Dupont, J. L.
Osaka J. Math.
19 (1982), 599-641

ALGEBRA OF POLYTOPES AND HOMOLOGY OF FLAG COMPLEXES

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(Received June 26, 1980)

1. Introduction

It has been known for some time that the socalled Hilbert's 3rd problem which deals with equidecomposability of polytypes ("scissors congruence") is related to homological algebra (see e.g. Jessen-Karpf-Thorup [14]). Recently C.H. Sah in his book [19] has demonstrated the close relationship between this problem and the group cohomology of the relevant Lie groups considered as discrete groups. The present approach carries this idea further and is actually dual to Sah's. It starts out from the observation that for any subgroup G of the group $E(n)$ of isometries of Euclidean space $\boldsymbol{R}^n$, the scissors congruence group $\mathcal{P}(\boldsymbol{R}^n, G)$ is a homology group to begin with (Theorem 2.3). (Although much of what follows is valid for more general fields, we shall throughout restrict to the ground field being $\boldsymbol{R}$.) For $G=\mathrm{T}(n)$, the group of translations, Jessen and Thorup [15] (see also Sah [19, chapter 4]) has shown that $\mathcal{P}(\boldsymbol{R}^n, \mathrm{T}(n))$ is separated by the Hadwiger invariants, and using their result we show (Section 3):

Theorem 1.1. *Let $\mathcal{T}(\boldsymbol{R}^n)$ be the Tits complex of flags of proper linear subspaces of $\boldsymbol{R}^n$. Let $\mathfrak{g}$ be the local coefficient system given by $\mathfrak{g}_{U_0 \supseteqq \cdots \supseteqq U_i} = U_i$, and put*

$$\mathcal{D}^q(\boldsymbol{R}^n) = \tilde{H}_{n-q-1}(\mathcal{T}(\boldsymbol{R}^n), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}))\,, \qquad q = 1, 2, \cdots, n\,.$$

Then there is a natural isomorphism

$$h\colon \mathcal{P}(\boldsymbol{R}^n, \mathrm{T}(n)) \xrightarrow{\cong} \bigoplus_{s=1}^{n} \mathcal{D}^q(\boldsymbol{R}^n)\,.$$

Furthermore h is equivariant with respect to the natural action of $\mathrm{E}(n)/\mathrm{T}(n)=\mathrm{O}(n)$ *on the left and the action on the right induced by* $\det(g)\Lambda^q \in (g)$ *for* $g \in \mathrm{O}(n)$.

Here $\tilde{H}$ denotes the homology of the chain-complex augmented to $\Lambda^q_{\boldsymbol{R}}(\boldsymbol{R}^n)$ in degree -1, so in particular $\mathcal{D}^n(\boldsymbol{R}^n)=\Lambda^n_{\boldsymbol{R}}(\boldsymbol{R}^n)$ corresponding to the Hadwiger invariant given by the volume. In general

$$\mathcal{D}^q(\boldsymbol{R}^n) \subseteq \coprod_{U_0\supset\cdots\supset U_{n-q-1}} \Lambda^q_{\boldsymbol{R}}(U_{n-q-1}),$$

where the summation is taken over all *strict* flags (i.e. codim $U_i = i+1$), and the injectivity of h is just the theorem of Jessen and Thorup. The surjectivity expresses the precise relations between the Hadwiger invariants (Proposition 3.16 below; see also Sah [19, chapter 5]).

The problem of determining $\mathcal{P}(\boldsymbol{R}^n) = \mathcal{P}(\boldsymbol{R}, \mathrm{E}(n))$ is thus a matter of computing the 0-th homology groups of the discrete group $\mathrm{O}(n)$ with coefficients in the modules $\mathcal{D}^q(\boldsymbol{R}^n)^t$ (where the index ^{t} indicates that we have twisted the natural action by the determinant). For this we use (in Section 4) some exact sequences due to Lusztig [16] to set up a spectral sequence converging to zero and with $H_0(\mathrm{O}(n), \mathcal{D}^q(\boldsymbol{R}^n)^t)$ in the E^1-term. The differentials are thus separating invariants and some of them can be interpreted as generalizations of the classical Dehn invariant. In fact

Corollary 1.2. a) *There is an exact sequence*

$$0 \to H_2(\mathrm{SO}(3), \boldsymbol{R}^3) \to \mathcal{P}(\boldsymbol{R}^3)/\mathcal{Z}_2(\boldsymbol{R}^3) \xrightarrow{D} \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} (\boldsymbol{R}/\boldsymbol{Z}) \to H_1(\mathrm{SO}(3), \boldsymbol{R}^3) \to 0$$

where SO(3) *acts in the natural way on* $\boldsymbol{R}^3$, *D is the Dehn invariant, and* $\mathcal{Z}_2(\boldsymbol{R}^3)$ *is the subgroup generated by all prisms.*

b) *In particular* $H_2(\mathrm{SO}(3), \boldsymbol{R}^3) = 0$ *by Sydler's theorem and* $H_1(\mathrm{SO}(3), \boldsymbol{R}^3)$ *is a vector space of dimension as the continuum.*

For the definition of D and Sydler's theorem see e.g. Jessen [12]. Apart from the identification of D, the above exact sequence is dual to Sah [19, chapter 5, proposition 7.5]. It would be nice to have a direct algebraic proof of b) thus proving Sydler's theorem via a).

In the case of spherical geometry which we consider in Section 5, the Steinberg module $\mathrm{St}(\boldsymbol{R}^n) = \tilde{H}_{n-2}(\mathcal{T}(\boldsymbol{R}^n), \boldsymbol{Z})$ plays a role analogous to $\mathcal{D}^q$ above. The analogue of Corollary 1.2 for $\mathcal{P}(\mathrm{S}^3)$ reads

Theorem 1.3. a) *There is an exact sequence*

$$0 \to A \to H_3(\mathrm{SU}(2), \boldsymbol{Z}) \to \mathcal{P}(\mathrm{S}^3)/\boldsymbol{Z} \xrightarrow{D} \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} (\boldsymbol{R}/\boldsymbol{Z}) \to H_2(\mathrm{SU}(2), \boldsymbol{Z}) \to 0.$$

Here D is the Dehn invariant, $\boldsymbol{Z} \subseteq \mathcal{P}(\mathrm{S}^3)$ *is generated by the class of the total sphere* $[\mathrm{S}^3]$ *and A is an abelian group satisfying* $2^N A = 0$ *for some integer N.*

b) *Let* $u\colon \mathcal{P}(S^3) \to \boldsymbol{R}$ *be given by* $u[P] = \mathrm{Vol}(P)/\mathrm{Vol}(S^3)$ *where* Vol *is the volume. Then the composite map*

$$H_3(\mathrm{SU}(2), \boldsymbol{Z}) \to \mathcal{P}(\mathrm{S}^3)/\boldsymbol{Z} \xrightarrow{u} \boldsymbol{R}/\boldsymbol{Z}$$

is the evaluation of the Cheeger-Simons class $\hat{C}_2 \in H^3(\mathrm{SU}(2), \boldsymbol{R}/\boldsymbol{Z})$.

For the definition of the Cheeger-Simons class, see Cheeger-Simons [6]. According to Cheeger [5], $H_3(\mathrm{SU}(2), \boldsymbol{Z})$ has infinite rank but it is not even known if $\hat{C}_2$ evaluated on a homology class can be irrational. In any case it follows from Theorem 1.3 that Vol and the Dehn invariant are separating invariants for $\mathcal{P}(\mathrm{S}^3)$ iff the kernel of $\hat{C}_2$: $H_3(\mathrm{SU}(2), \boldsymbol{Z}) \to \boldsymbol{R}/\boldsymbol{Z}$ is a 2-torsion group of finite exponent (see also Jessen [11]).

Similar remarks hold for the hyperbolic space $\mathcal{H}^3$ where we obtain

Theorem 1.4. *There is an exact sequence*

$$0 \to B \to H_3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \to \mathcal{P}(\mathcal{H}^3) \xrightarrow{D} \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} (\boldsymbol{R}/\boldsymbol{Z}) \to H_2(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \to 0$$

where D is the Dehn invariant, B an abelian group with $2^N B=0$ for some integer N, and $^-$ indicates the -1-eigenspace for the automorphism induced by complex conjugation.

Notice that $H_2(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z}) = K_2(\boldsymbol{C})$ (see Sah-Wagoner [20]) and so in particular the sequence in Theorem 1.4 answers a question raised in Sah [19, chapter 7, p. 148].

Acknowlegement. I would like to thank several people who have contributed to this paper: D. Sullivan for the original inspiration a few years ago, and C.-H. Sah for valuable comments and suggestions during the preparation of this paper. Also his book [19] is a useful general reference for the entire subject.

Most of this paper was written during a stay at Osaka University and I want to thank everybody there and especially Prof. M. Nakaoka for their hospitality and interest in this work. Finally thanks are also due to The Japan Society for the Promotion of Science which supported my visit in Japan.

2. Preliminaries

Algebra of polytopes.

Let X be either the Euclidean space $\boldsymbol{R}^n$, the sphere S^n or the hyperbolic space $\mathcal{H}^n$. A *k-simplex* σ in X is just an ordered set of points $\sigma=(a_0, a_1, \cdots, a_k)$, $a_i \in X$, where in the spherical case we assume the distance between any two of these points to be less than π. The underlying *geometric simplex* $|\sigma|$ is the geodesic convex hull (in the spherical case of diameter $<\pi$) and for $k=n$ $|\sigma|$ is said to be *proper* if it is not contained in a $(n-1)$-dimensional geodesic subspace. A *polytope* $P \subseteq X$ is a finite union $P = \bigcup_i |\sigma_i|$ of proper geometric n-simplices such that any two intersect in a common face of dimension less than n.

Now let G be a subgroup of the group of isometries of X and define $\mathcal{P}(X, G)$ to be the free abelian group generated by all polytopes P modulo the relations

$$(2.1)\qquad \begin{array}{lll} \text{(i)} & [P]=[P']+[P''] & \text{if } P=P'\cup P'' \text{ and } P'\cap P'' \\ & & \text{has no interior points,} \\ \text{(ii)} & [P]=[gP], & g\in G. \end{array}$$

$\mathcal{P}(X, G)$ is called the *polytope group* or *scissors congruence group* for (X, G), and the general problem is to determine the structure of this group.

For any group G and M a (left) G-module the *coinvariants* of G is the group

$$_GM=\boldsymbol{Z}\underset{\boldsymbol{Z}[G]}{\otimes}M=M/I,$$

where I is generated by all elements of the form $gx-x$, with $g\in G$, $x\in M$. With this notation we clearly have by relation (2.1) (ii):

Proposition 2.2. a) *For $H\subseteqq G$ an invariant subgroup*

$$\mathcal{P}(X, G)={}_{G/H}\mathcal{P}(X, H)$$

b) *In particular*

$$\mathcal{P}(X, G)={}_G\mathcal{P}(X, \{1\}).$$

Thus it is no wonder that the calculation of $\mathcal{P}(X, G)$ is related to homology of groups since after all $H_0(G, M)={}_GM$ and the higher homology functors are the derived of this functor in M (Cartan-Eilenberg [4, chapter 5]). However, also the relation (2.1) (i) can be expressed in terms of homology as follows:

The Eilenberg-MacLane chain complex.

Let $C_*(X)$ be the chain complex generated by simplices $\sigma=(a_0, \cdots, a_k)$ as defined above, with the usual boundary homomorphism

$$\partial(a_0, \cdots, a_k)=\sum_{i=0}^{k}(-1)^i(a_0, \cdots, \hat{a}_i, \cdots, a_k)$$

and for any p let $C_*(X)^p$ be the subcomplex generated by all simplices σ lying in geodesic subspaces of dimension at most p. Notice that a group G of isometries of X acts on $C_*(X)$ by

$$g(a_0, \cdots, a_k)=(ga_0, \cdots, ga_k)$$

and that $C_*(X)^p$ is stable under this action. Finally if M is any G-module we let $M^t=\boldsymbol{Z}^t\underset{\boldsymbol{Z}}{\otimes}M$, where $\boldsymbol{Z}^t$ is the integers with the action for $g\in G$ given by multiplication by $+1$ or -1 depending on g being orientation preserving or reversing. With this notation we have:

Theorem 2.3. *Given an orientation of X there are canonical isomorphisms*

$$\begin{aligned}\mathcal{P}(X, G) &\cong H_n({}_G[C_*(X)/C_*(X)^{n-1}]^t)\\ &\cong {}_GH_n(C_*(X)/C_*(X)^{n-1})^t .\end{aligned}$$

Proof. The second isomorphism is obvious from the fact that every n-chain in $C_*(X)/C_*(X)^{n-1}$ is a cycle. By proposition 2.2 it suffices to exhibit an equivariant isomorphism

$$(2.4) \qquad \varphi\colon H_n(C_*(X)/C_*(X)^{n-1}) \to \mathcal{P}(X, \{1\}):$$

Given an orientation of X, φ is simply defined by sending a *proper* n-simplex $\sigma=(a_0, \cdots, a_n)$ to $\varphi(\sigma)=\varepsilon_\sigma|\sigma|$ where $\varepsilon_\sigma=+1$ or -1 according to whether the orientation of X agrees with the ordering of the vertices $(a_0, \cdots, a_n)$ or not. To see that φ vanishes on boundaries is easiest using a topological argument: Let $\Delta^k \subseteq \boldsymbol{R}^{k+1}$ be the standard k-simplex $\Delta^k=(e_0, \cdots, e_k)$; then any k-simplex $\sigma=(a_0, \cdots, a_k)$ gives rise to a continuous map $f\colon \Delta^k \to |\sigma| \subseteq X$ constructed inductively by considering $|\sigma|$ as the geodesic cone on $|(a_1, \cdots, a_k)|$ with top point a_0. In particular for any $(n+1)$-simplex $\sigma=(a_0, \cdots, a_{n+1})$ we get that

$$f\colon \partial\Delta^{n+1} \to \bigcup_{\tau_i \subseteq \partial\sigma} |\tau_i| \subseteq X$$

is a map of degree 0. If $\tau_i=(a_0, \cdots, \hat{a}_i, \cdots, a_{n+1})$ is proper then $\varepsilon_\sigma=(-1)^i$ or $(-1)^{i+1}$ depending on $f \,|\, |(e_0, \cdots, \hat{e}_i, \cdots, e_{n+1})|$ beeing orientation preserving or reversing. Therefore if we subdivide any proper simplex $|\tau_i|$ by the bounding hyperplanes of the other ones, then every piece in the resulting subdivision occur in the sum

$$\sum_{\tau_i \text{ proper}} (-1)^i \varepsilon_{\tau_i} |\tau_i| = \varphi(\partial\sigma)$$

with multiplicity zero. It follows that φ is well defined, and obviously φ is surjective.

Now we want to construct the inverse map

$$(2.5) \qquad \psi\colon \mathcal{P}(X, \{1\}) \to H_n(C_*(X)/C_*(X)^{n-1}) .$$

For this first recall the well-known observation that $\mathcal{P}(X, \{1\})$ is generated by all *convex* polytopes (again in the spherical case of diameter $<\pi$) subject to the relations

$$(2.6) \qquad [P] = [P_1]+[P_2] ,$$

where $P=P_1 \cup P_2$ is a *simple* subdivision, i.e. The convex polytope P is divided into the convex polytopes P_1 and P_2 by a cutting geodesic hyperplane (c.f. Sah [19, chapter 1]). Next it is easy to see (by induction on n) that a convex polytope P can be triangulated as a simplicial complex $P=|K|$ such that all n-simplices in K are proper. Therefore in any such simplex $\sigma=(a_0^\sigma, \cdots, a_n^\sigma)$ we can order the vertices compatible with the given orientation of X and we want to associate with P the chain

$$[K] = \sum_{\sigma\in K_n} \sigma \in C_n(X).$$

To see that $\psi(P)=[K]$ gives a well-defined map in (2.5) we must prove.

Lemma 2.7. *Let $P=|K|=|L|$ be two triangulations of the convex polytope P. Then*

$$[L]-[K] = \partial c+d \quad \textit{with} \quad c\in C_{n+1}(X),\ d\in C_n(X)^{n-1}.$$

By this lemma $\psi(P)$ only depends on P and also ψ is compatible with the relation (2.6). In fact if P is simply subdivided into P_1 and P_2 it is easy to construct a triangulation of P such that P_1 and P_2 are subcomplexes hence clearly $\psi(P)=\psi(P_1)+\psi(P_2)$. Hence ψ in (2.5) is well-defined and is clearly an inverse to φ.

Proof of lemma 2.7. For this we shall again use a topological argument: Let K' be the barycentric subdivision of K. Then it is easy to see that $[K]-[K']$ is homologous to a chain in $C_n(X)^{n-1}$. Therefore by possibly replacing K by a suitable subdivision we can choose a simplicial approximation $f\colon K\to L$ to the identity: $|K|=P=|L|$ (see e.g. Spanier [23, chapter 3 section 4]). That is, f is a simplicial map such that for every vertex a in K the image $f(a)$ is a vertex of the smallest simplex of L containing a. Notice that if U is a supporting geodesic hyperplane for P then both K and L induce triangulations $K\cap U$ and $L\cap U$ of $P\cap U$ and f clearly maps $K\cap U$ to $L\cap U$. In particular f maps the boundary ∂K to the boundary ∂L. Topologically P is a ball so

$$H_n(K, \partial K)\cong H_n(L, \partial L)\cong \boldsymbol{Z},$$

and since $|f|$ is homotopic to the identity it is a map of degree one, so it follows that

$$f_*[K] = [L]+d' \tag{2.8}$$

where d' consists of degenerate simplices (i.e. simplices with repetions among the vertices). On the other hand consider the "mapping cylinder"

$$c(f) = \sum_{\sigma\in K_n} \sum_{i=0}^{n} (-1)^i (a_0^\sigma, \cdots, a_i^\sigma, f(a_i^\sigma), \cdots, f(a_n^\sigma))$$

where $\sigma=(a_0^\sigma, \cdots, a_n^\sigma)$ as above. Then

$$f_*[K]-[K] = \partial c(f)+c(f\,|\,\partial K). \tag{2.9}$$

Now observe that if $\sigma=(a_0^\sigma, \cdots, a_{n-1}^\sigma)\in\partial K$ then $|\sigma|\subseteqq U$ for some supporting geodesic hyperplane U to P and so all simplices of the form

$$(a_0^\sigma, \cdots, a_i^\sigma, f(a_i^\sigma), \cdots, f(a_{n-1}^\sigma))$$

lie in U. Therefore

$$d = c(f\,|\,\partial K)-d'\in C_n(X)^{n-1}$$

and so the lemma follows from (2.8) and (2.9).

