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RIGIDITY AND INFINITESIMAL DEFORMABILITY OF EINSTEIN METRICS

Dedicated to Professor Yozo Matsushima on his 60th birthday

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1. Introduction and results

Let (M,g) be a compact Einstein manifold. If all Einstein metrics on M near g are homothetic to g, then the Einstein metric g is said to be *rigid*. The first result concerning the rigidity of Einstein metrics is given by Berger [1; Proposition 6.4]. He proved that all Einstein metrics on the sphere S^n whose sectional curvature is $(\dim M-2)/(\dim M-1)$-pinched are homothetic to g. Berger and Ebin [2; §7] considered generalizations of this result and introduced "infinitesimal deformations". The result they gave is, roughly speaking, that the space of all Einstein metrics on M is locally finite dimensional. By their method, Koiso [7; Proposition 3.3] gave the following Proposition (for the definition, see **2**) and applied it to locally symmetric spaces of non-compact type without 2-dimensional factor ([7; Theorem 1.1]) and to some irreducible locally symmetric spaces of compact type ([7; Theorem 1.2]).

Proposition 2.5. *If there is no essential Einstein i-deformation of an Einstein metric g, then g is rigid.*

One of the purposes of this paper is to generalize Koiso [7; Theorem 1.2]. For that, we shall classify essential Einstein i-deformations on simply connected symmetric spaces of compact type (Theorem 5.7). The result is as follows.

Corollary 5.8. *Let (M,g) be a locally symmetric Einstein manifold of compact type. Let $(\tilde{M},\tilde{g})$ be its universal riemannian covering and $(\tilde{M},\tilde{g})=\prod_{a=1}^{N}(M_a,g_a)$ the irreducible decomposition as symmetric space. If $N=1$ and $(\tilde{M},\tilde{g})$ is neither $SU(p+q)/S(U(p)\times U(q))$ $(p\geqq q\geqq 2)$, $SU(l)/SO(l)$ $(l\geqq 3)$, $SU(2l)/Sp(l)$ $(l\geqq 3)$, E_6/F_4 nor $SU(l)$ $(l\geqq 3)$, then g is rigid. If $N=2$ and M_a are neither one of the above, G_2 nor any hermitian space except S^2, then g is rigid. If $N\geqq 3$ and M_a are neither one of the above nor S^2, then g is rigid.*

Another purpose is to decide whether the converse of Proposition 2.5 holds or not. We expect that the converse holds, because if so, we would get

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many examples of Einstein metrics by Theorem 5.7. In the case of Kähler metrics, i.e., if we consider only Kähler metrics on a compact complex manifold, then it is not difficult to show that the converse holds (cf. Yau [13]). But, unfortunately, we shall give counter-examples to the converse in the real case. To analyze this problem, we shall introduce "*infinitesimal deformations of second order*" (Definition 4.4) and check whether each essential Einstein i-deformation has an Einstein i-deformation of second order or not (Theorem 6.2). As a result, we shall give the following

Theorem 6.12. *There exist Einstein metrics which is infinitesimally deformable but rigid.*

This paper is organized as follows: after some preliminaries in **2**, we consider infinitesimal Einstein deformations in **3** and infinitesimal Einstein deformations of second order in **4**, in general case. We apply the results in **3** and **4** to symmetric spaces of compact type in **5** and **6**. Theorem 5.7 and Corollary 5.8 are proved in **5** and Theorem 6.12 in **6**.

2. Preliminaries

In this section, we recall some fundamental definitions and some known facts concerning the space of riemannian metrics and deformations of Einstein metrics. Let M be a compact connected C^∞-manifold with $n=\dim M\geqq 3$. Riemannian metrics on M, etc. are all to be in C^∞-category, unless otherwise stated. When we fix a riemannian metric on M, we identify covariant tensors and contravariant tensors with each other by the fixed metric as usual, and *denote by* T^pM, S^2M the p-tensor bundle over M, the symmetric 2-tensor bundle over M, respectively. Moreover, we *denote by* (,) the inner product on tensors on M and *by* $\langle\ ,\ \rangle$ the global inner product for tensor fields.

For a fibre bundle F over M, we *denote by* $H^s(F)$ the set of all H^s-cross sections of F. We *denote by* $\mathcal{M}^s$, $\mathcal{D}^s$ the Hilbert manifold of all H^s-riemannian metrics on M, the group of all H^s-diffeomorphisms of M, respectively. (Here, we assume that s is sufficiently large.) The group $\mathcal{D}^{s+1}$ acts on $\mathcal{M}^s$ by pull-back and this action admits a slice (Ebin [6;8.20 Théorème]). For a riemannian metric g on M, we *denote by* S^s_g this slice. Recall that S^s_g is a submanifold of $\mathcal{M}^s$ containing g such that $T_gS^s_g=\operatorname{Ker}\,\delta$, *where* δ is the differential operator: $H^s(S^2M)\to H^{s-1}(TM)$ defined by

$$(\delta h)_i=-\nabla^l h_{li}\,.$$

Denote by $\mathcal{M}^s_c$ the Hilbert manifold of all H^s-riemannian metrics on M with volume c. The tangent space of $\mathcal{M}^s_c$ at $g\in\mathcal{M}^s_c$ is given by $\operatorname{Ker}\,\int$, where *the function* $\int$ on $H^s(S^2M)$ is defined by $\int h=\langle h,g\rangle$.

DEFINITION 2.1. Let $g\in\mathcal{M}_c^\infty$ be an Einstein metric. If there exists a $\mathcal{D}^{s+1}$-invariant open set U of $\mathcal{M}_c^s$ containing g such that every H^s-Einstein metric in U is an element of $(\mathcal{D}^{s+1})^*g$, then g is said to be *rigid.*

If we use Ebin's slice, we get the following

Lemma 2.2 (Koiso [8; Lemma 3.1)]. *Let $g\in\mathcal{M}_c^\infty$ be an Einstein metric. If there exists an open neighbourhood V of g in $S_g^s\cap\mathcal{M}_c^s$ such that g is the unique H^s-Einstein metric in V, then g is rigid.*

For $g\in\mathcal{M}_c^s$, we define

$$T(g)=\int_M u_g v_g\,,$$
$$E(g)=S(g)-(T(g)/nc)\cdot g\,,$$

where u_g is the H^{s-2}-function on M defined by the scalar curvature of g *and* $S(g)$ the Ricci tensor of g. Remark that g is an Einstein metric if and only if $E(g)=0$. Following Lichnerowicz [10; (19.4)], the differential S_g' of the map $S\colon \mathcal{M}^s\to H^{s-2}(S^2M)$ at $g\in\mathcal{M}^s$ is given by

$$(2.2.1)\qquad 2S_g'(h)=(\bar\Delta+2L+2Q-2\delta^*\delta-\mathrm{Hess\ tr})h\,,$$

where

$$(\bar\Delta h)_{ij}=-\nabla^l\nabla_l h_{ij}\quad\text{for } h\in H^s(S^2M)\,,$$
$$(Lh)_{ij}=R_i{}^k{}_j{}^l h_{kl}\quad\text{for } h\in H^s(S^2M)\,,$$
$$2(Qh)_{ij}=S_i^k h_{kj}+S_j^k h_{ki}\quad\text{for } h\in H^s(S^2M)\,,$$
$$2(\delta^*\xi)_{ij}=\nabla_i\xi_j+\nabla_j\xi_i\quad\text{for } \xi\in H^s(TM)\,,$$

and the sign convention of the curvature tensor R is taken in such a way that $R_{ijij}\leqq 0$ for the standard sphere. Since an Einstein metric is a critical point of the function T on $\mathcal{M}_c^s$, the differential E' of E at an Einstein metric $g\in\mathcal{M}_c^\infty$ is given by

$$(2.2.2)\qquad 2E_g'(h)=(\bar\Delta+2L-2\delta^*\delta-\mathrm{Hess\ tr})h\,.$$

Since $T_g(S_g^s\cap\mathcal{M}_c^s)=\mathrm{Ker}\ \delta\cap\mathrm{Ker}\int$, if $h\in T_g(S_g^s\cap\mathcal{M}_c^s)$, then

$$(2.2.3)\qquad 2E_g'(h)=(\bar\Delta+2L-\mathrm{Hess\ tr})h\,.$$

DEFINITION 2.3. Let $g\in\mathcal{M}_c^\infty$ be an Einstein metric. We *denote by* EID(M) or simply EID the kernel of the map $E_g'\,|\,T_g(S_g^s\cap\mathcal{M}_c^s)$. A non-zero element $h\in$EID is called *an essential Einstein i-deformation.* If EID vanishes, then g is said to be *infinitesimally non-deformable,* otherwise *infinitesimally deformable.*

The Lichnerowicz operator Δ is defined by

$$\Delta\psi=\bar\Delta\psi+2L\psi+pQ\psi\quad\text{for } \psi\in H^s(T^pM)\,,$$

where
$$(\bar{\Delta}\psi)_{i_1\cdots i_p} = -\nabla^l\nabla_l\psi_{i_1\cdots i_p},$$
$$(L\psi)_{i_1\cdots i_p} = \sum_{a<b} R_{i_a}{}^k{}_{i_b}{}^l\psi_{i_1\cdots \overset{(a)}{k}\cdots \overset{(b)}{l}\cdots i_p},$$
and
$$p(Q\psi)_{i_1\cdots i_p} = \sum_a S_{i_a}{}^k\psi_{i_1\cdots \overset{(a)}{k}\cdots i_p}.$$

Remark that this definition does not contradict the previous definitions and the ordinary Laplace-Bertrami operator (Lichnerowicz [10; §10]). Moreover, we can check that Δ commutes with δ, δ^*, Hess, tr and d on an Einstein manifold.

Lemma 2.4 (Berger and Ebin [2; Lemma 7.1]). *Let $g\in\mathcal{M}_c^\infty$ be an Einstein metric. The space* EID(M) *coincides with* $\mathrm{Ker}(\Delta_S-2\varepsilon)\cap\mathrm{Ker}\,\mathrm{tr}\cap\mathrm{Ker}\,\delta$, *where Δ_S is the restriction of the Lichnerowicz operator Δ to $H^s(S^2M)$ and ε the Einstein constant, i.e., $S(g)=\varepsilon\cdot g$.*

Proposition 2.5 (Koiso [8; Proposition 3.3]). *Let g be an Einstein metric on M. If g is infinitesimally non-deformable, then g is rigid.*

3. Einstein *i*-deformation

Let $g\in\mathcal{M}^\infty$ be an Einstein metric with Einstein constant ε, i.e., $S(g)=\varepsilon\cdot g$. We define differential operators $\gamma: H^s(S^2M)\to H^{s-1}(TM)$ and $\beta: H^s(S^2M)\to H^{s-2}(S^2M)$ by
$$\gamma = \delta+\frac{1}{2}d\,\mathrm{tr},$$
$$\beta = \Delta_S-2\varepsilon-\mathrm{Hess}\,\mathrm{tr}.$$
Remark that β is an elliptic operator.

Lemma 3.1. $\beta(\mathrm{Ker}\,\delta\cap\mathrm{Ker}\int)=\mathrm{Im}\,\beta\cap\mathrm{Ker}\,\gamma\cap\mathrm{Ker}\int$.

Proof. Denote by Δ_1 the Lichnerowicz operator on $H^s(TM)$. By Koiso [8; Lemma 3.2],

$$\gamma\beta = (\Delta_1-2\varepsilon)\delta. \tag{3.1.1}$$

Since $\mathrm{tr}\,\beta=2(\Delta-\varepsilon)\mathrm{tr}$,
$$\beta(\mathrm{Ker}\,\delta\cap\mathrm{Ker}\int)\subset\mathrm{Im}\,\beta\cap\mathrm{Ker}\,\gamma\cap\mathrm{Ker}\int.$$
Let $\beta h\in\mathrm{Ker}\,\gamma\cap\mathrm{Ker}\int$ and decompose h into $\psi+\delta^*\xi$; $\delta\psi=0$, by Ebin [6;8.8 Proposition]. Then
$$0 = \gamma\beta h = (\Delta_1-2\varepsilon)\delta(\psi+\delta^*\xi) = \delta\delta^*(\Delta_1-2\varepsilon)\xi, \tag{3.1.1}$$
and so, $\delta^*(\Delta_1-2\varepsilon)\xi=0$, $\delta(\Delta_1-2\varepsilon)\xi=0$. Since we can easily check that

$$(3.1.2) \qquad \delta\delta^* = \frac{1}{2}(\Delta_1 - 2\varepsilon + d\delta),$$

$$0 = \delta\delta^*(\Delta_1 - 2\varepsilon)\xi = \frac{1}{2}(\Delta_1 - 2\varepsilon + d\delta)(\Delta_1 - 2\varepsilon)\xi = \frac{1}{2}(\Delta_1 - 2\varepsilon)^2\xi,$$

which implies that $(\Delta_1 - 2\varepsilon)\xi = 0$.

$$(3.1.3) \qquad \begin{aligned} \beta\delta^*\xi &= (\Delta_S - 2\varepsilon - \text{Hess tr})\delta^*\xi \\ &= \delta^*(\Delta_1 - 2\varepsilon)\xi + \text{Hess }\delta\xi = \text{Hess }\delta\xi. \end{aligned}$$

Set $\phi = \text{Hess }\delta\xi + \varepsilon\delta\xi \cdot g$. Then

$$(3.1.4) \qquad \begin{aligned} \delta\phi &= \delta\delta^* d\delta\xi - \varepsilon d\delta\xi \\ &= \frac{1}{2}(\Delta_1 - 2\varepsilon + d\delta)d\delta\xi - \varepsilon d\delta\xi && (3.1.2) \\ &= \frac{1}{2}d\delta(\Delta_1 - 2\varepsilon)\xi + \frac{1}{2}d\Delta\delta\xi - \varepsilon d\delta\xi \\ &= \frac{1}{2}d\delta(\Delta_1 - 2\varepsilon)\xi = 0, \end{aligned}$$

$$(3.1.5) \qquad \begin{aligned} \beta\phi &= (\Delta_S - 2\varepsilon - \text{Hess tr})(\text{Hess }\delta\xi + \varepsilon\delta\xi \cdot g) \\ &= \text{Hess }\Delta\delta\xi - n\varepsilon\text{ Hess }\delta\xi \\ &= (2-n)\varepsilon\text{ Hess }\delta\xi. \end{aligned}$$

