

EQUIVARIANT KO-RINGS AND J-GROUPS OF SPHERES WHICH HAVE LINEAR PSEUDOFREE S^1 -ACTIONS

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(Received January 30, 1980)

1. Introduction

In this paper, we consider the equivariant KO -rings and J -groups of spheres which have linear pseudofree circle actions.

Let S^1 be the circle group consisting of complex numbers of absolute value one. For a sequence $p=(p_1, p_2, \dots, p_m)$ of positive integers, we define the S^1 -action φ_p on the complex m -dimensional vector space C^m by

$$\varphi_p(s, (z_1, z_2, \dots, z_m)) = (s^{p_1}z_1, s^{p_2}z_2, \dots, s^{p_m}z_m)$$

and denote by

$$S^{2m-1}(p_1, p_2, \dots, p_m)$$

the unit sphere S^{2m-1} in C^m with this action φ_p . Then the S^1 -action on $S^{2m-1}(p_1, p_2, \dots, p_m)$ is said to be *pseudofree* (resp. *free*) if $(p_i, p_j)=1$ for $i \neq j$ and $p_i > 1$ for some $1 \leq i \leq m$ (resp. $p_1=p_2=\dots=p_m=1$) (see Montgomery-Yang [19], [20]).

The main results of our paper are as follows:

Theorem 4.7. *Let p_i ($1 \leq i \leq m$) be positive odd integers such that $(p_i, p_j)=1$ for $i \neq j$. Then there is a monomorphism of rings :*

$$\Phi: KO_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m)) \rightarrow KO(CP^{m-1}) \oplus \bigoplus_{i=1}^m RO(\mathbf{Z}_{p_i}).$$

(For details see §4.)

Let G_i ($i \geq 1$) denote the stable homotopy group $\pi_{n+i}(S^n)$ ($n \geq i+2$). We define $s(k) = \prod_{i=1}^k |G_i|$ for $k > 0$, where $|G_i|$ denotes the order of the group G_i and put $s(-1)=1$.

Theorem 5.4. *Let p_i ($1 \leq i \leq m$) be positive odd integers such that $(p_i, p_j)=1$ for $i \neq j$ and $(p_i, s(2m-3))=1$ for $1 \leq i \leq m$. Then there is a monomorphism of groups:*

$$\tilde{\Phi}: J_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m)) \rightarrow J(CP^{m-1}) \oplus_{i=1}^m J_{\mathbf{Z}_{p_i}}(*).$$

(For details see §5.)

The paper is organized as follows:

In §§2 and 3, we consider a generalization of the results due to Folkman [9] and Rubinsztein [23] and prove some preliminary results. In §§4 and 5, we study an isomorphism and an S^1 -fiber homotopy equivalence of real S^1 -vector bundles over the pseudofree S^1 -manifold $S^{2m-1}(p_1, p_2, \dots, p_m)$ respectively. In §6, we consider the problem on quasi-equivalence posed by Meyerhoff and Petrie ([18], [21]).

2. Equivariant homotopy

Let n be a positive integer. Denote by \mathbf{Z}_n the cyclic group $\mathbf{Z}/n\mathbf{Z}$ of order n . If V is a real representation space of \mathbf{Z}_n , we denote by $S(V)$ its unit sphere with respect to some \mathbf{Z}_n -invariant inner product. Denote by $[X, Y]$ the set of homotopy classes of maps from X to Y . In this section, we shall prove the following theorem (cf. Folkman [9; Proposition 2.3] and Rubinsztein [23; Corollary 5.3]).

Theorem 2.1. *Let V be a complex \mathbf{Z}_n -representation space such that \mathbf{Z}_n acts freely on $S(V)$ and $\dim_{\mathbf{R}} V=2m$. Let X be a \mathbf{Z}_n -space which satisfies the following conditions:*

- (i) X is path-connected and q -simple for $1 \leq q \leq 2m-1$,
- (ii) the map of X into itself given by the action of a generator of \mathbf{Z}_n is homotopic to the identity,

$$(iii) \begin{cases} \text{Hom}(\mathbf{Z}_n, \pi_{2i-1}(X))=0 & \text{for } 1 \leq i \leq m, \\ \text{Ext}(\mathbf{Z}_n, \pi_{2i}(X))=0 & \text{for } 1 \leq i \leq m-1. \end{cases}$$

If there exist \mathbf{Z}_n -maps $f_0, f_1: S(V) \rightarrow X$ such that $[f_0]=[f_1] \in [S^{2m-1}, X]$, then f_0 and f_1 are \mathbf{Z}_n -homotopic.

Before beginning the proof of Theorem 2.1, we require some notations and lemmas.

Let M be a \mathbf{Z}_n -space $S(V) \times [0, 1]$, where $[0, 1]$ is the unit interval with the trivial \mathbf{Z}_n -action. Then M is a compact smooth \mathbf{Z}_n -manifold with a free \mathbf{Z}_n -action. Let x_0 be a point of $S(V)$. We put $N=S(V) \times \{0, 1\} \cup \{x_0\} \times [0, 1]$ and $M'=M/\mathbf{Z}_n$. Let $\pi: M \rightarrow M'$ be the natural projection. We put $N'=\pi(N)$.

Let R be an arbitrary abelian group. By the universal-coefficient theorem, we have the following lemmas.

Lemma 2.2. *There are isomorphisms:*

$$\begin{aligned} H^q(M, N; R) &= 0 & \text{for } 0 \leq q \leq 2m-1, \\ H^{2m}(M, N; R) &\cong R. \end{aligned}$$

Lemma 2.3. *There are isomorphisms:*

$$\begin{aligned} H^0(M', N'; R) &= H^1(M', N'; R) = 0, \\ H^{2q-1}(M', N'; R) &\cong \text{Ext}(\mathbf{Z}_n, R) \quad \text{for } 2 \leq q \leq m, \\ H^{2q}(M', N'; R) &\cong \text{Hom}(\mathbf{Z}_n, R) \quad \text{for } 1 \leq q \leq m-1, \\ H^{2m}(M', N'; R) &\cong R. \end{aligned}$$

Since the \mathbf{Z}_n -action on M is free and orientation-preserving, we have

Lemma 2.4. *Assume that $\text{Hom}(\mathbf{Z}_n, R)=0$. Then the homomorphism*

$$\pi^*: H^{2m}(M', N'; R) \rightarrow H^{2m}(M, N; R)$$

is injective.

Proof of Theorem 2.1. In order to prove Theorem 2.1, it suffices to show that there exists a \mathbf{Z}_n -map $F: M \rightarrow X$ such that $F|S(V) \times \{0\} = f_0$ and $F|S(V) \times \{1\} = f_1$.

Since $[f_0] = [f_1] \in [S^{2m-1}, X]$, there exists a continuous map $F': M \rightarrow X$ such that $F'|S(V) \times \{0\} = f_0$ and $F'|S(V) \times \{1\} = f_1$. Since M is a compact smooth \mathbf{Z}_n -manifold and \mathbf{Z}_n acts freely on M , we can consider the fiber bundle \mathcal{B} :

$$X \rightarrow M \times_{\mathbf{Z}_n} X \rightarrow M'.$$

A cross-section s_0 of the part of \mathcal{B} over N' ($=\pi(N)$) is defined by

$$s_0(\pi(z)) = [z, F'(z)] \in M \times_{\mathbf{Z}_n} X \quad \text{for } z \in N.$$

To prove Theorem 2.1, it suffices to show that the cross-section s_0 defined on N' is extendable to a full cross-section of \mathcal{B} . Because there is a one-to-one correspondence between \mathbf{Z}_n -maps from M to X and cross-sections of \mathcal{B} .

Let K be a simplicial complex. Denote by K^q the q -skelton. Denote by $|K|$ the geometric realization of K in the weak topology. It is easy to see that there exist finite simplicial complexes K_1 and K_2 which satisfy the following:

- (2.5) $|K_1| = M$ and $|K_2| = M'$,
- (2.6) there exist subcomplexes $L_1 \subset K_1$ and $L_2 \subset K_2$ such that $|L_1| = N$ and $|L_2| = N'$,
- (2.7) there exists a simplicial map $\tau: (K_1, L_1) \rightarrow (K_2, L_2)$ such that $|\tau| = \pi: (|K_1|, |L_1|) \rightarrow (|K_2|, |L_2|)$.