REMARK 1. The content of theorem 2.3 has probably been known for a long time. Thus A. Thorup has given a completely combinatorial proof (unpublished) of an essentially equivalent result. See also Sah [19, chapter 2, proposition 2.2 and chapter 1, lemma 2.2] (however Sah has informed me that this last mentioned lemma is probably stated in too wide generality).

REMARK 2. Theorem 2.3 suggests generalizations of $\mathcal{P}(X, G)$ in at least 2 directions:

1) For X any Riemannian manifold it makes sense to consider the complex $C_*(X)$ of "small geodesic simplices" (simply determined by their vertices) and also the filtration $C_*(X)^p$ defined using geodesic subspaces makes sense. For G a group of isometries one should investigate the homological properties of ${}_GC_*(X)$ and the induced filtration. This seems to be of particular interest in the case X is a symmetric space and G the group of isometries (c.f. e.g. Dupont [9])

2) For k any field and $X=k^n$, again the right hand side of the isomorphism in theorem 2.3 makes sense as long as G is a group of affine transformations with determinant ±1.

3. Translation equivalence

In this section we shall prove theorem 1.1, that is, we take $X=\boldsymbol{R}^n$ and $G=\mathrm{T}(n)$ the group of translations of $\boldsymbol{R}^n$. This group is naturally isomorphic to the additive group $\boldsymbol{R}^n$. Let us write $V=\boldsymbol{R}^n$ and $T=\mathrm{T}(n)$ for short and we want to calculate

$$\mathcal{P}(V, T)\cong H_n({}_TC_*(V)/{}_TC_*(V)^{n-1}).$$

First notice

Lemma 3.1. *There is a natural isomorphism*

$$H_q({}_TC_*(V))\cong\Lambda_Z^q(V)$$

Proof. The chain complex $C_*(V)$ is just the usual "bar resolution" for computing homology of the additive group V (see e.g. MacLane [17, chapter IV, § 5]). Hence $H_*({}_TC_*(V))\cong H_*(V, \boldsymbol{Z})$ in a canonical way. Here at least $H_1(V, \boldsymbol{Z})=V$ and we also have a natural map of q factors

$$V\underset{\boldsymbol{Z}}{\otimes}\cdots\underset{\boldsymbol{Z}}{\otimes}V\cong H_1(V)\underset{\boldsymbol{Z}}{\otimes}\cdots\underset{\boldsymbol{Z}}{\otimes}H_1(V)\to H_1(V\times\cdots\times V)\to H_q(V)$$

where the last map is induced by addition of the components. This product is clearly anti-commutative, hence induces a natural map $\psi\colon \Lambda^q_{\boldsymbol{Z}}(V)\to H_q(V)$. Every finitely generated subgroup A of V is a free abelian group and the restriction of ψ to A is just the Kunneth isomorphism. Hence the result follows by writing V as a direct limit of such groups.

REMARK. Notice that the isomorphism in (3.1) can be made explicit using the Eilenberg-Zilber map (see e.g. MacLane [17, chapter VIII, § 8]). Geometrically for $v_1, \cdots, v_q\in V$, $\psi(v_1\wedge\cdots\wedge v_q)$ is represented by a triangulation of the "q-cube" with vertices $\{\delta_1v_1+\cdots+\delta_qv_q\}$, $\delta_i=0$ or 1, $i=1, \cdots, q$.

Flags of linear subspaces and a double complex.

Now consider the category of linear subspaces $U\subset V$ ($U\neq V$) and let $\overline{\mathfrak{T}}(V)$ be the nerve of this category, i.e. the simplicial set where a p-simplex σ is a flag $\sigma=(U_0\supseteqq U_1\supseteqq\cdots\supseteqq U_p)$. Here the face and degeneracy operators are given by

$$\varepsilon_i(\sigma)=(U_0\supseteqq\cdots\supseteqq\hat{U}_i\supseteqq\cdots\supseteqq U_p)$$

and

$$\eta_i(\sigma)=(U_0\supseteqq\cdots\supseteqq U_i\supseteqq U_i\supseteqq\cdots U_p)\,,$$

$i=0, \cdots, p$. Also let $\mathfrak{T}(V)\subseteqq\overline{\mathfrak{T}}(V)$ be the subcomplex (the *Tits complex*) of flags of *proper* linear subspaces, i.e. $U\neq 0$.

For $U\subset V$ let ${}_TC_*(U)$ as before be the coinvariants of the bar-resolution for the group $T(U)$ of translations of U. Then we have a double complex $A_{*,*}=A_{*,*}(V)$:

$$\begin{tikzcd}\cdots \arrow[r, "\partial'"] & A_{p,*} \arrow[r, "\partial'"] & A_{p-1,*} \arrow[r, "\partial'"] & \cdots \arrow[r, "\partial'"] & A_{0,*} \arrow[r, "\partial'"] & A_{-1,*}\end{tikzcd}$$

given by

$$A_{p,*}(V)=\begin{cases}{}_TC_*(V)\,, & p=-1\\ \coprod\limits_{U_0\supseteqq\cdots\supseteqq U_p}{}_TC_*(U_p)\,, & p=0, 1, \cdots.\end{cases}\tag{3.2}$$

Here $\partial'=\sum\limits_{i=0}^{p}(-1)^i(\varepsilon_i)_*$ for $p>0$ and $\partial'\colon\coprod\limits_{U_0\subset V}{}_TC_*(U_0)\to{}_TC^*(V)$ is just induced by the natural inclusions $U_0\subset V$. The "vertical" differential ∂'' in $A_{p,*}$ is given by $(-1)^p$ times the usual boundary map in the complex ${}_TC_*(U_p)$.

Lemma 3.3.

$$H_p(A_{*,q}(V), \partial') = \begin{cases} {}_TC_q(V)/{}_TC_q(V)^{n-1}, & p=-1 \\ 0, & p>-1 \end{cases}$$

Proof. Let α be any affine q-simplex in V which lies on an affine subspace of dimension strictly less than n. If we replace α by a translate then the affine subspace can be assumed to pass through 0, which shows the statement for $p=-1$. In general for α a simplex as above let $U_\alpha \subset V$ be the unique linear subspace of smallest dimension which contain a translate α' of α. Then we define

$$s_p\colon A_{p,q}(V) \to A_{p+1,q}(V), \qquad p=0, 1, \cdots,$$

by

$$s_p(\alpha(U_0 \supseteqq \cdots \supseteqq U_p)) = (-1)^p \alpha'(U_0 \supseteqq \cdots \supseteqq U_p \supseteqq U_\alpha),$$
$$\alpha \text{ a simplex of } U_p.$$

Similarly define s_{-1} on the subgroup ${}_TC_p(V)^{n-1}$ by $s_{-1}(\alpha) = -\alpha'(U_\alpha)$. Then it is easily checked that

$$\partial' \circ s_p + s_{p-1} \circ \partial' = \mathrm{id}, \qquad p=0, 1, \cdots. \tag{3.4}$$

This proves the lemma.

It follows that the spectral sequence for the "second filtration" of the total complex $A_*(V)$ collapses (see e.g. MacLance [17, chapter XI, § 6]) and so we have a natural isomorphism

$$H_k(A_*(V)) \cong H_{k+1}({}_TC_*(V)/{}_TC_*(V)^{n-1}), \qquad k=-1, 0, 1, \cdots. \tag{3.5}$$

For the "first spectral sequence" $(E^r_{p,q}, d_r)$ we have using lemma 3.1 a natural isomorphism

$$E^1_{p,q} \cong \begin{cases} \Lambda^q_Z(V), & p=-1 \\ \coprod_{U_0 \supseteqq \cdots \supseteqq U_p} \Lambda^q_Z(U_p), & p=0, 1, 2, \cdots \end{cases} \tag{3.6}$$

and the differential d^1 beeing induced by ∂' is the obvious differential in the chain complex

$$\to \coprod_{U_0 \supseteqq \cdots \supseteqq U_p} \Lambda^q_Z(U_p) \xrightarrow{d^1} \cdots \xrightarrow{d^1} \coprod_{U_0} \Lambda^q_Z(U_0) \xrightarrow{d^1} \Lambda^q_Z(V) \tag{3.7}$$

for the local coefficient system $\Lambda^q_Z(\mathfrak{g})$ on the complex $\overline{\mathcal{T}}(V)$, where

$$\mathfrak{g}_{U_0 \supseteqq \cdots \supseteqq U_p} = U_p.$$

Notice that (3.7) is augmented to $\Lambda^q_Z(V)$ sitting in degree -1, so

$$E^2_{p,q} = \tilde{H}_p(\mathcal{T}(V), \Lambda^q_{\mathbf{Z}}(\mathfrak{g}))\,, \qquad p = -1, 0, 1, 2, \cdots \tag{3.8}$$

by definition. In particular

$$E^2_{-1,q} = \Lambda^q_{\mathbf{Z}}(V)/[\sum_{U_0\subset V} \Lambda^q_{\mathbf{Z}}(U_0)]\,, \qquad \dim U_0<n\,. \tag{3.9}$$

Now

Lemma 3.10. a) $E^2_{p,0}=0$, $\forall p$.

b) *$E^r_{p,q}$ are vector spaces over $\mathbf{Q}$ for $r=2, 3, \cdots$.*

c) *$d^r=0$ for $r\geqq 2$.*

Proof. a) For $q=0$ the local coefficient system is constant $(=\mathbf{Z})$ but clearly $\overline{\mathcal{T}}(V)$ is contractible so $\tilde{H}_p(\overline{\mathcal{T}}(V), \mathbf{Z})=0$.

b) For $q>0$ allready $E^1_{p,q}$ are $\mathbf{Q}$-vector spaces by (3.6).

c) Choose any positive integer $\lambda>1$ and let $\mu_\lambda: V\to V$ be given by multiplication by λ. Now for $v\in V$ let t_v be the corresponding translation of V, i.e., $t_v(x)=x+v$, $x\in C$. Then clearly $\mu_\lambda\circ t_v=t_{\lambda v}\circ\mu_\lambda$. It follows that μ_λ induces an operation of the spectral sequence which on $E^1_{p,q}$ is given by $\Lambda^q_{\mathbf{Z}}(\mu_\lambda)$, i.e. by multiplication by λ^q. Since

$$d_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

commutes with μ_λ it follows that $d_r=0$ for $r\geqq 2$.

Thus we have proved that $E^\infty_{p,q}=E^2_{p,q}$ and it follows from the proof that $E^2_{p,q}$ can be naturally identified with the λ^q-eigenspace of μ_λ acting on $H_{p+q}(A_*(V))$. We collect these observations in the following

Proposition 3.11. a) *$H({}_TC_*(V)/{}_TC_*(V)^{n-1})$ and in particular $\mathcal{P}(V, T)$ are vector spaces over $\mathbf{Q}$.*

b) *There is a splitting into eigenspaces for the operator μ_λ:*

$$\mathcal{P}(V, T)\cong \bigoplus_{q=1}^{n} \tilde{H}_{n-q-1}(\mathcal{P}(V), \Lambda^q_{\mathbf{Z}}(\mathfrak{g}))$$

c) $\tilde{H}_p(\mathcal{T}(V), \Lambda^q_{\mathbf{Z}}(\mathfrak{g}))=0$ for *for $p<n-q-1$, $q>0$.*

Proof. a) clearly follows from (3.5) and lemma 3.10 b). For b) we have by the above remark

$$\begin{aligned}\mathcal{P}(V, T)\cong H_{n-1}(A_*)&\cong \bigotimes_{q=1}^{n} \tilde{H}_{n-q-1}(\overline{\mathcal{T}}(V), \Lambda^q_{\mathbf{Z}}(\mathfrak{g}))\\ &= \bigoplus_{q=1}^{n} \tilde{H}_{n-q-1}(\mathcal{T}(V), \Lambda^q_{\mathbf{Z}}(\mathfrak{g}))\,,\end{aligned}$$

where the last isomorphism is obvious from the definitions.

c) Since $H_k(A_*)=H_{k+1}({}_TC_*(V/{}_TC_*(V)^{n-1})=0$ for $k<n-1$ we get that $E^2_{p,q}=0$ for $p+q<n-1$ which gives the result.

REMARKS. 1. Notice that the isomorphism in proposition 3.11 b) depends on the given orientation of V as in theorem 2.3.

2. By Jessen and Thorup [15] one can identify the λ^q-eigenspace for the dilatation operator μ_λ with the quotient $\mathcal{Z}_q(V, T)/\mathcal{Z}_{q-1}(V, T)$, where $\mathcal{Z}_q(V, T)\subseteq \mathcal{P}(V, T)$ is generated by all q-cylinders, i.e. polytopes which are products (relative to some splitting $V=V_1\oplus\cdots\oplus V_q$) of lower dimensional polytopes. We shall however not use this geometric interpretation in any essential way.

Homology of the Tits complex and the Hadwiger invariants.

We can now state the main result of this section:

Theorem 3.12. *Let* ρ: $\tilde{H}_*(\mathcal{T}(V), \Lambda^q_{\boldsymbol{Z}}(\mathfrak{g}))\to\tilde{H}_*(\mathcal{T}(V), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}))$ *be induced by the natural map* $\Lambda^q_{\boldsymbol{Z}}(U)\to\Lambda^q_{\boldsymbol{R}}(U)$, $U\subseteq V$. *Then*

a) $\tilde{H}_p(\mathcal{T}(V), \Lambda^q_{\boldsymbol{Z}}(\mathfrak{g}))=\tilde{H}_p(\mathcal{T}(V), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}))=0$ *for* $p<n-q-1$,

b) *the map*

$$\rho: \tilde{H}_{n-q-1}(\mathcal{T}(V), \Lambda^q_{\boldsymbol{Z}}(\mathfrak{g})) \to \tilde{H}_{n-q-1}(\mathcal{T}(V), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g})) \tag{3.13}$$

is an isomorphism for $q=1, 2, \cdots, n$.

Before we prove this theorem let us note the relation with the Hadwiger invariants as defined is Jessen-Thorup [15].

A *Hadwiger invariant* H_Φ is defined for a *strict* flag $\Phi=(U_0\supset U_1\supset\cdots\supset U_{n-q-1})$, that is codim $U_i=i+1$, together with an orientation of each of the spaces $U_{-1}=V$, U_0, $\cdots$, U_{n-q-1}. Now let P be a polytope and suppose that $\bar{\Delta}=(\Delta_{-1}\supset\Delta_0\supset\cdots\supset\Delta_{n-q-1})$ is a string of geometric simplices of P (in a given triangulation) such that Δ_i spans an affine subspace parallel to U_i, $i=-1, 0, \cdots, n-q-1$. for such a string we write $\bar{\Delta}||\Phi$ and we let ε_i, $i=0, \cdots, n-q-1$, be the sign ±1 depending on whether the given orientation on U_i agrees or not with the induced orientation on Δ_i considered as a face of Δ_{i-1} (with the orientation from U_{i-1}). With this notation

$$H_\Phi(P)=\sum_{\bar{\Delta}||\Phi}\varepsilon_0\cdots\varepsilon_{n-q-1}\mathrm{Vol}(\Delta_{n-q-1}) \tag{3.14}$$

where Vol is the volume in the affine subspace containing Δ_{n-q-1} (and hence parallel to U_{n-q-1}). Notice that change of orientation of U_0, $\cdots$, U_{n-q-2} does *not* change $H_\Phi(P)$, whereas change of orientation of V or U_{n-q-1} change $H_\Phi(P)$ by a sign. Therefore if we let $\omega_{U_{n-q-1}}\in\Lambda^q_{\boldsymbol{R}}(U_{n-q-1})$ (which is 1-dimensional) be the volume element determined by the given orientation of U_{n-q-1} (and the Euclidean inner product induced from V) then

$$h_\Phi(P)=H_\Phi(P)\cdot\omega_{U_{n-q-2}}\in\Lambda^q_{\boldsymbol{R}}(U_{n-q-1}) \tag{3.15}$$

only depends on the choice of orientation of V. This all works for $q=1, \cdots, n-1$. For $q=n$ we shall think of the volume Vol in V as H_Φ for Φ the "empty" flag, and again $h_\Phi(P)$ given by (3.15) depends on the choice of orientation of V. We now observe

Proposition 3.16. a) *The vector space*

$$\mathcal{D}^q(V) = \tilde{H}_{n-q-1}(\mathcal{T}(V), \Lambda^q_{\mathbf{R}}(\mathfrak{g}))$$

is given by $\mathcal{D}^n(V)=\Lambda^q_{\mathbf{R}}(V)$, *and for* $q<n$ $\mathcal{D}^q(V)$ *is naturally identified with a subspace of the direct sum* $\coprod_\Phi \Lambda^q_{\mathbf{R}}(U_{n-q-1})(\Phi)$, *where* $\Phi=(U_0\supset\cdots\supset U_{n-q-1})$ *runs through all strict flags. Here a formal sum* $x=\sum x_\Phi(\Phi)$ *lies in* $\mathcal{D}^q(V)$ *iff for all* $i=0, \cdots, n-q-1$, *and all fixed* $(U_0, \cdots, U_{i-1}, U_{i+1}, \cdots, U_{n-q-1})$,

$$\sum_{U_i} x_{(U_0\supset\cdots\supset U_i\supset\cdots\supset U_{n-q-1})} = 0 \tag{3.17}$$

where this sum takes place in $\Lambda^q_{\mathbf{R}}(U_{n-q-1})\subset\Lambda^q_{\mathbf{R}}(V)$ (*respectively* $\Lambda^q_{\mathbf{R}}(U_{n-q-2})$) *for* $i<n-q-1$ (*respectively* $i=n-q-1$).

b) *For a fixed orientation of* V, *the homomorphism*

$$\mathcal{P}(V, T)\cong \bigoplus_{s=1}^{n} \tilde{H}_{n-q-1}(\mathcal{T}(V), \Lambda^q_{\mathbf{Z}}(\mathfrak{g})) \xrightarrow{\rho} \bigoplus_{s=1}^{n} \mathcal{D}^q(V)$$

$$\subseteqq \bigoplus_{s=1}^{n} \coprod_{\Phi} \Lambda^q_{\mathbf{R}}(U_{n-q-1})(\Phi)$$

agrees with the map given by the set of Hadwiger invariants

$$P \mapsto \sum h_\Phi(P)(\Phi)\,, \quad \Phi_0 = (U_0\supset\cdots\supset U_{n-q-1})\,, \qquad q = 1, \cdots, n\,.$$

Proof. Since the homology of $\mathcal{T}(V)$ can be computed using only non-degenerate "simplices", i.e. flags of the form $(U_0\supset U_1\supset\cdots\supset U_p)$ (see e.g. MacLane [17, chapter VIII, § 6]) a) clearly follows from the definitions. For b) consider an element $c\in H_n({}_TC_*(V)/{}_TC_*(V)^{n-1})$ which is of filtration $n-q$ in the spectral sequence for the double complex (3.2). Then c is represented in $A_{n-q-1,q}$ by

$$c' = s_{n-q-2}\circ\partial''\circ\cdots\circ s_0\circ\partial''\circ s_{-1}\circ\partial'' c$$

where s_p are the homomorphisms considered in the proof of lemma 3.3. Now for U q-dimensional the natural map

$$H_q({}_TC_*(U))\cong\Lambda^q_{\mathbf{Z}}(U) \xrightarrow{\rho} \Lambda^q_{\mathbf{R}}(U)$$

is clearly the oriented volume in U and so b) easily follows from the definition of s_p.

