UNIFORM APPROXIMATION BY ENTIRE FUNCTIONS
OF SEVERAL COMPLEX VARIABLES

Dedicated to Professor Yukinari Tōki on his 70th birthday

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Introduction. Let $G$ be a holomorphically convex open subset of $\mathbb{C}^n$ and $T$ a closed subset of $G$. We say that $T$ is totally real, if it is the zero set of a non-negative $C^2$ function $\rho$ which is strictly plurisubharmonic on $T$. It is known that a real $C^1$ submanifold $M$ is totally real if and only if it has no complex tangents (cf. [3]). The problem of uniform approximation on totally real submanifolds was studied to a great extent by many authors (cf. Wells [9], Hörmander and Wermer [4], Nirenberg and Wells [5], Harvey and Wells [2], [3] and Nunezach [6]). The result of [6] states that if $M$ is a totally real submanifold then there exists a holomorphically convex open neighborhood $B$ such that every continuous function on $M$ is uniformly approximated on $M$ by functions holomorphic in $B$. In [8], the author extended this result to the case of totally real sets with $C^\infty$ defining functions. (A totally real set is not necessarily a submanifold. The approximation theorem for totally real sets contains one for totally real analytic subvarieties which was conjectured by Wells [9].)

In this paper, we give a sufficient condition for $T$ and $G$ under which every continuous function on $T$ is uniformly approximated on $T$ by functions holomorphic in $G$. The theorem we prove contains the following result which is a straight generalization to higher dimensions of Carleman's theorem [1].

Every continuous function on $\mathbb{R}^n$, canonically imbedded in $\mathbb{C}^n$, is uniformly approximated on $\mathbb{R}^n$ by entire functions on $n$ complex variables.

We shall make use of the $L^2$-method due to Hörmander and Wermer [4] and the swelling method similar to one used in [8].

1. Statements. Let $S$ be a closed subset of an open set $U$ of $\mathbb{C}^n$. We denote by $H(S)$ (or $H(S, U)$) the algebra of uniform limits of restrictions of functions holomorphic in a neighborhood of $S$ (or in $U$, resp.).

We use an abbreviation $L[u; \xi]$ for the Levi form of a $C^\infty$ function $u$:

$$L[u; \xi] = \sum_{k} \frac{\partial^2 u}{\partial z_k \bar{z}_k} \xi_k \bar{\xi}_k,$$

$\xi \in \mathbb{C}^n$.

By an exhaustion function $\sigma$ of $G$ we mean a positive $C^\infty$ strictly plurisubharmonic
function which maps properly $G$ into $R$. We define a form

$$A[\sigma; \xi] = \frac{1}{2\sigma} L[\sigma^2; \xi]$$

$$= L[\sigma; \xi] + \frac{1}{\sigma} \left| \sum_j \frac{\partial \sigma}{\partial z_j} \xi_j \right|^2, \quad \xi \in C^n.$$

**Theorem.** Let $G$ be a holomorphically convex open subset of $C^n$ and $\sigma$ be an exhaustion function of $G$. If $T$ is the zero set of a nonnegative $C^\infty$ function $\rho$ on $G$ satisfying

$$L[\rho; \xi] \geq cA[\sigma; \xi], \quad \xi \in C^n,$$

for some constant $c > 0$, then $H(T, G) = C(T)$.

When $G$ is $C^n$, this is a uniform approximation theorem by entire functions. In this case, we can choose $\sigma(z) = |z|^2 + 1$ as an exhaustion function of $C^n$ and we have $|\xi|^2 \leq A[\sigma; \xi] \leq 2|\xi|^3, \xi \in C^n$. Therefore, we obtain

**Corollary 1.** If $T$ is the zero set of a nonnegative $C^\infty$ function $\rho$ on $C^n$ satisfying

$$L[\rho; \xi] \geq c|\xi|^2, \quad \xi \in C^n,$$

with some constant $c > 0$, then $H(T, C^n) = C(T)$.

If we write $R^n = \{z; y_j = 0, j = 1, \ldots, n\}$, then $\rho(z) = \sum_j |y_j|^2$ is a defining function of $R^n$ satisfying (2). Thus we obtain the following corollary.

**Corollary 2.** $H(R^n, C^n) = C(R^n)$.

The proof of Theorem is based on the following lemma essentially due to [4]. (For the proof, see Proposition 1 of [7].)

**Lemma 1.** Let $\delta$ be a nonnegative function defined in an open set $V$ in $C^n$. Suppose $K$ is a compact subset of $V$ satisfying the following condition: There exists a constant $\eta > 0$ such that for every sufficiently small $\varepsilon > 0$, we can find a holomorphically convex open set $V_\varepsilon$ satisfying

$$\{z: \text{dist} (z, K) < \varepsilon \} \subset V_\varepsilon \subset \{z: \delta(z) < \varepsilon \eta \}.$$

If $F$ is a a $C^\infty$ function on $V$ satisfying

$$|\partial F(z)| \leq c\delta(z)^{s+1}, \quad z \in V,$$

then $F|_K$ belongs to $H(K)$.