Let $\mathcal{B}(\pi_{q-1})$ ($1 \leq q \leq 2m$) be the bundles of coefficients associated with $\pi_{q-1}(X)$ (see Steenrod [27; §30]). By the assumption (ii), $\mathcal{B}(\pi_{q-1})$ ($1 \leq q \leq 2m$) are product bundles. Therefore the cohomology groups $H^q(M', N'; \mathcal{B}(\pi_{q-1}))$ are isomorphic to the ordinary cohomology groups $H^q(M', N'; \pi_{q-1}(X))$ for $1 \leq q \leq 2m$. By the assumption (iii) and Lemma 2.3, we have

$$H^q(M', N'; \pi_{q-1}(X)) = 0 \quad \text{for } 1 \leq q \leq 2m-1.$$

It follows from Steenrod [27; 34.2] that there exists a cross-section of \mathcal{B} defined on $|K_2^{2m-1}| (\supset |L_2|)$:

$$s_1: |K_2^{2m-1}| \rightarrow \underset{Z_n}{M \times X}$$

such that $s_1|_{|L_2|} = s_0$. There exists an obstruction cohomology class

$$\bar{c}(s_1) \in H^{2m}(M', N'; \pi_{2m-1}(X))$$

such that its vanishing is a necessary and sufficient condition for $s_1|_{|K_2^{2m-2} \cup L_2|}$ to be extendable over M' . Thus we shall show that $\bar{c}(s_1) = 0$. Consider the product bundle \mathcal{B}' :

$$X \rightarrow M \times X \rightarrow M.$$

Let $\mathcal{B}'(\pi_{q-1})$ ($1 \leq q \leq 2m$) be the bundles of coefficients associated with $\pi_{q-1}(X)$. Since \mathcal{B}' is a product bundle, $\mathcal{B}'(\pi_{q-1})$ ($1 \leq q \leq 2m$) are also product bundles. The natural projection $M \times X \rightarrow M \times X$ induces the bundle maps $\bar{\pi}: \mathcal{B}' \rightarrow \mathcal{B}$ and $\bar{\pi}_{q-1}: \mathcal{B}'(\pi_{q-1}) \rightarrow \mathcal{B}(\pi_{q-1})$ ($1 \leq q \leq 2m$) covering $\pi: (M, N) \rightarrow (M', N')$. Let $s_2: |K_1^{2m-1}| \rightarrow M \times X$ be the cross-section of \mathcal{B}' induced by s_1 and $\bar{\pi}$. It follows from (2.7) that we have

$$\pi^*(\bar{c}(s_1)) = \bar{c}(s_2) \in H^{2m}(M, N; \pi_{2m-1}(X)).$$

By the assumption (iii) and Lemma 2.4, π^* is a monomorphism. Hence $\bar{c}(s_1) = 0$ if and only if $\bar{c}(s_2) = 0$. Let $s_3: M = |K_1| \rightarrow M \times X$ be a cross-section of \mathcal{B}' defined by

$$s_3(z) = (z, F'(z)) \in M \times X \quad \text{for } z \in M.$$

We put

$$s_4 = s_3|_{|K_1^{2m-1}|}: |K_1^{2m-1}| \rightarrow M \times X.$$

Then s_2 and s_4 are cross-sections of \mathcal{B}' defined on $|K_1^{2m-1}| (\supset |L_1|)$ such that $s_2|_{|L_1|} = s_4|_{|L_1|}$. By Lemma 2.2, we have

$$H^q(M, N; \pi_q(X)) = 0 \quad \text{for } 0 \leq q \leq 2m-2.$$

It follows from Steenrod [27; 35.9] that

$$\bar{c}(s_2) = \bar{c}(s_4) \in H^{2m}(M, N; \pi_{2m-1}(X)).$$

It is obvious that $\bar{c}(s_2) = \bar{c}(s_4) = 0$. Hence we have $\bar{c}(s_1) = 0$. q.e.d.

Corollary 2.8. *Let X and V be as in Theorem 2.1. Suppose that*

- (i) $X^{\mathbf{Z}_n} \neq \phi$,
- (ii) *there exists a \mathbf{Z}_n -map $f: S(V) \rightarrow X$ such that $[f] = 0 \in [S^{2m-1}, X]$ ($\cong \pi_{2m-1}(X)$).*

Let y_0 be an arbitrary point of $X^{\mathbf{Z}_n}$. Then there exists a \mathbf{Z}_n -map

$$F: D(V) \rightarrow X$$

such that $F|_{S(V)} = f$ and $F(0) = y_0$. Here $D(V)$ denotes the unit disk.

3. Equivariant maps which are equivariantly homotopic to zero

Let n be a positive integer. Let V and W be real \mathbf{Z}_n -representation spaces with $\dim_{\mathbf{R}} V = \dim_{\mathbf{R}} W = k > 0$. Let

$$\rho_V, \rho_W: \mathbf{Z}_n \rightarrow GL(k, \mathbf{R})$$

be the \mathbf{Z}_n -representations afforded by V, W respectively. Then a \mathbf{Z}_n -action on $GL(k, \mathbf{R})$ is given by

$$s \circ A = \rho_W(s) A \rho_V(s)^{-1} \quad \text{for } s \in \mathbf{Z}_n, A \in GL(k, \mathbf{R}),$$

and denote by $GL(V, W)$ this \mathbf{Z}_n -space. Remark that $GL(k, \mathbf{R})$ has two connected components $GL^+(k, \mathbf{R})$ and $GL^-(k, \mathbf{R})$. If n is an odd integer, then we have

$$\rho_V(\mathbf{Z}_n), \rho_W(\mathbf{Z}_n) \subset GL^+(k, \mathbf{R}).$$

Hence $GL^+(k, \mathbf{R})$ and $GL^-(k, \mathbf{R})$ are \mathbf{Z}_n -subspaces of $GL(V, W)$ and are denoted by $GL^+(V, W)$ and $GL^-(V, W)$ respectively.

Let $F(S(V), S(W))$ denote the space of homotopy equivalent maps from $S(V)$ to $S(W)$ with the compact-open topology. A \mathbf{Z}_n -action on $F(S(V), S(W))$ is given by

$$(s \circ f)(v) = sf(s^{-1}v) \quad \text{for } s \in \mathbf{Z}_n, f \in F(S(V), S(W)), v \in S(V).$$

It is well-known that $F(S(V), S(W))$ has two connected components $F^+(S(V), S(W))$ and $F^-(S(V), S(W))$ representing maps of degree $+1$ and -1 respectively. If n is an odd integer, then $F^+(S(V), S(W))$ and $F^-(S(V), S(W))$ are \mathbf{Z}_n -subspaces of $F(S(V), S(W))$.

It is well-known that

(3.1) $GL^\varepsilon(V, W)$ and $F^\varepsilon(S(V), S(W))$ ($\varepsilon = \pm$) are path-connected and q -simple for $q > 0$.

Moreover it is easy to see that

(3.2) *If n is an odd integer, then the maps of $GL^\varepsilon(V, W)$ and $F^\varepsilon(S(V), S(W))$ ($\varepsilon = \pm$) into themselves given by the action of a generator of \mathbf{Z}_n are homotopic to the identity.*

Proposition 3.3. *Let n be a positive odd integer. Let V and W be real \mathbf{Z}_n -representation spaces with $\dim_{\mathbf{R}} V = \dim_{\mathbf{R}} W = k$. Let U be a complex \mathbf{Z}_n -representation space such that \mathbf{Z}_n acts freely on $S(U)$ and $\dim_{\mathbf{R}} U = 2m$. Assume that*

- (i) $k \geq 2m + 1$,
- (ii) *there exists a \mathbf{Z}_n -map $f: S(U) \rightarrow GL^\varepsilon(V, W)$ such that $[f] = 0 \in [S^{2m-1}, GL^\varepsilon(V, W)]$,*
- (iii) $GL^\varepsilon(V, W)^{\mathbf{Z}_n} \neq \phi$,

where $\varepsilon = +$ or $-$. Then there exists a \mathbf{Z}_n -map $F: D(U) \rightarrow GL^\varepsilon(V, W)$ such that $F|S(U) = f$.