Proof of theorem 3.12. The *injectivity* of ρ in b) now clearly follows from Jessen-Thorup [15, theorem 2] and proposition 3.16. For a) and the *surjectivity* of ρ in b) we consider the exact sequence of local coefficient systems on $\mathfrak{T}(V)$:

$$0 \to \hat{\Lambda}^q(\mathfrak{g}) \to \Lambda^q_{\boldsymbol{Z}}(\mathfrak{g}) \to \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}) \to 0 \tag{3.18}$$

where for $U \subseteqq V$ a linear subspace we define

$$\hat{\Lambda}^q(U) = \mathrm{Ker}(\Lambda^q_{\boldsymbol{Z}}(U) \to \Lambda^q_{\boldsymbol{R}}(U)).$$

We shall prove

Lemma 3.19. Both

$$\tilde{H}_i(\mathfrak{T}(V), \hat{\Lambda}^q(\mathfrak{g})) = 0 \quad \text{and} \quad \tilde{H}_i(\mathfrak{T}(V), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g})) = 0$$

for $-1 \leqq i < n-q-1$, $q > 0$.

From this lemma and the long exact homology sequence for the sequence of coefficients (3.18) we clearly get the surjectivity of ρ. For the proof of lemma 3.19 we follow the ideas of Lusztig [16, chapter 1]. Let $\mathfrak{s}$ denote the local coefficient system defined by the exact sequence

$$0 \to \mathfrak{g} \to V \to \mathfrak{s} \to 0$$

where V is the constant coefficient system with group V at every simplex. Further let $r_1, \cdots, r_s$ be some set of non-negative integers, and write for short

$$\Lambda^{(r_1,\cdots,r_s)}_{\boldsymbol{Z}} = \Lambda^{r_1}_{\boldsymbol{Z}} \underset{\boldsymbol{Z}}{\otimes} \cdots \underset{\boldsymbol{Z}}{\otimes} \Lambda^{r_s}_{\boldsymbol{Z}}$$

and

$$\Lambda^{(r_1,\cdots,r_s)}_{\boldsymbol{R}} = \Lambda^{r_1}_{\boldsymbol{R}} \underset{\boldsymbol{R}}{\otimes} \cdots \underset{\boldsymbol{R}}{\otimes} \Lambda^{r_s}_{\boldsymbol{Z}}$$

Then we have in analogy to (3.18) an exact sequence of local coefficients

$$\begin{aligned} 0 \to (\Lambda^q(\mathfrak{g}) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s}))^{\sim} &\to \Lambda^q_{\boldsymbol{Z}}(\mathfrak{g}) \underset{\boldsymbol{Z}}{\otimes} \Lambda^{(r_1,\cdots,r_s)}_{\boldsymbol{Z}}(\mathfrak{s}) \to \\ &\to \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}) \underset{\boldsymbol{R}}{\otimes} \Lambda^{(r_1,\cdots,r_s)}_{\boldsymbol{R}}(\mathfrak{s}) \to 0 \end{aligned} \tag{3.20}$$

and so lemma 3.19 is clearly a special case of

Lemma 3.21. *Let* $\Lambda^q(\mathfrak{g}) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})$ *denote any of the* 3 *coefficient systems in* (3.20). *Then*

$$\tilde{H}_i(\bar{\mathfrak{T}}(V), \Lambda^q(\mathfrak{g}) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})) = 0$$

for $-1 \leqq i < n-q-1$, $q \geqq 0$.

(Notice that again the change from $\mathfrak{T}(V)$ to $\bar{\mathfrak{T}}(V)$ makes no difference for $q > 0$.) For the proof of this lemma we need the following lemma which is a trivial extension of Lusztig [16, §1. 12, Proposition]:

Lemma 3.22. *Let F be any functor from the category of vector spaces to the category of abelian groups satisfying $F(0)=0$. Then*

$$H_i(\bar{\mathfrak{T}}(V),\, F(\mathfrak{s})) = 0,\quad 0 \leqq i < n-1$$

Proof of lemma 3.21. Let us show the lemma for $\Lambda^q_Z(\mathfrak{g}) \underset{Z}{\otimes} \Lambda^{(r_1,\cdots,r_s)}_Z(\mathfrak{s})$. For the other two systems the proofs are similar. Let us write $\Lambda^q = \Lambda^q_Z$ for short and let us filtre the constant coefficient system $\Lambda^q(V)$ by

$$\Lambda^q(V) = \Lambda^q(V)^0 \supseteq \cdots \supseteq \Lambda^q(V)^l \supseteqq \cdots \supseteqq \Lambda^q(V)^q = \Lambda^q(\mathfrak{g})$$

where $\Lambda^q(V)^l$ at the flag $(U_0 \supseteqq \cdots \supseteqq U_p)$ is generated by by elements of the form $v_1 \wedge \cdots \wedge v_q$ with $v_1, \cdots, v_l \in U_p$. Then we have exact sequences

$$\begin{gathered}
0 \to \Lambda^q(V)^q \to \Lambda^q(V) \to \Lambda^q(V)/\Lambda^q(V)^q \to 0 \\
0 \to \Lambda^{q-1}(\mathfrak{g}) \otimes \mathfrak{s} \to \Lambda^q(V)/\Lambda^q(V)^q \to \Lambda^q(V)/\Lambda^q(V)^{q-1} \to 0 \\
0 \to \Lambda^{q-2}(\mathfrak{g}) \otimes \Lambda^2(\mathfrak{s}) \to \Lambda^q(V)/\Lambda^q(V)^{q-1} \to \Lambda^q(V)/\Lambda^q(V)^{q-2} \to 0 \\
\cdots\cdots \\
0 \to \mathfrak{g} \otimes \Lambda^{q-1}(\mathfrak{s}) \to \Lambda^q(V)/\Lambda^q(V)^2 \to \Lambda^q(V)/\Lambda^q(V)^1 \to 0
\end{gathered}$$

where

$$\Lambda^q(V)^q = \Lambda^q(\mathfrak{g}) \quad\text{and}\quad \Lambda^q(V)/\Lambda^q(V)^1 = \Lambda^q(\mathfrak{s})\,.$$

We get similar exact sequences after tensoring with $\Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})$ and we can now show the lemma by induction in q: For $q=0$ it is clear from lemma 3.22, and if we assume the lemma for $\Lambda^j(\mathfrak{g})$, $j<q$, $q>0$, then both

$$\begin{aligned}
&H_i(\bar{\mathfrak{T}}(V),\, \Lambda^j(\mathfrak{g}) \otimes \Lambda^{q-j,r_1,\cdots,r_s}(\mathfrak{s})) = 0\,, \\
&H_{i+1}(\bar{\mathfrak{T}}(V),\, \Lambda^j(\mathfrak{g}) \otimes \Lambda^{q-j,r_1,\cdots,r_s}(\mathfrak{s})) = 0 \\
&\qquad\qquad\qquad \text{for } -1 \leqq i < n-j-1\,.
\end{aligned}$$

Hence from the exact sequences above we obtain

$$\begin{aligned}
&H_i(\bar{\mathfrak{T}}(V),\, \Lambda^q(\mathfrak{g}) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})) \\
\cong{}& H_{i-1}(\bar{\mathfrak{T}}(V),\, (\Lambda^q(V)/\Lambda^q(V)^q) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})) \\
\cong{}& H_{i+1}(\bar{\mathfrak{T}}(V),\, (\Lambda^q(V)/\Lambda^q(V)^{q-1}) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})) \\
&\cdots\cdots \\
\cong{}& H_{i+1}(\bar{\mathfrak{T}}(V),\, \Lambda^q(\mathfrak{s}) \otimes \Lambda^{(r_1,\cdots,r_s)}(\mathfrak{s})) = 0
\end{aligned}$$

as long as $i<n-q-1$. This proves the lemma and so ends the proof of theorem 3.12.

Proof of theorem 1.1. The first statement clearly follows from theorem 3.12

and proposition 3.16 b) once we have chosen the natural orientation on $\boldsymbol{R}^n$. The second statement follows from the fact that the action of $g \in \mathrm{O}(n)$ on $\mathscr{P}(\boldsymbol{R}^n, \mathrm{T}(n))$ clearly commute with the dilatation operator μ_λ in the proof of lemma 3.10.

REMARKS. 1. Notice that $\tilde{H}_i(\mathcal{T}(V), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}))=0$ for $i>n-q-1$ since clearly the local coefficient system vanishes on the non-degenerate simplices in these dimensions. Thus $\tilde{H}_{n-q-1}(\mathcal{T}(V), \Lambda^q_{\boldsymbol{R}}(\mathfrak{g}))=\mathscr{D}^q(V)$ is the only non-trivial homology group for $\Lambda^q_{\boldsymbol{R}}(\mathfrak{g})$.

2. Notice that theorem 3.12 a) and the surjectivity of ρ in b) are valid more generally for V a finite dimensional vector space over any field k (even of finite characteristic) if $\Lambda^q_{\boldsymbol{R}}$ is replaced by Λ^q_k. The injectivity of ρ however depends on the results of Jessen-Thorup [15] which requires k to be an ordered field. Probably one can show the injectivity of ρ for more general fields (including char $k=0$). Thus for $q=n$

$$\tilde{H}_{-1}(\mathcal{T}(V), \Lambda^n_{\boldsymbol{Z}}(\mathfrak{g})) = \Lambda^n_{\boldsymbol{Z}}(V)/[\sum_{U\subset V} \Lambda^n_{\boldsymbol{Z}}(U)]$$

is easily seen to be isomorphic to $\Lambda^n_{\boldsymbol{R}}(V)$ (by use of the equation

$$v_1\wedge\lambda v_2-\lambda v_1\wedge v_2=(v_1+v_2)\wedge\lambda(v_1+v_2)-v_1\wedge\lambda v_1-v_2\wedge\lambda v_2$$

for $v_1, v_2 \in V$, $\lambda\in k$). Otherwise at the moment I can only prove the injectivity of ρ directly for the cases dim $V\leqq 3$, char $k\neq 2$, and dim $V\leqq 4$, char $k\neq 2,3$.

Let us end this section by an example:

EXAMPLE 3.24. Consider $h^1\colon \mathscr{P}(\boldsymbol{R}^3, T(3))\to\mathscr{D}^1(\boldsymbol{R}^3)$ in theorem 1.1 and let us calculate $h^1(\Delta)$ where $\Delta=(a_0, a_1, a_2, a_3)$ is a simplex. The natural orientation in $\boldsymbol{R}^3$ induce orientations on the faces as indicated by the arrows in fig. 1. Let $U_{(\check{i})}$, $i=0,\cdots,3$, be the plane (through the origin) parallel to the face

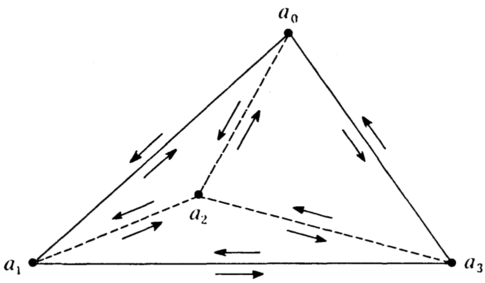

Fig. 1

opposite a_i, let $U_{(jk)}$, $j<k$, be the line (through the origin) spanned by the vector $\overrightarrow{a_j a_k}$, and let Φ_{ijk}, $i\neq j,k$ be the strict flag $\Phi_{ijk}=(U_{(\check{i})}\supset U_{(jk)})$. Then $h^1(\Delta)$ is given by the formel sum

$$h^1(\Lambda) = \sum_{\substack{j<k \\ i\neq j}} \varepsilon_{ijk} \overrightarrow{a_j a_k}(\Phi_{ijk}) \tag{3.25}$$

where $\varepsilon_{ijk}=\pm 1$ depending on the direction of the arrow on fig. 1 corresponding to the pair $U_{(\check{i})}\supset U_{(jk)}$. Notice that the relations in (3.17) are immediate from the figure. Thus it follows from corollary 3.23 that any element $\sum x_\Phi(\Phi)\in \coprod_{\Phi=(U_0\supset U_1)} U_1(\Phi)$ satisfying (3.17) is a linear combination of elements of the form (3.25).

4. The Euclidean case

We shall now apply the results in the previous section to obtain information about $\mathcal{P}(\boldsymbol{R}^n)$. For this we shall use homological algebra in a similar way as Sah [19, chapter 5] and in fact many of our results are exactly dual to his.

Homology of groups.

For G any group and M a (left) $\boldsymbol{Z}[G]$-module $H_i(G,M)$, $i=0,1,2,\cdots$, are the derived functors of the right exact functor $M\to {}_GM=H_0(G,M)$. Explicitly $H_*(G,M)$ can be computed as the homology of the standard "bar complex" $C_*(G,M)$, where a q-chain is a formal sum of elements of the form $(g_1,\cdots,g_q)x$, $g_1,\cdots,g_q\in G$, $x\in M$ and where

$$\begin{aligned}\partial(g_1,\cdots,g_q)x &= (g_2,\cdots,g_q)x+ \\ &+\sum_{i=1}^{q-1}(-1)^i(g_1,\cdots,g_ig_{i+1},\cdots,g_q)x+(-1)^q(g_1,\cdots,g_{q-1})g_qx\,.\end{aligned} \tag{4.1}$$

We shall need the following easy fact (see Cartan-Eilenberg [4, chapter X, proposition 7.4]):

Lemma 4.2 (Shapiro's lemma). *For $K\subseteq G$ a subgroup and N any (left) K-module there is a natural isomorphism*

$$H_*(G,\boldsymbol{Z}[G]\underset{K}{\otimes}N)\cong H_*(K,N)$$

Also we shall use the following lemma (cf. Sah [18, proposition 2.7 c] or Cartan-Eilenberg [4, chapter X, exercise 6]):

Lemma 4.3 ("Center kills"). *Let M be a G-module and $\sigma\in G$ an element in the center such that $\sigma x=-x$, $\forall x\in M$. Then $2H_*(G,M)=0$.*

The modules $\mathcal{D}^q(V)$ and the Lusztig exact sequence.

Now let us return to $\mathcal{P}(\boldsymbol{R}^n)$. As in section 3 let us consider V a real vector space of dimension n with Euclidean inner product, and let $O(V)$ be the associated orthogonal group. Again let $\mathcal{T}(V)$ be the Tits-complex of flags of proper linear subspaces of V with the local coefficient system $\mathfrak{g}$. Then clearly $O(V)$ acts on $\mathscr{T}(V)$ and $\mathfrak{g}$ and we define

DEFINITION 4.4. For $q=1,\cdots,n$, let $\mathscr{D}^q(V)$ be the $O(V)$-module given by

$$\mathscr{D}^q(V) = \tilde{H}_{n-q-1}(\mathscr{T}(V), \Lambda^q_{\mathbf{R}}(\mathfrak{g})) .$$

Here $\mathscr{D}^n(V)=\Lambda^n_{\mathbf{R}}(V)$ and $\mathscr{D}^{n-1}(V)=\mathrm{Ker}(H_0(\mathscr{T}(V), \Lambda^{n-1}_{\mathbf{R}}(\mathfrak{g}))\to\Lambda^{n-1}_{\mathbf{R}}(V))$.

Notice that the element $-id\in \mathrm{O}(n)$ acts on the "twisted" module $\mathscr{D}^q(\mathbf{R}^n)^t$ as multiplication by $(-1)^q(-1)^n=(-1)^{n-q}$. Therefore by Theorem 1.1

Corollary 4.6. $\mathscr{P}(\mathbf{R}^n)\cong \bigoplus_{q\equiv n \bmod 2} H_0(\mathrm{O}(n), \mathscr{D}^q(\mathbf{R}^n)^t)$

For the study of these homology groups we shall use the following exact sequence which for $q=1$ is due to Lusztig [16]:

Proposition 4.7. *Let $V=\mathbf{R}^n$ and let $V^i\subseteq V$ run through all i-dimensional subspaces, $i=1,2,\cdots,n-1$. Then for $q=1,\cdots,n$ there is an exact sequence of* $\mathrm{O}(n)$-*modules*

$$0 \to \mathscr{D}^q(V) \xrightarrow{\partial_n} \coprod_{V^{n-1}} \mathscr{D}^q(V^{n-1}) \xrightarrow{\partial_{n-1}} \cdots \coprod_{V^{q+1}} \mathscr{D}^q(V^{q+1}) \xrightarrow{\partial_{q+1}}$$
$$\to \coprod_{V^q} \Lambda^q_{\mathbf{R}}(V^q) \to \Lambda^q_{\mathbf{R}}(V) \to 0$$

where ∂_p: $\mathscr{D}^q(V^p)\to \coprod_{V^{p-1}} \mathscr{D}^q(V^{p-1})$ is induced by sending a flag $(U_0\supset\cdots\supset U_{p-q-1})$ in V^p to the flat $(U_1\supset\cdots\supset U_{p-q-1})$ in $V^{p-1}=U_0$.