Since $\Delta\xi = 2\varepsilon\xi$ and so $\Delta\delta\xi = 2\varepsilon\delta\xi$, if $\varepsilon = 0$ then $\Delta\xi = 0$ and $\delta\xi = 0$. Therefore $2\delta\delta^*\xi = (\Delta_1 - 2\varepsilon + d\delta)\xi = 0$, which implies that $\delta^*\xi = 0$. In this case the equalities $\delta h = 0$ and $\beta h = \beta(h - (\int h/nc)\cdot g)$ hold, and so $\beta(\text{Ker }\delta \cap \text{Ker }\int) \supset \text{Im }\beta \cap \text{Ker }\gamma \cap \text{Ker }\int$. Thus we may assume that $\varepsilon \neq 0$. Then

$$\begin{aligned} \beta h &= \beta\psi + \beta\delta^*\xi = \beta\psi + \text{Hess }\delta\xi && (3.1.3) \\ &= \beta(\psi + \phi/(2-n)\varepsilon), && (3.1.5) \end{aligned}$$

and

$$\begin{aligned} &\delta(\psi + \phi/(2-n)\varepsilon) = 0, && (3.1.4) \\ &\int\psi = \int h - \int\delta^*\xi = -\frac{1}{2\varepsilon}\int\beta h = 0, \\ &\int\phi = \int\text{Hess }\delta\xi + \varepsilon\int\delta\xi\cdot g = 0. && \text{Q.E.D.} \end{aligned}$$

Proposition 3.2. *Let g be an Einstein metric on M. Then*

$$\text{Im}(E'_g|\text{Ker }\textstyle\int) \oplus \text{EID} = \text{Ker }\gamma \cap \text{Ker }\textstyle\int$$

(*orthogonal direct sum*), *where* $\text{Im}(E'_g|\text{Ker }\int)$ *is a closed subspace.*

Proof. First we see that $\text{Im}(E'_g|\text{Ker }\int) = E'_g(\text{Ker }\delta \cap \text{Ker }\int) \oplus E'_g(\text{Im }\delta^*)$ and $E'_g(\text{Ker }\delta \cap \text{Ker }\int) = \beta(\text{Ker }\delta \cap \text{Ker }\int)$ by (2.2.4) and $E'_g(\text{Im }\delta^*) = 0$, and so

$\mathrm{Im}(E'_g|\mathrm{Ker}\,\int)=\beta(\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)$. Next we remark that the formal adjoint β^* of β is given by $\Delta_S-2\varepsilon-g\cdot\delta\delta$ and see, by Lemma 2.4, that

$$\langle\beta(\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int),\ \mathrm{EID}\rangle=\langle\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int,\ \beta^*\mathrm{EID}\rangle=0\,.$$

Moreover, by Lemma 2.4 and Lemma 3.1, it is easy to see that $\beta(\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)\oplus\mathrm{EID}\subset\mathrm{Ker}\,\gamma\cap\mathrm{Ker}\,\int$.

Now, let $h\in\mathrm{Ker}\,\gamma\cap\mathrm{Ker}\,\int$. Since β is elliptic, we can decompose h into $\beta\phi+\psi$; $\beta^*\psi=0$. Then

$$\begin{aligned}0&=\delta\delta\beta^*\psi=\delta\delta(\Delta_S-2\varepsilon-g\cdot\delta\delta)\psi\\&=(\Delta-2\varepsilon)\delta\delta\psi+\delta d\delta\delta\psi\\&=2(\Delta-\varepsilon)\delta\delta\psi\,.\end{aligned}$$

But here $\varepsilon=0$ or ε is not an eigenvalue of Δ on a compact Einstein manifold (Lichnerowicz [9; p. 135]). Then $\delta\delta\psi=0$, and so $(\Delta_S-2\varepsilon)\psi=0$.

$$\begin{aligned}0&=\delta\gamma h=\delta\gamma\beta\phi+\delta\gamma\psi\\&=\delta(\Delta_1-2\varepsilon)\delta\phi+\delta(\delta+\frac{1}{2}\,d\,\mathrm{tr})\psi\\&=(\Delta-2\varepsilon)\delta\delta\phi+\frac{1}{2}\,\Delta\,\mathrm{tr}\,\psi\,.\end{aligned}\tag{3.1.1}$$

Therefore $(\Delta-2\varepsilon)^2\delta\delta\phi=-\frac{1}{2}\Delta\,\mathrm{tr}(\Delta_S-2\varepsilon)\psi$, and so $(\Delta-2\varepsilon)\delta\delta\phi=0$, $0=\Delta\,\mathrm{tr}\,\psi=2\varepsilon\,\mathrm{tr}\,\psi$. If $\varepsilon\neq0$, then $\mathrm{tr}\,\psi=0$. Even if $\varepsilon=0$, $\int\psi=\int h-\int\beta\phi=0$ implies that $\mathrm{tr}\,\psi=0$. Thus

$$\begin{aligned}0&=\gamma h=\gamma\beta\phi+\gamma\psi\\&=(\Delta_1-2\varepsilon)\delta\phi+\delta\psi\,,\end{aligned}\tag{3.1.1}$$

which implies that $(\Delta_1-2\varepsilon)^2\delta\phi=-\delta(\Delta_1-2\varepsilon)\psi=0$ and so $(\Delta_1-2\varepsilon)\delta\phi=0$ and $\delta\psi=0$. These formulae implies that $\psi\in\mathrm{EID}$ and $\beta\phi\in\mathrm{Ker}\,\gamma\cap\mathrm{Ker}\,\int$, and so $\beta\phi\in\beta(\mathrm{Ker}\,\delta\cap\mathrm{Ker}\,\int)$ by Lemma 3.1. Q.E.D.

Proposition 3.3. *Let g be an Einstein metric with Einstein constant ε. Then* dim EID

$$=\dim(\mathrm{Ker}(\Delta_S-2\varepsilon)\cap\mathrm{Ker}\,\mathrm{tr})-\dim(\mathrm{Ker}(\Delta_1-2\varepsilon))+\dim(\mathrm{Ker}\,\delta^*)\,.$$

Proof. Define a differential operator $\theta\colon H^s(TM)\to H^{s-1}(S^2M)$ by

$$\theta\xi=\delta^*\xi+\frac{1}{n}\,\delta\xi\cdot g\,.$$

Remark that $\mathrm{tr}\,\theta=0$ and the formal adjoint θ^* of θ is given by

$$\theta^*h=\delta h+\frac{1}{n}d\,\mathrm{tr}\,h\,.$$

Let $h\in\mathrm{Ker}(\Delta_S-2\mathcal{E})\cap\mathrm{Ker}\,\mathrm{tr}$. Since θ has injective symbol, we can decompose h into $\theta\xi+\psi$; $\theta^*\psi=0$ (Ebin [6; 8.5 Théorème]). Then $0=\mathrm{tr}\,h=\mathrm{tr}\,\theta\xi+\mathrm{tr}\psi=\mathrm{tr}\,\psi$, and

$$\delta\psi=\theta^*\psi-\frac{1}{n}d\,\mathrm{tr}\,\psi=0\,.$$

Moreover

$$\begin{aligned}0&=(\Delta_S-2\mathcal{E})h\\&=(\Delta_S-2\mathcal{E})\theta\xi+(\Delta_S-2\mathcal{E})\psi\\&=\theta(\Delta_1-2\mathcal{E})\xi+(\Delta_S-2\mathcal{E})\psi\,,\\ \theta^*(\Delta_S-2\mathcal{E})\psi&=(\Delta_1-2\mathcal{E})\theta^*\psi=0\,,\end{aligned}$$

and so $\theta^*\theta(\Delta_1-2\mathcal{E})\xi=0$, $\theta(\Delta_1-2\mathcal{E})\xi=0$ and $(\Delta_S-2\mathcal{E})\psi=0$, which implies that $\psi\in\mathrm{EID}$. In this correspondence: $h\to\psi$, if $h\in\mathrm{EID}$ then $\psi=h$. Thus we have a projection P: $\mathrm{Ker}(\Delta_S-2\mathcal{E})\cap\mathrm{Ker}\,\mathrm{tr}\to\mathrm{EID}$; $P(h)=\psi$. Then

$$\dim\mathrm{EID}=\dim(\mathrm{Ker}(\Delta_S-2\mathcal{E})\cap\mathrm{Ker}\,\mathrm{tr})-\dim(\mathrm{Ker}\,P)\,.$$

Here, if we remark that $\mathrm{tr}\,\theta=0$, then we see that

$$\mathrm{Ker}\,P=\mathrm{Im}\,\theta\cap\mathrm{Ker}(\Delta_S-2\mathcal{E})\,.$$

We easily see that $\theta(\mathrm{Ker}(\Delta_1-2\mathcal{E}))\subset\mathrm{Ker}\,P$. Conversely, let $\theta\xi\in\mathrm{Ker}(\Delta_S-2\mathcal{E})$ for $\xi\in H^s(TM)$ and decompose ξ into $\zeta+(\Delta_1-2\mathcal{E})\eta$; $(\Delta_1-2\mathcal{E})\zeta=0$. Then $0=(\Delta_S-2\mathcal{E})\theta\xi=(\Delta_S-2\mathcal{E})\theta\zeta+(\Delta_S-2\mathcal{E})^2\theta\eta=(\Delta_S-2\mathcal{E})^2\theta\eta$, and so $\theta(\Delta_1-2\mathcal{E})\eta=(\Delta_1-2\mathcal{E})\theta\eta=0$. Therefore $\xi\in\mathrm{Ker}(\Delta_1-2\mathcal{E})+\mathrm{Ker}\,\theta$, which implies that θ gives a surjection from $\mathrm{Ker}(\Delta_1-2\mathcal{E})$ to $\mathrm{Ker}\,P$. Thus

$$\dim\mathrm{Ker}\,P=\dim\mathrm{Ker}(\Delta_1-2\mathcal{E})-\dim(\mathrm{Ker}(\Delta_1-2\mathcal{E})\cap\mathrm{Ker}\,\theta)\,.$$

Here we easily see that $\mathrm{Ker}\,\delta^*\subset\mathrm{Ker}(\Delta_1-2\mathcal{E})\cap\mathrm{Ker}\,\theta$ by (3.1.2). Conversely, if $\xi\in\mathrm{Ker}(\Delta_1-2\mathcal{E})\cap\mathrm{Ker}\,\theta$, then

$$\begin{aligned}0=\delta\theta\xi&=\delta\Big(\delta^*+\frac{1}{n}g\cdot\delta\Big)\xi\\&=\frac{1}{2}(\Delta_1-2\mathcal{E}+d\delta)\xi-\frac{1}{n}d\delta\xi\\&=\Big(\frac{1}{2}-\frac{1}{n}\Big)d\delta\xi\,,\end{aligned}\tag{3.1.2}$$

and so $\delta\xi=0$, $\delta^*\xi=0$, which implies that $\mathrm{Ker}\,\delta^*\supset\mathrm{Ker}(\Delta_1-2\mathcal{E})\cap\mathrm{Ker}\,\theta$. Q.E.D.

4. Infinitesimal Einstein deformation of second order

In this section, we discuss about the second derivative of the map E. Let

$g \in \mathcal{M}^s$ and $h \in H^s(S^2M)$. Regarding h as an infinitesimal deformation of g, i.e., $h \in T_g\mathcal{M}^s$, we set

$$X(\xi, \eta) = (\nabla_\xi \eta)' \quad \text{for } \xi, \eta \in TM .$$

Then X is a well-defined 3-tensor field (of type (1,2)) and given by

$$X_{ij}{}^k = \frac{1}{2}(\nabla_i h_j{}^k + \nabla_j h_i{}^k - \nabla^k h_{ij})$$

(see Lichnerowicz [9; (17.2)]).

Lemma 4.1. *Let g be an Einstein metric and h an essential Einstein i-deformation of g. Then we have*

$$g^{kl} X_{kl}{}^i = 0 , \tag{4.1.1}$$

$$\nabla^k X_{ki}{}^j = (Lh)_i{}^j , \tag{4.1.2}$$

$$(R_{ijk}{}^l)' = \nabla_i X_{jk}{}^l - \nabla_j X_{ik}{}^l , \tag{4.1.3}$$

and the symmetric part of X_{ikj} with respect to i and j is $(1/2)\nabla_k h_{ij}$.

Proof. That is easy to check by tensor calculas. For (4.1.3), see Lichnerowicz [9; (17.5)].