Proof. It is well-known that

$$\pi_i(GL^\varepsilon(V, W)) \cong \begin{cases} \mathbf{Z}_2 & \text{if } i \equiv 0, 1 \pmod{8}, \\ 0 & \text{if } i \equiv 2, 4, 5, 6 \pmod{8}, \\ \mathbf{Z} & \text{if } i \equiv 3, 7 \pmod{8}, \end{cases}$$

for $1 \leq i \leq k - 2$. Since n is odd, we have

$$\begin{cases} \text{Hom}(\mathbf{Z}_n, \pi_{2i-1}(GL^\varepsilon(V, W))) = 0 & \text{for } 1 \leq i \leq m, \\ \text{Ext}(\mathbf{Z}_n, \pi_{2i}(GL^\varepsilon(V, W))) = 0 & \text{for } 1 \leq i \leq m - 1. \end{cases}$$

Therefore the result follows from Corollary 2.8.

q.e.d.

Proposition 3.4. *Let n be a positive odd integer. Let V and W be real \mathbf{Z}_n -representation spaces with $\dim_{\mathbf{R}} V = \dim_{\mathbf{R}} W = k$. Let U be a complex \mathbf{Z}_n -representation space such that \mathbf{Z}_n acts freely on $S(U)$ and $\dim_{\mathbf{R}} U = 2m$. Assume that*

- (i) $(n, s(2m - 1)) = 1$,
- (ii) $k \geq 2m + 2$,
- (iii) *there exists a \mathbf{Z}_n -map $f: S(U) \rightarrow F^\varepsilon(S(V), S(W))$ such that $[f] = 0 \in [S^{2m-1}, F^\varepsilon(S(V), S(W))]$,*
- (iv) $F^\varepsilon(S(V), S(W))^{\mathbf{Z}_n} \neq \phi$,

where $\varepsilon = +$ or $-$. Let φ be an arbitrary element of $F^\varepsilon(S(V), S(W))^{\mathbf{Z}_n}$. Then there exists a \mathbf{Z}_n -map $F: D(U) \rightarrow F^\varepsilon(S(V), S(W))$ such that $F|S(U) = f$ and $F(0) = \varphi$.

Proof. It follows from Atiyah [4; p. 294] that there exist isomorphisms

$$\pi_i(F^\varepsilon(S(V), S(W))) \cong G_i \quad \text{for } 1 \leq i \leq k - 3.$$

By the assumptions (i) and (ii), we have

$$\begin{cases} \text{Hom}(\mathbf{Z}_n, \pi_{2i-1}(F^\varepsilon(S(V), S(W)))) = 0 & \text{for } 1 \leq i \leq m, \\ \text{Ext}(\mathbf{Z}_n, \pi_{2i}(F^\varepsilon(S(V), S(W)))) = 0 & \text{for } 1 \leq i \leq m - 1. \end{cases}$$

Therefore the result follows from Corollary 2.8.

q.e.d.

4. Equivariant KO-rings

In this section, we consider an isomorphism of S^1 -vector bundles over $S^{2m-1}(p_1, p_2, \dots, p_m)$ when the S^1 -action is free or pseudofree.

Let V be a real S^1 -representation space. Let X be a compact S^1 -space. Denote by \underline{V} the trivial S^1 -vector bundle

$$V \rightarrow X \times V \rightarrow X.$$

Let ξ and η be real S^1 -vector bundles over X with $\dim_{\mathbb{R}} \xi = \dim_{\mathbb{R}} \eta$. Let

$$p: \text{Hom}(\xi, \eta) \rightarrow X$$

be the S^1 -vector bundle defined by Atiyah [3; §1.2] and Segal [25; §1]. Let $\text{Iso}(\xi, \eta) \subset \text{Hom}(\xi, \eta)$ be the subspace of all isomorphisms from ξ_x to η_x for $x \in X$, where ξ_x (resp. η_x) denotes the fiber of ξ (resp. η) over x . Clearly, $\text{Iso}(\xi, \eta)$ is an S^1 -subspace of $\text{Hom}(\xi, \eta)$ and

$$(4.1) \quad q = p|_{\text{Iso}(\xi, \eta)}: \text{Iso}(\xi, \eta) \rightarrow X$$

is a surjective S^1 -map. We remark that ξ and η are equivalent as S^1 -vector bundles over X if and only if there exists an S^1 -cross-section of q defined on X .

Let $p = (p_1, p_2, \dots, p_m)$ be a sequence of positive integers. Denote by $D^{2m}(p_1, p_2, \dots, p_m)$ the unit disk in \mathbb{C}^m with the S^1 -action φ_p (see §1).

Let $m > 1$ be an integer. We put

$$\begin{aligned} M_k &= S^{2m-1}(p_1, p_2, \dots, p_k, 1, \dots, 1) && \text{for } 1 \leq k \leq m, \\ S_k &= S^{2m-3}(p_1, p_2, \dots, p_{k-1}, 1, \dots, 1) && \text{for } 2 \leq k \leq m, \\ D_k &= D^{2m-2}(p_1, p_2, \dots, p_{k-1}, 1, \dots, 1) && \text{for } 2 \leq k \leq m, \\ M_0 &= S^{2m-1}(1, 1, \dots, 1), \\ S_1 &= S^{2m-3}(1, 1, \dots, 1), \\ D_1 &= D^{2m-2}(1, 1, \dots, 1). \end{aligned}$$

Here we remark that $\partial D_k = S_k$ for $1 \leq k \leq m$.

In the following, for every positive integer n , we always regard the cyclic group \mathbb{Z}_n as the subgroup of S^1 and regard an S^1 -space as a \mathbb{Z}_n -space in respective context.

We define a \mathbb{Z}_{p_k} -map $j_k: D_k \rightarrow M_k$ by

$$j_k(z_1, \dots, z_{k-1}, z_k, \dots, z_{m-1}) = (z_1, \dots, z_{k-1}, \sqrt{1 - |z_1|^2 - \dots - |z_{m-1}|^2}, z_k, \dots, z_{m-1}).$$

It is easy to see that j_k is a \mathbb{Z}_{p_k} -embedding and $j_k|_{S_k}: S_k \rightarrow M_k$ is an S^1 -embedding. In the following, D_k and S_k are regarded as a \mathbb{Z}_{p_k} -invariant subspace of M_k and an S^1 -invariant subspace of M_k by j_k respectively. Let e_j ($1 \leq j \leq m$) be

the j -th unit vector of C^m . Then we see that $e_1, e_2, \dots, e_{k-1} \in S_k$ and $e_k \in D_k$ as the center of the disk.

We define a continuous map $\alpha: S^1 \times D_k \rightarrow M_k$ by

$$\alpha(s, z) = sz \quad \text{for } s \in S^1, z \in D_k.$$

Then we have

Lemma 4.2. α is an identification map.

The proof is easy.

Lemma 4.3. Let X be an S^1 -space and let $p: X \rightarrow M_k$ be a surjective S^1 -map. If there exists a Z_{p_k} -cross-section $t_1: D_k \rightarrow X$ of $p|_{p^{-1}(D_k)}$ such that $t_1|_{S_k}: S_k \rightarrow X$ is an S^1 -cross-section of $p|_{p^{-1}(S_k)}$, then there exists an S^1 -cross-section $t: M_k \rightarrow X$ of p such that $t|_{D_k} = t_1$.

Proof. By Lemma 4.2, $\alpha: S^1 \times D \rightarrow M_k$ is surjective. Thus, given $z \in M_k$, there exists $s \in S^1$ such that $s^{-1}z \in D_k$. Define $t: M_k \rightarrow X$ by

$$t(z) = st(s^{-1}z),$$

where $s \in S^1$ is chosen as $s^{-1}z \in D_k$. Then it is easy to see that t is a well-defined S^1 -cross-section of p such that $t|_{D_k} = t_1$. q.e.d.