Proof. For $q=1$ this is just the sequence in Lusztig [16, §1.13(c)] and for $q>1$ the proof is similar: Let $\mathscr{T}(V)^p\subseteq\mathscr{T}(V)$ be the subcomplex of flags $U_0\supseteq\cdots\supseteq U_i$ with $\dim U_0\leq p$. Then by Lemma 3.19 and Lusztig [16, §1.6, lemma] we get $\tilde{H}_i(\mathscr{T}(V)^p, \Lambda^q_{\mathbf{R}}(\mathfrak{g}))=0$ for $i\neq p-q-1$. Furthermore from a chain description involving only non-degenerate flags it is readily seen that there is a natural identification

$$\tilde{H}_i(\mathscr{T}(V)^p, \mathscr{T}(V)^{p-1}; \Lambda^q_{\mathbf{R}}(\mathfrak{g}))\cong\begin{cases}\coprod_{V^p} \mathscr{D}^q(V^p), & i=p-q-1\,, \\ & p\geq q; \\ 0 & \text{otherwise.}\end{cases}$$

The exact sequence now easily follows from the spectral sequence for the filtration on the chain complex $C_*(\mathscr{T}(V), \Lambda^q_{\mathbf{R}}(\mathfrak{g}))$ correspondong to the filtration of simplicial sets

$$\mathscr{T}(V) = \mathscr{T}(V)^n\supset\mathscr{T}(V)^{n-1}\supset\cdots\supset\mathscr{T}(V)^0 = \emptyset$$

(cf. MacLane [17, chapter XI, §3]).

The Lusztig exact sequence links the homology of O(n) for the module $\mathscr{D}^q(\boldsymbol{R}^n)^{\mathrm{t}}$ to the homology for the module $\Lambda^q_{\boldsymbol{R}}(\boldsymbol{R}^n)$ through homology groups of smaller orthogonal groups. In fact

Lemma 4.8. *For* $n \geqq l \geqq q$:

a) $$H_*(\mathrm{O}(n), [\coprod_{V^l} \mathscr{D}^q(V^l)]^{\mathrm{t}})$$
$$\cong H_*(\mathrm{O}(l)\ \mathscr{D}^q(\boldsymbol{R}^l)^{\mathrm{t}}) \underset{\boldsymbol{Z}}{\otimes} H_*(\mathrm{O}(n-l), \boldsymbol{Z}^{\mathrm{t}})$$

b) *In particular*

$$H_*(\mathrm{O}(n), [\coprod_{V^l} \mathscr{D}^q(V^l)]^{\mathrm{t}}) = 0 \quad \textit{if} \ \ l \not\equiv n \bmod 2.$$

Proof. Clearly

$$\coprod_{V^l} \mathscr{D}^q(V^l) = \boldsymbol{Z}[\mathrm{O}(n)] \underset{\mathrm{O}(l)\times \mathrm{O}(n-l)}{\otimes} \mathscr{D}^q(\boldsymbol{R}^l)$$

where O($n-l$) acts trivially on the module $\mathscr{D}^q(\boldsymbol{R}^l)$. Twisting the action with the determinant we get

$$[\coprod_{V^l} \mathscr{D}^q(V^l)]^{\mathrm{t}} = \boldsymbol{Z}[\mathrm{O}(n)] \underset{\mathrm{O}(l)\times \mathrm{O}(n-l)}{\otimes} (\mathscr{D}^q(\boldsymbol{R}^l)^{\mathrm{t}} \otimes \boldsymbol{Z}^{\mathrm{t}})\,.$$

Then a) clearly follows from lemma 4.2 and the Künneth formula (MacLane [17, chapter X, §7]). b) now clearly follows from Lemma 4.3.

$\mathscr{P}(\boldsymbol{R}^n)$ in low dimensions.

EXAMPLE 4.9. $n=1$. By corollary 4.6,

$$\mathscr{P}(\boldsymbol{R}^1) \cong H_0(\mathrm{O}(1), \mathscr{D}^1(\boldsymbol{R}^1)^{\mathrm{t}}) = H_0(\mathrm{O}(1), (\boldsymbol{R}^1)^{\mathrm{t}}) = \boldsymbol{R}$$

and the isomorphism is clearly given by "Length".

EXAMPLE 4.10. $n=2$. By corollary 4.6,

$$\mathscr{P}(\boldsymbol{R}^2) \cong H_0(\mathrm{O}(2), \mathscr{D}^2(\boldsymbol{R}^2)^{\mathrm{t}}) = H_0(\mathrm{O}(2), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^2)^{\mathrm{t}})$$
$$\cong \boldsymbol{R}$$

and the isomorphism is clearly given by "Area".

EXAMPLE 4.11. $n=3$. By corollary 4.6,

$$\mathscr{P}(\boldsymbol{R}^3) \cong H_0(\mathrm{O}(3), \mathscr{D}^1(\boldsymbol{R}^3)^{\mathrm{t}}) \oplus H_0(\mathrm{O}(3), \mathscr{D}^3(\boldsymbol{R}^3)^{\mathrm{t}})\,.$$

Here again

$$H_0(\mathrm{O}(3), \mathscr{D}^3(\boldsymbol{R}^3)^{\mathrm{t}}) = H_0(\mathrm{O}(3), \Lambda^3_{\boldsymbol{R}}(\boldsymbol{R}^3)^{\mathrm{t}}) \cong \boldsymbol{R}$$

and the isomorphism is given by "Volume". For the computation of $H_0(\mathrm{O}(3), \mathscr{D}^1(\boldsymbol{R}^2)^{\mathrm{t}})$ we split the Lusztig sequence in two exact sequences

$$\text{(i)} \qquad 0 \to \mathscr{D}^1(\boldsymbol{R}^3) \to \coprod_{V^2} \mathscr{D}^1(V^2) \to Z_1 \to 0$$

$$\text{(ii)} \qquad 0 \to Z_1 \to \coprod_{V^1} V^1 \to \boldsymbol{R}^3 \to 0\,.$$

From (i) and lemma 4.8 b) we obtain

$$H_0(\mathrm{O}(3),\ \mathscr{D}^1(\boldsymbol{R}^3)^{\mathrm{t}}) \cong H_1(\mathrm{O}(3),\ Z_1^{\mathrm{t}})$$

and

$$H_0(\mathrm{O}(3),\ Z_1^{\mathrm{t}}) = 0\,.$$

Furthermore by lemma 4.8 a)

$$\begin{aligned} H_*(\mathrm{O}(3),\ [\coprod V^1]^{\mathrm{t}}) &\cong H_*(\mathrm{O}(1),\ (\boldsymbol{R}^1)^{\mathrm{t}}) \underset{\boldsymbol{Z}}{\otimes} H_*(\mathrm{O}(2),\ \boldsymbol{Z}^{\mathrm{t}}) \\ &= \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} H_*(\mathrm{O}(2),\ \boldsymbol{Z}^{\mathrm{t}})\,. \end{aligned}$$

As in lemma 3.1 it is easy to see that

$$H_*(\mathrm{SO}(2),\ \boldsymbol{Z})/\mathrm{Torsion} \cong \Lambda_{\boldsymbol{Z}}^*(\mathrm{SO}(2)/\mathrm{Torsion})$$

and here the action of $\mathrm{O}(2)/\mathrm{SO}(2) = \boldsymbol{Z}/2$ is induced by $g \mapsto g^{-1}$ in SO(2). In particular

$$H_1(\mathrm{O}(3),\ [\coprod_{V^1} V^1]^{\mathrm{t}}) \cong \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} \mathrm{SO}(2)$$

and

$$H_2(\mathrm{O}(3),\ [\coprod_{V^1} V^1]^{\mathrm{t}}) = 0\,.$$

From the exact sequence (ii) we then get

Corollary 4.12. *There is an exact sequence*

$$\begin{aligned} 0 \to H_2(\mathrm{O}(3),\ (\boldsymbol{R}^3)^{\mathrm{t}}) \to H_0(\mathrm{O}(3),\ \mathscr{D}^1(\boldsymbol{R}^3)^{\mathrm{t}}) \to \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} \mathrm{SO}(2) \to \\ \to H_1(\mathrm{O}(3),\ (\boldsymbol{R}^3)^{\mathrm{t}}) \to 0\,. \end{aligned}$$

Furthermore if SO(2) *is identified with the additive group* $\boldsymbol{R}/\boldsymbol{Z}$ *in the usual way, then the composite*

$$D\colon \mathscr{P}(\boldsymbol{R}^3) \to {}_{\mathrm{O}(3)}\mathscr{D}^1(\boldsymbol{R}^3)^{\mathrm{t}} \to \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} (\boldsymbol{R}/\boldsymbol{Z})$$

is the classical Dehn-invariant.

Proof. It remains to identify D with the Dehn-invariant. Consider the simplex Δ as in example 3.24 and let us first determine the image of $h^1(\Delta)$ under the isomorphism ${}_{\mathrm{O}(3)}\mathscr{D}^1(\boldsymbol{R}^3)^{\mathrm{t}} \cong H_1(\mathrm{O}(3),\ Z_1^{\mathrm{t}})$ using the standard complex $C_*(\mathrm{O}(3),\ Z_1^{\mathrm{t}})$ as in (4.1): Let s_i be the reflection in the plane $U_{(\overset{\vee}{i})}$ and let us write for short

$$h^1(\Delta)=\sum_{i,j,k} v_{ijk}\Phi_{ijk}, \qquad v_{ijk}=\varepsilon_{ijk}\overrightarrow{a_j a_k}\,. \tag{4.13}$$

Then as an element in $C_0(\mathrm{O}(3), [\coprod_{V^2}\mathscr{D}^1(V^2)]^t)$

$$\begin{aligned} h^1(\Delta) &= \frac{1}{2}\sum_{i,j,k} v_{ijk}(\Phi_{ijk}+s_i\Phi_{ijk}) \\ &= \frac{1}{2}\,\partial[\sum_i (s_i)\,(\sum_{j,k} v_{ijk}\Phi_{ijk})] \end{aligned}$$

where ∂ is given by (4.1). Therefore the image in $H_1(\mathrm{O}(3), Z_1^t)$ is represented by

$$\partial^{-1}(h^1(\Delta))=\frac{1}{2}\sum_i (s_i)\,[\sum_{j,k} v_{ijk}(U_{(jk)})]$$

where $Z_1\subseteqq\coprod_{V^1}V^1$, and so the image in $H_1(\mathrm{O}(3), [\coprod_{V^1}V^1]^t$ is represented by

$$\sum_{j<k}\frac{1}{2}\,((s_{i_1})v_{i_1jk}+(s_{i_2})v_{i_2jk})\,(U_{(jk)}) \tag{4.14}$$

where i_1, i_2 are the two indices $\neq j,k$. Clearly $v_{i_2jk}=-v_{i_1jk}$ so

$$\partial[(s_{i_2}, s_{i_1})v_{i_1jk}]=(s_{i_1})v_{i_1jk}-(s_{i_2}s_{i_1})v_{i_1jk}+(s_{i_2})v_{i_2jk}\,.$$

It follows that the element (4.14) is homologous to

$$\sum_{j<k}\frac{1}{2}\,(s_{i_2}s_{i_1})\varepsilon_{i_1jk}\overrightarrow{a_j a_k} \tag{4.15}$$

in $H_1(\mathrm{O}(3), [\coprod_{V^1}V^1]^t)$. Here $s_{i_2}s_{i_1}$ is the rotation around $U_{(jk)}$ by twice the angle θ_{jk} between the planes $U_{(\check{i}_1)}$ and $U_{(\check{i}_2)}$. The isomorphism

$$H_1(\mathrm{O}(3), [\coprod_{V^1}V^1]^t)\cong\boldsymbol{R}\underset{\boldsymbol{Z}}{\otimes}\mathrm{SO}(2)\cong\boldsymbol{R}\underset{\boldsymbol{Z}}{\otimes}(\boldsymbol{R}/\boldsymbol{Z})$$

therefore clearly takes the element (4.15) to

$$\sum_{j<k}\varepsilon_{i_1jk}\,|a_j a_k|\otimes\theta_{jk}/2\pi$$

which is just the classical Dehn-invariant for the simplex Δ (cf. Jessen [12]).

Proof of corollary 1.2. Since the element $-\mathrm{id}\in\mathrm{O}(3)$ has $\det(-\mathrm{id})=-1$ it follows that

$$H_*(\mathrm{SO}(3), \boldsymbol{R}^3)\cong H_*(\mathrm{O}(3), \boldsymbol{R}^3)\,.$$

The exact sequence in corollary 1.2 therefore clearly follows from corollary 4.12 and the remark 2 following proposition 3.11. From this sequence and since

$$D\colon \mathcal{P}(\boldsymbol{R}^3)/\mathcal{Z}_2(\boldsymbol{R}^3) \to \boldsymbol{R}\underset{\boldsymbol{Z}}{\otimes}(\boldsymbol{R}/\boldsymbol{Z})$$

is injective by Sydler's theorem [24] (c.f. Jessen [12]) it follows that

$$(4.16) \qquad H_2(\mathrm{SO}(3), \boldsymbol{R}^3) = 0\,.$$

Furthermore

$$(4.17) \qquad H_1(\mathrm{SO}(3), \boldsymbol{R}^3) \cong [\boldsymbol{R}\underset{\boldsymbol{Z}}{\otimes}(\boldsymbol{R}/\boldsymbol{Z})]/D(\mathcal{P}(\boldsymbol{R}^3))\,.$$

Now the image of the Dehn invariant has been described by Jessen [12, §12–14], and it is easy to see from this description that the cokernel has dimension as the continuum.

Applications to homology of SO(n).

Corollary 4.18. $n \geqq 3$. *Then*

a) $H_1(SO(n), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^n)) \cong \begin{cases} H_1(\mathrm{SO}(3), \boldsymbol{R}^3), & n \neq 4 \\ H_1(\mathrm{SO}(3), \boldsymbol{R}^3) \oplus H_1(\mathrm{SO}(3), \boldsymbol{R}^3), & n = 4 \end{cases}$

are real vector spaces of dimension as the continuum.

b) $H_2(\mathrm{SO}(n), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^n)) = 0$.

Proof. The corollary is allready proved for $n=3$ since $\Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^3) \cong \boldsymbol{R}^3$ as SO(3)-modules. For $n=4$

$$H_*(\mathrm{SO}(4), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^4) \cong H_*(\mathrm{Spin}(4), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^4))\,.$$

Here $\mathrm{Spin}(4) \cong S_+ \times S_-$ with $S_\pm \cong \mathrm{Spin}(3)$, and $\Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^4) \cong \Lambda_+ \oplus \Lambda_-$ with $\Lambda_\pm \cong \boldsymbol{R}^3$ in such a way that S_+ acts trivially on Λ_-(respectively S_- acts trivially on Λ_+) and via SO(3) on Λ_+ (respectively S_- acts via SO(3) on Λ_-). By Künneth's theorem and since $H_1(S_\pm, \boldsymbol{Z})=0$ we have for $i \leqq 2$:

$$\begin{aligned} H_i(\mathrm{Spin}(4), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^4)) &\cong H_i(S_+, \Lambda_+) \oplus H_i(S_-, \Lambda_-) \\ &\cong H_i(\mathrm{SO}(3), \boldsymbol{R}^3) \oplus H_i(\mathrm{SO}(3), \boldsymbol{R}^3)\,. \end{aligned}$$

This takes care of $n=4$. Notice that change of orientation in $\boldsymbol{R}^4$ interchanges the factors of $\mathrm{Spin}(4) \cong S_+ \times S_-$. Therefore

$$H_i(\mathrm{O}(4), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^4)) \cong H_i(\mathrm{SO}(3), \boldsymbol{R}^3), \quad i = 1,2\,,$$

and it follows that the natural map

$$\mathrm{SO}(3) \to \mathrm{SO}(4) \to \mathrm{O}(4)$$

induces an isomorphism

$$H_i(\mathrm{SO}(3), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^3)) \xrightarrow{\cong} H_i(\mathrm{O}(4), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^4)), \quad i = 1,2\,.$$

It remains to show that for $n \geqq 4$ the map

$$H_i(\mathrm{O}(n), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^n)) \rightarrow H_i(\mathrm{SO}(n+1), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^{n+1})), \tag{4.19}$$

induced by sending $g \in \mathrm{O}(n)$ to the matrix $\begin{pmatrix} g & 0 \\ 0 & \det g \end{pmatrix}$ is an isomorphism for $i=1$ and a surjection for $i=2$. Equivalently consider the exact sequence of $\mathrm{SO}(n+1)$-modules

$$0 \rightarrow K_0 \rightarrow \coprod_{V^n} \Lambda^2_{\boldsymbol{R}}(V^n) \rightarrow \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^{n+1}) \rightarrow 0\ ; \tag{4.20}$$

we must show that $H_i(\mathrm{SO}(n+1), K_0)=0$, $i=0,1$. For this consider the diagram of Lusztig exact sequences:

$$\begin{tikzcd}
0 & 0 & 0 & \\
\coprod_{V^3} \mathscr{D}^2(V^3) \arrow[r] \arrow[u] & \coprod_{V^2} \Lambda^2_{\boldsymbol{R}}(V^2) \arrow[r] \arrow[u] & \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^{n+1}) \arrow[r] \arrow[u] & 0 \\
\coprod_{V^3 \subset V^n} \mathscr{D}^2(V^3) \arrow[r] \arrow[u] & \coprod_{V^2 \subset V^n} \Lambda^2_{\boldsymbol{R}}(V^2) \arrow[r] \arrow[u] & \coprod_{V^n} \Lambda^2_{\boldsymbol{R}}(V^n) \arrow[r] \arrow[u] & 0 \\
K_2 \arrow[r] \arrow[u] & K_1 \arrow[r] \arrow[u] & K_0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here it suffices to show

$$H_0(\mathrm{SO}(n+1), K_2) = 0 \tag{4.22}$$

and

$$H_i(\mathrm{SO}(n+1), K_1) = 0\,, \quad i = 0,1\,. \tag{4.23}$$

Let us show (4.23) ((4.22) is similar but easier). By Shapiro's lemma and Künneth's theorem

$$H_*(\mathrm{SO}(n+1), \coprod_{V^2} \Lambda^2_{\boldsymbol{R}}(V^2)) \cong_{Z/2} [H_*(\mathrm{SO}(2), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^2)) \otimes \otimes H_*(\mathrm{SO}(n-1), \boldsymbol{Z})] \tag{4.24}$$

where the covariants are with respect to the involution which changes the orientation on both factors. Similarly

$$H_*(\mathrm{SO}(n+1), \coprod_{V^2 \subset V^n} \Lambda^2_{\boldsymbol{R}}(V^2)) \cong H_*(\mathrm{O}(2), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^2)) \otimes \otimes H_*(\mathrm{O}(n-2), \boldsymbol{Z})\,. \tag{4.25}$$

Both (4.24) and (4.25) are zero in dimension 0 and are equal to $H_1(\mathrm{SO}(2), \Lambda^2_{\boldsymbol{R}}(\boldsymbol{R}^2))$ in dimension 1, so

$$H_i(\mathrm{SO}(n+1), \coprod_{V^2 \subset V^n} \Lambda^2_{\boldsymbol{R}}(V^2)) \rightarrow H_i(\mathrm{SO}(n+1), \coprod_{V^2} \Lambda^2_{\boldsymbol{R}}(V^2)) \tag{4.26}$$

is an isomorphism for $i=0,1$. It remains to see that the map (4.26) is surjective for $i=2$, i.e. that the map

$$H_2(\mathrm{O}(n-2), \boldsymbol{Q}) \to H_2(\mathrm{O}(n-1), \boldsymbol{Q}), \quad n \geqq 4,$$

is surjective. For small n this is true by a theorem of J. Mather (see Alperin-Dennis [2]) and then it follows by induction using a diagram similar to (4.21) of Lusztig exact sequences for the Steinberg module (c.f. section 5 below or Alperin [1]).

