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SUFFICIENCY AND PAIRWISE SUFFICIENCY IN STANDARD BOREL SPACES–II

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Introduction

Let $(X, \boldsymbol{A}, P_\theta\colon \theta\in\Theta)$ be an experiment and $\boldsymbol{B}$ a sub σ-algebra of $\boldsymbol{A}$. It is known and can be proved easily [9], that if $\{P_\theta\colon \theta\in\Theta\}$ is dominated by a σ-finite measure then pairwise sufficiency of $\boldsymbol{B}$ implies its sufficiency. There has been attempts to generalise this result and show that even in the undominated case paiwise sufficiency is related to sufficiency. Pitcher [11] introduced compact statistical structures, Basu and Ghosh [1] discrete statitiscal structures and finally Hasegawa and Perlman [6] coherent experiments. It is now known that [4] coherence is equivalent to compactness and the discrete structure a special case of both. That these concepts are natural generalisation of domination was established by Dipenbrock [3], who showed that compactness and coherence are both equivalent to domination by a localizable measure. Their theorems connecting pairwise sufficiency with sufficiency is of the form "if $\boldsymbol{B}$ is pairwise sufficient then $\bigcap_{\theta_1,\theta_2} \boldsymbol{B}\vee N_{\theta_1,\theta_2}$ is sufficient".

While experiments dominated by a σ-finite measure are coherent, Rogge [13] showed that if $\boldsymbol{A}$ is countably generated then any coherent experiment is necessarily dominated by a σ-finite measure. Thus in countably generated situations' in particular in the Standard Borel Case, compactness is not more general than domination by a σ-finite measure. However it is proved in [12] that, in the Standard Borel case if P_θ's are discrete then pairwise sufficiency is equivalent to sufficiency. Since $P_\theta(x)$ can be thought of as density with respect to the counting measure, a similar generalisation seems possible. This paper centres on such a generalisation.

This paper is motivated by the work of Hasegawa.—Perlman and the theorem of Dipenbrock. We define the notion of weak coherence, Borel localizable and Borel decomposable measures—all standard Borel adaptations of known concepts. It is then shown that experiments dominated by a Borel localizable measure satisfying an additional measurability condition are weakly coherent. For weakly coherent experinents we show that if $\boldsymbol{B}$ is countably generated and pairwise sufficient then $\bigcap_{\theta_1,\theta_2} \boldsymbol{B}\vee N_{\theta_1,\theta_2}$ is sufficient.

1. In this section we fix the notations and state some set theoretic results used in the sequel.

The pair $(S, \boldsymbol{S})$ where S is a set and $\boldsymbol{S}$ a σ-algebra of subsets of S is called Standard Borel if S is a Borel subset of a complete separable metric space and $\boldsymbol{S}$ is the relativized Borel σ-algebra on S. Suppose $(S, \boldsymbol{S})$ and $(T, \boldsymbol{T})$ are two standard Borel spaces then $(S\times T, \boldsymbol{S}\times \boldsymbol{T})$ will denote the product space equipped with the product σ-algebra. For subsets E of $S\times T$, E^t will stand for the t-section $\{s: (s, t)\in E\}$ of E. We need the following facts about standard Borel spaces, details of which can be found in [8].

(i) Let $(S, \boldsymbol{S})$ be a standard Borel space. A subset A of S is Analytic if it is the projection of a Borel set in $S\times T$ for some standard Borel space T. Further A is Borel in S iff both A and A^c are Analytic.

(ii) Suppose E is a Borel set in $S\times T$ such that E^s is countable for all s in S then the projection of E on S is Borel in S.

(iii) Suppose E is a Borel set in $S\times T$ with E^s countable for all s in S then there are measurable functions $g_1, g_2, \cdots$ defined on S taking values in T such that $E=\bigcup_i \{(s, g_i(s)): s\in S\}$.

2. An experiment consists of a set $(X, \boldsymbol{A})$ and a family of probability measures $\{P_\theta: \theta\in\Theta\}$ on $(X, \boldsymbol{A})$. Throughout this paper we assume that Θ is also equipped with a σ-algebra $\boldsymbol{C}$ and further

(i) $(X, \boldsymbol{A})$ and $(\Theta, \boldsymbol{C})$ are standard Borel.