Define S^1 -maps

$$h_k: M_k \rightarrow M_{k+1} \quad \text{for } 0 \leq k \leq m-1$$

by

$$h_k(z_1, \dots, z_k, z_{k+1}, z_{k+2}, \dots, z_m) = \frac{(z_1, \dots, z_k, z_{k+1}^{p_{k+1}}, z_{k+2}, \dots, z_m)}{\|(z_1, \dots, z_k, z_{k+1}^{p_{k+1}}, z_{k+2}, \dots, z_m)\|}$$

and we put $h_m = id: M_m \rightarrow M_m$. Moreover we define

$$\tilde{h}_k = h_m \circ h_{m-1} \circ \dots \circ h_k: M_k \rightarrow M_m \quad \text{for } 0 \leq k \leq m.$$

Then it follows that

$$\tilde{h}_k(e_j) = e_j \quad \text{for } 0 \leq k \leq m, 1 \leq j \leq m.$$

Let ξ and η be S^1 -vector bundles over M_m with $\dim_{\mathbb{R}} \xi = \dim_{\mathbb{R}} \eta = n$. We put

$$V_k = (\tilde{h}_k^* \xi)_{e_k} = \xi_{e_k}, \quad W_k = (\tilde{h}_k^* \eta)_{e_k} = \eta_{e_k} \quad \text{for } 1 \leq k \leq m.$$

Here V_k, W_k ($1 \leq k \leq m$) are regarded as Z_{p_k} -representation spaces. Let $q_k: \text{Iso}(\tilde{h}_k^* \xi, \tilde{h}_k^* \eta) \rightarrow M_k$ ($0 \leq k \leq m$) be S^1 -maps defined by (4.1). Then we have

Lemma 4.4. *There are \mathbf{Z}_{p_k} -homeomorphisms*

$$\varphi_k: q_k^{-1}(D_k) \rightarrow D_k \times GL(V_k, W_k) \quad \text{for } 1 \leq k \leq m$$

such that the following diagram commutes:

$$\begin{array}{ccc} q_k^{-1}(D_k) & \xrightarrow{\varphi_k} & D_k \times GL(V_k, W_k) \\ q_k | q_k^{-1}(D_k) & \searrow & \swarrow \pi_1 \\ & D_k & \end{array}$$

where π_1 denotes the projection on the first factor.

Proof. Since D_k is \mathbf{Z}_{p_k} -contractible, there exist isomorphisms of \mathbf{Z}_{p_k} -vector bundles:

$$\begin{cases} \alpha: (\tilde{h}_k^* \xi) | D_k \rightarrow D_k \times V_k, \\ \beta: (\tilde{h}_k^* \eta) | D_k \rightarrow D_k \times W_k. \end{cases}$$

Let $\tilde{q}_k: \text{Iso}(D_k \times V_k, D_k \times W_k) \rightarrow D_k$ be an S^1 -map defined by (4.1). Then we can define \mathbf{Z}_{p_k} -homeomorphisms

$$\begin{cases} \psi_1: \text{Iso}((\tilde{h}_k^* \xi) | D_k, (\tilde{h}_k^* \eta) | D_k) \rightarrow \text{Iso}(D_k \times V_k, D_k \times W_k), \\ \psi_2: \text{Iso}(D_k \times V_k, D_k \times W_k) \rightarrow D_k \times GL(V_k, W_k), \end{cases}$$

by

$$\begin{cases} \psi_1(f_x) = \beta_x \circ f_x \circ \alpha_x^{-1} & \text{for } x \in D_k, f_x \in q_k^{-1}(x), \\ \psi_2(g_x) = (x, g_x) & \text{for } x \in D_k, g_x \in \tilde{q}_k^{-1}(x), \end{cases}$$

respectively. It is obvious that a \mathbf{Z}_{p_k} -homeomorphism

$$\varphi_k = \psi_2 \circ \psi_1: q_k^{-1}(D_k) = \text{Iso}((\tilde{h}_k^* \xi) | D_k, (\tilde{h}_k^* \eta) | D_k) \rightarrow D_k \times GL(V_k, W_k)$$

satisfies our condition.

q.e.d.

Define an S^1 -map $h: M_0 \rightarrow M_m$ by

$$h(z_1, z_2, \dots, z_m) = \frac{(z_1^{p_1}, z_2^{p_2}, \dots, z_m^{p_m})}{\|(z_1^{p_1}, z_2^{p_2}, \dots, z_m^{p_m})\|}.$$

Lemma 4.5. *Let $m > 1$ be an integer and let p_i ($1 \leq i \leq m$) be positive odd integers with $(p_i, p_j) = 1$ for $i \neq j$. Let ξ and η be real S^1 -vector bundles over M_m such that $\dim_{\mathbf{R}} \xi = \dim_{\mathbf{R}} \eta = n \geq 2m - 1$ and $\xi \supset \underline{\mathbf{R}}^1$ as an S^1 -vector subbundle. Assume that*

- (i) $h^* \xi$ and $h^* \eta$ are equivalent as S^1 -vector bundles over M_0 ,
- (ii) ξ_{e_k} and η_{e_k} are equivalent as \mathbf{Z}_{p_k} -representation spaces for $1 \leq k \leq m$.

Then ξ and η are equivalent as S^1 -vector bundles over M_m .

Proof. Let $q_k: \text{Iso}(\tilde{h}_k^*\xi, \tilde{h}_k^*\eta) \rightarrow M_k$ ($0 \leq k \leq m$) be S^1 -maps defined by (4.1). We shall show that there exist S^1 -cross-sections of q_k ($0 \leq k \leq m$):

$$t_k: M_k \rightarrow \text{Iso}(\tilde{h}_k^*\xi, \tilde{h}_k^*\eta),$$

by induction. Then the existence of the last S^1 -cross-section t_m shows the result.

It follows from IBERKLEID [11; Theorem 3.4] that the S^1 -maps $\tilde{h}_0, h: M_0 \rightarrow M_m$ are S^1 -homotopic. Hence, by the assumption (i), we have

$$\tilde{h}_0^*\xi \cong h^*\xi \cong h^*\eta \cong \tilde{h}_0^*\eta,$$

where \cong stands for *is equivalent to*. Therefore there exists an S^1 -cross-section of q_0 :

$$t_0: M_0 \rightarrow \text{Iso}(\tilde{h}_0^*\xi, \tilde{h}_0^*\eta).$$

Let k be an integer greater than zero. We now assume that there exists an S^1 -cross-section of q_{k-1} :

$$t_{k-1}: M_{k-1} \rightarrow \text{Iso}(\tilde{h}_{k-1}^*\xi, \tilde{h}_{k-1}^*\eta).$$

Remark that

$$\tilde{h}_{k-1} = \tilde{h}_k \circ h_{k-1}: M_{k-1} \rightarrow M_m.$$

It follows that there exist S^1 -vector bundle maps

$$\begin{cases} \bar{h}_{k-1}: \tilde{h}_{k-1}^*\xi \rightarrow \tilde{h}_k^*\xi, \\ \bar{h}'_{k-1}: \tilde{h}_{k-1}^*\eta \rightarrow \tilde{h}_k^*\eta, \end{cases}$$

covering $h_{k-1}: M_{k-1} \rightarrow M_k$. We define an embedding $j'_k: D_k \rightarrow M_{k-1}$ by

$$j'_k(z_1, \dots, z_{k-1}, z_k, \dots, z_{m-1}) = (z_1, \dots, z_{k-1}, \sqrt{1 - |z_1|^2 - \dots - |z_{m-1}|^2}, z_k, \dots, z_{m-1}).$$

Then the restriction $j'_k|S_k: S_k \rightarrow M_{k-1}$ is an S^1 -embedding. Thus D_k and S_k are also regarded as a subspace of M_{k-1} and an S^1 -invariant subspace of M_{k-1} by j'_k respectively. We put $D'_k = j'_k(D_k)$ and $S'_k = j'_k(S_k)$. It is easy to see that

$$\begin{cases} h_{k-1}|D'_k: D'_k \rightarrow D_k \subset M_k, \\ h_{k-1}|S'_k: S'_k \rightarrow S_k \subset M_k, \end{cases}$$

are a homeomorphism and an S^1 -homeomorphism respectively. It follows that the restrictions

$$\begin{cases} \bar{h}_{k-1}| \{(\tilde{h}_{k-1}^*\xi)|D'_k\}: (\tilde{h}_{k-1}^*\xi)|D'_k \rightarrow (\tilde{h}_k^*\xi)|D_k, \\ \bar{h}'_{k-1}| \{(\tilde{h}_{k-1}^*\eta)|D'_k\}: (\tilde{h}_{k-1}^*\eta)|D'_k \rightarrow (\tilde{h}_k^*\eta)|D_k, \end{cases}$$

are isomorphisms of vector bundles. Moreover the restrictions

$$\begin{cases} \bar{h}_{k-1} | \{(\tilde{h}_{k-1}^* \xi) | S'_k\} : (\tilde{h}_{k-1}^* \xi) | S'_k \rightarrow (\tilde{h}_k^* \xi) | S_k, \\ \bar{h}'_{k-1} | \{(\tilde{h}'_{k-1}^* \eta) | S'_k\} : (\tilde{h}'_{k-1}^* \eta) | S'_k \rightarrow (\tilde{h}_k^* \eta) | S_k, \end{cases}$$