$\mathcal{P}(\boldsymbol{R}^n)$ in higher dimensions.

In general for $n \geqq 3$ there is a Dehn invariant

$$D\colon \mathcal{P}(\boldsymbol{R}^n) \to \Lambda_{\boldsymbol{R}}^{n-2}(\boldsymbol{R}^{n-2}) \underset{\boldsymbol{Z}}{\otimes} (\boldsymbol{R}/\boldsymbol{Z})$$

(cf. Sah [19, chapter 7, §2]) and similar to corollary 4.12 there is an exact sequence for $n \geqq 3$ (cf. Sah [19, chapter 5, proposition 7.5]):

$$\begin{aligned}(4.27)\qquad & 0 \to H_2(\mathrm{O}(n), \Lambda_{\boldsymbol{R}}^{n-2}(\boldsymbol{R}^n)^{\mathrm{t}}) \to H_0(\mathrm{O}(n), \mathcal{D}^{n-2}(\boldsymbol{R}^n)^{\mathrm{t}}) \xrightarrow{|D} \\ & \xrightarrow{D} \Lambda_{\boldsymbol{R}}^{n-2}(\boldsymbol{R}^{n-2}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z} \to H_1(\mathrm{O}(n), \Lambda^{n-2}(\boldsymbol{R}^n)^{\mathrm{t}}) \to 0\end{aligned}$$

Here as $\mathrm{O}(n)$-modules $\Lambda_{\boldsymbol{R}}^{n-2}(\boldsymbol{R}^n)^{\mathrm{t}} \cong \Lambda^2(\boldsymbol{R}^n)$ so in particular we obtain from corollary 4.18:

Corollary 4.28. *For $n \geqq 3$, the Dehn invariant $D\colon \mathcal{P}(\boldsymbol{R}^n) \to \Lambda_{\boldsymbol{R}}^{n-2}(\boldsymbol{R}^{n-2}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z}$ induces an injection on the component ${}_{\mathrm{O}(n)}\mathcal{D}^{n-2}(\boldsymbol{R}^n)^{\mathrm{t}}$ of the decomposition of corollary* 4.6.

REMARKS 1. In particular for $n=4$ this corollary contains the result of Jessen [13] that also in this dimension the volume and the Dehn invariant are separating.

2. There is a "generalized Dehn invariant" (cf. Sah [19, chapter 7, §2] and Hadwiger [10, kap. 2, §2)]

$$(4.29)\qquad \Psi^{(2)}\colon \mathcal{P}(\boldsymbol{R}^n) \to \mathcal{P}(\boldsymbol{R}^{n-2}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z}$$

such that with respect to the decomposition of corollary 4.6 $\Psi^{(2)} = \sum_q \Psi_q^{(2)}$ with

$$\Psi_q^{(2)}\colon {}_{\mathrm{O}(n)}\mathcal{D}^q(\boldsymbol{R}^n)^{\mathrm{t}} \to {}_{\mathrm{O}(n-2)}\mathcal{D}^q(\boldsymbol{R}^{n-2})^{\mathrm{t}} \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z}$$

and $\Psi_{n-2}^{(2)} = D$ considered above.

The exact sequence of corollary 4.12 generalizes in higher dimensions to a spectral sequence:

Theorem 4.30. *For integers $1 \leqq q \leqq n$ with $q \equiv n$* mod 2 *there is a spectral* *sequence* $\{E^r_{1,m}\}$, $q-1 \leqq 1 \leqq n$, $m \geqq 0$, *satisfying*

a) $E^\infty_{*,*}=0$

b) $$E^1_{l,*} \cong \begin{cases} H_*(\mathrm{O}(n), \Lambda^q_{\boldsymbol{R}}(\boldsymbol{R}^n)^{\mathrm{t}}), & 1=q-1 \\ H_*(\mathrm{O}(l), \mathscr{D}^q(\boldsymbol{R}^l)^{\mathrm{t}}) \underset{\boldsymbol{Z}}{\otimes} H_*(\mathrm{O}(n-l), \boldsymbol{Z}^{\mathrm{t}}), & q \leqq l < n, \\ H_*(\mathrm{O}(n), \mathscr{D}^q(\boldsymbol{R}^n)^{\mathrm{t}}), & l=n \end{cases}$$

c) *Here* $E^1_{l,*}=0$ *if* $q \leqq l < n$ *and* $l \not\equiv n \bmod 2$.

d) *If* r *is odd then* $d^r=0$ *except possibly on* $E^r_{q+r-1,*}$. *In particular* d^1 *vanishes on* $E^1_{n,0} \cong H_0(\mathrm{O}(n), \mathscr{D}^q(\boldsymbol{R}^n)^{\mathrm{t}})$.

e) $E^1_{n-2,1} \cong H_0(\mathrm{O}(n-2), \mathscr{D}^q(\boldsymbol{R}^{n-2})^{\mathrm{t}}) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z}$ *and* $d_2\colon E^1_{n,0} \to E^1_{n-2,1}$ *is identified with the component* $\Psi^{(2)}_q$ *of the generalized Dehn invariant* (4.29).

Proof. The spectral sequence is just the hyperhomology spectral sequence in the sense of Cartan-Eilenberg [4, chapter XVII] for the Lusztig exact sequence (proposition 4.7 above). Explicitly it is the spectral sequence associated with the double complex $A_{1,m}$ defined by

$$A_{l,m} = \begin{cases} C_m(\mathrm{O}(n), \Lambda^q_{\boldsymbol{R}}(\boldsymbol{R}^n)^{\mathrm{t}}), & l=q-1 \\ C_m(\mathrm{O}(n), [\coprod_{V^l} \mathscr{D}^q(V^l)]^{\mathrm{t}}), & q \leqq l \leqq n \end{cases}$$

where $C_*(\mathrm{O}(n), -)$ is the standard bar complex for computing group homology (cf. (4.1)).

a) is obvious since the Lusztig sequence is exact. Clearly

$$E^1_{l,*} \cong \begin{cases} H_*(\mathrm{O}(n), \Lambda^q_{\boldsymbol{R}}(\boldsymbol{R}^n)^{\mathrm{t}}), & l=q-1 \\ H_*(\mathrm{O}(n), [\coprod_{V^l} \mathscr{D}^q(V^l)^{\mathrm{t}}), & q \leqq l \leqq n\,, \end{cases}$$

and hence b) and c) are just lemma 4.8.

d) is obvious from c), and e) is similar to the proof of corollary 4.12.

EXAMPLE 4.31. $\mathscr{P}(\boldsymbol{R}^5)$. Here

$$\mathscr{P}(\boldsymbol{R}^5) = H_0(\mathrm{O}(5), \mathscr{D}^1(\boldsymbol{R}^5)^{\mathrm{t}} \oplus \mathscr{D}^3(\boldsymbol{R}^5)^{\mathrm{t}} \oplus \mathscr{D}^5(\boldsymbol{R}^5)^{\mathrm{t}})\,.$$

Again the isomorphism $H_0(\mathrm{O}(5), \mathscr{D}^5(\boldsymbol{R}^5)^{\mathrm{t}}) \cong \boldsymbol{R}$ is given by "Volume" and also according to corollary 4.28 the Dehn invariant

$$D\colon {}_{\mathrm{O}(5)}\mathscr{D}^3(\boldsymbol{R}^5)^{\mathrm{t}} \to \Lambda^3_{\boldsymbol{R}}(\boldsymbol{R}^3) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z}$$

is injective. It remains to investigate $H_0(\mathrm{O}(5), \mathscr{D}^1(\boldsymbol{R}^5)^{\mathrm{t}})$ for which we have the spectral sequence in theorem 4.30. Here d_2 is again a generalized Dehn invariant, but we also have a d_4 and d_5. We shall in the next section (example 5.39) consider a slightly different spectral sequence for this case in which d_2 and d_4 above are replaced by a single generalized Dehn invariant.

5. The spherical case

In this section we shall study $\mathcal{P}(\mathrm{S}^n)=\mathcal{P}(\mathrm{S}^n, \mathrm{O}(n+1))$ and again we shall use theorem 2.3 as starting point. So we want to study the $\mathrm{O}(n+1)$-module

$$(5.1) \qquad \mathcal{P}(\mathrm{S}^n, \{1\}) \cong H_n(C_*(\mathrm{S}^n)/C_*(\mathrm{S}^n)^{n-1})^{\mathrm{t}}$$

(using the standard orientation of $\mathrm{S}^n \subseteqq \boldsymbol{R}^{n+1}$). Again it turns out that the Tits-complex of flags naturally occurs:

The Steinberg module.

This time let V be an $(n+1)$-dimensional real vector space with Euclidean inner product and let $S(V)$ be the unit sphere. Notice that the geodesic subspaces of $S(V)$ are all of the form $S(U)$, $U \subseteqq V$ a linear subspace, except that for U a line $S(U)$ is a pair of antipodal points. As in section 3 let $\mathcal{T}(V)$ denote the Tits-complex of flags of proper linear subspaces of V (i.e. subspaces different from 0 and V).

DEFINITION 5.2. For $\dim V=n+1 \geqq 2$ the *Steinberg-module* $\mathrm{St}(V)$ is the $\mathrm{O}(V)$-module given by

$$\mathrm{St}(V) = \tilde{H}_{n-1}(\mathcal{T}(V), \boldsymbol{Z}) .$$

For $\dim V=1$ define $\mathrm{St}(V)=\boldsymbol{Z}$ with the trivial action.

For completeness we include a proof of the following wellknown

Proposition 5.3. *If V has dimension $n+1 \geqq 2$ then*

a) *$\tilde{H}_q(\mathcal{T}(V), \boldsymbol{Z})=0$ for $q \neq n-1$.*

b) *Furthermore $\tilde{H}_{n-1}(\mathcal{T}(V), \boldsymbol{Z})=\mathrm{St}(V)$ is the following subgroup of $\coprod_{\Phi} \boldsymbol{Z}(\Phi)$, where $\Phi=(U_0 \supset \cdots \supset U_{n-1})$ runs through all strict flags: $x=\sum x_\Phi(\Phi)$ lies in $\mathrm{St}(V)$ iff for all $i=0,\cdots,n-1$, and all fixed $(U_0 \supset \cdots \supset U_{i-1} \supset U_{i+1} \supset \cdots \supset U_{n-1})$*

$$(5.4) \qquad \sum_{U_i} x_{(U_0 \supset \cdots \supset U_i \supset \cdots \supset U_{n-1})} = 0$$

Proof. b) clearly follows since $\tilde{H}_*(\mathcal{T}(V), \boldsymbol{Z})$ can be computed by the normalized chain complex. Also $H_q(\mathcal{T}(V), \boldsymbol{Z})=0$ for $q>n-1$ for the same reason. For the vanishing of $\tilde{H}_q(\mathcal{T}(V), \boldsymbol{Z})$ for $q<n-1$ we use a double complex argument similar to the proof of proposition 3.11:

First define for any set Y the standard chain complex $\bar{C}_*(Y)$ over $\boldsymbol{Z}$ where a k-simplex is just a k-tuple $(a_0, a_1, \cdots, a_k)$, $a_k \in Y$ and where the boundary homomorphism is given by the usual formula

$$\partial(a_0, \cdots, a_k) = \sum_i (-1)^i (a_0, \cdots, \hat{a}_i, \cdots, a_k) .$$

$\bar{C}_*(Y)$ is always acyclic and $H_0(\bar{C}_*(Y))=\boldsymbol{Z}$.

Now for the vector space V consider the double complex $\bar{A}_{p,q}$, $p\geqq -1$, $q\geqq 0$:

$$(5.5)\qquad \bar{A}_{p,q}=\begin{cases}\bar{C}_q(S(V)), & p=-1\,,\\ \coprod_{U_0\supseteqq\cdots\supseteqq U_p}\bar{C}_q(S(U_p)), & p\geqq 0\,,\end{cases}$$

where as usual $(U_0\supseteqq\cdots\supseteqq U_p)$ runs through all flags of proper subspaces of V. The differentials ∂' and ∂'' are defined by the same formulas as in section 3 for the complex (3.2). Again if we put

$$\bar{C}_*(S(V))^{n-1}=\sum_{U_0\subset V}\bar{C}_*(S(U_0))\subseteqq\bar{C}_*(S(V))$$

we obtain

$$H_p(\bar{A}_{*,q})=\begin{cases}\bar{C}_q(S(V))/\bar{C}_q(S(V))^{n-1}, & p=-1\,,\\ 0, & p\geqq 0\,,\end{cases}$$

so the total complex $\bar{A}_*$ satisfies

$$(5.6)\qquad H_k(\bar{A}_*)\cong H_{k+1}(\bar{C}_*(S(V))/\bar{C}_*(S(V))^{n-1})$$

which in particular is zero for $k<n-1$. On the other hand $H_q(\bar{A}_{p,*})=0$ for $q>0$, and

$$H_0(\bar{A}_{p,*})=\begin{cases}\boldsymbol{Z}, & p=-1\\ \coprod_{U_0\supseteqq\cdots\supseteqq U_p}\boldsymbol{Z}, & p\geqq 0\end{cases}$$

is just the augmented chain complex for $\mathcal{T}(V)$, hence

$$\tilde{H}_k(\mathcal{T}(V),\boldsymbol{Z})\cong H_k(\bar{A}_*)$$

and the result follows from (5.6).

The polytope module.

In view of (5.1) we now make the following

Definition 5.7. For dim $V=n+1\geqq 1$ define the *polytope module* Pt(V) to be the O(V)-module

$$\mathrm{Pt}(V)=H_n(C_*(S(V))/C_*(S(V))^{n-1})$$

where $C_*(S(V))$ is the chain complex of simplices of diameter $<\pi$ as in section 2.

Hence for $G\subseteqq \mathrm{O}(n+1)$

$$(5.8)\qquad \mathcal{P}(S^{n+1},G)\cong H_0(G,\mathrm{Pt}(\boldsymbol{R}^{n+1})^t)\,.$$

Also let us use the notation $\Omega(V)$ for the *orientation module*, i.e. $\Omega(V)\subseteqq\Lambda_{\boldsymbol{R}}^{n+1}(V)$ is the lattice of elements with norm an integer (with respect to the metric in V). Thus $\Omega(\boldsymbol{R}^{n+1})=\boldsymbol{Z}^t$ as $\mathrm{O}(n+1)$-module for $n\geqq 0$ and $\Omega(0)=\boldsymbol{Z}$. With this notation we prove

Theorem 5.9. *For* $\dim V=n+1\geqq 1$ *we have*

a) $H_q(C_*(S(V))/C_*(S(V))^{n-1})=0$ *for* $q\neq n$

b) *There is a filtration of* $\mathrm{O}(V)$*-modules*

$$\Omega(V) = F_{-1}\subseteqq F_0\subseteqq\cdots\subseteqq F_p\subseteqq\cdots\subseteqq F_n = \mathrm{Pt}(V)$$

such that

$$F_p/F_{p-1}\cong \coprod_{V^{n-p}} \mathrm{St}(V/V^{n-p})\otimes\Omega(V^{n-p}),\quad p=0,1,2,\cdots$$

where V^q *runs through all* q*-dimensional subspaces of* V.

Proof. This is another double complex argument: This time let

$$A_{p,q}=\begin{cases} C_q(S(V)), & p=-1\\ \coprod\limits_{\sigma_0\supseteqq\cdots\supseteqq\sigma_p} C_q(S(U_p)), & p\geqq 0\,, \end{cases}$$

and again

$$H_p(A_{*,q})=\begin{cases} C_q(S(V))/C_q(S(V))^{n-1}, & p=-1\\ 0, & p\geqq 0 \end{cases}$$

so for the total complex A_* we have

$$H_k(A_*)\cong H_{k+1}(C_*(S(V))/C_*(S(V))^{n-1}),\quad \forall k\,. \tag{5.10}$$

Thus the double complex gives rise to a spectral sequence converging to $H(A_*)$ and

$$E^1_{p,q}=\begin{cases} H_q(C_*(S(V))), & p=-1\,,\\ \coprod\limits_{\sigma_0\supseteqq\cdots\supseteqq\sigma_p} H_q(C_*(S(U_p))), & p\geqq 0\,. \end{cases} \tag{5.11}$$

Now the complex $C_*(S(V))$ is chain equivalent to the singular complex $C^{\mathrm{top}}_*(S(V))$ of the topological space $S(V)$. In fact as mentioned in the proof of theorem 2.3 every simplex $\sigma=(a_0,\cdots,a_k)$ of diameter $<\pi$ gives rise to a continuous map $f\colon \Delta^k\rightarrow S(V)$ defined by geodesic arcs and thus there is a natural inclusion

$$C_*(S(V))\subseteqq C^{\mathrm{top}}_*(S(V))$$

which is easily seen to be a homology isomorphism (cf. Dupont [8, chapter 1, exercise 4]). Hence also for $U\subseteqq V$ a subspace

$$H_q(C_*(S(U)))\cong H_q(S(U)) \tag{5.12}$$

which is zero except for $q=0$ or $q=\dim U-1$. Notice that

$$\tilde{H}_{\dim U-1}(S(U)) = \Omega(U)\,.$$

Therefore for $p\geqq 0$ and $q>0$ we have

$$E^1_{p,q} = \coprod_{V^{q+1}} \coprod_{U_0\supseteqq\cdots\supseteqq U_p=V^{q+1}} \Omega(V^{q+1})$$

where V^{q+1} runs through all subspaces of dimension $q+1$. Also the differential d^1 is easily identified with the direct sum of the boundary homomorphisms for the Tits-complexes $\mathcal{T}(V/V^{q+1})$. Hence

$$E^2_{p,q} = \coprod_{V^{q+1}} \tilde{H}_{p-1}(\mathcal{T}(V/V^{q+1}), \Omega(V^{q+1})), \quad p\geqq 0, q>0\,,$$

which by proposition 4.3 is zero for $p\neq n-q-1$ and

$$(5.13) \qquad E^2_{p,q}\cong \coprod_{V^{q+1}} \mathrm{St}(V/V^{q+1})\otimes\Omega(V^{q+1}) \quad \text{for } p+q=n-1\,.$$

For $q=0$ and $p\geqq 0$ the augmentation

$$C_0(S(U_p)) \to \mathbf{Z}$$

gives rise to an exact sequence

$$0\to \coprod_{V^1}\coprod_{U_0\supseteqq\cdots\supseteqq U_p=V^1}\Omega(V^1)\to E^1_{p,q}\to \coprod_{U_0\supseteqq\cdots\supseteqq U_p}\mathbf{Z}\to 0$$

and hence to an exact sequence

$$(5.14) \qquad 0\to \coprod_{V^1}\mathrm{St}(V/V^1)\otimes\Omega(V^1)\to E^2_{n-1,0}\to \mathrm{St}(V)\to 0$$

whereas $E^1_{p,0}=0$ for $p\neq n-1$. Finally $E^2_{-1,q}=0$ for $q\neq n$ and

$$(5.15) \qquad E^2_{-1,n}\cong H_n(S(V))\cong\Omega(V)\,.$$

It follows that $E^2_{p,q}$ is concentrated along the counter-diagonal $p+q=n-1$ and hence the theorem follows by (5.13)–(5.15).