Proposition 4.2. *Let $g \in \mathcal{M}_c^\infty$ be an Einstein metric and h an essential Einstein i-deformation of g. Then the second derivative $E_g''(h,h)$ is given by*

$$\begin{aligned}
&2E_g''(h,h)_{ij} \\
&= 2h^{kl}\nabla_k\nabla_l h_{ij} + 2\nabla_k h_i{}^l \cdot \nabla^k h_{jl} - 2\nabla^l h_{ik} \cdot \nabla^k h_{jl} - 4R_i{}^m{}_j{}^l h_m{}^k h_{kl} \\
&\quad - 2(h^{kl}\nabla_i\nabla_k h_{jl} + h^{kl}\nabla_j\nabla_k h_{il}) - \nabla_i h^k{}_m \cdot \nabla_j h_k{}^m \\
&\quad + 2((Lh)_i{}^k h_{kj} + (Lh)_j{}^k h_{ki}) + \nabla_i\nabla_j(h,h) .
\end{aligned}$$

Proof. Since g is a critical point of the function T on M_c^s, $T_g'(h)=0$. Moreover, (Hess T) $(h,h)=0$ by Koiso [7; Theorem 2.4, Theorem 2.5]. Thus we see $E''(h,h)=S''(h,h)$. We calculate $S''(h,h)$ by Lemma 4.1.

$$\begin{aligned}
(\bar{\Delta} h)'_{ij} &= -(g^{kl}\nabla_k\nabla_l h_{ij})' \\
&= h^{kl}\nabla_k\nabla_l h_{ij} + g^{kl}(X_{kl}{}^m\nabla_m h_{ij} + X_{ki}{}^m \nabla_l h_{mj} + X_{kj}{}^m \nabla_l h_{im}) \\
&\quad + g^{kl}\nabla_k(X_{li}{}^m h_{mj} + X_{lj}{}^m h_{im}) && \text{(4.1.1), (4.1.2)} \\
&= h^{kl}\nabla_k\nabla_l h_{ij} + 2X_{ki}{}^m \nabla^k h_{mj} + 2X_{kj}{}^m \nabla^k h_{im} + \nabla^k X_{ki}{}^m \cdot h_{mj} + \nabla^k X_{kj}{}^m \cdot h_{im} \\
&= h^{kl}\nabla_k\nabla_l h_{ij} + (\nabla_k h_i{}^m + \nabla_i h_k{}^m - \nabla^m h_{ki}) \cdot \nabla^k h_{mj} && \text{(4.1.2)} \\
&\quad + (\nabla_k h_j{}^m + \nabla_j h_k{}^m - \nabla^m h_{kj}) \cdot \nabla^k h_{im} + (Lh)_i{}^m h_{mj} + (Lh)_j{}^m h_{im} \\
&= h^{kl}\nabla_k\nabla_l h_{ij} + 2\nabla_k h_i{}^m \cdot \nabla^k h_{mj} + (\nabla_i h_k{}^m \cdot \nabla^k h_{mj} + \nabla_j h_k{}^m \cdot \nabla^k h_{im}) \\
&\quad - 2\nabla^m h_{ki} \cdot \nabla^k h_{mj} + ((Lh)_i{}^m h_{mj} + (Lh)_j{}^m h_{im}) ,
\end{aligned}$$

$$\begin{aligned}
(Lh)'_{ij} &= (g^{km}R_{imj}{}^{l}h_{kl})' \\
&= -h^{km}R_{imj}{}^{l}h_{kl}+g^{km}(\nabla_i X_{mj}{}^{l}-\nabla_m X_{ij}{}^{l})\cdot h_{kl} \\
&= -R_i{}^m{}_j{}^l h_m{}^k h_{kl}+\nabla_i X_{mj}{}^{l}\cdot h^m{}_l-\nabla_m X_{ij}{}^{l}\cdot h^m{}_l \\
&= -R_i{}^m{}_j{}^l h_m{}^k h_{kl}+\frac{1}{2}\nabla_i\nabla_j h_{ml}\cdot h^{ml}-\frac{1}{2}\nabla_m(\nabla_i h_j{}^l+\nabla_j h_i{}^l-\nabla^l h_{ij})\cdot h^m{}_l \\
&= -R_i{}^m{}_j{}^l h_m{}^k h_{kl}+\frac{1}{2}\nabla_i\nabla_j h_m{}^l\cdot h^m{}_l+\frac{1}{2}h^{ml}\nabla_m\nabla_l h_{ij} \\
&\quad -\frac{1}{2}(R_{mi}{}^k{}_j h_k{}^l+R_{mik}{}^l h_j{}^k+R_{mj}{}^k{}_i h_k{}^l+R_{mjk}{}^l h_i{}^k)h^m{}_l \\
&\quad -\frac{1}{2}(\nabla_i\nabla_m h_j{}^l\cdot h^m{}_l+\nabla_j\nabla_m h_i{}^l\cdot h^m{}_l) \\
&= -2R_i{}^m{}_j{}^l h_m{}^k h_{kl}+\frac{1}{2}\nabla_i\nabla_j h_m{}^l\cdot h^m{}_l+\frac{1}{2}h^{ml}\nabla_m\nabla_l h_{ij} \\
&\quad +\frac{1}{2}((Lh)_{ik}h^k{}_j+(Lh)_{jk}h^k{}_i)-\frac{1}{2}(\nabla_i\nabla_m h_j{}^l\cdot h^m{}_l+\nabla_j\nabla_m h_i{}^l\cdot h^m{}_l)\,,
\end{aligned} \tag{4.1.3}$$

$$\begin{aligned}
(Qh)'_{ij} &= \frac{1}{2}(g^{kl}S_{il}h_{kj}+g^{kl}S_{jl}h_{ki})' \\
&= -\frac{1}{2}(h^{kl}S_{il}h_{kj}+h^{kl}S_{jl}h_{ki})+\frac{1}{2}(S'_{ik}h^k{}_j+S'_{jk}h^k{}_i)=0\,,
\end{aligned}$$

$$(\text{Hess tr } h)'=(\text{Hess})' \text{ tr } h+\text{Hess }(\text{tr } h)'=\text{Hess }(\text{tr } h)'$$

$$(\text{tr } h)'=(g^{kl}h_{kl})'=-h^{kl}h_{kl}=-(h,h)\,,$$

$$(\text{Hess tr } h)'_{ij}=-\nabla_i\nabla_j(h,h),$$

$$(\delta^*\delta h)'=(\delta^*)'\delta h+\delta^*(\delta h)'=\delta^*(\delta h)'\,,$$

$$\begin{aligned}
(\delta h)'_i &= -(g^{kl}\nabla_k h_{li})' \\
&= h^{kl}\nabla_k h_{li}+g^{kl}(X_{kl}{}^m h_{mi}+X_{ki}{}^m h_{lm}) \\
&= h^{kl}\nabla_k h_{li}+\frac{1}{2}\nabla_i h_k{}^m\cdot h^k{}_m\,,
\end{aligned} \tag{4.1.1}$$

$$\begin{aligned}
(\delta^*\delta h)'_{ij} &= \frac{1}{2}\nabla_i(h^{kl}\nabla_k h_{lj}+\frac{1}{2}\nabla_j h_k{}^m\cdot h^k{}_m)+\frac{1}{2}\nabla_j(h^{kl}\nabla_k h_{li}+\frac{1}{2}\nabla_i h_k{}^m\cdot h_m{}^k) \\
&= \frac{1}{2}\nabla_i h^{kl}\cdot\nabla_k h_{lj}+\frac{1}{2}h^{kl}\nabla_i\nabla_k h_{lj}+\frac{1}{2}\nabla_j h^{kl}\cdot\nabla_k h_{li}+\frac{1}{2}h^{kl}\nabla_j\nabla_k h_{li} \\
&\quad +\frac{1}{4}\nabla_i\nabla_j h_k{}^m\cdot h_m{}^k+\frac{1}{4}\nabla_j h_k{}^m\cdot\nabla_i h^k{}_m+\frac{1}{4}\nabla_j\nabla_i h_k{}^m\cdot h^k{}_m+\frac{1}{4}\nabla_i h_k{}^m\cdot\nabla_j h^k{}_m \\
&= \frac{1}{2}\nabla_i h^{kl}\cdot\nabla_k h_{lj}+\frac{1}{2}\nabla_j h^{kl}\cdot\nabla_k h_{li}+\frac{1}{2}\nabla_i h^k{}_m\cdot\nabla_j h_k{}^m+\frac{1}{2}h^{kl}\nabla_i\nabla_k h_{lj}
\end{aligned}$$

$$+\frac{1}{2}h^{kl}\nabla_j\nabla_k h_{li}+\frac{1}{2}\nabla_i\nabla_j h_k{}^m\cdot h^k{}_m+\frac{1}{4}(R_{ji}{}^l{}_k h_l{}^m+R_{jil}{}^m h_k{}^l)h^k{}_m$$

$$=\frac{1}{2}(\nabla_i h^{kl}\cdot\nabla_k h_{lj}+\nabla_j h^{kl}\cdot\nabla_k h_{li})+\frac{1}{2}\nabla_i h^k{}_m\cdot\nabla_j h_k{}^m$$

$$+\frac{1}{2}(h^{kl}\nabla_i\nabla_k h_{lj}+h^{kl}\nabla_j\nabla_k h_{li})+\frac{1}{2}\nabla_i\nabla_j h_k{}^m\cdot h^k{}_m\,.$$

Therefore, $2E''(h,h)_{ij}$

$$\begin{aligned}
&= h^{kl}\nabla_k\nabla_l h_{ij}+2\nabla_k h_i{}^m\cdot\nabla^k h_{mj}+(\nabla_i h_k{}^m\cdot\nabla^k h_{mj}+\nabla_j h_k{}^m\cdot\nabla^k h_{im})\\
&\quad -2\nabla^m h_{ki}\cdot\nabla^k h_{mj}+((Lh)_i{}^m h_{mj}+(Lh)_j{}^m h_{mi})\\
&\quad -4R_i{}^m{}_j{}^l h_m{}^k h_{kl}+\nabla_i\nabla_j h_m{}^l\cdot h^m{}_l+h^{ml}\nabla_m\nabla_l h_{ij}\\
&\quad +((Lh)_{ik}h^k{}_j+(Lh))_{jk}h^k{}_i)-(\nabla_i\nabla_m h_j{}^l\cdot h^m{}_l+\nabla_j\nabla_m h_i{}^l\cdot h^m{}_l)\\
&\quad -(\nabla_i h^{kl}\cdot\nabla_k h_{lj}+\nabla_j h^{kl}\cdot\nabla_k h_{li})-\nabla_i h^k{}_m\cdot\nabla_j h_k{}^m\\
&\quad -(h^{kl}\nabla_i\nabla_k h_{lj}+h^{kl}\nabla_j\nabla_k h_{li})-\nabla_i\nabla_j h_k{}^m\cdot h^k{}_m\\
&\quad +\nabla_i\nabla_j(h,h)\,.
\end{aligned}$$

Q.E.D.

Now, we calculate $\langle E''(h,h), h\rangle$ which we use in **6**.

Lemma 4.3. *Let g and h be as in Proposition* 4.2. *Then*

$$2\langle E''(h,h), h\rangle = 2\varepsilon\langle h_{ij}, h_i{}^k h_{kj}\rangle+3\langle\nabla_i\nabla_j h_{kl}, h_{ij}h_{kl}\rangle-6\langle\nabla_i\nabla_j h_{kl}, h_{ik}h_{jl}\rangle\,.$$

Proof.

$$\begin{aligned}
&\langle h^{kl}\nabla_k\nabla_l h_{ij}, h_{ij}\rangle=\langle\nabla_i\nabla_j h_{kl}, h_{ij}h_{kl}\rangle\,,\\
&\langle\nabla_k h_i{}^l\cdot\nabla^k h_{jl}, h_{ij}\rangle=-\langle h_i{}^l\nabla_k\nabla^k h_{jl}, h_{ij}\rangle-\langle h_i{}^l\nabla^k h_{jl}, \nabla^k h_{ij}\rangle\,,\\
&\langle\nabla_k h_i{}^l\cdot\nabla^k h_{jl}, h_{ij}\rangle=\frac{1}{2}\langle h_i{}^l(\bar\Delta h)_{jl}, h_{ij}\rangle=-\langle(Lh)_{ij}, h_i{}^k h_{kj}\rangle\,,\\
&\langle\nabla^l h_{ik}\cdot\nabla^k h_{jl}, h_{ij}\rangle=-\langle h_{ik}\nabla^l\nabla^k h_{jl}, h_{ij}\rangle-\langle h_{ik}\nabla^k h_{jl}, \nabla_l h_{ij}\rangle\\
&\quad=-\langle h_{ik}(R^{lkm}{}_j h_{ml}+R^{lkm}{}_l h_{jm}), h_{ij}\rangle+\langle h_{ik}h_{jl}, \nabla_k\nabla_l h_{ij}\rangle\\
&\quad=\langle\nabla_i\nabla_j h_{kl}, h_{ik}h_{jl}\rangle-\langle(Lh)_{ij}, h_i{}^k h_{kj}\rangle-\varepsilon\langle h_{ij}, h_i{}^k h_{kj}\rangle\,,\\
&\langle R_i{}^m{}_j{}^l h_m{}^k h_{kl}, h_{ij}\rangle=\langle(Lh)_{ij}, h_i{}^k h_{kj}\rangle\,,\\
&\langle h^{kl}\nabla_i\nabla_k h_{jl}+h^{kl}\nabla_j\nabla_k h_{il}, h_{ij}\rangle=2\langle h^{kl}\nabla_i\nabla_k h_{jl}, h_{ij}\rangle\\
&\quad=2\langle\nabla_i\nabla_j h_{kl}, h_{ik}h_{jl}\rangle\,,\\
&\langle\nabla_i h^k{}_m\cdot\nabla_j h_k{}^m, h_{ij}\rangle=-\langle h^k{}_m\nabla_i\nabla_j h_k{}^m, h_{ij}\rangle=-\langle\nabla_i\nabla_j h_{kl}, h_{ij}h_{kl}\rangle\\
&\langle(Lh)_i{}^k h_{kj}+(Lh)_j{}^k h_{ki}, h_{ij}\rangle=2\langle(Lh)_{ij}, h_i{}^k h_{kj}\rangle\,,\\
&\langle\mathrm{Hess}(h,h), h\rangle=0.
\end{aligned}$$

Q.E.D.

Now, let $g\in\mathcal{M}_c^\infty$ be an Einstein metric and $g(t)$ a curve in $S_g^k\cap\mathcal{M}_c^\infty$ such that $g(0)=g$ and each $g(t)$ is an Einstein metric. Then,

$$E(g(0)) = 0\,,$$
$$\left.\frac{d}{dt}\right|_0 E(g(t)) = 0,\ \text{i.e.,}\ E'_g(g'(0)) = 0\,,$$
$$\left.\frac{d^2}{dt^2}\right|_0 E(g(t)) = 0,\ \text{i.e.,}\ E''_g(g'(0), g'(0)) + E'_g(g''(0)) = 0\,.$$

Therefore, for an Einstein metric g, we call a pair $(h,h')\in C^\infty(S^2M)\times C^\infty(S^2M)$ an essential Einstein i-deformation of second order of g if h is an essential Einstein i-deformation of g and h' satisfies that $E''(h,h)+E'(h')=0$.

