots x_r]$, then we have $B\supset A'$. Let ω be a primitive a-th root of unity and σ_i be the automorphism of B defined by the following manner:

$$\sigma_i(x_i^{p^n}) = \omega x_i^{p^n}\,, \quad \sigma_i(x_j^{p^n}) = x_j^{p^n} \qquad (1\leq j\leq r-1,\ j\neq i)\,,$$
$$\sigma_i(x_r^{p^n}) = \omega^{-1}x_r^{p^n} \quad \text{and,} \quad \sigma_i(x_1\cdots x_r) = x_1\cdots x_r$$

for $1\leq i\leq r-1$. Then σ_i is well-defined. Let G be the subgroup of Aut B generated by σ_i $(1\leq i\leq r-1)$. Then we get $B^G=A'$. In order to use Galois descent, we must prove that B is divisorially unramified over A'. We shall set

$$K_i\colon = k(x_1^d, \cdots, x_i^d, x_{i+1}^{p^n}, \cdots, x_r^{p^n}, x_1\cdots x_r)$$

for $1\leq i\leq r-1$. Then $F_s(T)=T^a-x_s^d$ is the minimal polynomial for a primitive element $x_s^{p^n}$ of K_{s-1} over K_s and $F'_s(x_s^{p^n})=a(x_s^{p^n})^{a-1}$ for $1\leq s\leq r$ where $K_0\colon=Q(B)$ and $K_r\colon=Q(A')$. Therefore every prime ideal $\mathcal{P}$ in $P(B)$ except $\mathcal{P}_s=(x_s^{p^n}, x_1\cdots x_r)$ $(1\leq s\leq r)$ is unramified over $\mathcal{P}\cap A'$. By a direct calculation the ramification index of $\mathcal{P}_s$ over $\mathcal{P}_s\cap A'$ is one. Hence B is divisorially unramified over A'. By Galois descent we get the following exact sequence:

$$0 \to H^1(G, B^*) \to \mathrm{Cl}(B^G) \to \mathrm{Cl}(B)\,.$$

Since G acts trivially on $B^*=k^*$, we know that $H^1(G, B^*)\cong \mathrm{Hom}_{\boldsymbol{Z}}(G, k^*)$. Furthermore it is easily verified that $\mathrm{Hom}_{\boldsymbol{Z}}(G, k^*)\cong(\boldsymbol{Z}/a\boldsymbol{Z})^{r-1}$ because ω is in k. On the other hand, Proposition 3.8 shows that $\mathrm{Cl}(B)\cong(\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1}$. Let $\mathcal{Q}_i$ be a prime ideal in $P(A')$ generated by x_i^d and $x_1\cdots x_r$ for $1\leq i\leq r-1$. Then we have $\mathcal{P}_i\cap A'=\mathcal{Q}_i$ and $j(\mathcal{Q}_i)=\mathcal{P}_i$ where $j\colon \mathrm{Div}(A')\to\mathrm{Div}(B)$. Besides, $\mathrm{cl}(\mathcal{P}_i)$ $(1\leq i\leq r-1)$ generate $\mathrm{Cl}(B)\cong(\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1}$. Finally we get the following exact sequence:

$$0 \to (\boldsymbol{Z}/a\boldsymbol{Z})^{r-1} \to \mathrm{Cl}(A') \to (\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1} \to 0\,.$$

Since a and p^n are relatively prime, $\mathrm{Ext}^1_{\boldsymbol{Z}}((\boldsymbol{Z}/p^n\boldsymbol{Z})^{r-1}, (\boldsymbol{Z}/a\boldsymbol{Z})^{r-1})$ vanishes and the above sequence splits ([3], p. 290, Theorem 1.1). This implies that $\mathrm{Cl}(A')\cong(\boldsymbol{Z}/d\boldsymbol{Z})^{r-1}$. Q.E.D.