The suspension homomorphism.

Geometrically the relation between the modules Pt and St can be described in terms of the *suspension homomorphism*:

For $W\subset V$ a hyperplane there is a natural map

$$(5.16) \qquad \Sigma_W\colon C_k(S(W))\otimes\Omega(V/W)\to C_{k+1}(S(V))$$

defined for a given orientation of V/W by sending a simplex $\sigma=(a_0.\cdots,a_k)$ of $S(W)$ to the chain

$$(5.17) \qquad \Sigma_W(a_0,\cdots,a_k) = (e,a_0,\cdots,a_k)-(-e,a_0,\cdots,a_k)$$

where $e\in S(V)\cap W^\perp$ is positive with respect to the given orientation of $W^\perp$.

Notice that Σ_W maps $C_*(S(W))^p \otimes \Omega(V/W)$ to $C_*(S(V))^{p+1}$ and in particular we obtain

$$\Sigma_W \colon \mathrm{Pt}(W) \otimes \Omega(V/W) \to \mathrm{Pt}(V)$$

Corollary 5.18. a) *There is an exact sequence of* $\mathrm{O}(V)$*-modules*

$$\coprod_{V^n} \mathrm{Pt}(V^n) \xrightarrow{\Sigma} \mathrm{Pt}(V) \xrightarrow{h} \mathrm{St}(V) \to 0$$

where Σ *is the sum of all* Σ_{V^n}*'s,* V^n *running through all hyperplanes.*

b) *In particular there is an exact sequence*

$$\mathcal{P}(\mathrm{S}^{n-1}) \xrightarrow{\Sigma} \mathcal{P}(\mathrm{S}^n) \xrightarrow{h} H_0(\mathrm{O}(n+1), \mathrm{St}(\boldsymbol{R}^{n+1})^{\mathrm{t}}) \to 0$$

where Σ *is the suspension map sending a polytope in* S^{n-1} *to its join with the north and south pole.*

Proof. b) clearly follows from a) and (5.8). For a) we proceed as follows:

We shall use the notation $A_{*,*}(V)$ for the double complex considered in the proof of theorem 5.9 and similarly $E^r_{*,*}(V)$ for the associated spectral sequences etc.

Now fix a hyperplane $W \subseteqq V$ and let the line $L = W^\perp$ be oriented by the vector e. Then for σ a simplex of $S(W)$ we clearly have

$$(5.19) \qquad (\partial \circ \Sigma_W - \Sigma_W \circ \partial)(\sigma) = \begin{cases} 0, & \dim \sigma > 0\,, \\ (e) - (-e), & \dim \sigma = 0\,. \end{cases}$$

More generally we extend the definition of Σ_W to a map

$$\Sigma'' = \Sigma''_W \colon A_{p,q}(W) \otimes \Omega(V/W) \to A_{p,q+1}(V)$$

by sending a simplex $\sigma \in C_q(S(U_p))\,(U_0 \supseteqq \cdots \supseteqq U_p)$ to

$$\Sigma_{U_p}(\sigma) \in C_{q+1}(S(U_p \oplus L))\,(U_0 \oplus L \supseteqq \cdots \supseteqq U_p \oplus L)$$

and here Σ'' satisfies

$$(5.20) \qquad (\partial'' \circ \Sigma'' - \Sigma'' \circ \partial'')(\sigma(U_0 \supseteqq \cdots \supseteqq U_p))$$
$$= \begin{cases} 0, & \dim \sigma > 0\,, \\ (-1)^p[(e) - (-e)]\,(U_0 \oplus L \supseteqq \cdots \supseteqq U_p \oplus L), & \dim \sigma = 0\,. \end{cases}$$

On the other hand we can define a map

$$\Sigma' = \Sigma'_W \colon A_{p,q}(W) \otimes \Omega(V/W) \to A_{p+1,q}(V)$$

by

$$\Sigma'(\sigma(U_0 \supseteqq \cdots \supseteqq U_p))$$

$$= \begin{cases} 0, & \dim \sigma > 0\,, \\ [(e)-(-e)]\,(U_0 \oplus L \supseteqq \cdots \supseteqq U_p \oplus L \supseteqq L), & \dim \sigma = 0\,, \end{cases}$$

and this time

$$(5.21) \qquad (\partial' \circ \Sigma' - \Sigma' \circ \partial')\,(\sigma(U_0 \supseteqq \cdots \supseteqq U_p))$$
$$= \begin{cases} 0, & \dim \sigma = 0\,, \\ (-1)^{p+1}[(e)-(-e)]\,(U_0 \oplus L \supseteqq \cdots \subseteqq U_p \oplus L), & \dim \sigma = 0\,. \end{cases}$$

It follows easily that the map of total complexes

$$\Sigma_W = \Sigma'_W + \Sigma''_W \colon A_k(W) \otimes \Omega(V/W) \to A_{k+1}(V)$$

commutes with the total differential $\partial' + \partial''$. Now checking through the spectral sequences one sees that Σ_W induces a map of filtrations

$$\Sigma_W \colon F_p(W) \otimes \Omega(V/W) \to F_{p+1}(V)$$

such that there is a commutative diagram

$$\begin{tikzcd}
(F_p(W)/F_{p-1}(W)) \otimes \Omega(V/W) \arrow[r, "\Sigma_W"] \arrow[d, "\cong"] & F_{p+1}(V)/F_p(V) \arrow[d, "\cong"] \\
\coprod_{W^{n-p-1}} \mathrm{St}(W/W^{n-p-1}) \otimes \Omega(W^{n-p-1}) \otimes \Omega(V/W) \arrow[r] & \coprod_{V^{n-p}} \mathrm{St}(V/V^{n-p}) \otimes \Omega(V^{n-p})
\end{tikzcd}$$

where the bottom map is induced by sending a flag $W \supset U_0 \supseteqq \cdots \supseteqq U_p = W^{n-p-1}$ to the flag $V \supset U_0 \oplus L \supseteqq \cdots \supseteqq U_p \oplus L = W^{n-p-1} \oplus L = V^{n-p}$ (together with the isomorphism

$$\Omega(W^{n-p-1}) \otimes \Omega(L) \cong \Omega(W^{n-p-1} \oplus L))\,.$$

This map is clearly an isomorphism onto the sum

$$\coprod_{V^{n-p} \supseteqq L} \mathrm{St}(V/V^{n-p}) \otimes \Omega(V^{n-p})\,.$$

Hence by induction over the filtration $F_p(V)$ it follows easily that every element of $F_{n-1}(V)$ is a sum of suspensions, and hence a) follows from

$$\mathrm{Pt}(V)/F_{n-1}(V) \cong \mathrm{St}(V)$$

REMARKS. 1. The map

$$h \colon \mathscr{P}(\mathrm{S}^n,\ \{1\}) = \mathrm{Pt}(\boldsymbol{R}^n)^{\mathrm{t}} \to \mathrm{St}(\boldsymbol{R}^{n+1})^{\mathrm{t}}$$

can be explicitly described similarly to the Hadwiger map in section 3: Thus let $\Phi = (U_0 \supset \cdots \supset U_{n-1})$ be a strict flag (so in particular dim $U_{n-1} = 1$; then the component $h_\Phi(P) \in \boldsymbol{Z}$ for P a spherical polytope is given as follows: Give $U_{-1} =$

$\boldsymbol{R}^{n+1}$ the standard orientation and choose orientations of $U_0,\cdots,U_{n-1}$. Now for any sequence of geometric simplices in a triangulation of P, $\bar{\Delta}=(\Delta_{-1}\supset\Delta_0\supset\cdots\supset\Delta_{n-1})$ (so that Δ_{n-1} is a point) for which $\Delta_i\subseteq S(U_i)$, $i=-1,\cdots,n-1$, define $\varepsilon_i=\pm 1$ as in (3.14) for $i=0,\cdots,n-1$ and $\varepsilon_n=\pm 1$ depending on whether the point Δ_{n-1} is the positive or negative vector in U_{n-1}. Then the integer

$$h_\Phi(P)=\sum_{\bar{\Delta}\| \Phi}\varepsilon_1\cdots\varepsilon_n$$

is easily seen to be independent of choice of orientations of $U_0,\cdots,U_{n-1}$.

2. For *n even* the element $-\mathrm{id}\in \mathrm{O}(n+1)$ acts by multiplication by -1 on $\mathrm{St}(\boldsymbol{R}^{n+1})^{\mathrm{t}}$. Hence by corollary 5.18 b) the cokernel of

$$(5.22)\qquad \Sigma\colon \mathcal{P}(\mathrm{S}^{n-1})\to\mathcal{P}(\mathrm{S}^n),\quad n \text{ even},$$

is annihilated by 2, and since by a classical argument $\mathcal{P}(\mathrm{S}^n)$ is 2-divisible (see Sah [19, chapter 1, proposition 4.3]) we have thus reproved the result of Sah [19, chapter 6, proposition 2.2] that Σ is surjective in this case. It has recently been proved by Sah [20] that (5.22) is actually an isomorphism.

The Lusztig exact sequence.

Corollary 5.19 suggests an inductional proceedure for calculating $\mathcal{P}(\mathrm{S}^n)$ by calculating $H_0(\mathrm{O}(n+1),\ \mathrm{St}(\boldsymbol{R}^{n+1})^{\mathrm{t}})$. For this we can use the Lusztig exact sequence (Lusztig [16, §1.13 (b)])

$$(5.23)\qquad 0\to\mathrm{St}(\boldsymbol{R}^{n+1})\to\coprod_{V^n}\mathrm{St}(V^n)\to\cdots\to\coprod_{V^1}\mathrm{St}(V^1)\to\boldsymbol{Z}\to 0$$

as in section 4. However it is just as convenient to use the module Pt directly:

Proposition 5.24. *There is an exact sequence of* $\mathrm{O}(n+1)$*-modules*

$$0\to\boldsymbol{Z}^{\mathrm{t}}\to\mathrm{Pt}(\boldsymbol{R}^{n+1})\to\coprod_{V^n}\mathrm{Pt}(V^n)\to\cdots\to\coprod_{V^1}\mathrm{Pt}(V^1)\to\boldsymbol{Z}\to 0\,.$$

Proof. Consider the first quadrant spectral sequence for the chain complex $C_*(\mathrm{S}^n)$ with the filtration $C_*(\mathrm{S}^n)^p$ (see MacLane [17, chapter XI, §3]). Now there is a natural isomorphism of chain complexes

$$C_*(\mathrm{S}^n)^p/C_*(\mathrm{S}^n)^{p-1}\cong\coprod_{V^{p+1}}C_*(S(V^{p+1}))/C_*(S(V^{p+1}))^{p-1}\,.$$

Now by theorem 5.9

$$H_i(C_*(S(V^{p+1}))/C_*(S(V^{p+1}))^{p-1})=\begin{cases}0, & i\neq p\,,\\ \mathrm{Pt}(V^{p+1}), & i=p\,.\end{cases}$$

Hence for the spectral sequence we have

$$E^1_{p,q}=\begin{cases}0, & q\neq 0\,,\\ \coprod_{V^{p+1}}\mathrm{Pt}(V^{p+1}), & q=0\,,\end{cases}$$

and it follows that the only non-zero differential is d^1. On the other hand the spectral sequence converges to $H(C_*(\mathrm{S}^n))=H_*(\mathrm{S}^n)$ by (5.12). It follows that the spectral sequence reduces to an exact sequence

$$0 \to H_n(\mathrm{S}^n) \to E^1_{n,0} \xrightarrow{d^1} E^1_{n-1,0} \xrightarrow{d^1} \cdots \xrightarrow{d^1} E^1_{0,0} \to H_0(\mathrm{S}^n) \to 0$$

which for $n>0$ is just the sequence (5.24). For $n=0$ (5.24) is simply the sequence

$$0 \to \boldsymbol{Z}^t \to \mathrm{Pt}(\boldsymbol{R}^1) \to \boldsymbol{Z} \to 0$$

which is obviously exact.

The Serre class C_2.

Before we start using the exact sequences in concrete cases it is convenient to introduce

DEFINITION 5.25. Let C_2 denote the Serre class (see e.g. Spanier [23, chapter 9, § 6]) of abelian 2-primary torsion groups of finite exponent, i.e., $A \in C_2$ iff for some integer N, $2^N A=0$.

REMARKS. 1. Observe that C_2 is an ideal of abelian groups.

2. If $A \to B \to C$ is exact mod C_2 and $A \approx A'$, $B \approx B'$, and $C \approx C'$ mod C_2 then there is an exact sequence mod C_2: $A' \to B' \to C'$.

3. If B is 2-divisible and f: $A \to B$ is surjective mod C_2 then clearly f is surjective. Similarly if A is 2-divisible, B is 2-torsion free, and f is injective mod C_2, then f is injective. These remarks are particularly useful in connection with $\mathcal{P}(\mathrm{S}^n)$ since this group is 2-divisible (Sah [19, chapter 1, proposition 4.3]).

We now have as usual by Shapiro's lemma and "center kills" (lemma 4.2–4.3):

Lemma 5.26.

a) $H_*(\mathrm{O}(n+1), [\coprod_{V^p} \mathrm{Pt}(V^p)]^t) \cong H_*(\mathrm{O}(p)\times \mathrm{O}(n-p+1), \mathrm{Pt}(\boldsymbol{R}^p)^t \otimes \boldsymbol{Z}^t)$

b) *If* $p \equiv n$ mod 2 *then*

$$H_*(\mathrm{O}(n+1), [\coprod_{V^p} \mathrm{Pt}(V^p)^t] \cong 0 \bmod C_2$$

With this lemma we obtain from the exact sequence in corollary 5.24 a hyperhomology spectral sequence as in section 4 relating $\mathcal{P}(\mathrm{S}^n)$ to $H_*(\mathrm{O}(n+1), \boldsymbol{Z}^t)$.

$\mathcal{P}(S^n)$ in low dimensions.

For $n=1, 2$, $\mathcal{P}(\mathrm{S}^n)$ are of course well-known (cf. Sah [19, chapter 6]), but we include these cases as an illustration:

EXAMPLE 5.27. $\mathscr{P}(\mathrm{S}^1)$. From the exact sequence

$$0 \to \boldsymbol{Z} \to \mathrm{Pt}(\boldsymbol{R}^2)^{\mathrm{t}} \to [\coprod_{V^1} \mathrm{Pt}(V^1)]^{\mathrm{t}} \to \boldsymbol{Z}^{\mathrm{t}} \to 0$$

and lemma 5.26 b) we immediately conclude

$$\mathscr{P}(S^1)/\boldsymbol{Z} \cong H_0(\mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^{\mathrm{t}}/\boldsymbol{Z}) \approx H_1(\mathrm{O}(2), \boldsymbol{Z}^{\mathrm{t}}) \approx \boldsymbol{R}/\boldsymbol{Z} \bmod \mathcal{C}_2$$

where $\boldsymbol{Z} \subseteqq \mathscr{P}(\mathrm{S}^1)$ is the subgroup generated by the class $[S^1]$. From this it is easy to deduce the expected conclusion that $\mathscr{P}(\mathrm{S}^1) \cong \boldsymbol{R}$ with the isomorphism given by "Length".

EXAMPLE 5.28. $\mathscr{P}(\mathrm{S}^2)$. By the remark 2 following corollary 5.18 it now clearly follows that "Area" induces an isomorphism $\mathscr{P}(\mathrm{S}^2) \cong \boldsymbol{R}$.