DEFINITION 4.4. Let g be an Einstein metric and h an essential Einstein i-deformation of g. If there exists $h'\in C^\infty(S^2M)$ such that (h,h') is an essential Einstein i-deformation of second order, h is said to be *integrable up to second order*. If there is an Einstein deformation $g(t)$ of g such that $g'(0)=h, h$ is said to be *integrable*.

We easily see the following

Proposition 4.5. *Let g be an Einstein metric and h an essential Einstein i-deformation of g. If h is not integrable up to second order, then h is not integrable.*

Moreover the following proposition holds.

Proposition 4.6. *Let $g\in\mathcal{M}_c^\infty$ be an Einstein metric. If all essential Einstein i-deformations are not integrable up to second order, then g is rigid.*

Proof. By Lemma 2.2, it is sufficient to prove that g is isolated in $S_g^s\cap\mathcal{M}_c^s$. Consider the map $E\,|\,S_g^s\cap\mathcal{M}_c^s\colon S_g^s\cap\mathcal{M}_c^s\to H^{s-2}(S^2M)$. By formula (2.2.3) and Lemma 3.1, $\mathrm{Im}(E\,|\,S_g^s\cap\mathcal{M}_c^s)'_g$ is closed in $H^{s-2}(S^2M)$. Denote by P the orthogonal projection: $H^{s-2}(S^2M)\to\mathrm{Im}(E\,|\,S_g^s\cap\mathcal{M}_c^s)'_g$. Then $\mathrm{Im}(P\circ E\,|\,S_g^s\cap\mathcal{M}_c^s)'_g=\mathrm{Im}(E\,|\,S_g^s\cap\mathcal{M}_c^s)'_g$ and, by the implicit function theorem, there is an open neighbourhood U of g in $S_g^s\cap\mathcal{M}_c^s$ such that all H^s-Einstein metrics in U are elements of $(P\circ E\,|\,S_g^s\cap\mathcal{M}_c^s)^{-1}(0)\cap U$. Here, since the operator β is elliptic, $(P\circ E\,|\,S_g^s\cap\mathcal{M}_c^s)^{-1}(0)\cap U$ becomes a finite dimensional submanifold of $S_g^s\cap\mathcal{M}_c^s$. If we apply the condition to the map $E\,|\,(P\circ E\,|\,S_g^s\cap\mathcal{M}_c^s)^{-1}(0)\cap U$, then the result is obvious. Q.E.D.

Lemma 4.7. *Let $g\in\mathcal{M}_c^\infty$ be an Einstein metric and h an essential Einstein i-deformation. Then h is integrable up to second order if and only if $E''(h,h)$ is orthogonal to* EID.

Proof. By the definition, h is integrable up to second order if and only if

$E''(h,h) \in \mathrm{Im}(E \mid \mathcal{M}^s)'_g$. Remark that the formulae $\gamma E=0$ (by the Bianchi identity) and $\int E=0$ on $\mathcal{M}^s$ hold. By differentiating the formulae, we get that

$$\gamma''_g(h,h)(E(g))+2\gamma'_g(h)(E'_g(h))+\gamma E''_g(h,h) = 0\,,$$
$$\int''_g(h,h)(E(g))+2\int'_g(h)(E'_g(h))+\int E''_g(h,h) = 0$$

for all $g \in \mathcal{M}^s_c$ and $h \in H^s(S^2M)$. Therefore the assumption of g and h implies that $\gamma E''(h,h)=0$ and $\int E''(h,h)=0$, i.e., $E''(h,h) \in \mathrm{Ker}\,\gamma \cap \mathrm{Ker}\int$. Thus by Proposition 3.2, the result is obvious. Q.E.D.

5. Classification of essential Einstein *i*-deformations

In this section and the following, we use the representation theory of Lie groups. For fundamental data concerning root systems of simple Lie algebras (resp. of irreducible symmetric pairs), see Bourbaki [4; Planche I-IX] (resp. Murakami [11]).

First we show some facts concerning a compact semi-simple Lie group G and G-modules. Modules are all taken to be complex modules, unless otherwise stated. Let $\mathfrak{g}$ be the Lie algebra of G with a G-invariant inner product B and $\mathfrak{a}$ a Cartan subalgebra of $\mathfrak{g}$ with a linear order. We denote by $2\delta_{\mathfrak{g}}$ the sum of all positive roots of $\mathfrak{g}^C$ and by $V(\lambda)$ the irreducible G-module with highest weight λ. Then the Casimir operator on $V(\lambda)$ coincides with the scalar operator $e(V(\lambda))=B(\lambda+2\delta_{\mathfrak{g}}, \lambda)$. If G is decomposed into $\Pi_i G_i$ where G_i are simple groups, we denote by $\mathfrak{g}_i$ the Lie algebra of G_i and B_i the restriction of B on $\mathfrak{g}_i$. An irreducible G-module V has the form $\otimes_i V_i$ where each V_i is an irreducible G_i-module or a trivial G_i-module $\boldsymbol{C}$. Then we see $e(V)=\sum_i e(V_i)$. Assume that all B_i satisfy $e(\mathfrak{g}_i^C)=2\varepsilon$. By an easy computation, we can check

Lemma 5.1. *Let G be a compact simple Lie group. Then for any irreducible G-module, $e(V)>(2/3)\varepsilon$ holds.*

By this lemma, we can classify irreducible G-modules V such that $e(V)=2\varepsilon$, for a semi-simple Lie group G. Assume that V has the form $\otimes V_i$ and that each V_i is not trivial. Then the equality $e(V)=2\varepsilon$ implies that G has at most two simple factors. For the case that G is simple, we can check

Lemma 5.2. *Let G be a compact simple Lie group and V an irreducible G-module. If $e(V)=2\varepsilon$, then V is isomorphic to $\mathfrak{g}^C$.*

For the case where G has two simple factors, we list all pairs of irreducible G_i-modules $V_i (i=1,2)$ such that $e(V_1)+e(V_2)=2\varepsilon$ and $e(V_1) \leqq e(V_2)$. In the following table, ω_i means the highest weight of V_1 and V_2.

Table 5.3.

V_1	V_2	
$\omega_1,\ \omega_{2l}/\mathrm{A}_2$	$\omega_1/\mathrm{B}_{2l^2+2l+1}$	$(l\geqq 1)$
$\omega_1,\ \omega_{2l-1}/\mathrm{A}_{2l-1}$	$\omega_1/\mathrm{D}_{2l^2+1}$	$(l\geqq 2)$
ω_1/C_l	$\omega_1/\mathrm{D}_{l+2}$	$(l\geqq 1)$
ω_2/B_2	$\omega_3,\ \omega_4/\mathrm{D}_4$	
ω_1/C_9	ω_3/B_3	
ω_1/G_2	ω_1/G_2	

Next, we show some facts concerning a simply connected irreducible symmetric space G/K of compact type. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ the canonical decomposition. We compute the dimension of $\mathrm{Hom}_K(\mathfrak{g}^C, S_0^2(\mathfrak{m}^C))$, where S_0^2 means the traceless part of the symmetric tensor product. If G/K is of group type, then $\mathfrak{g}^C=\mathfrak{k}^C+\mathfrak{k}^C$, $\mathfrak{m}^C=\mathfrak{k}^C$ as K-modules. So we have to compute $\dim_C\mathrm{Hom}_K(\mathfrak{k}^C, S_0^2(\mathfrak{k}^C))$, where K is a compact simple Lie group.

Lemma 5.4. *If K is not of type* A_l $(l\geqq 2)$, *then* $\dim_C\mathrm{Hom}_K(\mathfrak{k}^C, S_0^2(\mathfrak{k}^C))=0$. *If K is of type* A_l $(l\geqq 2)$, *then* $\dim_C\mathrm{Hom}_K(\mathfrak{k}^C, S_0^2(\mathfrak{k}^C))=1$.

If G/K is not of group type, we can check

Lemma 5.5. *The dimension of* $\mathrm{Hom}_K(\mathfrak{g}^C, S_0^2(\mathfrak{m}^C))$ *is* $(H1)$ *two if* $(G,K)=(SU(p+q),\ S(U(p)\times U(q)))$ [AIII] $(p\geqq q\geqq 2)$, $(H2)$ *zero if* $(G,K)=(SU(2),\ S(U(1)\times U(1)))$ $[S^2]$, $(H3)$ *one if* (G,K) *is of another hermitian type,* $(N1)$ *one if* $(G,K)=(SU(l),\ SO(l))$ [AI] $(l\geqq 3)$, $(SU(2l),\ Sp(l))$ [AII] $(l\geqq 3)$ *or* (E_6, F_4) [EIV] *and* $(N2)$ *zero if* (G,K) *is of another non-hermitian type.*

Now, we come back to our Einstein manifold (M,g) and assume that (M,g) is a simply connected symmetric space G/K. The tangent space T_0M of M at the origin is identified with $\mathfrak{m}$ and the metric g is induced by a G-invariant inner product B on $\mathfrak{g}$.

Generally, for a finite dimensional K-module U, a cross section s of the homogeneous vector bundle $G\times_K U$ over M may be identified with a U-valued function s on G such that $s(xy)=y^{-1}s(x)$ for all $x\in G$ and $y\in K$. Let $C^\infty(G,U)_K$ be the space of all such s and enlarge this space to $H^0(G,U)_K$. Then $C^\infty(G,U)_K$ and $H^0(G,U)_K$ canonically become G-modules and $H^0(G,U)_K$ is decomposed into $\oplus_i V_i$ as Hilbert space, where V_i are irreducible G-modules contained in $C^\infty(G,U)_K$. Let V be an irreducible G-module and denote by W the direct sum of all irreducible G-modules V_i which are isomorphic to V. Then we see, by the Frobenius reciprocity theorem (cf. Wallach [12; Theorem 8.2]), that

$$\begin{aligned}\dim W &= \dim V\cdot\dim \mathrm{Hom}_G(V, C^\infty(G,U)_K)\\ &= \dim V\cdot\dim \mathrm{Hom}_K(V,U)\,.\end{aligned}$$

Lemma 5.6 (Koiso [8; Proposition 5.3]). *The Lichnerowicz operator* Δ *regarded as an endomorphism of* $C^\infty(G, \otimes^p \mathfrak{m}^C)_K$ *coincides with the Casimir operator.*

Let $M=\prod_{a=1}^N M_a$ be the irreducible decomposition of the symmetric space M. Remark that all (M_a, g_a) are Einstein manifolds with the same Einstein constant ε. Let (G_a, K_a) be the symmetric pair of each M_a, $\mathfrak{g}_a$ (resp. $\mathfrak{k}_a$) the Lie algebra of G_a (resp. K_a) and $\mathfrak{g}_a=\mathfrak{k}_a+\mathfrak{m}_a$ the canonical decomposition. Since $\operatorname{Ker}\delta^*\subset\operatorname{Ker}(\Delta_1-2\varepsilon)$, Lemma 5.6 implies that $e(\mathfrak{g}_a^C)=2\varepsilon$. Therefore we see, combining Proposition 3.3, that

$$\begin{aligned}(5.6.1)\qquad \dim_R \mathrm{EID} &= \textstyle\sum_\alpha \dim_C V^\alpha\cdot\dim_C \operatorname{Hom}_K(V^\alpha, S_0^2(\mathfrak{m}^C))\\ &\quad-\textstyle\sum_\alpha \dim_C V^\alpha\cdot\dim_C\operatorname{Hom}_K(V^\alpha,\mathfrak{m}^C)+\dim_C\mathfrak{g}^C\,,\end{aligned}$$

where V^α runs through the set of all equivalence classes of irreducible G-modules whose Casimir operators are 2ε. Let

$$V^\alpha = C^{\nu_\alpha}\oplus\oplus_{i=1}^{n_\alpha}V_i^\alpha$$

be the irreducible decomposition of V^α as K-module. Each V_i^α has the form

$$\bigotimes_{a\in I_i^\alpha} V_{i,a}^\alpha\,,$$

where I_i^α is a subset of $\{b\in Z;\ 1\leqq b\leqq N\}$ and $V_{i,a}^\alpha$ are irreducible K_a-modules. Then we see that

$$\begin{aligned}&\operatorname{Hom}_K(V^\alpha, S_0^2(\mathfrak{m}^C))\\ &=\operatorname{Hom}_K(V^\alpha, \bigoplus_{a=1}^N S_0^2(\mathfrak{m}_a^C)\oplus\bigoplus_{a<b}\mathfrak{m}_a^C\otimes\mathfrak{m}_b^C+C^{N-1})\\ &=\bigoplus_{a=1}^N\operatorname{Hom}_K(V^\alpha, S_0^2(\mathfrak{m}_a^C))\oplus\bigoplus_{a<b}\operatorname{Hom}_K(V^\alpha,\mathfrak{m}_a^C\otimes\mathfrak{m}_b^C)\oplus\bigoplus^{N-1}\operatorname{Hom}_K(V^\alpha, C)\,.\end{aligned}$$

Here, by Frobenius reciprocity, if $\operatorname{Hom}_K(V^\alpha,\mathfrak{m}_a^C\otimes\mathfrak{m}_b^C)$ does not vanish, then there is a non-zero 2-tensor field h on M such that $\Delta h=2\varepsilon h$ and $h\in T(M_a)^C\otimes T(M_b)^C$ at each point of M. Then $\bar\Delta h=-2Lh=0$ and so h is parallel. But a parallel symmetric 2-tensor field is a linear combination of the metrics g_a on M_a. Therefore

$$\operatorname{Hom}_K(V^\alpha, \mathfrak{m}_a^C\otimes\mathfrak{m}_b^C)=0\quad\text{for } a\neq b\,.$$