are isomorphisms of S^1 -vector bundles. Using the S^1 -cross-section $t_{k-1}: M_{k-1} \rightarrow \text{Iso}(\tilde{h}_{k-1}^* \xi, \tilde{h}'_{k-1}^* \eta)$, we can define a continuous cross-section of $q_k | q_k^{-1}(D_k)$:

$$u_k: D_k \rightarrow q_k^{-1}(D_k) \subset \text{Iso}(\tilde{h}_k^* \xi, \tilde{h}_k^* \eta)$$

by putting $u_k(x) = \{\bar{h}'_{k-1} | (\tilde{h}'_{k-1}^* \eta)_x\} \circ t_{k-1}((h_{k-1} | D_k)^{-1}(x)) \circ \{h_{k-1} | (\tilde{h}_{k-1}^* \xi)_x\}$ for $x \in D_k \subset M$. Then the restriction

$$v_k = u_k | S_k: S_k \rightarrow q_k^{-1}(S_k) \subset \text{Iso}(\tilde{h}_k^* \xi, \tilde{h}_k^* \eta)$$

is an S^1 -cross-section of $q_k | q_k^{-1}(S_k)$. Let $\pi_2: D_k \times GL^\varepsilon(V_k, W_k) \rightarrow GL^\varepsilon(V_k, W_k)$ be the projection on the second factor. It follows from Lemma 4.4 that v_k yields a \mathbf{Z}_{p_k} -map

$$\vartheta_k: S_k \rightarrow GL^\varepsilon(V_k, W_k)$$

by $\vartheta_k(x) = \pi_2(\varphi_k(v_k(x)))$ for $x \in S_k$, where $\varepsilon = +$ or $-$. Since $v_k = u_k | S_k$, we have

$$[\vartheta_k] = 0 \in [S^{2m-3}, GL^\varepsilon(V_k, W_k)].$$

By the assumption (ii), $V_k (= (\tilde{h}_k^* \xi)_{e_k} = \xi_{e_k})$ and $W_k (= (\tilde{h}_k^* \eta)_{e_k} = \eta_{e_k})$ are equivalent as \mathbf{Z}_{p_k} -representation spaces and $V_k \supset \mathbf{R}^1$. This shows that

$$GL^\varepsilon(V_k, W_k)^{\mathbf{Z}_{p_k}} \neq \phi.$$

Moreover we remark that p_k is an odd integer and \mathbf{Z}_{p_k} acts freely on S_k . Therefore it follows from Proposition 3.3 that there exists a \mathbf{Z}_{p_k} -map

$$\bar{w}_k: D_k \rightarrow GL^\varepsilon(V_k, W_k)$$

such that $\bar{w}_k | S_k = \vartheta_k$. By Lemma 4.4, we can define a \mathbf{Z}_{p_k} -cross-section of $q_k | q_k^{-1}(D_k)$:

$$w_k: D_k \rightarrow q_k^{-1}(D_k) \subset \text{Iso}(\tilde{h}_k^* \xi, \tilde{h}_k^* \eta)$$

by $w_k(x) = \varphi_k^{-1}(x, \bar{w}_k(x))$ for $x \in D_k$. Since $w_k | S_k = \vartheta_k$, it follows from Lemma 4.3 that there exists an S^1 -cross-section of q_k :

$$t_k: M_k \rightarrow \text{Iso}(\tilde{h}_k^* \xi, \tilde{h}_k^* \eta).$$

In this way, we obtain S^1 -cross-sections t_0, t_1, \dots, t_m . q.e.d.

The following lemma is due to Segal (see [25; Proposition 2.1]).

Lemma 4.6. *Let G be a compact Lie group and let X be a compact Hausdorff G -space such that G acts freely on X . Then the projection $\text{pr}: X \rightarrow X/G$ induces*

an isomorphism of rings

$$pr^*: KO(X/G) \rightarrow KO_G(X).$$

We put

$$\mu = (pr^*)^{-1}: KO_{S^1}(M_0) \xrightarrow{\cong} KO(CP^{m-1}).$$

Denote by $RO(G)$ the real representation ring of G . We define a homomorphism of rings

$$\Phi: KO_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m)) \rightarrow KO(CP^{m-1}) \oplus \bigoplus_{i=1}^m RO(\mathbb{Z}_{p_i})$$

by putting

$$\Phi(\xi - \eta) = \mu(h^*\xi - h^*\eta) \oplus \bigoplus_{i=1}^m (\xi_{e_i} - \eta_{e_i}).$$

Then we have

Theorem 4.7. *Let p_i ($1 \leq i \leq m$) be positive odd integers such that $(p_i, p_j) = 1$ for $i \neq j$. Then the homomorphism Φ is injective.*

Proof. If $m=1$, then $KO_{S^1}(S^1(p_1)) = KO_{S^1}(S^1/\mathbb{Z}_{p_1}) \cong RO(\mathbb{Z}_{p_1})$. Therefore we assume that $m > 1$. If $\Phi(\xi - \eta) = 0$, then $h^*\xi - h^*\eta = 0$ in $KO_{S^1}(M_0)$ and $\xi_{e_i} - \eta_{e_i} = 0$ in $RO(\mathbb{Z}_{p_i})$ for $1 \leq i \leq m$. Thus there exists an S^1 -representation space U such that $h^*(\xi \oplus \underline{U})$ is equivalent to $h^*(\eta \oplus \underline{U})$. Then we put

$$\xi' = \xi \oplus \underline{\mathbb{R}^{2m}} \oplus \underline{U} \quad \text{and} \quad \eta' = \eta \oplus \underline{\mathbb{R}^{2m}} \oplus \underline{U}.$$

Since ξ' and η' satisfy the assumption of Lemma 4.5, ξ' is equivalent to η' . It follows that

$$\xi - \eta = \xi' - \eta' = 0 \quad \text{in } KO_{S^1}(M_m).$$

Hence Φ is injective. q.e.d.

Next we consider the condition (i) of Lemma 4.5. Let ES^1 (resp. BS^1) be a universal S^1 -space (resp. a classifying space for S^1). Let $\pi_k: ES^1 \times_{S^1} M_k \rightarrow BS^1$ ($0 \leq k \leq m$) be the natural projection.

Lemma 4.8. *The homomorphism*

$$\pi_k^*: H^q(BS^1; \mathbb{Z}) \rightarrow H^q(ES^1 \times_{S^1} M_k; \mathbb{Z})$$

is an isomorphism for $0 \leq q \leq 2m - 2$. Moreover the integral cohomology ring of $ES^1 \times_{S^1} M_k$ is

$$H^*(ES^1 \times_{S^1} M_k; \mathbb{Z}) = \mathbb{Z}[c]/(qc^m),$$

where $\deg c = 2$ and $q = \prod_{i=1}^k p_i$.

Proof. The map π_k is a projection of a sphere bundle associated with the complex m -plane bundle $\eta^{p_1} \oplus \dots \oplus \eta^{p_k} \oplus \eta \oplus \dots \oplus \eta$, where η is the canonical complex line bundle over BS^1 . Then the result follows from the Thom-Gysin exact sequence. q.e.d.

Lemma 4.9. *Let $\tau: ES^1 \times_{S^1} M_0 \rightarrow M_0/S^1 = CP^{m-1}$ be the natural projection. Then*

$$\tau^*: H^*(CP^{m-1}; \mathbf{Z}) \rightarrow H^*(ES^1 \times_{S^1} M_0; \mathbf{Z})$$

is an isomorphism.

Proof. The result follows from the Vietoris-Begle Mapping Theorem (see Bredon [6; p. 371], Spanier [26; p. 344]).

Lemma 4.10. *The homomorphism*

$$(1 \times h)^*: H^q(ES^1 \times_{S^1} M_m; \mathbf{Z}) \rightarrow H^q(ES^1 \times_{S^1} M_0; \mathbf{Z})$$

is an isomorphism for $0 \leq q \leq 2m - 2$.