(ii) For all A in $\boldsymbol{A}$, $P_\theta(A)$ is measurable in θ.

Such experiments will be called standard Borel experiments.

DEFINITION 2.1. Let $(X, \boldsymbol{A}, P_\theta: \theta\in\Theta)$ be an experiment. A sub σ-algebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is said to be sufficient for $(X, \boldsymbol{A}, P_\theta: \theta\in\Theta)$ if given any bounded $\boldsymbol{A}$ measurable function f, there is a $\boldsymbol{B}$ measurable function f^* such that

$$f^* = E_\theta(f\mid \boldsymbol{B}) \qquad \text{for all } \theta \text{ in } \Theta$$

$\boldsymbol{B}$ is pairwise sufficient if $\boldsymbol{B}$ is sufficient for $(X, \boldsymbol{A}, P_{\theta_1}, P_{\theta_2})$ for every pair θ_1, θ_2 in Θ.

DEFINITION 2.2. A family of functions $f_\theta(x)$ jointly measurable in θ and x is weakly pairwise coherent if given θ_1 and θ_2, there is an $\boldsymbol{A}$ measurable function $f_{\theta_1,\theta_2}(x)$ such that

$$f_{\theta_1,\theta_2}(x) = f_{\theta_1}(x)\ [P_{\theta_1}]$$
$$f_{\theta_1,\theta_2}(x) = f_{\theta_2}(x)\ [P_{\theta_2}]\,.$$

DEFINITION 2.3. $f_\theta(x)$ is weakly coherent if there is an $\boldsymbol{A}$-measurable function $f(x)$ such that

$$f(x)=f_\theta(x)\,[P_\theta] \qquad \text{for all } \theta \text{ in } \Theta .$$

DEFINITION 2.4. An experiment $(X, \boldsymbol{A}, P_\theta\colon \theta\in\Theta)$ is weakly coherent if every family of weakly pairwise coherent functions is weakly coherent.

Our interest in weakly coherent experiments is due to the following theorem. Let $N_\theta=\{A\in\boldsymbol{A}\colon P_\theta(A)=0\}$.

Theorem 2.1. *Let $(X, \boldsymbol{A}, P_\theta\colon \theta\in\Theta)$ be a weakly coherent experiment. If a countably generated sub σ-algebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is pairwise sufficient then $\boldsymbol{B}=\bigcup_\theta \boldsymbol{B}\vee N_\theta$ is sufficient.*

Proof. Let f be any bounded measurable function. Get $f_\theta(x)$ a jointly measurable version of $E_\theta(f\,|\,\boldsymbol{B})$ (see proposition 2.3 in [14]). Since $\boldsymbol{B}$ is pairwise sufficient, $f_\theta(x)$ is weakly pairwise coherent. Now since $(X, \boldsymbol{A}, P_\theta\colon \theta\in\Theta)$ is weakly coherent there is an $\boldsymbol{A}$ measurable function f^* such that

$$f^*(x)=f_\theta(x)\,[P_\theta] \qquad \text{for all } \theta \text{ in } \Theta .$$

Since for each $\theta, f_\theta(x)$ is $\boldsymbol{B}$-measurable, $f^*(x)$ is $\boldsymbol{B}\vee N_\theta$ measurable for each θ.

REMARK. Since $\boldsymbol{B}$ is pairwise sufficient

$$\bigcap_\theta \boldsymbol{B}\vee N_\theta=\bigcap_{\theta_1,\theta_2} \boldsymbol{B}\vee N_{\theta_1,\theta_2} .$$

Therefore in the above theorem one can assert that $\bigcap_{\theta_1,\theta_2} \boldsymbol{B}\vee N_{\theta_1,\theta_2}$ is itself sufficient.

3. In this section we introduce Borel localizable and Borel decomposable measures. These notions correspond to the well known (see for enstance [15]) localizable and strictly localizable measures, and unlike them Borel localizability turns out to be equivalent to Borel decomposability.