Thus
$$\begin{aligned}&\operatorname{Hom}_K(V^\alpha, S_0^2(\mathfrak{m}^C))\\ &=\bigoplus_{i=1}^{n_\alpha}\bigoplus_{a=1}^N\operatorname{Hom}_K(V_i^\alpha, S_0^2(\mathfrak{m}_a^C))\oplus\bigoplus_{a=1}^N\operatorname{Hom}_K(C^{\nu_\alpha}, S_0^2(\mathfrak{m}_a^C))\\ &\quad\oplus\bigoplus^{N-1}\bigoplus_{i=1}^{n_\alpha}\operatorname{Hom}_K(V_i^\alpha, C)\oplus\bigoplus^{N-1}\operatorname{Hom}_K(C^{\nu_\alpha}, C)\,.\end{aligned}$$

If $\operatorname{Hom}_K(C, S_0^2(\mathfrak{m}_a^C))\neq 0$, then there is a G-invariant symmetric 2-tensor field h such that $h\in S_0^2(M_a)^C$ at each point. Since there is no such h,

$$\mathrm{Hom}_K(C^{\nu_\alpha}, S_0^2(\mathfrak{m}_a^C)) = 0\,.$$

Thus
$$\begin{aligned}
&\mathrm{Hom}_K(V^\alpha, S_0^2(\mathfrak{m}^C)) \\
&= \bigoplus_{i=1}^{n_\alpha} \bigoplus_{a=1}^{N} \mathrm{Hom}_K\big(\bigotimes_{b\in I_i^\alpha} V_{i,b}^\alpha, S_0^2(\mathfrak{m}_a^C)\big) \oplus \bigoplus^{\nu_\alpha(N-1)} \mathrm{Hom}_K(C, C) \\
&= \bigoplus_{a,i\,;\,I_i^\alpha=\{a\}} \mathrm{Hom}_{K_a}(V_i^\alpha, S_0^2(\mathfrak{m}_a^C)) \oplus C^{\nu_\alpha(N-1)}\,.
\end{aligned}$$

Moreover,

$$\mathrm{Hom}_K(V^\alpha, \mathfrak{m}^C) = \mathrm{Hom}_K\big(\bigoplus_{i=1}^{n_\alpha} V_i^\alpha, \bigoplus_{a=1}^{N} \mathfrak{m}_a^C\big) + \mathrm{Hom}_K\big(C^{\nu_\alpha}, \bigoplus_{a=1}^{N} \mathfrak{m}_a^C\big)\,.$$

Here, since there is no parallel 1-tensor field on M, $\mathrm{Hom}_K(C, \oplus_{a=1}^N \mathfrak{m}_a^C)=0$. Therefore,

$$\begin{aligned}
\mathrm{Hom}_K(V^\alpha, \mathfrak{m}^C) &= \bigoplus_{i=1}^{n_\alpha} \bigoplus_{a=1}^{N} \mathrm{Hom}_K(V_i^\alpha, \mathfrak{m}_a^C) \\
&= \bigoplus_{i=1}^{n_\alpha} \bigoplus_{a=1}^{N} \mathrm{Hom}_K\big(\bigotimes_{b\in I_i^\alpha} V_{i,b}^\alpha, \mathfrak{m}_a^C\big) \\
&= \bigotimes_{a,i\,;\,I_i^\alpha=\{a\}} \mathrm{Hom}_{K_a}(V_i^\alpha, \mathfrak{m}_a^C)\,.
\end{aligned}$$

Thus we see

$$\dim \mathrm{EID} = \textstyle\sum_\alpha N(V^\alpha)\cdot \dim V^\alpha\,,$$

where
$$\begin{aligned}
&N(V^\alpha) \\
&= \sum_{a,i\,;\,I_i^\alpha=\{a\}} [\dim_C \mathrm{Hom}_{K_a}(V_i^\alpha, S_0^2(\mathfrak{m}_a^C)) - \dim_C \mathrm{Hom}_{K_a}(V_i^\alpha, \mathfrak{m}_a^C)] \\
&\quad + \nu^\alpha(N-1) + \kappa^\alpha\,,
\end{aligned}$$

and $\kappa^\alpha=1$ if V^α or $V^\alpha\oplus V^\alpha$ is isomorphic to some $\mathfrak{g}_b^C$, $\kappa^\alpha=0$ if not. (The case $V^\alpha\oplus V^\alpha$ occurs if M_b is of group type.)

Now, we compute $N(V^\alpha)$. By Lemma 5.1 and remarks following it, the number of elements of $I^\alpha=\bigcup_{i=1}^{n_\alpha} I_i^\alpha$ is one or two.

Case 1: the number of elements of I^α is one. We may assume that $I^\alpha=\{1\}$. First we assume that M_1 is not of group type. Then Lemma 5.2 implies that V^α is isomorphic to $\mathfrak{g}_1^C$.

Case 1-H (M_1 is hermitian). The module V^α is decomposed into $\mathfrak{k}_1'^C \oplus \mathfrak{m}_1^+ \oplus \mathfrak{m}_1^- \oplus C$ as K_1-module, where $\mathfrak{k}_1'$ is the semisimple part of $\mathfrak{k}_1$, $\mathfrak{m}_1^\pm$ is the $\pm\sqrt{-1}$-eigenspace of $\mathfrak{m}_1^C$ with respect to the almost complex structure of M_1. Then $\dim \mathrm{Hom}_{K_1}(V^\alpha, \mathfrak{m}_1^C)=2$, $\nu^\alpha=1$, $\kappa^\alpha=1$. Therefore,

$$N(V^\alpha) = \dim \mathrm{Hom}_{K_1}(\mathfrak{g}_1^C, S_0^2(\mathfrak{m}_1^C)) + N - 2\,.$$

Combining with Lemma 5.5 (H), we see that

$N(V^\alpha)=N$ if M_1 is of type AIII ($p\geqq q\geqq 2$),

$N(V^\alpha)=N-2$ if M_1 is S^2,

$N(V^\alpha)=N-1$ if M_1 is of another hermitian type.

Case 1-N (M_1 is not hermitian). The module V^{α} is irreducibly decomposed into $\mathfrak{k}_1^C \oplus \mathfrak{m}_1^C$ as K_1-module. Then $\dim_C \mathrm{Hom}_{K_1}(V^{\alpha}, \mathfrak{m}_1^C)=1$, $\nu^{\alpha}=0$, $\kappa^{\alpha}=1$. Therefore,

$$N(V^{\alpha}) = \dim \mathrm{Hom}_{K_1}(\mathfrak{g}_1^C, S_0^2(\mathfrak{m}_1^C)) .$$

By Lemma 5.5(N), we see that

$N(V^{\alpha})=1$ if M_1 is of type AI ($l\geqq 3$), AII ($l\geqq 3$) or EIV,

$N(V^{\alpha})=0$ if M_1 is of another non-hermitian type.

Next we assume that M_1 is of group type. Then Lemma 5.2 implies that V^{α} is isomorphic to $\mathfrak{k}_1^C$ or to $W_1\otimes W_2$ as G_1-module, where W_1 and W_2 are irreducible modules of simple factors of G_1.

Case 1-G (V^{α} is isomorphic to $\mathfrak{k}_1^C$). The modules V^{α}, $\mathfrak{m}_1^C$ and $\mathfrak{k}_1^C$ are isomorphic to each other as K_1-modules. Then $\dim \mathrm{Hom}_{K_1}(V^{\alpha}, \mathfrak{m}_1^C)=1$, $\nu^{\alpha}=0$ and $\kappa^{\alpha}=1$. Therefore,

$$N(V^{\alpha}) = \dim \mathrm{Hom}_{K_1}(\mathfrak{k}_1^C, S_0^2(\mathfrak{k}_1^C)) .$$

By Lemma 5.4, we see that

$N(V^{\alpha})=1$ if M_1 is $SU(l)$ ($l\geqq 3$),

$N(V^{\alpha})=0$ if M_1 is another group.

Case 1'-G (V^{α} is isomorphic to $W_1\otimes W_2$). Table 5.3 implies that this case occurs only if M_1 is the group of type G_2. By computing, we see that $\dim \mathrm{Hom}_{K_1}(V^{\alpha}, S_0^2(\mathfrak{m}_1^C))=1$, $\dim \mathrm{Hom}_{K_1}(V^{\alpha}, \mathfrak{m}_1^C)=1$, $\nu^{\alpha}=1$ and $\kappa^{\alpha}=0$. Therefore,

$N(V^{\alpha})=N-1$ if M_1 is of type G_2,

$N(V^{\alpha})=0$ if M_1 is another group.

Case 2: the number of elements of I^{α} is two. We may assume that $I^{\alpha}=\{1,2\}$ and $V^{\alpha}=W_1\otimes W_2$, where W_a is an irreducible G_a-module such that $e(W_1)\leqq e(W_2)$. Then, since the first non-zero eigenvalue of Δ on $C^{\infty}(M_1)$ is greater than $\mathcal{E}$ (Lichnerowicz [9; p. 135]), $\mathrm{Hom}_{G_1}(W_1, C^{\infty}(G_1, \boldsymbol{C})_{K_1})=0$ and so $\mathrm{Hom}_{K_1}(W_1, \boldsymbol{C})=0$. Let $W_1=\oplus_i W_{1,i}$ and $W_2=\boldsymbol{C}^{\mu}\oplus \oplus_i W_{2,i}$ be the irreducible decompositions as K_1 and K_2-modules. Then V^{α} is irreducibly decomposed into

$$\bigoplus_i \overset{\mu}{\oplus} W_{1,i} \oplus \bigoplus_{i,j} W_{1,i}\otimes W_{2,j}$$

as $K_1\times K_2$-module. Therefore, since $\nu^{\alpha}=0$ and $\kappa^{\alpha}=0$, we see that

$$N(V^{\alpha}) = \mu\cdot[\dim \mathrm{Hom}_{K_1}(W_1, S_0^2(\mathfrak{m}_1^C)) - \dim \mathrm{Hom}_{K_1}(W_1, \mathfrak{m}_1^C)] .$$

If M_2 is of group type, then W_2 is irreducible as K_2-module, and so $\mu=0$, which implies that $N(V^{\alpha})=0$. Let G_2 and W_2 be in the list of V_2 in Table 5.3 and assume that (G_2, K_2) is a symmetric pair. We can check that if $\mathrm{Hom}_{K_2}(W_2, \boldsymbol{C})\neq 0$, then G_2/K_2 is the standard sphere, i.e., of type B or D, and $W_2=V(\omega_1)$. On the other hand, if G_1 is of type A_l and $W_1=V(\omega_1)$ or $V(\omega_l)$, or G_1 is of type C_l and $W_1=V(\omega_1)$, then we can check that there is no symmetric pair (G_1, K_1) such that $\mathrm{Hom}_{K_1}(W_1, S_0^2(\mathfrak{m}_1^C))\neq 0$ or $\mathrm{Hom}_{K_1}(W_1, \mathfrak{m}_1^C)\neq 0$. Moreover if

M_1 is of group type, we easily see that the K_1-module W_1 does not admit zero as weight and $S_0^2(\mathfrak{m}_1^C)$ and $\mathfrak{m}_1^C$ admits zero as weight, and so $\mathrm{Hom}_{K_1}(W_1, S_0^2(\mathfrak{k}_1^C))=0$ and $\mathrm{Hom}_{K_1}(W_1, \mathfrak{k}_1^C)=0$. Thus in this case we see that $N(V^\alpha)=0$.

Let M, M_a and G_a be as above. Assume that M_1 is a hermitian space or the group of type G_2. Then there is a unique irreducible G_1-module V_1 such that $e(V_1)=2\varepsilon$ and $\mathrm{Hom}_{K_1}(V_1, \boldsymbol{C})\neq 0$. Moreover $\dim \mathrm{Hom}_{K_1}(V_1, \boldsymbol{C})=1$. Therefore 2ε is an eigenvalue of Δ on $C^\infty(M_1)$ and the corresponding eigenspace F becomes an irreducible real G_1-module. Let g_a be the metric on each M_a and $f_a \in F$ and set

$$h = \mathrm{Hess}\, f_1 + \varepsilon \sum_{a=1}^{N} f_a \cdot g_a .$$

Then,

$$\Delta h = \mathrm{Hess}\, \Delta f_1 + \varepsilon \sum_{a=1}^{N} 2\varepsilon f_a \cdot g_a = 2\varepsilon h ,$$

$$\delta h = \delta(\mathrm{Hess}\, f_1 + \varepsilon f_1 \cdot g_1) + \varepsilon \sum_{a=2}^{N} \delta(f_a \cdot g_a) = 0 ,$$

$$\mathrm{tr}\, h = -\Delta f_1 + \varepsilon \sum_{a=1}^{N} n_a f_a = -2\varepsilon f_1 + \varepsilon \sum_{a=2}^{N} n_a f_a ,$$

where $n_a = \dim M_a$. If $\sum_{a=1}^{N} n_a f_a - 2f_1 = 0$, then $h \in \mathrm{EID}(M)$. Remark that if $M_1 = S^2$, then $\mathrm{Hess}\, f_1 + \varepsilon f_1 \cdot g_1 = 0$. Since $\mathrm{EID}(M_1) \subset \mathrm{EID}(M)$, we get the following

Theorem 5.7. *Let (M, g) be a compact simply connected symmetric Einstein*

type	V_1	N_1	form of $h \in W_1$
$SU(p+q)/S(U(p)\times U(q))$ $(p\geqq q\geqq 2)$	$\mathfrak{su}(p+q)^C$	N	$h_0 + \mathrm{Hess} f_1 + \varepsilon \sum_{a=1}^{N} f_a g_a$ $(\sum_{a=1}^{N} n_a f_a - 2f_1 = 0)$
S^2	$\mathfrak{su}(2)^C$	$N-2$	$\sum_{a=2}^{N} f_a g_a$ $(\sum_{a=2}^{N} n_a f_a = 0)$
other hermitian	$\mathfrak{g}_1^C$	$N-1$	$\mathrm{Hess} f_1 + \varepsilon \sum_{a=1}^{N} f_a g_a$ $(\sum_{a=1}^{N} n_a f_a - 2f_1 = 0)$
G_2	$V(\omega_1)\otimes V(\omega_1)$	$N-1$	$\mathrm{Hess} f_1 + \varepsilon \sum_{a=1}^{N} f_a g_a$ $(\sum_{a=1}^{N} n_a f_a - 2f_1 = 0)$
$SU(l)$ $(l\geqq 3)$	$\mathfrak{su}(l)^C$	2	$h_0{}^{(*)}$
$SU(l)/SO(l)$ $(l\geqq 3)$	$\mathfrak{su}(l)^C$	1	h_0
$SU(2l)/Sp(l)$ $(l\geqq 3)$	$\mathfrak{su}(2l)^C$	1	h_0
$\mathrm{E}_6/\mathrm{F}_4$	$\mathfrak{e}_6$	1	h_0
other type		0	

(*) decomposes into right invariant form and left invariant form *manifold,* $(M,g)=\prod_{a=1}^{N}(M_a,g_a)$ *its irreducible decomposition as symmetric space* $(\dim M_a=n_a)$ *and* (G_a,K_a) *the symmetric pair attached to* M_a. *Then* $\mathrm{EID}(M)$ *becomes a real G-module and is decomposed into* $\oplus_a W_a$ *where each* W_a^C *is a* G_a*-module (which may be* 0). *Each* W_a^C *is the direct sum of* N_a *copies of an irreducible* G_a*-module* V_a (N_a *may be* 0). *The* G_a*-module* V_a, N_a *and the form of elements of* W_a *are listed above (we may assume that* $a=1$). *There* h_0 *means an element of* $\mathrm{EID}(M_1)$ $(\subset \mathrm{EID}(M))$, f_a *eigenfunctions of* Δ *on* $C^\infty(M_1)$ *with eigenvalues* $2\mathcal{E}$.