REMARK 3.13. In the notations of the proof of Proposition 3.12, $p^n\mathrm{cl}(\mathcal{Q}_i)$ $(1\leq i\leq r-1)$ generate $\mathrm{Ker}(\bar{j})$ because $j(p^n\mathcal{Q}_i)=\mathrm{div}_B(x_i^{p^n})$ and $\mathrm{Ker}(\bar{j})\cong \mathrm{Hom}_{\boldsymbol{Z}}(G, k^*)\cong(\boldsymbol{Z}/a\boldsymbol{Z})^{r-1}$. Furthermore it follows from Proposition 3.8 that $\mathrm{cl}(\mathcal{Q}_i)$ $(1\leq i\leq r-1)$ generate $\mathrm{Cl}(A')$ modulo $\mathrm{Ker}(\bar{j})$. Hence $\mathrm{cl}(\mathcal{Q}_i)$ $(1\leq i\leq r-1)$ generate $\mathrm{Cl}(A')$.

Proposition 3.14. *Let k be a field of arbitrary characteristic. Then the divisor class group of the homogeneous coordinate ring of a Veronese transform $v_d(\boldsymbol{P}^r)$ of a projective space $\boldsymbol{P}^r$ over k $(d\geq 2)$ is a cyclic group of order d.*

Proof. Let $x_0, x_1, \cdots, x_r$ be independent variables over k. We shall set $A:=k[x_0, x_1, \cdots, x_r]$. Let A' be the subring of A generated by monomials with degree d. Then A' is isomorphic to the homogeneous coordinate ring of $v_d(\boldsymbol{P}^r)$. We may assume that k is algebraically closed by Lemma 3.11. Let p denote the characteristic of k. In the case $p=0$, we have $\mathrm{Cl}(A')\cong\boldsymbol{Z}/d\boldsymbol{Z}$ by [8], p. 85, (1). Assume that $p>0$ and d is a power of p, say, $d=p^n$. Let $\underline{D}$ be the higher derivation on $Q(A)$ over k of rank $d-1$ defined by $\underline{D}(x_i)=x_i(1+t)$ $(0\leq i\leq r)$. Then we see easily that A' is the ring of $\underline{D}$-constants and $[K: K']=d$ where $K:=Q(A)$ and $K':=Q(A')$. Since $J(\underline{D}: x_i)=x_i(0\leq i\leq r)$, the height one property is satisfied. Hence by Theorem 1.6, $\mathrm{Cl}(A')\cong\mathrm{Ker}(\bar{j})\cong\mathcal{L}_A/\mathcal{L}_A'\cong\mathcal{L}_A$. Let θ be the homomorphism of $\boldsymbol{Z}/d\boldsymbol{Z}$ into $\mathcal{L}_A$ satisfying θ (the residue class of $j)=\underline{D}((x_0)/x_0)^j$. It is easily seen that θ is well-defined and bijective. Hence we have $\mathrm{Cl}(A')\cong\boldsymbol{Z}/d\boldsymbol{Z}$. If d is not a power of p, write $d=ap^n$, $p\nmid a$ and let B be the subring of A generated by monomials with degree p^n. Let ω be a primitive a-th root of unity and let σ be the automorphism of B defined by $\sigma(M)=\omega M$ for every monomial M with degree p^n. Let G be the subgroup of Aut B generated by σ. Then we have $A'=B^G$. Since $x_i^{p^n}$ is a primitive element of $Q(B)$ over $Q(A')$ for $0\leq i\leq r$, it is easily seen that B is divisorially unramified over A'. By the similar device to the proof of Proposition 3.12, we get $\mathrm{Cl}\,(A')\cong\boldsymbol{Z}/d\boldsymbol{Z}$. Q.E.D.

All rings appeared in the above Propositions are generated by monomials. The coordinate ring of the following surface is not generated by monomials:

Proposition 3.15. *Let n be a positive integer and s be a non-negative integer with $0\leq s\leq n$. Then the divisor class group of a surface S: $Z^{p^n}=X^{p^s}Y^{p^n}-Y$ is isomorphic to $\boldsymbol{Z}/p^{n-s}\boldsymbol{Z}$.*