DEFINITION 3.1. Let $(X, \boldsymbol{A})$ be a standard Borel space. A measure m on $(X, \boldsymbol{A})$ is Borel localizable if there is a standard Borel space $(T, \boldsymbol{T})$ and a Borel subset E of $T\times X$ satisfying

(i) $0<m(E^t)<\infty$

(ii) $t_1\neq t_2$ then $m(E^{t_1}\cap E^{t_2})=0$

(iii) for all A in $\boldsymbol{A}$, $m(A)=\sum_{t\in T} m(A\cap E^t)$

(iv) If B is a Borel subset of E, then $\{B^t\colon t\in T\}$ has an m essential supremum in $\boldsymbol{A}$.

DEFINTION 3.2. A Borel localizable measure m on a standard Borel space is Borel decomposable if there is an E satisfying (i), (ii) and (iii) of Definition 3.1 and also (ii)′ $t_1 \neq t_2$ then $E^{t_1} \cap E^{t_2} = \emptyset$.

Any set E satisfying (i), (ii)′ and (iii) will be referred to as a Borel decomposition of $(X, \boldsymbol{A}, m)$.

We note that in case of Borel decomposability condition (iv) of Definition 3.1 is automatically satisfied. For if E is a Borel decomposition of $(X, \boldsymbol{A}, m)$ then for any Borel set $B \subset E$, $\bigcup_t B^t$ is itself Borel and acts as an essential supremum of $\{B^t: t \in T\}$.

Theorem 3.1. *If $(X, \boldsymbol{A}, m)$ is Borel localizable then it is Borel decomposable.*

Proof. Since m is Borel localizable there is an E satisfying

(i) $0 < m(E^t) < \infty$

(ii) $t_1 \neq t_2$ then $m(E^{t_1} \cap E^{t_2}) = 0$

(iii) $m(A) = \sum_{t \in T} m(A \cap E^t)$

(iv) for every Borel set $B \subset E$, $\{B^t: t \in T\}$ has an essential supremum in $\boldsymbol{A}$.

We will construct an E^*, Borel subset of $T \times X$, such that

(i) for all $t \in T$, $E^t = E^{*t}[m]$

(ii) $E^{*t_1} \cap E^{*t_2} = \emptyset$.

It is easy to see then that E^* will serve as a Borel decomposition of $(X, \boldsymbol{A}, m)$. Let $\{C_1, C_2, \cdots\}$ be a countable algebra generating $\boldsymbol{T}$. For each i define $F_i = \operatorname{ess\,sup}_{t \in C_i} E^t$. We now define E^* by

$$E^{*t} = \bigcap_{\substack{i \\ t \in C_i}} F_i - \bigcup_{\substack{j \\ t \notin C_j}} F_j$$

Then E^* is Borel in $T \times X$ and satisfies the required properties.

EXAMPLES *of Borel decomposable measures.*

(i) $(X, \boldsymbol{A})$ standard Borel and m a σ-finite measure an $(X, \boldsymbol{A})$. Choose $T = N$ and $\{E^n: n \in N\}$ any decomposition of $(X, \boldsymbol{A})$ into sets of positive finite measure.

(ii) $(X, \boldsymbol{A})$ standard Borel; m counting measure. Choose $T = X$ and E to be the diagonal in $X \times X$.

(iii) $X = [0, 1] \times [0, 1]$, $\boldsymbol{A}$ Borel σ-algebra on X. $(T, \boldsymbol{T}) = ([0, 1]$, Borel σ-algebra) $m(A) = \sum_t \lambda(A^t)$ where λ is the Lebsgue measure on [0, 1].

Let m be a Borel decomposable measure on $(X, \boldsymbol{A})$ and E be a Borel decomposition of $(X, \boldsymbol{A}, m)$. For each t, let m_t be the measure m restricted to E^t.

DEFINITION 3.3. We say that m is strongly Borel decomposable if there is a Borel decomposition E of $(X, \boldsymbol{A}, m)$, such that for all B in $\boldsymbol{A}$.

$$t \to m_t(B) = m(B \cap E^t) \quad \text{is measurable in } t.$$

Note that examples (i), (ii) and (iii) above are indeed strongly decomposable.