Proof. Consider the following commutative diagram:

$$\begin{CD} H^q(BS^1) @>id>> H^q(BS^1) \\ @V\pi_m^*VV @VV\pi_0^*V \\ H^q(ES^1 \times_{S^1} M_m) @>(1 \times h)^*>> H^q(ES^1 \times_{S^1} M_0) . \end{CD}$$

Since π_m^* and π_0^* are isomorphisms for $0 \leq q \leq 2m - 2$, $(1 \times h)^*$ is an isomorphism for $0 \leq q \leq 2m - 2$. q.e.d.

Lemma 4.11. *Let ξ and η be real S^1 -vector bundles over M_m with $\dim_{\mathbf{R}} \xi = \dim_{\mathbf{R}} \eta = k$. Assume that $m \equiv 2 \pmod{4}$. Then the following two conditions are equivalent:*

- (i) $\mu(h^*\xi) = \mu(h^*\eta)$ in $KO(CP^{m-1})$,
- (ii) $p_i(ES^1 \times_{S^1} \xi) = p_i(ES^1 \times_{S^1} \eta)$ in $H^{4i}(ES^1 \times_{S^1} M_m; \mathbf{Z})$ for $1 \leq i \leq \min([k/2], [(m-1)/2])$.

Here $p_i(ES^1 \times_{S^1} \xi)$ (resp. $p_i(ES^1 \times_{S^1} \eta)$) denotes the i -th Pontrjagin class of the bundle $ES^1 \times_{S^1} \xi \rightarrow ES^1 \times_{S^1} M_m$ (resp. $ES^1 \times_{S^1} \eta \rightarrow ES^1 \times_{S^1} M_m$).

Proof. Remark that $\tau^*(\mu(h^*\xi)) = ES^1 \times_{S^1} h^*\xi$, where $\tau: ES^1 \times_{S^1} M_0 \rightarrow M_0/S^1 = CP^{m-1}$ is the natural projection. Then we have

$$\tau^*(p_i(\mu(h^*\xi))) = p_i(ES^1 \times_{S^1} h^*\xi)$$

and

$$(1 \times h)^*(p_i(ES^1 \times_{S^1} \xi)) = p_i(ES^1 \times_{S^1} h^* \xi).$$

Hence it follows from Lemmas 4.9 and 4.10 that the condition (ii) is equivalent to the following:

$$p_i(\mu(h^* \xi)) = p_i(\mu(h^* \eta)) \text{ in } H^{4i}(CP^{m-1}; \mathbf{Z})$$

for $1 \leq i \leq \min([k/2], [(m-1)/2])$. Since $m \equiv 2 \pmod 4$, $KO(CP^{m-1})$ is a free abelian group (see Sanderson [24; Theorem 3.9]). It follows from Hsiang [10; §3] that

$$p_i(\mu(h^* \xi)) = p_i(\mu(h^* \eta)) \quad \text{for } 1 \leq i \leq \min([k/2], [(m-1)/2])$$

if and only if

$$\mu(h^* \xi) = \mu(h^* \eta) \quad \text{in } KO(CP^{m-1}). \quad \text{q.e.d.}$$

By Theorem 4.7 and Lemma 4.11, we have

Theorem 4.12. *Let m be a positive integer such that $m \equiv 2 \pmod 4$. Let p_i ($1 \leq i \leq m$) be positive odd integers with $(p_i, p_j) = 1$ for $i \neq j$. Let ξ and η be real S^1 -vector bundles over $S^{2m-1}(p_1, p_2, \dots, p_m)$ with $\dim_{\mathbf{R}} \xi = \dim_{\mathbf{R}} \eta = k$. Then $\xi = \eta$ in $KO_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m))$ if and only if the following two conditions are satisfied:*

- (i) $\xi_{e_i} = \eta_{e_i}$ in $RO(\mathbf{Z}_{p_i})$ for $1 \leq i \leq m$,
- (ii) $p_i(ES^1 \times_{S^1} \xi) = p_i(ES^1 \times_{S^1} \eta)$ for $1 \leq i \leq \min([k/2], [(m-1)/2])$.

REMARK 4.13. Let G be a compact Lie group and let X be a finite G -CW-complex in the sense of Matumoto [17]. Let ξ and η be G -vector bundles over X such that they are stably equivalent. But, in general, ξ and η are not equivalent even if $\dim \xi = \dim \eta > \dim X$ (cf. Sanderson [24; Lemma 1.2]). For example, for an arbitrary integer $n \geq 0$, we put

$$\begin{cases} \xi = S^3(7, 11) \times t^2 \oplus t \oplus nt, \\ \eta = S^3(7, 11) \times t^9 \oplus t^{78} \oplus nt, \end{cases}$$

where t^d ($d \in \mathbf{Z}$) denotes the complex one-dimensional S^1 -representation space defined by $t^d(s)z = s^d z$ for $s \in S^1$, $z \in \mathbf{C}^1$. It follows from Lemma 4.5 that

$$\xi \oplus \underline{\mathbf{R}}^1 \cong \eta \oplus \underline{\mathbf{R}}^1.$$

Now we assume that there exists an isomorphism of S^1 -vector bundles:

$$\omega: \xi \rightarrow \eta.$$

Since ξ (resp. η) is a complex vector bundle, ξ (resp. η) has a canonical orientation. Then the isomorphism of \mathbf{Z}_7 -representation spaces $\omega_{e_1}: \xi_{e_1} \rightarrow \eta_{e_1}$

is orientation-preserving, but the isomorphism of Z_{11} -representation spaces $\omega_{e_2}: \xi_{e_2} \rightarrow \eta_{e_2}$ is orientation-reversing. Since $S^3(7, 11)$ is connected, this is a contradiction. Therefore ξ and η are not equivalent.

5. Equivariant J-groups

In [12] and [14], Kawakubo has defined the notion of the equivariant J -group as follows:

Let G be a compact Lie group and let X be a compact G -space. Let ξ and η be real G -vector bundles over X . Denote by $S(\xi)$ (resp. $S(\eta)$) the unit sphere bundle associated with ξ (resp. η) with respect to some S^1 -invariant metric. $S(\xi)$ and $S(\eta)$ are said to be G -fiber homotopy equivalent if $S(\xi)$ and $S(\eta)$ are homotopy equivalent by fiber-preserving G -maps and G -homotopies. Let $T_G(X)$ be the additive subgroup of $KO_G(X)$ generated by elements of the form $\xi - \eta$, where ξ and η are G -vector bundles over X whose associated sphere bundles are G -fiber homotopy equivalent. We define the equivariant J -group $J_G(X)$ by

$$J_G(X) = KO_G(X)/T_G(X)$$

and define the equivariant J -homomorphism J_G by the natural epimorphism

$$J_G: KO_G(X) \rightarrow J_G(X).$$

When X is a point, $J_G(X)$ is denoted by $J_G(*)$.

In this section, we shall consider the equivariant J -group of $S^{2m-1}(p_1, p_2, \dots, p_m)$ when the S^1 -action is free or pseudofree. We shall use freely the notations in §§3 and 4.

Let X be a compact S^1 -space. Let ξ and η be real S^1 -vector bundles over X with $\dim_{\mathbb{R}} \xi = \dim_{\mathbb{R}} \eta$. Let $E(S(\xi), S(\eta))$ denote the disjoint union of the function spaces $F(S(\xi_x), S(\eta_x))$ (see §3) and define

$$(5.1) \quad q': E(S(\xi), S(\eta)) \rightarrow X$$

by

$$q'(F(S(\xi_x), S(\eta_x))) = x.$$

Then there exists a canonical topology for $E(S(\xi), S(\eta))$ so that $E(S(\xi), S(\eta))$ is the total space of a fiber bundle with projection q' and with fibers $F(S(\xi_x), S(\eta_x))$. An S^1 -action

$$\rho: S^1 \times E(S(\xi), S(\eta)) \rightarrow E(S(\xi), S(\eta)),$$

is given by $\rho(s, f)(v) = sf(s^{-1}v)$ for $s \in S^1$, $f \in F(S(\xi_x), S(\eta_x))$, $v \in S(\xi_x)$. Then $q': E(S(\xi), S(\eta)) \rightarrow X$ is an S^1 -map.