Proof. Let x, y be independent variables over k. Then it is easily seen that the affine coordinate ring of the surface S is given by $A':=k[x^{p^n}, y^{p^n}, x^{p^s}y^{p^n}-y]$. Set $A:=k[x, y]$ and let D be the higher derivation of rank $m:=p^n-1$ on $Q(A)$ over k defined by $\underline{D}(x)=x+t$, $\underline{D}(y)=y+y^{p^n}t^{p^s}$. Then it is easily checked that the assumptions in Theorem 1.6 are satisfied. Define the homomorphism of $\boldsymbol{Z}/p^{n-s}\boldsymbol{Z}$ into $\mathcal{L}_A$ by θ (the residue class of i)$=(\underline{D}(y)/y)^i$. Then θ is well-defined and injective. We shall show that θ is surjective. Suppose that $\underline{D}(f)/f\in A[t\colon m]$ $(f\in A-\{0\})$, then there exists an element $g(T)$ of $A[T]$ such that $\underline{D}(f)/f=g(t)$. Since the degree with respect to x of the coefficient of t^j in $\underline{D}(f)$ is not more than that of f for $0\leq j\leq m$, we have $g(t)\in k[y]\,[t\colon m]$. Write

$$\begin{aligned} &f=a_0(y)+a_1(y)x+\cdots+a_h(y)x^h\,,\\ &a_\nu(y)\in k[y]\quad (0\leq\nu\leq h)\quad \text{and}\quad a_h(y)\neq 0\,. \end{aligned}$$

From $\underline{D}(f)=fg(t)$, we get

$$\begin{aligned} &\underline{D}(a_0(y))+\underline{D}(a_1(y))(x+t)+\cdots+\underline{D}(a_h(y))(x+t)^h\\ &\quad=a_0(y)g(t)+a_1(y)g(t)x+\cdots+a_h(y)g(t)x^h\,. \end{aligned}$$

Comparing the coefficients of x^h on both sides, we have $\underline{D}(a_h(y))=a_h(y)g(t)$ because x, y and T are algebraically independent over k. By Lemma 3.17, there exists an integer i such that $g(t)=(\underline{D}(y)/y)^i$. Hence θ is surjective and $\mathrm{Cl}(A')\cong$ $\boldsymbol{Z}/p^{n-s}\boldsymbol{Z}$. Q.E.D.

REMARK 3.16. Let $\mathcal{G}$ be the prime ideal in $P(A')$ generated by y^{p^n} and $x^{p^s}y^{p^n}-y$. Then $\mathrm{cl}(\mathcal{G})$ generates $\mathrm{Cl}(A')$. The q-th symbolic power $\mathcal{G}^{(q)}$ of $\mathcal{G}$ is a principal ideal generated by $y^{p^{n-s}}$ where $q\colon=p^{n-s}$.

Lemma 3.17. *Let $A=k[y]$ be a one-dimensional polynomial ring over k. Let n be a positive integer and s be a non-negative integer with $0\leq s\leq n$. Let $\underline{D}$ be the higher derivation of rank $m\colon=p^n-1$ on $Q(A)$ over k defined by $\underline{D}(y)=y+y^{p^n}t^{p^s}$. If $\underline{D}(f)/f$ $(f\in A-\{0\})$ is in $A[t\colon m]$, there exists an integer i such that $\underline{D}(f)/f=(\underline{D}(y)/y)^i$.*

Proof. Set $A'\colon=k[y^{p^{n-s}}]$, then we have $A'=A\cap K'$ where K' is the field of D-constants. Notice that $\mathcal{P}\colon=yA$ is the only prime ideal in $P(A)$ such that $D_q(A)\subset\mathcal{P}$ $(q\colon=p^s)$. Then we have $e(\mathcal{P})=p^{n-s}$ and $s(\mathcal{P})=1$. Hence we get the following exact sequence by Theorem 2.6.

$$0\to \mathrm{Ker}\,(\bar{j})\to \mathcal{L}_A/\mathcal{L}'_A\xrightarrow{\eta}\boldsymbol{Z}/p^{n-s}\boldsymbol{Z}\to 0\,.$$

Notice that η (the residue class of $(\underline{D}(y)/y)^j$)$=$the residue class of j. Further more $\mathrm{Ker}(\bar{j})\cong\mathrm{Cl}(A')=0$ and $\mathcal{L}'_A=\{1\}$. So we have the desired result.

Q.E.D.

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