Example (ii) can be modified to get a decomposable but not strongly decomposable measure. For this choose a non measurable positive function ϕ and set $m(x)=\phi(x)$.

DEFINITION 3.4. An experiment $(X, \boldsymbol{A}, P_\theta: \theta\in\Theta)$ where $(X, \boldsymbol{A})$ and $(\Theta, \boldsymbol{C})$ are standard Borel is dominated by a strongly Borel decomposable measure m if

(i) For each θ in Θ P_θ is dominated by m and $\frac{dP_\theta}{dm}$ exists.

(ii) $\{P_\theta: \theta\in\Theta\}\equiv m$ i.e. $P_\theta(A)=0$ for all θ in Θ iff $m(A)=0$.

We have assumed "strong" Borel decomposability rather than Borel decomposability to ensure the measurability of certain functions. This is exemplified by the following lemma.

Lemma 1. *Assume that $(X, \boldsymbol{A}, P_\theta: \theta\in\Theta)$ is dominated by a "strongly" Borel decomposable measure m and let E be a strong Borel decomposition of $(X, \boldsymbol{A}, m)$. Then for each Borel subset B of $\Theta\times T\times X$, the following functions are measurable in (θ, t).*

(i) $(\theta, t)\to P_\theta(B^{\theta,t})$ *where* $B^{\theta,t}=\{x: (\theta, t, x)\in B\}$.

(ii) $(\theta, t)\to m(B^{\theta,t}\cap E^t)$.

Proof. (i) Let $\boldsymbol{M}=\{B\subset\Theta\times T\times X: P_\theta(B^{\theta,t})$ is measurable in $(\theta, t)\}$.

$\boldsymbol{M}$ contains all rectangles, is closed under finite disjoint unions and is further a monotone class. Consequently $\boldsymbol{M}$ contains all Borel sets in $\Theta\times T\times X$.

(ii) Let $\boldsymbol{M}'=\{B\subset\Theta\times T\times X: m(B^{\theta,t}\cap E^t)$ is measurable in $(\theta, t)\}$.

That $\boldsymbol{M}'$ contains all rectangles follows from 'strong' decomposability of m. $\boldsymbol{M}'$ is closed under finite disjoint unions. Further, since for all t, $m(E^t)<\infty$, $\boldsymbol{M}'$ is also a monotone class.

Lemma 2. *Let $D=\{(\theta, t, x): P_\theta(E^t)>0$ and $x\in E^t\}$ and D_1 be the projection of D to the $\Theta\times X$ space. Then the function $\theta\to m(D_1^\theta\cap A)$ is meausrable in θ for every Borel subset A of X.*

Proof. D is Borel in $\Theta\times T\times X$. (by lemma 1). Further, for each (θ, x) there is at most one t such that $(\theta, t, x)\in D$. Therefore D_1 is Borel in $\Theta\times X$.

Let $D_2=\{(\theta, t): P_\theta(E^t)>0\}$.

D_2 is a Borel set in $\Theta\times T$ such that each θ section of D_2 is at most countable. Therefore (see section 1) there are measurable functions $g_1, g_2, \cdots$ defined on Θ, taking values in T such that

$$D_2=\bigcup_{i=1}^{\infty}\{(\theta, g_i(\theta));\ \theta\in\Theta\}$$

Fix any A in $\boldsymbol{A}$. Define a sequence of functions $\phi_1(\theta), \phi_2(\theta), \cdots$ by

$$\phi_1(\theta)=m(E^{g_1(\theta)}\cap A)$$

$$\begin{aligned}\phi_2(\theta) &= m(E^{g_2(\theta)} \cap A) \quad \text{if} \quad g_2(\theta) \neq g_1(\theta)\\ &= 0 \quad\quad\quad\quad\quad\quad \text{if} \quad g_2(\theta) = g_1(\theta)\end{aligned}$$

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$$\begin{aligned}\phi_n(\theta) &= m(E^{g_n(\theta)} \cap A) \quad \text{if} \quad g_n(\theta) \neq g_1(\theta) \quad \text{for} \quad i=1, \cdots, n-1\\ &= 0 \quad\quad\quad\quad\quad\quad \text{otherwise.}\end{aligned}$$

....