Let p_i ($1 \leq i \leq m$) be positive integers. Let ξ and η be real S^1 -vector bundles over $M_m (= S^{2m-1}(p_1, p_2, \dots, p_m))$ with $\dim_{\mathbb{R}} \xi = \dim_{\mathbb{R}} \eta$. We choose and fix some S^1 -invariant metrics on ξ and η . Then the S^1 -vector bundles $h^*\xi$, $h^*\eta$, $\tilde{h}_k^*\xi$ and $\tilde{h}_k^*\eta$ ($0 \leq k \leq m$) have canonical S^1 -invariant metrics induced by the S^1 -invariant metrics on ξ and η . We put

$$V_k = (\tilde{h}_k^*\xi)_{e_k} = \xi_{e_k}, \quad W_k = (\tilde{h}_k^*\eta)_{e_k} = \eta_{e_k} \quad \text{for } 1 \leq k \leq m.$$

Here V_k and W_k ($1 \leq k \leq m$) are regarded as orthogonal \mathbb{Z}_{p_k} -representation spaces. Let $q'_k: E(S(\tilde{h}_k^*\xi), S(\tilde{h}_k^*\eta)) \rightarrow M_k$ ($0 \leq k \leq m$) be S^1 -maps defined by (5.1). Then we have

Lemma 5.2. *There are \mathbb{Z}_{p_k} -homeomorphisms*

$$\varphi'_k: q_k'^{-1}(D_k) \rightarrow D_k \times F(S(V_k), S(W_k)) \quad \text{for } 1 \leq k \leq m$$

such that the following diagram commutes :

$$\begin{array}{ccc} q_k'^{-1}(D_k) & \xrightarrow{\varphi'_k} & D_k \times F_k(S(V_k), S(W_k)) \\ & \searrow q'_k|_{q_k'^{-1}(D_k)} & \swarrow \pi_1 \\ & & D_k \end{array}$$

where π_1 denotes the projection on the first factor and the restriction

$$\begin{aligned} \varphi'_k|_{q_k'^{-1}(e_k)}: q_k'^{-1}(e_k) = F(S(V_k), S(W_k)) &\rightarrow \\ \{e_k\} \times F(S(V_k), S(W_k)) \subset D_k \times F(S(V_k), S(W_k)) & \end{aligned}$$

is the identity.

The proof is parallel to that of Lemma 4.4, so we omit it.

Lemma 5.3. *Let $m > 1$ be an integer and let p_i ($1 \leq i \leq m$) be positive odd integers such that $(p_i, p_j) = 1$ for $i \neq j$ and $(p_i, s(2m-3)) = 1$ for $1 \leq i \leq m$. Let ξ and η be real S^1 -vector bundles over M_m such that $\dim_{\mathbb{R}} \xi = \dim_{\mathbb{R}} \eta = n \geq 2m$ and $\xi \supset \underline{\mathbb{R}}^1$ as an S^1 -vector subbundle. Assume that*

- (i) $S(h^*\xi)$ and $S(h^*\eta)$ are S^1 -fiber homotopy equivalent,
- (ii) $S(\xi_{e_i})$ and $S(\eta_{e_i})$ are \mathbb{Z}_{p_i} -homotopy equivalent for $1 \leq i \leq m$.

Then $S(\xi)$ and $S(\eta)$ are S^1 -fiber homotopy equivalent.

Proof. We put

$$V_i = (\tilde{h}_i^*\xi)_{e_i} = \xi_{e_i} \quad \text{and} \quad W_i = (\tilde{h}_i^*\eta)_{e_i} = \eta_{e_i} \quad \text{for } 1 \leq i \leq m.$$

By the assumption (ii), there exist \mathbb{Z}_{p_i} -homotopy equivalences

$$f_i: S(V_i) \rightarrow S(W_i) \quad \text{for } 1 \leq i \leq m.$$

Since $\xi \supset \underline{\mathbf{R}}^1$, there exist \mathbf{Z}_{p_i} -homeomorphisms

$$\tau_i: S(V_i) \rightarrow S(W_i) \quad \text{for } 1 \leq i \leq m$$

such that $\deg \tau_i = -1$. Remark that $f_i \circ \tau_i: S(V_i) \rightarrow S(W_i)$ is also a \mathbf{Z}_{p_i} -homotopy equivalence.

First we shall show that, for each $0 \leq k \leq m$, there exists an S^1 -cross-section of q'_k :

$$t'_k: M_k \rightarrow E(S(\tilde{h}_k^* \xi), S(\tilde{h}_k^* \eta))$$

such that $t'_k(e_j) = f_j$ or $f_j \circ \tau_j$ for $1 \leq j \leq k$.

Since $\tilde{h}_0, h: M_0 \rightarrow M_m$ are S^1 -homotopic, it follows from the assumption (i) that

$$S(\tilde{h}_0^* \xi) \sim S(h^* \xi) \sim S(h^* \eta) \sim S(\tilde{h}_0^* \eta),$$

where \sim stands for *is S^1 -fiber homotopy equivalent to*. Thus there exists an S^1 -cross-section of q'_0 :

$$t'_0: M_0 \rightarrow E(S(\tilde{h}_0^* \xi), S(\tilde{h}_0^* \eta)).$$

Let k be an integer greater than zero. Suppose that we are given an S^1 -cross-section of q'_{k-1} :

$$t'_{k-1}: M_{k-1} \rightarrow E(S(\tilde{h}_{k-1}^* \xi), S(\tilde{h}_{k-1}^* \eta))$$

such that $t'_{k-1}(e_j) = f_j$ or $f_j \circ \tau_j$ for $1 \leq j \leq k-1$. Then there exist a continuous cross-section of $q'_k | q'^{-1}(D_k)$:

$$u'_k: D_k \rightarrow q'^{-1}(D_k) \subset E(S(\tilde{h}_k^* \xi), S(\tilde{h}_k^* \eta))$$

and an S^1 -cross-section of $q'_k | q'^{-1}(S_k)$:

$$v'_k: S_k \rightarrow q'^{-1}(S_k) \subset E(S(\tilde{h}_k^* \xi), S(\tilde{h}_k^* \eta))$$

such that $v'_k = u'_k | S_k$ and $u'_k(e_j) = f_j$ or $f_j \circ \tau_j$ for $1 \leq j \leq k-1$. This is proved similarly as Lemma 4.6, but we need give care to the condition $v'_k(e_j) = f_j$ or $f_j \circ \tau_j$ for $1 \leq j \leq k-1$. Let $\pi_2: D_k \times F^e(S(V_k), S(W_k)) \rightarrow F^e(S(V_k), S(W_k))$ denote the projection on the second factor. By Lemma 5.2, v'_k yields a \mathbf{Z}_{p_k} -map

$$\vartheta'_k: S_k \rightarrow F^e(S(V_k), S(W_k))$$

by putting $\vartheta'_k(x) = \pi_2(\varphi'_k(v'_k(x)))$ for $x \in S_k$, where $\varepsilon = +$ or $-$. Since $v'_k = u'_k | S_k$, we have

$$[\vartheta'_k] = 0 \in [S^{2m-3}, F^e(S(V_k), S(W_k))].$$

Moreover $f_k \in F^e(S(V_k), S(W_k))^{\mathbf{Z}_{p_k}}$ or $f_k \circ \tau_k \in F^e(S(V_k), S(W_k))^{\mathbf{Z}_{p_k}}$. It follows

from Proposition 3.4 that there exists a Z_{p_k} -map

$$\bar{w}'_k: D_k \rightarrow F^e(S(V_k), S(W_k))$$

such that $\bar{w}'_k|_{S_k} = \vartheta'_k$ and $\bar{w}'_k(e_k) = f_k$ or $f_k \circ \tau_k$. Using Lemma 5.2, we define a Z_{p_k} -cross-section of $q'_k|q'^{-1}_k(D_k)$:

$$w'_k: D_k \rightarrow q'^{-1}_k(D_k) \subset E(S(\tilde{h}^*_k \xi), S(\tilde{h}^*_k \eta))$$

by putting $w'_k(x) = \varphi'^{-1}_k(x, \bar{w}'_k(x))$ for $x \in D_k$. Since $w'_k|_{S_k} = \vartheta'_k$ and $w'_k(e_k) = f_k$ or $f_k \circ \tau_k$, it follows from Lemma 4.3 that there exists an S^1 -cross-section of q'_k :

$$t'_k: M_k \rightarrow E(S(\tilde{h}^*_k \xi), S(\tilde{h}^*_k \eta))$$

such that $t'_k(e_j) = w'_k(e_j) = f_j$ or $f_j \circ \tau_j$ for $1 \leq j \leq k$.