EXAMPLE 5.29. $\mathscr{P}(\mathrm{S}^3)$. Split the exact sequence (5.24) into short exact sequences

(i) $0 \to \boldsymbol{Z}^{\mathrm{t}} \to \mathrm{Pt}(\boldsymbol{R}^4) \to Z_3 \to 0$
(ii) $0 \to Z_3 \to \coprod_{V^3} \mathrm{Pt}(V^3) \to Z_2 \to 0$
(iii) $0 \to Z_2 \to \coprod_{V^2} \mathrm{Pt}(V^2) \to Z_1 \to 0$
(iv) $0 \to Z_1 \to \coprod_{V^1} \mathrm{Pt}(V^1) \to \boldsymbol{Z} \to 0.$

By (i), $\mathscr{P}(\mathrm{S}^3)/\boldsymbol{Z} \cong H_0(\mathrm{O}(4), Z_3^{\mathrm{t}})$ where $\boldsymbol{Z} \subseteqq \mathscr{P}(S^3)$ is the subgroup generated by $[\mathrm{S}^3]$. Furthermore by lemma 5.26 we conclude from (ii) and (iv):

$$H_i(\mathrm{O}(4), Z_3^{\mathrm{t}}) \approx H_{i+1}(\mathrm{O}(4), Z_2^{\mathrm{t}}) \bmod \mathcal{C}_2, \ \forall i, \tag{5.30}$$

$$H_i(\mathrm{O}(4), Z_1^{\mathrm{t}}) \approx H_{i+1}(\mathrm{O}(4), \boldsymbol{Z}^{\mathrm{t}}) \bmod \mathcal{C}_2, \ \forall i. \tag{5.31}$$

Also using the splitting $\mathrm{Spin}(4) \cong S_+ \times S_-$ with $S_\pm \cong \mathrm{Spin}(3) \cong \mathrm{SU}(2)$ as in the proof of corollary 4.18, it is easy to see that the natural inclusion $\mathrm{SU}(2) \to \mathrm{O}(4)$ induces an equivalence

$$H_i(\mathrm{SU}(2), \boldsymbol{Z}) \approx H_i(\mathrm{O}(4), \boldsymbol{Z}^{\mathrm{t}}) \bmod \mathcal{C}_2, \qquad i = 1, 2, 3. \tag{5.35}$$

Finally by lemma 5.26 a)

$$H_*(\mathrm{O}(4), [\coprod_{V^2} \mathrm{Pt}(V^2)^{\mathrm{t}}]) \cong H_*(\mathrm{O}(2) \times \mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^{\mathrm{t}} \otimes \boldsymbol{Z}^{\mathrm{t}}).$$

Here as in example 5.27

$$\begin{aligned} &H_0(\mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^{\mathrm{t}}) \cong \boldsymbol{R}, \\ &H_1(\mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^{\mathrm{t}}) \approx H_1(\mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^{\mathrm{t}}/\boldsymbol{Z}) \\ &\qquad\qquad \approx H_2(\mathrm{O}(2), \boldsymbol{Z}^{\mathrm{t}}) \approx 0 \qquad \bmod \mathcal{C}_2. \end{aligned}$$

Hence by Künneth's theorem

$$\begin{aligned} &H_1(\mathrm{O}(2)\times \mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^t\otimes \boldsymbol{Z}^t)\approx \boldsymbol{R}\otimes H_1(\mathrm{O}(2), \boldsymbol{Z}^t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \approx \boldsymbol{R}\otimes \boldsymbol{R}/\boldsymbol{Z} \quad \bmod \mathcal{C}_2, \\ &H_2(\mathrm{O}(2)\times \mathrm{O}(2), \mathrm{Pt}(\boldsymbol{R}^2)^t\otimes \boldsymbol{Z}^t)\approx 0 \qquad\qquad \bmod \mathcal{C}_2. \end{aligned}$$

The exact sequence (iii) together with (5.30)–(5.32) thus yields

Proposition 5.33. *There is an exact sequence* mod $\mathcal{C}_2$:

$$0 \to H_3(\mathrm{SU}(2), \boldsymbol{Z}) \to \mathscr{P}(S^3)/\boldsymbol{Z} \xrightarrow{D} \boldsymbol{R}\otimes \boldsymbol{R}/\boldsymbol{Z} \to H_2(\mathrm{SU}(2), \boldsymbol{Z}) \to 0\,.$$

Here $\boldsymbol{Z}\subseteq \mathscr{P}(S^3)$ *is generated by* $[S^3]$ *and* D *is the spherical Dehn invariant.*

REMARK. The identification of D in the above sequence with the Dehn invariant is completely analogous to the proof of corollary 4.12.

Proof of theorem 1.3. a) It is easy to see that the last map in the sequence factorizes as

(5.34)
$$\begin{tikzcd} \boldsymbol{R}\otimes \boldsymbol{R}/\boldsymbol{Z} \arrow[r] & \boldsymbol{R}/\boldsymbol{Z}\otimes \boldsymbol{R}/\boldsymbol{Z} \arrow[r] & \Lambda^2_{\boldsymbol{Z}}(\boldsymbol{R}/\boldsymbol{Z}) \arrow[r] \arrow[d, dash, "\cong"] & H_2(\mathrm{SU})(2), \boldsymbol{Z}) \\ & & H_2(\mathrm{U}(1), \boldsymbol{Z}) \arrow[ur] & \end{tikzcd}$$

where U(1)→SU(2) is the inclusion as diagonal matrices. This map induces a surjection by a theorem of J. Mather (see Alperin-Dennis [2]), and hence also the composite map (5.34) is surjective (not just mod $\mathcal{C}_2$). Since $\boldsymbol{R}\otimes \boldsymbol{R}/\boldsymbol{Z}$ is divisible and torsion free and since $\mathscr{P}(\mathrm{S}^3)$ is 2-divisible it follows easily that exactness mod $\mathcal{C}_2$ implies actual exactness at the places $\mathscr{P}(S^3)/\boldsymbol{Z}$ and $\boldsymbol{R}\otimes \boldsymbol{R}/\boldsymbol{Z}$.

b) The map

(5.35)
$$H_3(\mathrm{SU}(2), \boldsymbol{Z}) \to \mathscr{P}(\mathrm{S}^3)/\boldsymbol{Z}$$

can be described on the chain level as follows: Let $\bar{C}_*(\mathrm{SU}(2))$ denote the homogeneous bar complex. For $i=0, 1, 2, 3$ one can construct by successive choices a SU(2)-equivariant homomorphism

$$\varphi\colon\ \bar{C}_i(\mathrm{SU}(2)) \to C_i(\mathrm{S}^3)$$

which commutes with the boundary maps. It is easily checked that the map (5.35) is induced by φ. On the other hand the Cheeger-Simons class $\hat{C}_2\in H^3(\mathrm{SU}(2), \boldsymbol{R}/\boldsymbol{Z})$ can be represented by the SU(2)-invariant homomorphism

$$\bar{C}_3(\mathrm{SU}(2)) \xrightarrow{\varphi} C_3(\mathrm{S}^3) \xrightarrow{u} \boldsymbol{R} \to \boldsymbol{R}/\boldsymbol{Z}$$

where u: $C_3(\mathrm{S}^3)\to \boldsymbol{R}$ is given by integration of the volume form divided by the volume of S^3 (c.f. Cheeger-Simons [6, § 8] or Dupont [8, chapter 9, exercise 3]).

This proves the statement.

In the following let $A \subseteq H_3(\mathrm{SU}(2), \boldsymbol{Z})$ be the subgroup in $\mathcal{C}_2$ as in corollary 1.3.

Corollary 5.36. a) *A spherical polyhedron P in which all dihedral angles are rational multiples of* π *determines a homology class* $\hat{P} \in H_3(\mathrm{SU}(2), \boldsymbol{Z})/A$. *Furthermore*

$$\hat{C}_2(\hat{P}) = \mathrm{Vol}(P)/\mathrm{Vol}(\mathrm{S}^3)\,.$$

b) *Let* $\mathcal{K} \subseteq \mathcal{P}(\mathrm{S}^3)$ *be the kernel of D. Then* $\mathcal{K}$ *has infinite rank.*

c) *The restriction of* Vol: $\mathcal{K} \to \boldsymbol{R}$ *has countable image.*

d) Vol *and D are separating invraiants for* $\mathcal{P}(\mathrm{S}^2)$ *iff the evaluation* $\langle \hat{C}_2, - \rangle$: $H_3(\mathrm{SU}(2), \boldsymbol{Z}) \to \boldsymbol{R}/\boldsymbol{Z}$ *has kernel in* $\mathcal{C}_2$.

Proof. a) is obvious since $D(P)=0$ when all dihedral angles are rational multiples of π.

b) is a restatement of the result of Cheeger [5] that $H_3(\mathrm{O}(4), \boldsymbol{Z})$ and hence also $H_3(\mathrm{SU}(3), \boldsymbol{Z})$ has infinite rank.

c) By the rigidity of the class $\hat{C}_2$ (Cheeger-Simons [6, proposition 8.10]) it is easy to see that the evaluation on $H_3(\mathrm{SU}(2), \boldsymbol{Z})$ only depends on the algebraic points of SU(2) (considered as an algebraic group over $\boldsymbol{R}$), and hence has at most a countable image (cf. Cheeger [5]). On the other hand $\boldsymbol{Q}/\boldsymbol{Z}$ is contained in the image of the evaluation of $\hat{C}_2$.

d) The "only if" part is obvious. Now assume that $\langle \hat{C}_2, - \rangle$ has kernel in $\mathcal{C}_2$. Then also the kernel of u: $\mathcal{K}/\boldsymbol{Z} \to \boldsymbol{R}/\boldsymbol{Z}$ is in $\mathcal{C}_2$ and let 2^N be the exponent. Now suppose $x \in \mathcal{P}(\mathrm{S}^3)$ satisfies $\mathrm{Vol}(x)=0$, $D(x)=0$. Since $\mathcal{P}(\mathrm{S}^3)$ is 2-divisible, $x=2^N y$ and clearly $\mathrm{Vol}(y)=0$, $D(y)=0$. Therefore $y \in \mathcal{K}$ and $u(y)=0$, hence

$$x = 2^N y \equiv 0 \qquad \mathrm{mod}\ [\mathrm{S}^3]$$

by assumption, and hence $x=0$ since $\mathrm{Vol}(x)=0$.

REMARKS. 1. Corollary 5.36 a) generalizes a theorem of Thurston (see Cheeger-Simons [6, theorem 8.18]) associating homology classes to "almost all" spherical simplices with rational dihedral angles.

2. As mentioned in the proof of b), $H_3(\mathrm{SU}(2), \boldsymbol{Z})$ contains a copy of $\boldsymbol{Q}/\boldsymbol{Z}$. Here the cyclic subgroup generated by $1/k$, $k \in \boldsymbol{Z}$, is the image in H_3 under the inclusion of the cyclic subgroup of $\mathrm{U}(1) \subset \mathrm{SU}(2)$ generated by a k'th root of unity. As an element of $\mathcal{K} \in \mathcal{P}(S^3)/\boldsymbol{Z}$ the generator is represented by the double suspension of an arc of length $2\pi/k$. One can actually show that every polyhedron which is a fundamental region for a finite subgroup of O(4) is scissors congruent to one of these double suspensions.

3. The volumes of spherical polyhedra, especially simplices, has been studied by Schläfli [22] (see also Coxeter [7] and Aomoto [3]). However, it is difficult to see from his formulas whether the volume of a polyhedron with rational dihedral angles is a rational multiple of $\mathrm{Vol}(S^3)$ $(=2\pi^2)$ or not except in the obvious cases of a fundamental region for a finite group.

4. Corollary 1.3 also gives a presentation of $H_2(\mathrm{SU}(2), \boldsymbol{Z})$ as a quotient of $\Lambda^2_{\boldsymbol{Z}}(\boldsymbol{R}/\boldsymbol{Z})$ by the image of the composite

$$(5.37)\qquad \mathscr{P}(\mathrm{S}^3) \xrightarrow{D} \boldsymbol{R}\otimes\boldsymbol{R}/\boldsymbol{Z} \to \Lambda^2_{\boldsymbol{Z}}(\boldsymbol{R}/\boldsymbol{Z}),$$

thus answering a question stated by R. Alperin [1, p. 284]: Since $\mathscr{P}(\mathrm{S}^3)$ is generated by *orthoschemes* (cf. Schläfli [22, p. 229]) we must compute $D(\Delta)$ for Δ being such a simplex in which 3 of the dihedral angles are right. If α, β, γ are the remaining angles at the edges of length a, b, c respectively, then by Schäfli we must have $\sin^2\alpha\ \sin^2\gamma > \cos^2\beta$ and

$$\cos a = \frac{\sin\alpha\cos\gamma}{\sqrt{\sin^2\alpha-\cos^2\beta}}, \qquad \cos c = \frac{\cos\alpha\sin\gamma}{\sqrt{\sin^2\gamma-\cos^2\beta}}$$

$$\cos b = \frac{\cos\alpha\cos\beta\cos\gamma}{\sqrt{\sin^2\alpha-\cos^2\beta}\,\sqrt{\sin^2\gamma-\cos^2\beta}}\,.$$

Therefore the image in $\Lambda^2_{\boldsymbol{Z}}(\boldsymbol{R}/\boldsymbol{Z})$ of the map (5.37) is generated by all elements of the form

$$\frac{a}{2\pi}\wedge\frac{\alpha}{2\pi}+\frac{b}{2\pi}\wedge\frac{\beta}{2\pi}+\frac{c}{2\pi}\wedge\frac{\gamma}{2\pi}$$

where α, β, γ and a, b, c satisfy the above conditions.

Relation with Euclidean case.

We end this section with an application to the Euclidean case. Recall from Sah [19, chapter 7, § 2] that there are generalized Dehn invariants

$$\Psi^{(p)}\colon\ \mathscr{P}(\boldsymbol{R}^n) \to \mathscr{P}(\boldsymbol{R}^{n-p})\otimes\mathscr{P}(\mathrm{S}^{p-1})/\Sigma\mathscr{P}(\mathrm{S}^{p-2})\,.$$

Via the isomorphisms of the corollaries 4.6 and 5.18 b) this decomposes into a sum of homomorphisms

$$\begin{aligned}\Psi^{(p)}_q\colon\ & H_0(\mathrm{O}(n), \mathscr{D}^q(\boldsymbol{R}^n)^t) \to \\ & \to H_0(\mathrm{O}(n-p), \mathscr{D}^q(\boldsymbol{R}^{n-p})^t)\otimes H_0(\mathrm{O}(p), \mathrm{St}(\boldsymbol{R}^p)^t)\,.\end{aligned}$$

In particular for $p=n-q$ this map is easily seen to be induced by the first homomorphism in the following exact sequence:

Proposition 5.38. *Let $V=\boldsymbol{R}^n$ and $1\leqq q\leqq n$. Then there is an exact sequence* *of* O(n)*-modules:*

$$0 \to \mathscr{D}^q(V) \to \coprod_{V^q} \Lambda^q_{\boldsymbol{R}}(V^q) \otimes \mathrm{St}(V/V^q) \to \coprod_{V^{q+1}} \Lambda^q_{\boldsymbol{R}}(V^{q+1}) \otimes \mathrm{St}(V/V^{q+1}) \to$$
$$\to \cdots \to \coprod_{V^{n-1}} \Lambda^q_{\boldsymbol{R}}(V^{n-1}) \to \Lambda^q_{\boldsymbol{R}}(V) \to 0\,.$$

Proof. The maps in this sequence are rather obvious, e.g. the first one is defined by sending a strict flag $(U_0 \supset \cdots \supset U_{n-q-1})$ to the flag $U_0/V^q \cdots U_{n-q-2}/V^q$ where $V^q = U_{n-q-1}$. To prove the exactness we use induction on $n \geqq q$: For $n=q$ the sequence reduces to $\mathscr{D}^q(V) = \Lambda^q_{\boldsymbol{R}}(V)$. In general the sequences for V, V^{n-1}, etc. fit into a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{D}^q(V) \arrow[r] \arrow[d] & \coprod_{V^q} \Lambda^q(V^q) \otimes \mathrm{St}(V/V^q) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \coprod_{V^{n-1}} \mathscr{D}^q(V^{n-1}) \arrow[r] \arrow[d] & \coprod_{V^q \subset V^{n-1}} \Lambda^q(V^q) \otimes \mathrm{St}(V^{n-1}/V^q) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

where the vertical sequences are the Lusztig exact sequences (4.7) and (5.23). The exactness of the top row now easily follows from the exactness of the other rows.

Again this exact sequence gives rise to a hyperhomology spectral sequence which can be used for studying $\mathscr{P}(\boldsymbol{R}^n)$. Here we just consider the following

EXAMPLE 5.39. $\mathscr{P}(\boldsymbol{R}^5)$. In example 4.31 we were left to study $H_0(\mathrm{O}(5), \mathscr{D}^1(\boldsymbol{R}^5)^t)$, and for this we now consider the spectral sequence corresponding to the exact sequence in proposition 5.38 for $n=5$, $q=1$. The differentials d^r, $r=1, \cdots, 4$, defined on subgroups of $H_0(\mathrm{O}(5), \mathscr{D}^1(\boldsymbol{R}^5)^t)$ lands in subquotients of $H_{r-1}(\mathrm{O}(r) \times \mathrm{O}(5-r), (\boldsymbol{R}^r)^t \otimes \mathrm{St}(\boldsymbol{R}^{5-r})^t)$. For $r=2$ and 4 this group is clearly zero by "center kills". For $r=3$ it is also zero since $H_2(\mathrm{O}(3), (\boldsymbol{R}^3)^t = 0$ (corollary 1.2 b)) and since

$$H_1(\mathrm{O}(2), (\mathrm{St}(\boldsymbol{R}^2)^t) \approx H_2(\mathrm{O}/(2), \boldsymbol{Z}^t) \approx 0 \qquad \mathrm{mod}\, C_2\,.$$

Now clearly d^1 coincides with the generalized Dehn invariant $\Psi_1^{(4)}$ and the only other possibly non-zero differential is d^5 landing in a quotient of $H_4(\mathrm{O}(5), (\boldsymbol{R}^5)^t)$. Thus we obtain

Proposition 5.40. *The subgroup of* $\mathscr{P}(\boldsymbol{R}^5)$ *of elements in the kernel of "Volume" and the Dehn-invariants* $\Psi^{(2)}$ *and* $\Psi^{(4)}$, *is isomorphic to the quotient* $H_4(\mathrm{O}(5), (\boldsymbol{R}^5)^t)/J$, *where* J *is the subgroup generated by the images of the two maps*

i) $\boldsymbol{R} \otimes H_4(\mathrm{O}(4), \boldsymbol{Z}^t) = H_4(\mathrm{O}(1) \times \mathrm{O}(4), (\boldsymbol{R}^1)^t \otimes \boldsymbol{Z}^t) \to H_4(\mathrm{O}(5), (\boldsymbol{R}^5)^t)$

ii) $H_3(\mathrm{O}(3), (\boldsymbol{R}^3)^t) \otimes H_1(\mathrm{O}(2), \boldsymbol{Z}^t) \to H_4(\mathrm{O}(3) \times \mathrm{O}(2), (\boldsymbol{R}^3)^t \otimes \boldsymbol{Z}^t) \to H_4(\mathrm{O}(5), (\boldsymbol{R}^5)^t)$.

In i) and ii) $\boldsymbol{R}^1 \to \boldsymbol{R}^5$ and $\boldsymbol{R}^3 \to \boldsymbol{R}^5$ are the inclusions on the first coordinates of $\boldsymbol{R}^5$. Proposition 5.40 shows that Hilbert's 3rd problem in $\boldsymbol{R}^5$ is more or less equivalent with the calculation of the homology group $H_4(\mathrm{SO}(5), \boldsymbol{R}^5)$, a group which however seems rather out of reach with the present knowledge.