Then $m(A \cap D_1^\theta) = \sum_{n=1}^{\infty} \phi_n(\theta)$ which is measurable in θ.

Theorem 3.2. *If $(X, A, P_\theta \colon \theta \in \Theta)$ is dominated by a strongly Borel decomposable measure m, then there is a version of $\frac{dP_\theta}{dm}(x)$ which is jointly measurable in θ and x.*

Proof. Let $D = \{(\theta, t, x) \colon P_\theta(E^t) > 0 \text{ and } x \in E^t\}$ and D_1 be the projection of D to the $\Theta \times X$ space. Then D_1 has the following properties.

(i) $P_\theta(D_1^\theta) = 1$ for all θ in Θ

(ii) $m(D_1^\theta)$ is σ-finite for all θ in Θ.

To see (ii) note that $D_1^\theta = \bigcup_{\substack{t \\ P_\theta(E^t)>0}} E^t$ and $\{t \colon P_\theta(E^t) > 0\}$ is at most countable.

Now fix finite algebras A_n, generating A and denote the atoms by $A_n^1, A_n^2, \cdots, A_n^{k(n)}$. Define

$$f_n^\theta(x) = \sum_{i=1}^{k(n)} \sum_{j=1}^{\infty} \frac{P_\theta(A_n^i \cap E^{\phi_j'(\theta)})}{m(A_n^i \cap E^{\phi_j'(\theta)})} I_{A_n^i \cap E^{\phi_j'(\theta)}}(x) .$$

where $\phi_j'(\theta)$ are obtained from $g_1(\theta), g_2(\theta), \cdots$ of lemma 2 as follows. Fix some ξ outside T and declare $E^\xi = \phi$.

$$\begin{aligned}\phi_1'(\theta) &= g_1(\theta)\\ \phi_2'(\theta) &= g_2(\theta) \quad \text{if} \quad g_2(\theta) \neq g_1(\theta)\\ &= \xi \quad\quad\quad \text{if} \quad g_2(\theta) = g_1(\theta)\end{aligned}$$

....

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$$\begin{aligned}\phi_n'(\theta) &= g_n(\theta) \quad \text{if} \quad g_n(\theta) \neq g_i(\theta) \quad \text{for} \quad i=1, 2, \cdots, n-1\\ &= \xi \quad\quad\quad\quad \text{otherwise.}\end{aligned}$$

....

Then by a well known theorem (see [10]), since m is finite on $E^{\phi_j'(\theta)}$, $f_n^\theta(x)$ converges] to $\frac{dP_\theta}{dm}$. Since, for each n, $f_\theta^n(x)$ is jointly measurable in (θ, x), $f_\theta(x)$ defined by

$$f_\theta(x) = \lim_{n\to\infty} f_\theta^n(x) \quad \text{if it exists}$$
$$= 0 \qquad \text{otherwise}$$

is a required version.

We use the next lemma in the proof of the theorem that follows it.

Lemma 3. *Let D be a Borel subset of $\Theta \times T$ whose projection on T is whole of T Suppose g is measurable function defined on D which is constant on each t section of D, then*

$$g^*(t) = \sup_{D^t} g(\theta, t)$$

is measurable in t.

Proof. $$\{t: g^*(t) > a\} = P_T[\{(\theta, t): g(\theta, t) > a\} \cap D]$$
$$\{t: g^*(t) \leq a\} = P_T[\{(\theta, t): g(\theta, t) \leq a\} \cap D].$$

where P_T is the projection on the T-space. Thus being projections of Borel sets $\{t: g^*(t) > a\}$ and $\{t: g^*(t) \leq a\}$ are both Analytic and consequently Borel, [See section 1]. Hence g^* is $\boldsymbol{T}$ measurable.