Corollary 5.8. *Let* (M,g) *be a locally symmetric Einstein manifold of compact type and* $\prod_{a=1}^{N}M_a$ *be the irreducible decomposition of the universal riemannian covering manifold* $\tilde{M}$ *of* M. *If* $N=1$ *and* $\tilde{M}$ *is neither* $SU(p+q)/S(U(p)\times U(q))$ $(p\geqq q\geqq 2)$, $SU(l)/SO(l)$ $(l\geqq 3)$, $SU(2l)/Sp(l)$ $(l\geqq 3)$, $\mathrm{E}_6/\mathrm{F}_4$ *nor* $SU(l)$ $(l\geqq 3)$, *then* g *is rigid. If* $N=2$ *and* M_a *are neither one of the above, the group of type* G_2 *nor any hermitian space except* S^2, *then* g *is rigid. If* $N\geqq 3$ *and* M_a *are neither one of the above nor* S^2, *then* g *is rigid.*

Proof. It is obvious that infinitesimal non-deformability of an Einstein metric reduces to that of its riemannian covering. So Proposition 2.5 implies this result. Q.E.D.

6. Second order Einstein *i*-deformation on symmetric spaces

Let (M,g) be a compact simply connected symmetric space G/K where g is an Einstein metric with Einstein constant $\mathcal{E}$. Let $M=\prod_{a=1}^{N}M_a$ be its irreducible decomposition and (G_a,K_a) the symmetric pair of M_a. By Theorem 5.7, $\mathrm{EID}(M)=\oplus_{a=1}^{N}W_a$ where each W_a is a real G_a-module (which may be 0). By Lemma 4.7, if we denote by $\psi(h_1,h_2)$ the EID-component of $E''(h_1,h_2)$ for h_1, $h_2\in\mathrm{EID}$, then h is integrable up to second order if and only if $\psi(h,h)=0$. We easily see that ψ is a G-homomorphism. Therefore we get

Lemma 6.1. *In the above situation, if* $\mathrm{Hom}_G(S^2(\oplus_a W_a), \oplus_a W_a)=0$, *then all essential Einstein i-deformations are integrable up to second order.*

$$\begin{aligned}
&\mathrm{Hom}_G(S^2\oplus_a W_a), \oplus_a W_a)\\
&\quad=\mathrm{Hom}_G(\bigoplus_a S^2(W_a)\oplus \bigoplus_{a<b} W_a\otimes W_b, \bigoplus_c W_c)\\
&\quad=\bigoplus_{a,b}\mathrm{Hom}_G(S^2(W_a), W_b)\oplus \bigoplus_{\substack{a<b\\ c}}\mathrm{Hom}_G(W_a\otimes W_b, W_c)\,.
\end{aligned}$$

Since each W_a has no trivial component as G_a-module, the last form equals to $\oplus_{a=1}^{N}\mathrm{Hom}_G(S^2(W_a), W_a)$. Thus the integrability of $h\in\mathrm{EID}(M)$ up to second order reduces to the integrability of its components in each W_a.

If M_1 is $\mathrm{E}_6/\mathrm{F}_4$, then by Theorem 5.7, W_1 is isomorphic to $\mathfrak{g}_1$ and Lemma 5.4 implies that $\mathrm{Hom}_G(S^2(W_1), W_1)=0$.

Let M_1 be the group of type G_2 or a hermitian space except AIII $(p\geqq q\geqq 2)$

and denote by F the 2ε-eigenspace of Δ on $C^\infty(M_1)$. Then by Theorem 5.7, an element h of W_1 has the form

$$h_1+\sum_{a=2}^{N} f_a\cdot g_a\,,$$

where $h_1\in C^\infty(S^2M_1)$ and $f_a\in F$. We calculate $E''(h,h)$.

$$\begin{aligned}
&h^{kl}\nabla_i\nabla_j h_{kl}=(h_1)^{kl}\nabla_i\nabla_j(h_1)_{kl}+\sum_{a=2}^{N}(h_1,\operatorname{Hess} f_a)\cdot g_a\,,\\
&\nabla^k h_{il}\cdot\nabla_k h_j{}^l=\nabla^k(h_1)_{il}\cdot\nabla_k(h_1)_j{}^l+\sum_{a=2}^{N}(df_a,df_a)\cdot g_a\,,\\
&\nabla^k h_{il}\cdot\nabla^l h_{jk}=\nabla^k(h_1)_{il}\cdot\nabla^l(h_1)_{jk}\,,\\
&R_i{}^k{}_j{}^l h_{km}h^m{}_l=R_i{}^k{}_j{}^l(h_1)_{km}(h_1)^m{}_l-\varepsilon\sum_{a=2}^{N}(f_a)^2\cdot g_a\,,\\
&h^{kl}\nabla_i\nabla_k h_{jl}=(h_1)^{kl}\nabla_i\nabla_k(h_1)_{jl}\,,\\
&\nabla_i h^{kl}\cdot\nabla_j h_{kl}=\nabla_i(h_1)^{kl}\cdot\nabla_j(h_1)_{kl}+\sum_{a=2}^{N}n_a\nabla_i f_a\cdot\nabla_j f_a\,,\\
&R_i{}^l{}_k{}^m h_{lm}h^k{}_j=R_i{}^l{}_k{}^m(h_1)_{lm}(h_1)^k{}_j-\varepsilon\sum_{a=2}^{N}(f_a)^2\cdot g_a\,,\\
&\operatorname{Hess}(h,h)=\operatorname{Hess}(h_1,h_1)+2\sum_{a=2}^{N}n_a df_a\otimes df_a+2\sum_{a=2}^{N}n_a f_a\cdot\operatorname{Hess} f_a\,,
\end{aligned}$$

and so $2E''(h,h)$

$$\begin{aligned}
&=2E''(h_1,h_1)+2\sum_{a=2}^{N}(h_1,\operatorname{Hess} f_a)\cdot g_a\\
&\quad+2\sum_{a=2}^{N}(df_a,df_a)\cdot g_a+\sum_{a=2}^{N}n_a df_a\otimes df_a+2\sum_{a=2}^{N}n_a f_a\cdot\operatorname{Hess} f_a\,.
\end{aligned}$$

Let $h'=h_1'+\sum_{a=2}^{N}f_a'\cdot g_a\in W_1$. Then

$$\begin{aligned}
&\langle E''(h,h),h'\rangle=\langle E''(h_1,h_1),h_1'\rangle\\
&\quad+\sum_{a=2}^{N}n_a\langle df_a\otimes df_a+2f_a\cdot\operatorname{Hess} f_a,h_1'\rangle\\
&\quad+2\sum_{a=2}^{N}n_a\langle(h_1,\operatorname{Hess} f_a)+(df_a,df_a),f_a'\rangle\,.
\end{aligned}$$

Assume that M_1 is not of type AIII ($p+q\geqq 3$). Then we can set $h_1=\operatorname{Hess} f+\varepsilon f\cdot g_1$ and $h_1'=\operatorname{Hess} f'+\varepsilon f'\cdot g_1$, where $f,f'\in F$. Moreover, by Lemma 5.4, $\operatorname{Hom}_{G_1}(S^2(\mathfrak{g}_1^C),\mathfrak{g}_1^C)=0$ holds. Therefore

$$\begin{aligned}
&\langle E''(h_1,h_1),h_1'\rangle=0\,,\\
&\langle df_a\otimes df_a+2f_a\cdot\operatorname{Hess} f_a,h_1'\rangle=0\,,\\
&\langle(df_a,df_a),f_a'\rangle=0\,,\\
&\langle(h_1,\operatorname{Hess} f_a),f_a'\rangle=\langle(\operatorname{Hess} f,\operatorname{Hess} f_a),f_a'\rangle-\varepsilon^2\langle f\cdot f_a,f_a'\rangle=0\,,
\end{aligned}$$

which implies that $\psi(h,h)=0$ for $h\in W_1$.

Theorem 6.2. *Let (M,g) be a compact simply connected symmetric Einstein* *manifold. If all irreducible factors of M are neither $SU(p+q)/S(U(p)\times U(q))$ $(p+q\geqq 3)$, $SU(l)$ $(l\geqq 3)$, $SU(l)/SO(l)$ $(l\geqq 3)$ nor $SU(2l)/Sp(l)$ $(l\geqq 3)$, then all essential Einstein i-deformations are integrable up to second order.*

Now, we treat the case where $M_1=P^l(\boldsymbol{C})$ $(l\geqq 2)$. For $f,f'\in F$, we decompose $f\cdot f'$ into eigenfunctions of Δ and denote by $\psi(f,f')$ the F-component. The map ψ becomes a real $SU(l+1)$-homomorphism: $S^2(F)\to F$.

Lemma 6.3. *Let ψ and F be as above. Then $\psi\neq 0$. Moreover, if l is even, $\psi(f,f)\neq 0$ for all non-zero $f\in F$.*

Proof. Let $S^{2l+1}\subset \boldsymbol{C}^{l+1}$ be the unit sphere. Then $U(1)=\{w\in\boldsymbol{C};|w|=1\}$ acts on S^{2l+1} and $\boldsymbol{C}^{l+1}$ by $w(z)=w\cdot z$ and $S^{2l+1}/U(1)$ becomes the projective space $P^l(\boldsymbol{C})$. The spectrum of Δ on $C^\infty(P^l(\boldsymbol{C}))$ is given by $\{2m(l+m)\varepsilon/(l+1);\ m\in\boldsymbol{Z}, m\geqq 0\}$. Denote by F^m the eigenspace with eigenvalue $2m(l+m)\varepsilon/(l+1)$ and $H^m(\boldsymbol{C}^{l+1})$ the space of all homogeneous harmonic polynomials of degree $2m$ on $\boldsymbol{C}^{l+1}$ which are invariant under the action of $U(1)$. If $f\in F^m$, then f is extended canonically to an element $\tilde{f}\in H^m(\boldsymbol{C}^{l+1})$. This correspondence $\tilde{}$ is an $SU(l+1)$-isomorphism (cf. Berger, Gauduchon and Mazet [3; pp. 172–173]). Let $f\in F$. Since F is isomorphic to $\mathfrak{su}(l+1)$ as a real $SU(l+1)$-module, we may assume that f is an element of the subspace of F which corresponds to a Cartan subalgebra of $\mathfrak{su}(l+1)$. That is,

$$\tilde{f}(z)=\sum_{i=1}^{l+1}a_i|z^i|^2;\ a_i\in\boldsymbol{R},\ \sum_{i=1}^{l+1}a_i=0\,.$$

Set $\Delta'=\Delta/4$ on $\boldsymbol{C}^{l+1}$. Then $\Delta'=\sum_{i=1}^{l+1}\partial^2/\partial z^i\partial\bar{z}^i$.

$$\begin{aligned}\Delta'\tilde{f}^2&=\Delta'\sum_i a_i^2|z^i|^4+\Delta'\sum_{i\neq j}a_ia_j|z^i|^2|z^j|^2\\&=4\sum_i a_i^2|z^i|^2+2\sum_{i\neq j}a_ia_j|z^i|^2\\&=2\sum_i a_i^2|z^i|^2,\end{aligned}$$

and,

$$\begin{aligned}\Delta'(\sum_i b_i|z^i|^2\cdot\sum_j|z^j|^2)&=\Delta'\sum_i b_i|z^i|^4+\Delta'\sum_{i\neq j}b_i|z^i|^2|z^j|^2\\&=4\sum_i b_i|z^i|^2+\sum_{i\neq j}b_i(|z^i|^2+|z^j|^2)\\&=\sum_i((l+3)b_i+\sum_j b_j)|z^i|^2\,.\end{aligned}$$

Therefore,

$$\tilde{f}^2-\frac{1}{l+3}\sum_i(2a_i^2-\frac{1}{l+2}\sum_k a_k^2)|z^i|^2\cdot\sum_j|z^j|^2\in H^2(\boldsymbol{C}^{l+1})\,,$$

and
$$\psi(f,f)=\frac{2}{l+3}\sum_i(a_i^2-\frac{1}{l+1}\sum_k a_k^2)|z^i|^2\,.$$

Thus $\psi(f,f)=0$ if and only if $|a_i|$ is independent of i. Q.E.D.