6. The hyperbolic case

We now come to the scissors congruence group for the hyperbolic space $\mathcal{H}^n$. Let us write $G(n)$ for the group of all isometries. Thus if as a model for $\mathcal{H}^n$ we take the hypersurface in $\boldsymbol{R}^{n+1}$ given by $Q(x)=-1$, $x_0>0$, with $Q(x)=-x_0^2+x_1^2+\cdots+x_n^2$, then $G(n)=\mathrm{O}(1,n)^+$ is the subgroup of the orthogonal group for Q stabilizing the halfspace $x_0>0$. Again we want to study

$$\mathcal{P}(\mathcal{H}^n) = {}_{G(n)}\mathcal{P}(\mathcal{H}^n, \{1\})\,.$$

Not surprisingly we shall use the Tits-complex $\mathcal{T}(\mathcal{H}^n)$ of flags of proper hyperbolic subspaces (including the zero-dimensional ones, i.e. the points), and exactly similar to proposition 5.3 we have

Proposition 6.1. *$\tilde{H}_q(\mathcal{T}(\mathcal{H}^n), \boldsymbol{Z})=0$ for $q\neq n-1$.*

Therefore we make the following

DEFINITION 6.2. Let

$$\mathrm{St}(\mathcal{H}^n) = \tilde{H}_{n-1}(\mathcal{T}(\mathcal{H}^n), \boldsymbol{Z})\,, \qquad n\geqq 0\,,$$

and $\mathrm{St}(\mathcal{H}^0)=\boldsymbol{Z}$ be the *hyperbolic Steinberg module* for $G(n)$.

Similarly to corollary 5.18 and proposition 5.24, we obtain

Proposition 6.3. i) $\mathcal{P}(\mathcal{H}^n)\cong H_0(G(n), \mathrm{St}(\mathcal{H}^n)^t)$.

ii) *There is an exact sequence of $G(n)$-modules*

$$0 \to \mathrm{St}(\mathcal{H}^n) \to \coprod_{V^{n-1}} \mathrm{St}(V^{n-1}) \to \cdots \to \coprod_{V^0} \mathrm{St}(V^0) \to \boldsymbol{Z} \to 0$$

where V^p runs through all p-dimensional hyperbolic subspaces of $\mathcal{H}^n$.

For $V^p\subset\mathcal{H}^n$ a hyperbolic subspace let G_{V^p} denote the stabilizer for V^p, i.e. the subgroup of $G(n)$ mapping V^p into itself. Notice that if $\mathcal{H}^p\subset\mathcal{H}^n$ is the natural inclusion on the first $p+1$ coordinates in the above model then the stabilizer for $\mathcal{H}^p$ is

$$G_{\mathcal{H}^p} = G(p)\times \mathrm{O}(n-p)\,. \tag{6.4}$$

Therefore again by Shapiro's lemma and "center kills" we have

Lemma 6.5.

a) $H_*(G(n), [\coprod_{V^p} \mathrm{St}(V^p)]^t)\cong H_*(G(p)\times \mathrm{O}(n-p), \mathrm{St}(\mathcal{H}^p)^t\otimes \boldsymbol{Z}^t)$.

b) *In particular for $n-p$ odd*

$$H_*(G(n), [\coprod_{V^p} \mathrm{St}(V^p)]^{\mathrm{t}}) \equiv 0 \qquad \bmod \mathscr{C}_2 .$$

Also let us recall (Sah [19, chapter 1, proposition 4.3]) that $\mathscr{P}(\mathscr{H}^n)$ is 2-divisible. Then we can study $\mathscr{P}(\mathscr{H}^n)$ in low dimensions using the exact sequence in proposition 6.3. For $n=1, 2$, the results are of course already known (cf. Sah [19, chapter 8]).

EXAMPLE 6.6. $\mathscr{P}(\mathscr{H}^1)$. For $n=1$ we obtain from proposition 6.3

$$H_i(G(1), \mathrm{St}(\mathscr{H}^1)^{\mathrm{t}}) \approx H_{i+1}(G(1), \boldsymbol{Z}^{\mathrm{t}}) \qquad \bmod \mathscr{C}_2 .$$

Here $G(1)=\mathrm{O}(1,1)^+$ is the semidirect product of $\boldsymbol{R}$ and $\boldsymbol{Z}/2$ where $\boldsymbol{Z}/2$ acts by -1. In particular

$$\mathscr{P}(\mathscr{H}^1) \approx H_1(G(1), \boldsymbol{Z}^{\mathrm{t}}) \approx \boldsymbol{R} \qquad \bmod \mathscr{C}_2$$

so $\mathscr{P}(\mathscr{H}^1) \cong \boldsymbol{R}$ and the isomorphism is given by "length".

EXAMPLE 6.7. $\mathscr{P}(\mathscr{H}^2)$. Using proposition 6.3 and lemma 6.5 we obtain an exact sequence moe $\mathscr{C}_2$

$$\begin{aligned} (6.8) \qquad & H_2(\mathrm{O}(2), \boldsymbol{Z}^{\mathrm{t}}) \to H_2(G(2), \boldsymbol{Z}^{\mathrm{t}}) \to \mathscr{P}(\mathscr{H}^2) \to \\ & \to H_1(\mathrm{O}(2), \boldsymbol{Z}^{\mathrm{t}}) \to H_1(G(2), \boldsymbol{Z}^{\mathrm{t}}) . \end{aligned}$$

If we use the upper halfplane model for $\mathscr{H}^2$, i.e. the set of complex numbers z with $\mathrm{Im}\, z>0$, then $G(2)$ is the semidirect product of $\mathrm{PSl}(2, \boldsymbol{R})$ acting as $z \mapsto (az+b)/(cz+d)$ and the $\boldsymbol{Z}/2$ given by $z \mapsto -\bar{z}$. Since $\mathrm{Sl}(2, \boldsymbol{R})$ is perfect the last term in (6.8) vanishes. Also clearly the first term lies in $\mathscr{C}_2$. Now $H_2(G(2), \boldsymbol{Z}^{\mathrm{t}})$ is equivalent ($\bmod\, \mathscr{C}_2$) to the (-1)-eigenspace in $H_2(\mathrm{Sl}(2, \boldsymbol{R}), \boldsymbol{Z})$ for the automorphism given by conjugating with the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This eigenspace however is contained in the kernel for the map

$$(6.9) \qquad H_2(\mathrm{Sl}(2, \boldsymbol{R}), \boldsymbol{Z}) \to H_2(\mathrm{Sl}(3, \boldsymbol{R}), \boldsymbol{Z})$$

since on the latter group conjugation by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

clearly induces the identity. Now the kernel of (6.9) is infinite cyclic (cf. Sah-Wagoner [21]) corresponding to the central extension given by the universal covering of $\mathrm{Sl}(2, \boldsymbol{R})$. Hence (6.8) reduces to an exact sequence $\bmod\, \mathscr{C}_2$

$$(6.10)\qquad 0 \to \boldsymbol{Z} \to \mathcal{P}(\mathcal{H}^2) \xrightarrow{\sigma} \boldsymbol{R}/\boldsymbol{Z} \to 0$$

and it is easy to see that σ is given (up to a factor a power of 2) by

$$\sigma(\Delta) = -\frac{1}{2\pi}(\alpha+\beta+\gamma)$$

where Δ is a triangle with angles α, β, γ. From (6.10) and the classical formula

$$\text{Area}(\Delta) = \pi-(\alpha+\beta+\gamma)$$

it now easily follows that "Area": $\mathcal{P}(\mathcal{H}^2) \to \boldsymbol{R}$ is an isomorphism.

Notice that if $\Gamma \subset \text{PSl}(2, \boldsymbol{R})$ is a discrete subgroup with $\Gamma\backslash\mathcal{H}^2$ a surface of genus $g>1$, then the image of the fundamental class under the composite

$$H_2(\Gamma\backslash\mathcal{H}^2, \boldsymbol{Z}) = H_2(\Gamma, \boldsymbol{Z}) \to H_2(G(2), \boldsymbol{Z}^t) \to \mathcal{P}(\mathcal{H}^2)$$

can be represented by the regular polygon with $4g$ sides and angle $\pi/2g$ at the vertices.

EXAMPLE 6.11. $\mathcal{P}(\mathcal{H}^3)$. We now prove theorem 1.4:

Using the exact sequence of proposition 6.3 in the same way as in example (5.29) we obtain an exact sequence mod $\mathcal{C}_2$:

$$(6.12)\qquad \begin{aligned} &0 \to H_3(G(3), \boldsymbol{Z}^t) \to \mathcal{P}(\mathcal{H}^3) \to \\ &\to H_1(G(3), [\textstyle\coprod_{V^1} \text{St}(V^1)]^t) \to H_2(G(3), \boldsymbol{Z}^t) \to 0\,. \end{aligned}$$

Here

$$\begin{aligned} H_1(G(3), [\textstyle\coprod_{V^1} \text{St}(V^1)]^t) &\approx H_0(G(1), \text{St}(\mathcal{H}^1)^t) \otimes_{\boldsymbol{Z}} H_1(\text{O}(2), \boldsymbol{Z}^t) \\ &\approx \boldsymbol{R} \otimes_{\boldsymbol{Z}} \boldsymbol{R}/\boldsymbol{Z} \qquad \text{mod } \mathcal{C}_2 \end{aligned}$$

and again the map

$$D\colon \mathcal{P}(\mathcal{H}^3) \to \boldsymbol{R} \otimes_{\boldsymbol{Z}} \boldsymbol{R}/\boldsymbol{Z}$$

is identified with the Dehn invariant.

Now let us use as model for $\mathcal{H}^3$ the upper halfspace bounded by the Riemann sphere $\boldsymbol{C} \cup \{\infty\}$. Then $G(3)$ is the semidirect product of $\text{PSl}(2, \boldsymbol{C})$ and $\boldsymbol{Z}/2$, where $\text{PSl}(2, \boldsymbol{C})$ acts on $\boldsymbol{C} \cup \{\infty\}$ as the Möbius transformations

$$z \mapsto (az+b)/(ca+d)$$

and $\boldsymbol{Z}/2$ acts by complex conjugation. It follows that

$$(6.13)\qquad H_*(G(3), \boldsymbol{Z}^t) \approx H_*(\text{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \qquad \text{mod } \mathcal{C}_2$$

where $^-$ indicates the (-1)-eigenspace for the automorphism induced by complex conjugation. Thus (6.9) is equivalent to the following exact sequence mod C_2:

$$(6.14)\qquad 0 \to H_3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \to \mathcal{P}(\mathcal{H}^3) \xrightarrow{D} \boldsymbol{R} \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}/\boldsymbol{Z} \to$$
$$\to H_2(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \to 0\,.$$

Now by Sah-Wagoner [21]

$$H_2(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z}) \cong K_2(\boldsymbol{C})$$

and this group is divisible, hence the last map of (6.14) is surjective. Since $\mathcal{P}(\mathcal{H}^3)$ is 2-divisible and $\boldsymbol{R}\otimes\boldsymbol{R}/\boldsymbol{Z}$ is divisible and torsion free we conclude that (6.14) is exact at these two places. This ends the proof of theorem 1.4.

REMARKS 1. This time $H_2(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- = K_2(\boldsymbol{C})^-$ is the cokernel of the hyperbolic Dehn invariant D, and similar to the spherical case the image of D in $\boldsymbol{R}\otimes\boldsymbol{R}/\boldsymbol{Z}$ is generated by elements of the form

$$a\otimes\frac{\alpha}{2\pi} + b\otimes\frac{\beta}{2\pi} + c\otimes\frac{\gamma}{2\pi}$$

where $\sin^2\alpha\sin^2\gamma < \cos^2\beta$ and where $i^{-1}\cosh a$, $i^{-1}\cosh b$, $i^{-1}\cosh c$ are given by the same expression as in the spherical case in section 5 remark 4. However Sah-Wagoner [21] has already described this group in a simple way as generated by all elements of the form

$$-\log(2\cos\alpha)\otimes\frac{\alpha}{2\pi}\,.$$

2. If $\Gamma\subseteqq \mathrm{PSl}(2, \boldsymbol{C})$ acts discontinuously on $\mathcal{H}^3$ with compact quotient then the image of the fundamental class under the composite map.

$$H_3(\Gamma\backslash\mathcal{H}^3, \boldsymbol{Z}) = H_3(\Gamma, \boldsymbol{Z}) \to H_3(G(3), \boldsymbol{Z}^t) \to \mathcal{P}(\mathcal{H}^3)$$

is represented by a polyhedron which is a fundamental domain for Γ acting on $\mathcal{H}^3$.

3. If u: $\mathcal{P}(\mathcal{H}^3)\to\boldsymbol{R}$ is given by $u(P) = \mathrm{Vol}(P)/\mathrm{Vol}(S^3)$ then the composite map

$$H_3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \to \mathcal{P}(\mathcal{H}^3) \xrightarrow{u} \boldsymbol{R}$$

is the evaluation of the Cheeger-Simons class $\frac{1}{i}\hat{C}_2 \in H^3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{C}/i\boldsymbol{Z})$ or rather the class corresponding to the imaginary part of the Chern polynomial C_2 (the real part obviously vanishes on the (-1)-eigenspace for conjugation). Notice that this time there is a natural choice for the chain map of Eilenberg-

MacLane chain complexes:

$$\bar{C}_*(\mathrm{Sl}(2, \boldsymbol{C})) \to C_*(\mathscr{H}^3)$$

(cf. Dupont [9]).

As in the spherical case we now finally obtain

Corollary 6.15. *Let* $B \subseteq H_3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^-$ *be as in the theorem* 1.4. *Then*

a) *A hyperbolic polyhedron P in which all dihedral angles are rational multiples of* π *determines a homology class* $\hat{P} \in H_3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^-/B$. *Furthermore*

$$\left\langle \frac{1}{i} \hat{C}_2, \hat{P} \right\rangle = \mathrm{Vol}(P)/\mathrm{Vol}(S^3)$$

b) *Let* $\mathscr{K} \subseteq \mathscr{P}(\mathscr{H}^3)$ *be the kernel of D. Then* $\mathscr{K}$ *has infinite rank.*

c) *The restriction of* Vol: $\mathscr{K} \to \boldsymbol{R}$ *has countable image.*

d) Vol *and D are separating invariants for* $\mathscr{P}(\mathscr{H}^3)$ *iff the evaluation*

$$\left\langle \frac{1}{i} \hat{C}_2, - \right\rangle : H_3(\mathrm{Sl}(2, \boldsymbol{C}), \boldsymbol{Z})^- \to \boldsymbol{R}$$

has kernel in C_2.

In another paper with C.-H.Sah we shall extend the results on $\mathscr{P}(\mathscr{H}^3)$ to the extended hyperbolic space where the polyhedra may have some (or all) vertices lying on the boundary of $\mathscr{H}^3$.

References

[1] R.C. Alperin: *Stability for* $H_2(SU_n)$, in Algebraic K-theory. Evanston 1976, ed. M.R. Stein (Lecture Notes in Mathematics 551) pp. 283–289, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] R.C. Alperin and R.K. Dennis: K_2 *of quaternion algebras,* J. Algebra **56** (1979), 262–273.

[3] K. Aomoto: *Analytic structure of Schläfli function,* Nagoya Math. J. **68** (1977), 1–16.

[4] H. Cartan and S. Eilenberg: Homological algebra (Princeton Mathematical Series 19), Princeton University Press, Princeton, 1956.

[5] J. Cheeger: *Invariants of flat bundles,* Proceedings of the International Congress of Mathematicians, Vancoucer, 1974, pp. 3–6.

[6] J. Cheeger and J. Simons: *Differential characters and geometric invariants,* preprint.

[7] H.S.M. Coxeter: *The functions of Schläfli and Lobatschefsky,* Quart. J. Math. (Oxford) **6** (1935), 13–29.

[8] J.L. Dupont: Curvature and characteristic classes (Lecture Notes in Mathematics 640), Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[9] J.L. Dupont: *Simplicial De Rham cohomology and characteristic classes of flat bundles,* Topology **15** (1976), 233–245.

[10] H. Hadwiger: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie (Grundlehren Math. Wissensch. 93), Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[11] B. Jessen: *Einige Bemerkungen zur Algebra der Polyeder in nicht-Euclideschen Räumen,* Comment. Math. Helv. **53** (1978), 525–528.

[12] B. Jessen: *The algebra of polyhedra and the Dehn-Sydler theorem,* Math. Scand. **22** (1968), 241–256.

[13] B. Jessen: *Zur Algebra der Polytope,* Nachr. Akad. Wiss. Göttingen II, (1972), 47–53.

[14] B. Jessen, J. Karpf and A. Thorup: *Some functional equations in groups and rings,* Math. Scand. **22** (1968), 257–265.

[15] B. Jessen and A. Thorup: *The algebra of polytopes in affine* spaces, Math. Scand. **43** (1978), 211–240.

[16] G. Lusztig: The discrete series of Gl_n over a finite field. (Annals of Mathematics Studies 81) Princeton University Press, Princeton, 1974.

[17] S. MacLane: Homology (Grundlehren Math. Wissensch. 114), Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[18] C.-H. Sah: *Automorphisms of finite groups,* J. Algebra **10** (1968), 47–68.

[19] C.-H. Sah: Hilbert's third problem: scissors congruence (Research Notes in Mathematics 33), Pitman Publishing Ltd., San Francisco-London-Melbourne, 1979.

[20] C.-H. Sah: *Scissors congruences,* I: *The Gauss-Bonnet map,* Math. Scand. **49** (1981), 181–210.

[21] C.-H. Sah and J.B. Wagoner: *Second homology of Lie groups made discrete,* Comm. Algebra **5** (1977), 611–643.

[22] L. Schläfli: *On the multiple integral* $\int^n dx\,dy\cdots dz$, *whose limits are* $p_1=a_1x+b_1y+\cdots+h_1z>0$, $p_2>0, \cdots, p_n>0$, *and* $x^2+y^2+\cdots+z^2<1$, in Gesammelte Mathematische Abhandlungen II, pp. 219–270, Verlag Birkhäuser, Basel, 1953.

[23] E.H. Spanier: Algebraic topology, McGraw-Hill, Inc., New York-London-Sydney, 1966.

[24] J.P. Sydler: *Conditions nécessaires et suffisantes pour l'équivalence des polyèdres l'espace euclidien à trois dimensions,* Comment. Math. Helv. **40** (1965), 43–80.

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