Theorem 3.3. *If $(X, \boldsymbol{A}, P_\theta: \theta \in \Theta)$ is dominated by a strongly Borel decomposable measure then it is weakly coherent.*

Proof. Let m be the dominating measure and E be a strong Borel decomposition. By theorem 1, we choose a jointly measurable version of $\frac{dP_\theta}{dm}$.

Denote by S, $\left\{(\theta, x): \frac{dP_\theta}{dm} > 0\right\}$.

Suppose $f_\theta(x)$ is weakly pairwise coherent, then by letting $f_\theta(x)$ to be zero outside S, it is possible to extend $f_\theta(x)$ as a weakly pairwise coherent family of functions on $(X, \boldsymbol{A}, P_{\bar\theta}: \bar\theta \in (\bar\Theta)$

where $\bar\Theta = \{a_i > 0: \sum a_i = 1\} \times \Theta \times \Theta \cdots\cdots$

and $P_{\bar\theta} = \sum_1^\infty a_i P_{\theta_i}$.

Therefore we will assume without loss of generality that $\{P_\theta: \theta \in \Theta\}$ is closed under countable convex combinations. We will also assume for simplicity that $f_\theta(x) = I_{B_\theta}(x)$.

We will briefly describe the idea of the proof. On each E^t, P_θ is a family of measures dominated by, in fact, equivalent to the finite measure m_t. Therefore there is some θ' such that $P_{\theta'}\equiv m_t$. Also since $I_{B_\theta}(x)I_{E^t}(x)$ is coherent, $I_{B_\theta}(x)I_{E^t}(x)$ will be a P_θ equivalent version of $I_{B_{\theta'}}I_{E^t}$ for all θ. Our proof shows that $B_{\theta'}$ on E^t can be defined independently of θ' and also can be done meeasurably in t. Having got B^t's we piece them together to get a B.

We now give the details of the proof. Define h on $D_2=\{(\theta, t): P_\theta(E^t)>0\}$ by

$$h(\theta, t)=\frac{m(E^t\cap S^\theta)}{m(E^t)}.$$

It is then measurable in (θ, t) and therefore, $D_0\{(\theta, t): h(\theta, t)=1\}$ is Borel in $\Theta\times T$.

Note that $(\theta, t)\in D_0$ iff P_θ is equivalent to m on E^t. By a theorem of Halmos-Savage [5], for every t there is at least one θ such that $(\theta, t)\in D_0$. It can be easily seen that $I_{E^t}I_{B^\theta}=I_{E^t}I_{B^{\theta'}}$ $[P_\theta]$ if $(\theta', t)\in D_0$:

As before choose $\boldsymbol{A}_n$ finite algebras generating $\boldsymbol{A}$ and let $A_n^1, A_n^2, \cdots, A_n^{k(n)}$ denote the atoms of $\boldsymbol{A}_n$. For fixed (θ, t) in D_0

$$I_{B_\theta}(x)I_{E^t}(x)=\lim_{n\to\infty}\sum_{i=1}^{k(n)}\frac{m(A_n^i\cap B_\theta\cap E^t)}{m(A_n^i\cap E^t)}I_{E^t\cap A_n^i}(x)\,[P_\theta]\,.$$

We will show that for each i and n, $\frac{m(A_n^i\cap B_\theta\cap E^t)}{m(A_n^i\cap E^t)}$ is independent of θ and is further a measurable function of t. Towards this first note that, since I_{B^θ} is paiirwise coherent

$$(\theta_1, t), (\theta_2, t)\in D_0\Rightarrow I_{B_{\theta_1}}I_{E^t}=I_{B_{\theta_2}}I_{E^t}\,. \qquad [m]$$

and hence

$$\frac{m(A_n^i\cap B^{\theta_1}\cap E^t)}{m(A_n^i\cap E^t)}=\frac{m(A_n^i\cap B^{\theta_2}\cap E^t)}{m(A_n^i\cap E^t)}\,.$$

On D_0 look at the function $g(\theta, t)=\frac{m(A_n^i\cap B^\theta\cap E^t)}{m(A_i^n\cap E^t)}$. Then $g(\theta, t)$ is measurable in (θ, t) and is constant on each t-section of D_0. By lemma 3 $g^*(t)=\sup_{D_0^t} g(\theta, t)$ is measurable in t. Since $g^*(t)=g(\theta, t)$ for $(\theta, t)\in D_0$ our claim is established.