Lemma 6.4. *Let ψ' be any real $SU(l+1)$-homomorphism: $S^2(F)\to F$. If $\langle\psi'(f,f), f\rangle = c\langle\psi(f,f), f\rangle$ for all $f\in F$, then $\psi'=c\psi$.*

Proof. That is easy to see by Lemma 5.4 and the fact that F is isomorphic to $\mathfrak{su}(l+1)$ as real $SU(l+1)$-module. Q.E.D.

Lemma 6.5. *The Lichnerowicz operator Δ commutes with the covariant derivative ∇ on a locally symmetric space.*

Proof. The operators Δ and ∇ may be regarded as the Casimir operator (Lemma 5.6) and a G-homomorphism, respectively. Q.E.D.

Denote by $D^p f$ the p-tensor field defined by

$$(D^p f)_{i_1\cdots i_p} = \nabla_{i_1}\cdots\nabla_{i_p} f\,.$$

Lemma 6.6. *Let N be a locally symmetric Einstein manifold with Einstein constant ε. If $f\in C^\infty(N)$ satisfies $\Delta f = 2\varepsilon f$, then*

$$\langle D^{p+1}f, df\otimes D^p f\rangle = \varepsilon\langle (D^p f, D^p f), f\rangle\,, \tag{6.6.1}$$

$$\begin{aligned}&\langle (D^{p+1}f, D^{p+1}f), f\rangle \\ &= (1-p)\varepsilon\langle (D^p f, D^p f), f\rangle - 2\langle (LD^p f, D^p f), f\rangle\,.\end{aligned} \tag{6.6.2}$$

Proof.

[6.6.1]

$$\begin{aligned}&\langle \nabla_i\nabla_{i_1}\cdots\nabla_{i_{1p}}f, \nabla_i f\cdot\nabla_{i_1}\cdots\nabla_{i_p}f\rangle \\ &= -\langle \nabla_{i_1}\cdots\nabla_{i_p}f, \nabla^i\nabla_i f\cdot\nabla_{i_1}\cdots\nabla_{i_p}f + \nabla_i f\cdot\nabla^i\nabla_{i_1}\cdots\nabla_{i_p}f\rangle \\ &= \langle D^p f, \Delta f\cdot D^p f\rangle - \langle df\otimes D^p f, D^{p+1}f\rangle\,.\end{aligned}$$

[6.6.2]

$$\begin{aligned}&\langle \nabla_i\nabla_{i_1}\cdots\nabla_{i_p}f, f\cdot\nabla_i\nabla_{i_1}\cdots\nabla_{i_p}f\rangle \\ &= -\langle \nabla_{i_1}\cdots\nabla_{i_p}f, \nabla^i f\cdot\nabla_i\nabla_{i_1}\cdots\nabla_{i_p}f + f\cdot\nabla^i\nabla_i\nabla_{i_1}\cdots\nabla_{i_p}f\rangle \\ &= -\langle df\otimes D^p f, D^{p+1}f\rangle + \langle D^p f, f\bar{\Delta} D^p f\rangle\,,\end{aligned}$$

and so

$$\begin{aligned}&\langle (D^{p+1}f, D^{p+1}f), f\rangle \\ &= -\varepsilon\langle (D^p f, D^p f), f\rangle + \langle fD^p f, (\Delta - 2L - pQ)D^p f\rangle && (6.6.1) \\ &= -\varepsilon\langle (D^p f, D^p f), f\rangle + \langle fD^p f, D^p\Delta f\rangle && (6.5) \\ &\quad -2\langle fD^p f, LD^p f\rangle - p\varepsilon\langle fD^p f, D^p f\rangle\,.\end{aligned}$$

Q.E.D.

Lemma 6.7. *If $f\in C^\infty(P^l(\boldsymbol{C}))$ satisfies $\Delta f = 2\varepsilon f$, then*

$$L\,\mathrm{Hess}\,f = -c(\mathrm{Hess}\,f - \varepsilon f\cdot g)\,, \tag{6.7.1}$$

$$R_{is\ t}^{\ \ k}R_j^{\ tls}\nabla_k\nabla_l f = 2c^2(\mathrm{Hess}\,f - \varepsilon f\cdot g)_{ij}\,, \tag{6.7.2}$$

where $2c$ is the holomorphic sectional curvature.

Proof. Denote by z^{α}, z^{β} etc. holomorphic coordinate functions. Since ∇f is a holomorphic vector field, $\nabla_{\alpha}\nabla_{\beta}f=0$. We know that the curvature tensor has the form

$$R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta} = c(\delta_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta}+\delta_{\alpha}{}^{\delta}\delta_{\beta}{}^{\gamma})$$

(cf. Calabi and Vesentini [5; (3.5)]). Therefore,

$$[6.7.1] \qquad \begin{aligned} (L\ \mathrm{Hess}\, f)_{\alpha\beta} &= R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}\nabla_{\gamma}\nabla_{\delta}f = 0\,, \\ (L\ \mathrm{Hess}\, f)^{\bar\alpha}{}_{\beta} &= R^{\bar\alpha}{}_{\gamma\beta}{}^{\delta}\nabla^{\gamma}\nabla_{\delta}f = -R_{\gamma}{}^{\bar\alpha}{}_{\beta}{}^{\delta}\nabla^{\gamma}\nabla_{\delta}f \\ &= -c(\delta_{\gamma}{}^{\bar\alpha}\delta_{\beta}{}^{\delta}+\delta_{\gamma}{}^{\delta}\delta_{\beta}{}^{\bar\alpha})\nabla^{\gamma}\nabla_{\delta}f \\ &= -c(\nabla^{\bar\alpha}\nabla_{\beta}f+\nabla^{\gamma}\nabla_{\gamma}f\cdot\delta_{\beta}{}^{\bar\alpha}) \\ &= -c(\mathrm{Hess}\, f+\varepsilon f\cdot g)^{\bar\alpha}{}_{\beta}\,. \end{aligned}$$

And if we set $\phi_{ij}=R_{is}{}^{k}{}_{t}R_{j}{}^{tls}\nabla_{k}\nabla_{l}f$, then

$$[6.7.2] \qquad \begin{aligned} \phi_{\alpha\beta} &= R_{\alpha\bar\varepsilon}{}^{\gamma}{}_{\zeta}R_{\beta}{}^{\zeta\delta\bar\varepsilon}\nabla_{\gamma}\nabla_{\delta}f = 0\,, \\ \phi^{\bar\alpha}{}_{\beta} &= R^{\bar\alpha}{}_{\varepsilon}{}^{\gamma}{}_{\zeta}R_{\beta}{}^{\zeta\bar\delta\varepsilon}\nabla_{\gamma}\nabla_{\bar\delta}f = R_{\varepsilon}{}^{\bar\alpha}{}_{\zeta}{}^{\gamma}R_{\beta}{}^{\zeta}{}_{\delta}{}^{\varepsilon}\nabla_{\gamma}\nabla^{\delta}f \\ &= c^2(\delta_{\varepsilon}{}^{\bar\alpha}\delta_{\zeta}{}^{\gamma}+\delta_{\varepsilon}{}^{\gamma}\delta_{\zeta}{}^{\bar\alpha})\,(\delta_{\beta}{}^{\zeta}\delta_{\delta}{}^{\varepsilon}+\delta_{\beta}{}^{\varepsilon}\delta_{\delta}{}^{\zeta})\nabla_{\gamma}\nabla^{\delta}f \\ &= 2c^2(\delta_{\delta}{}^{\bar\alpha}\delta_{\beta}{}^{\gamma}+\delta_{\delta}{}^{\gamma}\delta_{\beta}{}^{\bar\alpha})\nabla_{\gamma}\nabla^{\delta}f \\ &= 2cR_{\delta}{}^{\bar\alpha}{}_{\beta}{}^{\gamma}\nabla_{\gamma}\nabla^{\delta}f \\ &= -2c(L\ \mathrm{Hess}\, f)^{\bar\alpha}{}_{\beta}\,. \end{aligned}$$

Q.E.D.

Lemma 6.8. *Let f and c be as above. Then*

$$(6.8.1) \qquad \langle (df, df), f\rangle = \varepsilon\langle f^2, f\rangle\,,$$

$$(6.8.2) \qquad \langle (\mathrm{Hess}\, f,\ \mathrm{Hess}\, f), f\rangle = 0\,,$$

$$(6.8.3) \qquad \langle \mathrm{Hess}\, f, df\otimes df\rangle = \varepsilon^2\langle f^2, f\rangle\,,$$

$$(6.8.4) \qquad \langle D^3 f, df\otimes \mathrm{Hess}\, f\rangle = 0\,,$$

$$(6.8.5) \qquad \langle (D^3 f, D^3 f), f\rangle = 4c\varepsilon^2\langle f^2, f\rangle\,,$$

$$(6.8.6) \qquad \langle (L\ \mathrm{Hess}\, f,\ \mathrm{Hess}\, f), f\rangle = -2c\varepsilon^2\langle f^2, f\rangle\,,$$

$$(6.8.7) \qquad \langle D^4 f, df\otimes D^3 f\rangle = 4c\varepsilon^3\langle f^2, f\rangle\,,$$

$$(6.8.8) \qquad \langle L\ \mathrm{Hess}\, f, df\otimes df\rangle = 0\,,$$

$$(6.8.9) \qquad \langle D^3 f, df\otimes L\ \mathrm{Hess}\, f\rangle = -2c\varepsilon^3\langle f^2, f\rangle\,,$$

$$(6.8.10) \qquad \langle \nabla_i\nabla_j f\cdot\nabla^j\nabla_k f,\ \nabla_i\nabla_k f\rangle = \varepsilon^3\langle f^2, f\rangle\,,$$

$$(6.8.11) \qquad \bar\Delta df = \varepsilon df\,,$$

$$(6.8.12) \qquad \bar\Delta \mathrm{Hess}\, f = 2c(\mathrm{Hess}\, f-\varepsilon f\cdot g)$$

Proof. Except (6.8.10), that is easy to show by Lemma 6.6 and Lemma 6.7.

[6.8.10] $\langle \nabla_i\nabla_j f\cdot\nabla^j\nabla_k f,\ \nabla_i\nabla_k f\rangle$

$$= -\langle \nabla^i\nabla_i\nabla_j f\cdot\nabla^j\nabla_k f+\nabla_i\nabla_j f\cdot\nabla^i\nabla^j\nabla_k f,\ \nabla_k f\rangle$$

$$= \langle \bar{\Delta} df\otimes df, \text{Hess} f\rangle - \langle \nabla_i\nabla_j\nabla_k f,\ \nabla_k f\cdot\nabla_i\nabla_j f\rangle \qquad (6.8.11)$$

$$= \varepsilon\langle df\otimes df, \text{Hess} f\rangle - \langle R_{ik}{}^l{}_j\nabla_l f+\nabla_k\nabla_i\nabla_j f,\ \nabla_k f\cdot\nabla_i\nabla_j f\rangle \qquad (6.8.3)$$

$$= \varepsilon^3\langle f^2, f\rangle + \langle L\ \text{Hess} f, df\otimes df\rangle - \langle D^3 f, df\otimes \text{Hess} f\rangle \qquad (6.8.8), (6.8.4)$$

$$= \varepsilon^3\langle f^2, f\rangle\,.$$

Q.E.D.

Lemma 6.9. *Let f and c be as above. Then*

$$(6.9.1)\qquad \langle \nabla_i\nabla_j\nabla_k\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle = -2c\varepsilon^3\langle f^2, f\rangle\,,$$

$$(6.9.2)\qquad \langle \nabla_i\nabla_k\nabla_j\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle = -c\varepsilon^3\langle f^2, f\rangle\,.$$

Proof.

[6.9.1] $\langle \nabla_i\nabla_j\nabla_k\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle$

$$= -\langle \nabla_j\nabla_k\nabla_l f,\ \nabla^i\nabla_i\nabla_j f\cdot\nabla_k\nabla_l f+\nabla_i\nabla_j f\cdot\nabla^i\nabla_k\nabla_l f\rangle$$

$$= \langle D^3 f, \bar{\Delta} df\otimes \text{Hess} f\rangle$$
$$+\langle \nabla^i\nabla_j\nabla_k\nabla_l f,\ \nabla_j f\cdot\nabla^i\nabla_k\nabla_l f\rangle + \langle \nabla_j\nabla_k\nabla_l f,\ \nabla_j f\cdot\nabla_i\nabla^i\nabla_k\nabla_l f\rangle \qquad (6.8.11)$$

$$= \varepsilon\langle D^3 f, df\otimes \text{Hess} f\rangle$$
$$+\langle R_{ij}{}^m{}_k\nabla_m\nabla_l f+R_{ij}{}^m{}_l\nabla_k\nabla_m f+\nabla_j\nabla_i\nabla_k\nabla_l f,\ \nabla_j f\cdot\nabla_i\nabla_k\nabla_l f\rangle$$
$$-\langle D^3 f, df\otimes\bar{\Delta}\,\text{Hess} f\rangle \qquad (6.8.4), (6.8.12)$$

$$= 2\langle R_{ij}{}^m{}_k\nabla_m\nabla_l f,\ \nabla_j f\cdot\nabla_i\nabla_k\nabla_l f\rangle + \langle D^4 f, df\otimes D^3 f\rangle$$
$$-2c\langle D^3 f, df\otimes(\text{Hess} f-\varepsilon f\cdot g)\rangle \qquad (6.8.7), (6.8.4)$$

$$= 2\langle R_{ij}{}^m{}_k\nabla^i\nabla^l\nabla^k f,\ \nabla_j f\cdot\nabla^m\nabla^l f\rangle + 4c\varepsilon^3\langle f^2, f\rangle$$
$$-2c\varepsilon\langle d\Delta f, f\cdot df\rangle \qquad (6.8.1)$$