By induction, we obtain S^1 -cross-sections t'_0, t'_1, \dots, t'_m . The last S^1 -cross-section t'_m gives a fiber-preserving S^1 -map

$$\omega: S(\xi) \rightarrow S(\eta)$$

such that $\omega_{e_j} = f_j$ or $f_j \circ \tau_j$ for $1 \leq j \leq m$. It is easy to see that, for every $x \in M_m$, $\omega_x: S(\xi_x) \rightarrow S(\eta_x)$ is an S^1_x -homotopy equivalence, where S^1_x denotes the isotropy group at $x \in M_m$. Therefore it follows from the equivariant Dold theorem that ω gives an S^1 -fiber homotopy equivalence (cf. Kawakubo [12; Theorem 2.1] and [24; Theorem 2.1]). q.e.d.

By the same argument as in §2 of Segal [25], we obtain an isomorphism of groups:

$$pr^*: J(CP^{m-1}) \rightarrow J_{S^1}(M_0)$$

and the following diagram commutes:

$$\begin{CD} KO(CP^{m-1}) @>pr^*>> KO_{S^1}(M_0) \\ @VJVV @VVJ_{S^1}V \\ J(CP^{m-1}) @>pr^*>> J_{S^1}(M_0) \end{CD}$$

(cf. Lemma 4.6). We define

$$\tilde{\mu} = (pr^*)^{-1}: J_{S^1}(M_0) \xrightarrow{\cong} J(CP^{m-1}).$$

Now we define a homomorphism of groups

$$\tilde{\Phi}: J_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m)) \rightarrow J(CP^{m-1}) \oplus \bigoplus_{i=1}^m J_{Z_{p_i}}(*)$$

by putting

$$\tilde{\Phi}(J_{S^1}(\xi - \eta)) = \tilde{\mu}(J_{S^1}(h^*\xi - h^*\eta)) \oplus \bigoplus_{i=1}^m J_{Z_{p_i}}(\xi_{e_i} - \eta_{e_i}).$$

Then we have

Theorem 5.4. *Let p_i ($1 \leq i \leq m$) be positive odd integers such that $(p_i, p_j) = 1$ for $i \neq j$ and $(p_i, s(2m-3)) = 1$ for $1 \leq i \leq m$. Then the homomorphism Φ is injective.*

Proof. We see easily that $J_{S^1}(S^1/Z_{p_1}) \cong J_{Z_{p_1}}(*)$. Hence Theorem 5.4 will follow from Lemma 5.3 by the same argument as in the proof of Theorem 4.7.

Let ψ^b denote the Adams operation on equivariant KO -theory.

Corollary 5.5. (cf. [18; Theorem 6.8].) *Let a and b be integers with $(a, b) = (ab, p_i) = 1$ for $1 \leq i \leq m$. For an arbitrary element α of $KO_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m))$, we have*

$$J_{S^1}((\psi^a - 1)(\psi^b - 1)(\alpha)) = 0 \text{ in } J_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m)).$$

Proof. By tom Dieck [7; Theorem 1] and tom Dieck-Petrie [8; Theorem 5], we have

$$J_{Z_{p_i}}((\psi^a - 1)(\psi^b - 1)(\alpha)_{e_i}) = 0 \text{ in } J_{Z_{p_i}}(*) \quad \text{for } 1 \leq i \leq m.$$

On the other hand, by the solution of the Adams conjecture ([1], [22]), we see that

$$\tilde{\mu}(J_{S^1}(h^*(\psi^a - 1)(\psi^b - 1)(\alpha))) = J((\psi^a - 1)(\psi^b - 1)(\mu(h^*(\alpha))) = 0 \text{ in } J(CP^{m-1}).$$

Therefore the result follows from Theorem 5.4. q.e.d.

REMARK 5.7. i) The ring structure of $KO(CP^{m-1})$ and the group structure of $J(CP^{m-1})$ have been determined by Sanderson [24; Theorem 3.9] and Adams-Walker [2] (see also Suter [28]). ii) The group structure of $J_{Z_n}(*)$ has been determined by Kawakubo [13] and [15].

6. Quasi-equivalence

Let G be a compact Lie group and let X be a compact G -space. Let ξ and η be real G -vector bundles of the same dimension over X . In [18] and [21], a G -map $\omega: \xi \rightarrow \eta$ which is proper, fiber-preserving and degree one on fibers is called a *quasi-equivalence*. Let $\alpha = \eta - \xi \in KO_G(X)$ and define $\alpha \geq 0$ to mean there exist a G -vector bundle θ over X and a quasi-equivalence $\omega: \xi \oplus \theta \rightarrow \eta \oplus \theta$.

Problem 6.1. ([18], [21].) Given $\alpha \in KO_G(X)$, given necessary and sufficient conditions for $\alpha \geq 0$.

In this section, we consider the above problem when $G = S^1$ and $X = S^{2m-1}(p_1, p_2, \dots, p_m)$ with a free or pseudofree S^1 -action.

We have

Theorem 6.2. *Let $p_i (1 \leq i \leq m)$ be positive odd integers such that $(p_i, p_j) = 1$ for $i \neq j$ and $(p_i, s(2m-3)) = 1$ for $1 \leq i \leq m$. Let ξ and η be real S^1 -vector bundles of the same dimension over $S^{2m-1}(p_1, p_2, \dots, p_m)$. Then $\alpha = \eta - \xi \geq 0$ if and only if ξ and η satisfy the following two conditions:*

- (i) $J(\mu(h^*\xi)) = J(\mu(h^*\eta))$ in $J(CP^{m-1})$,
- (ii) $\alpha_{e_i} = \eta_{e_i} - \xi_{e_i} \geq 0$ for $1 \leq i \leq m$,

where we regard α_{e_i} as an element of $KO_{\mathbb{Z}_{p_i}}(*) \cong RO(\mathbb{Z}_{p_i})$ for $1 \leq i \leq m$.

Proof. It is obvious that $\alpha \geq 0$ if and only if there exist an S^1 -vector bundle θ over $S^{2m-1}(p_1, p_2, \dots, p_m)$ and a fiber-preserving S^1 -map $\zeta: S(\xi \oplus \theta) \rightarrow S(\eta \oplus \theta)$ such that $\deg \zeta_x = 1$ for $x \in S^{2m-1}(p_1, p_2, \dots, p_m)$. Then the proof is parallel to that of Lemma 5.3. q.e.d.

Corollary 6.3. (cf. [21; Corollary 1.13].) *Let α be an arbitrary element of $KO_{S^1}(S^{2m-1}(p_1, p_2, \dots, p_m))$ such that $\alpha_{e_i} \geq 0$ for $1 \leq i \leq m$. Then there exists a non-negative integer n so that*

$$n\alpha \geq 0.$$

Proof. Remark that $\mu(h^*\alpha) \in \widetilde{KO}(CP^{m-1})$. It is well-known that $\widetilde{J}(CP^{m-1})$ is a finite abelian group. Hence there exists an integer n such that

$$J(\mu(h^*(n\alpha))) = nJ(\mu(h^*\alpha)) = 0 \quad \text{in } J(CP^{m-1}).$$

Thus the result follows from Theorem 6.2. q.e.d.

Corollary 6.4. *Let k be an integer with $(k, p_i) = 1$ for $1 \leq i \leq m$. Let α be an arbitrary element of $KO_{S^1}(S^{2m-1}(p_1, \dots, p_m))$. Then there exists a non-negative integer $e = e(k, \alpha)$ such that*

$$k^e(\psi^k - 1)(\alpha) \geq 0.$$

Proof. By the solution of the Adams conjecture (see [1], [22]), there exists a non-negative integer e such that

$$J(\mu(h^*(k^e(\psi^k - 1)(\alpha)))) = J(k^e(\psi^k - 1)(\mu(h^*\alpha))) = 0 \quad \text{in } J(CP^{m-1}).$$

On the other hand, by Lee-Wasserman [16; Corollaries 3.3 and 4.8] and Atiyah-Tall [5; V. Theorem 2.8], we have

$$k^e(\psi^k - 1)(\alpha_{e_i}) \geq 0 \quad \text{for } 1 \leq i \leq m.$$

Therefore the result follows from Theorem 6.2. q.e.d.

REMARK 6.5. When X is a point and $\alpha \in K_c(X) \cong R(G)$, Problem 6.1 is solved by the main theorem of [18; Theorem 5.1] (see also Atiyah-Tall [5] and Lee-Wasserman [16]).

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