Therefore for each (i, n), $\frac{m(A_n^i\cap B^\theta\cap E^t)}{m(A_n^i\cap E^t)}I_{A_n^i\cap E^t}(x)$ is a measurable function of only t and x. Hence the function $f_t(x)$ defined by

$$\begin{aligned} f_t(x) &= \lim_n \sum_1^{k(n)}\frac{m(A_n^i\cap B^\theta\cap E^t)}{m(A_n^i\cap E^t)}I_{A_n^i\cap E^t}(x) \qquad \text{if the limit exists} \\ &= 0 \qquad \text{Otherwise} \end{aligned}$$

is also measurable in (t, x). Further since for each t_0 there is some θ_0 such that $(\theta_0, t_0) \in D_0$

$$f_t(x) = I_{B^{\theta_0}}(x) I_{E^t} = I_{B^{\theta}}(x) I_{E^t}(x)[P_\theta] \qquad \text{for all } \theta.$$

We can now define $f(x) = \sum_t f_t(x) I_{E^t}(x)$ and then

$$f(x) = I_{B^\theta}(x)[P_\theta] \qquad \text{for all } \theta \text{ in } \Theta.$$

This completes the proof of the theorem.

Combining Theorem 2.1 and 3.3 we get

Theorem 3.4. *If* $(X, \boldsymbol{A}, P_\theta: \theta \in \Theta)$ *is dominated by a strongly Borel decomposable measure, then for any countably generated* σ*-algebra* $\boldsymbol{B}$ *which is pairwise sufficient, the completion* $\hat{\boldsymbol{B}} = \bigcap_{\theta_1, \theta_2} \boldsymbol{B} \vee N_{\theta_1, \theta_2}$ *is sufficient.*

REMARK. Suppose $\boldsymbol{B}$ is countably generated and pairwise sufficient and further if $\boldsymbol{m}$ admits a decomposition E such that for each t, E^t is $\boldsymbol{B}$ measurable, then $\boldsymbol{B}$ is itself sufficient. This follows from the construction of $f_t(x)$ and a theorem of Blackwell [2]. In fact this is precisely what happens in the discrete case. For in the discrete case given a countably generated pairwise sufficient σ-algebra $\boldsymbol{B}$, it is easy to see that the atoms of $\boldsymbol{B}$ are countable. Hence for T one can take the Quotient space of atoms of $\boldsymbol{B}$, and for each t take E^t to be the t-atom. T is in general Analytic. However Theorem 3.3 goes through even when T is Analytic.

We will now give an example to show that Theorem 3.4 cannot be improved in the sense that while $\hat{\boldsymbol{B}}$ is sufficient $\boldsymbol{B}$ itself may not be.

EXAMPLE. $X = [0, 1] \times [0, 1]$ $\quad$ $\boldsymbol{A}$: Borel σ-algebra on X

$\Theta = [0, 1] \cup \{2\}$ $\quad$ $\boldsymbol{C}$: Borel σ-algebra on Θ

for $\theta \in [0, 1]$: P_θ = Lebesgue measure on $\{\theta\} X [0, 1]$

P_2 = Lebesgue measure on the diagonal in X.

To construct $\boldsymbol{m}$, take $T = [0, 1] \cup \{2\}$.

For $t \in [0, 1]$ define $E^t = \{t\} \times [0, 1] - \{(t, t)\}$

$t = 2$ $\quad$ E^t = diagonal in $[0, 1] \times [0, 1]$

We now define $\boldsymbol{m}$ by $m(A) = \sum_{t \in [0,1]} \lambda(A^t) + \lambda'(A \cap D)$ where λ is the Lebesgue measure on $[0, 1]$ and λ' the Lebesgue measure on the diagonal.

In this example, the σ-algebra of vertical Borel sets, i.e. sets of the form $B \times [0, 1]$, is pairwise sufficient but not sufficient.

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