$$= 2\langle R^i{}_{jm}{}^k(R_{il}{}^s{}_k\nabla_s f+\nabla_l\nabla_i\nabla_k f),\ \nabla_j f\cdot\nabla_m\nabla_l f\rangle$$

$$= 2\langle R^i{}_j{}^{mk}R_i{}^l{}_{sk}\nabla_m\nabla_l f,\ \nabla_s f\cdot\nabla_j f\rangle$$
$$+2\langle \nabla_l(R^i{}_{jm}{}^k\nabla_i\nabla_k f),\ \nabla_j f\cdot\nabla_m\nabla_l f\rangle$$

$$= 2\langle R_{ji}{}^m{}_k R_s{}^{kli}\nabla_m\nabla_l f,\ \nabla_j f\cdot\nabla_s f\rangle - 2\langle \nabla_l(L\ \text{Hess} f)_{jm},\ \nabla_j f\cdot\nabla_m\nabla_l f\rangle \qquad (6.7.2), (6.7.1)$$

$$= 4c^2\langle \text{Hess} f-\varepsilon f\cdot g,\ df\otimes df\rangle$$
$$+2c\langle \nabla_l(\text{Hess} f-\varepsilon f\cdot g)_{jm},\ \nabla_j f\cdot\nabla_m\nabla_l f\rangle \qquad (6.8.3)$$

$$= 4c^2\varepsilon^2\langle f^2, f\rangle - 4c^2\varepsilon\langle f, (df, df)\rangle$$
$$+2c\langle \nabla_l\nabla_j\nabla_m f,\ \nabla_j f\cdot\nabla_l\nabla_m f\rangle - 2c\varepsilon\langle df\otimes df, \text{Hess} f\rangle \qquad (6.8.1), (6.8.3)$$

$$= 2c\langle R_{lj}{}^k{}_m\nabla_k f+\nabla_j\nabla_l\nabla_m f,\ \nabla_j f\cdot\nabla_l\nabla_m f\rangle - 2c\varepsilon^3\langle f^2, f\rangle$$

$$= -2c\langle df\otimes df, L\,\mathrm{Hess}\,f\rangle + 2c\langle D^3 f, df\otimes \mathrm{Hess}\,f\rangle - 2c\varepsilon^3\langle f^2, f\rangle \qquad (6.8.4)$$
$$= 2c^2\langle df\otimes df, \mathrm{Hess}\,f - \varepsilon f\cdot g\rangle - 2c\varepsilon^2\langle f^2, f\rangle \qquad (6.8.3)$$
$$= 2c^2\varepsilon^2\langle f^2, f\rangle - 2c^2\varepsilon\langle (df, df), f\rangle - 2c\varepsilon^3\langle f^2, f\rangle \qquad (6.8.1)$$
$$= -2c\varepsilon^3\langle f^2, f\rangle\,.$$

[6.9.2] $\langle \nabla_i\nabla_k\nabla_j\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle$

$$= \langle \nabla_i(R_{kj}{}^m{}_l\nabla_m f + \nabla_j\nabla_k\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle$$
$$= \langle R_{kj}{}^m{}_l\nabla_i\nabla_m f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle + \langle \nabla_i\nabla_j\nabla_k\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle \qquad (6.9.1)$$
$$= -\langle \nabla_i\nabla_m f\cdot\nabla^i\nabla_j f,\ (L\,\mathrm{Hess}\,f)_{mj}\rangle - 2c\varepsilon^3\langle f^2, f\rangle \qquad (6.7.1)$$
$$= c\langle \nabla_i\nabla_m f\cdot\nabla^i\nabla_j f,\ \nabla_m\nabla_j f - \varepsilon f\cdot g_{mj}\rangle - 2c\varepsilon^3\langle f^2, f\rangle \qquad (6.8.10)$$
$$= c\varepsilon^3\langle f^2, f\rangle - c\varepsilon\langle (\mathrm{Hess}\,f, \mathrm{Hess}\,f), f\rangle - 2c\varepsilon^3\langle f^2, f\rangle \qquad (6.8.2)$$
$$= -c\varepsilon^3\langle f^2, f\rangle\,.$$

Q.E.D.

Now, we come back to our symmetric space (M,g) where $M_1=P^l(\boldsymbol{C})$ $(l\geqq 2)$ (below Theorem 6.2). We assume that $N\geqq 2$. Set $h=\psi+\phi$; $\psi=\mathrm{Hess}\,f+\varepsilon f\cdot g_1$, $\phi=\varepsilon f'\cdot g_2$, where $f, f'\in F$. Remark that $\delta\psi=0$. In the following calculation, we use Lemma 6.4, Lemma 6.8 and Lemma 6.9. If tr $h=0$, then $h\in \mathrm{EID}(M)$ and

$$2\langle E''(h,h), h\rangle \qquad (4.3)$$
$$= 2\varepsilon\langle h_{ij}, h_i{}^k h_{kj}\rangle + 3\langle \nabla_i\nabla_j h_{kl}, h_{ij}h_{kl}\rangle - 6\langle \nabla_i\nabla_k h_{jl}, h_{ij}h_{kl}\rangle$$
$$= 2\varepsilon\langle \psi_{ij}, \psi_i{}^k\psi_{kj}\rangle + 2\varepsilon\langle \phi_{ij}, \phi_i{}^k\phi_{kj}\rangle + 3\langle \nabla_i\nabla_j\psi_{kl}, \psi_{ij}\psi_{kl}\rangle$$
$$+3\langle \nabla_i\nabla_j\phi_{kl}, \psi_{ij}\phi_{kl}\rangle - 6\langle \nabla_i\nabla_k\psi_{jl}, \psi_{ij}\psi_{kl}\rangle\,.$$

Here, $\langle \psi_{ij}, \psi_i{}^k\psi_{kj}\rangle$

$$= \langle \nabla_i\nabla_j f,\ \nabla_i\nabla^k f\cdot\nabla_k\nabla_j f\rangle + 3\varepsilon\langle \nabla_i\nabla_j f, f\cdot\nabla_i\nabla_j f\rangle$$
$$+3\varepsilon^2\langle \nabla_i\nabla_j f, f^2\cdot(g_1)_{ij}\rangle + \varepsilon^3\langle f\cdot(g_1)_{ij}, f^2\cdot(g_1)_{ij}\rangle \qquad (6.8.10),\ (6.8.2)$$
$$= \varepsilon^3\langle f^2, f\rangle - 6\varepsilon^3\langle f^2, f\rangle + n_1\varepsilon^3\langle f^2, f\rangle$$
$$= (n_1-5)\varepsilon^3\langle f^2, f\rangle\,,$$
$$\langle \phi_{ij}, \phi_i{}^k\phi_{kj}\rangle = \varepsilon^3\langle f'\cdot(g_2)_{ij}, (f')^2\cdot(g_2)_{ij}\rangle$$
$$= n_2\varepsilon^3\langle (f')^2, f'\rangle\,,$$
$$\langle \nabla_i\nabla_j\psi_{kl}, \psi_{ij}\psi_{kl}\rangle$$
$$= \langle \nabla_i\nabla_j\psi_{kl},\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle + \varepsilon\langle \nabla_i\nabla_j\psi_{kl},\ \nabla_i\nabla_j f\cdot f\cdot(g_1)_{kl}\rangle$$
$$+\varepsilon\langle \nabla_i\nabla_j\psi_{kl}, f\cdot(g_1)_{ij}\nabla_k\nabla_l f\rangle + \varepsilon^2\langle \nabla_i\nabla_j\psi_{kl}, f^2\cdot(g_1)_{ij}(g_1)_{kl}\rangle$$
$$= \langle \nabla_i\nabla_j\nabla_k\nabla_l f,\ \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle + \varepsilon\langle \nabla_i\nabla_j f,\ \nabla_i\nabla_j f\cdot\nabla^k\nabla_k f\rangle$$
$$+\varepsilon\langle \nabla_i\nabla_j\psi^k{}_k, f\cdot\nabla_i\nabla_j f\rangle - \varepsilon\langle \bar\Delta\psi, f\cdot\mathrm{Hess}\,f\rangle - \varepsilon^2\langle \Delta\psi^k{}_k, f^2\rangle \qquad (6.9.1)$$
$$= -2c\varepsilon^3\langle f^2, f\rangle - 2\varepsilon^3\langle \mathrm{Hess}\,f, f\cdot\mathrm{Hess}\,f\rangle$$
$$+(n_1-2)\varepsilon^2\langle \mathrm{Hess}\,f, f\cdot\mathrm{Hess}\,f\rangle$$

$$\begin{aligned}
&-\varepsilon\langle\bar{\Delta}\,\mathrm{Hess}\,f+\varepsilon\Delta f\cdot g_1, f\cdot\mathrm{Hess}\,f\rangle-2(n_1-2)\varepsilon^4\langle f^2,f\rangle && (6.8.2), (6.8.12)\\
&= -2c\varepsilon^3\langle f^2,f\rangle-2c\varepsilon\langle\mathrm{Hess}\,f-\varepsilon f\cdot g_1, f\cdot\mathrm{Hess}\,f\rangle\\
&\quad +4\varepsilon^4\langle f,f^2\rangle-2(n_1-2)\varepsilon^4\langle f^2,f\rangle && (6.8.2)\\
&= -6c\varepsilon^3\langle f^2,f\rangle-2(n_1-4)\varepsilon^4\langle f^2,f\rangle\,,\\
&\quad \langle\nabla_i\nabla_j\phi_{kl}, \psi_{ij}\phi_{kl}\rangle=\varepsilon^2\langle\nabla_i\nabla_j f'\cdot(g_2)_{kl}, \psi_{ij}\cdot f'\cdot(g_2)_{kl}\rangle\\
&= n_2\varepsilon^2\langle f'\cdot\mathrm{Hess}\,f', \mathrm{Hess}\,f+\varepsilon f\cdot g_1\rangle && (6.8.2)\\
&= -2n_2\varepsilon^4\langle (f')^2,f\rangle\,,\\
&\quad \langle\nabla_i\nabla_k\psi_{jl}, \psi_{ij}\psi_{kl}\rangle\\
&= \langle\nabla_i\nabla_k\psi_{jl}, \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle+\varepsilon\langle\nabla_i\nabla_k\psi_{jl}, f\cdot(g_1)_{ij}\nabla_k\nabla_l f\rangle\\
&= \langle\nabla_i\nabla_k\nabla_j\nabla_l f, \nabla_i\nabla_j f\cdot\nabla_k\nabla_l f\rangle+\varepsilon\langle\nabla_i\nabla_k f, \nabla_i\nabla^l f\cdot\nabla_k\nabla_l f\rangle\\
&\quad +\varepsilon\langle\nabla^j\nabla_k\nabla_j\nabla_l f, f\cdot\nabla_k\nabla_l f\rangle+\varepsilon^2\langle\nabla^j\nabla_k f\cdot(g_1)_{jl}, f\cdot\nabla_k\nabla_l f\rangle && (6.9.2), (6.8.10)\\
&= -c\varepsilon^3\langle f^2,f\rangle+\varepsilon^4\langle f^2,f\rangle\\
&\quad +\varepsilon\langle R^j{}_k{}^m{}_j\nabla_m\nabla_l f+R^j{}_k{}^m{}_l\nabla_j\nabla_m f+\nabla_k\nabla^j\nabla_j\nabla_l f, f\cdot\nabla_k\nabla_l f\rangle\\
&\quad +\varepsilon^2\langle(\mathrm{Hess}\,f, \mathrm{Hess}\,f), f\rangle && (6.8.2)\\
&= -c\varepsilon^3\langle f^2,f\rangle+\varepsilon^4\langle f^2,f\rangle+\varepsilon^2\langle(\mathrm{Hess}\,f, \mathrm{Hess}\,f), f\rangle\\
&\quad +\varepsilon\langle(L\ \mathrm{Hess}\,f, \mathrm{Hess}\,f), f\rangle-\varepsilon\langle\nabla\bar{\Delta}df, f\cdot\mathrm{Hess}\,f\rangle && (6.8.6),\\
&= -c\varepsilon^3\langle f^2,f\rangle+\varepsilon^4\langle f^2,f\rangle-2c\varepsilon^3\langle f^2,f\rangle-2\varepsilon^2\langle\mathrm{Hess}\,f, f\cdot\mathrm{Hess}\,f\rangle && (6.8.11)\\
&= -3c\varepsilon^3\langle f^2,f\rangle+\varepsilon^4\langle f^2,f\rangle\,. && (6.8.2)
\end{aligned}$$

Thus, $\langle E''(h,h), h\rangle$

$$= -2(n_1-2)\varepsilon^4\langle f^2,f\rangle-3n_2\varepsilon^4\langle (f')^2,f\rangle+n_2\varepsilon^4\langle (f')^2,f'\rangle\,.$$

Since $f'=-((n_1-2)/n_2)f$, we have

$$\langle E''(h,h), h\rangle=-\frac{(n_1-2)\,(n_1+n_2-2)\,(n_1+2n_2-2)}{n_2^2}\cdot\varepsilon^4\langle f^2,f\rangle\,.$$

Therefore, by Lemma 6.4, we get

Lemma 6.10. *Let h be as above and h'' have the same form defined by f''. Then $\langle E''(h,h),h''\rangle=r\cdot\langle f^2,f''\rangle$, where r is a non-zero constant.*

Theorem 6.11. *Let $P^l(C)\times M'$ ($l\geqq 2$) be a symmetric Einstein manifold. Then there exists an essential Einstein i-deformation which is not integrable.*

Proof. That is easy to see by Proposition 4.5, Lemma 4.7, Lemma 6.3 and Lemma 6.10. Q.E.D.

Moreover, we have the following

Theorem 6.12. *There exist rigid Einstein metrics which are infinitesimally deformable.*

Proof. For example, let M be $P^{2l}(\boldsymbol{C})\times S^2$. Then, by Theorem 5.7, all elements $h\in\mathrm{EID}(M)$ have the form introduced above Lemma 6.10. Thus Proposition 4.6, Lemma 6.3 and Lemma 6.10 complete the proof. Q.E.D.

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