2. **Construction of an exhaustion $\{K_m\}$ of $G$.** Let $\sigma$ and $\rho$ be func-
tions satisfying the assumption of the theorem. For every positive number $r$, the open set $G_r = \{ z \in G : \sigma(z) < r \}$ is relatively compact in $G$.

Let $\lambda$ be a $C^\infty$ function: $R \to [0, 1]$ such that $\lambda(t) = 1$ ($t < 0$) and $\lambda(t) = 0$ ($t > 2$). For every positive number $m$, we set

$$\lambda_m(z) = \lambda(\sigma(z)/m).$$

Then we have

$$L[\lambda_m; \xi] = \frac{1}{m} \left\{ \lambda' L[\sigma; \xi] + \frac{\lambda''}{m} \sum \frac{\partial \sigma}{\partial z_j} \xi_j \right\} \leq \frac{a}{m} A[\sigma; \xi], \quad \xi \in C^n,$$

with $a = \sup \{ |\lambda'| + 2|\lambda''| + 1 \}$, since $\lambda_m(z) = 0$ for $z \in G \setminus G_{2m}$.

We set $\rho_0 = \rho$ and $\rho_m = \rho - m\lambda_m$ for $m > 1$. Since we may assume that $L[\rho; \xi] \geq 2aA[\sigma; \xi]$, multiplying $\rho$ by a constant if necessary, we have

$$L[\rho_m; \xi] \geq aA[\sigma; \xi], \quad \xi \in C^n.$$

For each nonnegative integer $m$, we define the compact set $K_m = \{ z \in \bar{G}_{2m+3} : \rho_m(z) \leq 0 \}$. It is easy to show that $K_m \subset K_{m+1}$ and $\bigcup K_m = G$.

### 3. Approximation on $K_m$. In this section, we fix a nonnegative integer $m$. We shall prove the following lemma.

**Lemma 2.** If $f$ is a $C^\infty$ function, then $f |_{K_m} \in H(K_m, G)$. If $f$ is a $C^\infty$ function which is holomorphic in an open neighborhood of $\bar{G}_{2m}$, $m > 0$, then $f |_{K_m} \in H(K_m, G)$.

Proof. Since $\rho_m$ is strictly plurisubharmonic in $G$ and since $K_m = \{ \rho_m \leq 0 \} \cap \{ \sigma \leq 2m+3 \}$, $K_m$ is $C_0$-convex and therefore we have $H(K_m) = H(K_m, G)$. It suffices to prove that $f |_{K_m} \in H(K_m)$.

Let $\varphi$ be a $C^\infty$ function satisfying $\varphi = 1$ in an open neighborhood of $\bar{G}_{2m}$ and $\varphi = 0$ in $G \setminus \bar{G}_{2m+1}$. We consider the function

$$\delta_m = \varphi \rho_m + (1 - \varphi) \sum \left| \frac{\partial \rho}{\partial z_j} \right|^2.$$

If $z \in G_{2m}$, we have $L[\delta_m; \xi] = L[\rho_m; \xi], \xi \neq 0$. If $z \in T \setminus G_{2m}$ then $\rho_m = 0$ and $d\rho = 0$. Hence we have

$$L[\delta_m; \xi] \geq \varphi L[\rho; \xi] + (1 - \varphi) L[\rho; \xi] |\xi|^2 > 0, \quad \xi \neq 0.$$

Therefore we can find an open neighborhood $\Omega_m$ of $K_m$ so that $\delta_m$ is strictly plurisubharmonic in $\Omega_m$. There exists a constant $\eta > 0$ such that $\delta_m(z) \leq \eta \text{ dist}(z, K_m)$ and $\sigma(z) \leq 2m+3 + \eta \text{ dist}(z, G_{2m+3})$. If we set $\delta(z) = \max \{ 0, \delta_m(z) \}$
and $V_\epsilon = \{ z \in \Omega_n : \delta_m(z) < \epsilon \eta \} \cap G_{2m+3+\epsilon}$, then, for sufficiently small $\epsilon > 0$, $V_\epsilon$ is holomorphically convex and satisfies

$$\{ z : \text{dist} (z, K_m) < \epsilon \} \subset V_\epsilon \subset \{ z : \delta(z) < \epsilon \eta \}.$$ 

We can now find a $C^\infty$ extension $F$ of $f|_T$ on $G$ which satisfies

$$|\overline{\delta} F(z)| \leq c\delta(z)^{p+1}, \quad z \in V \setminus K_m$$

for an open neighborhood $V$ of $K_m$ and for some positive constant $c$. The way of construction of $F$ is the same as one in Lemma 6 of [7]. We note that, if $f$ is holomorphic in an open neighborhood $U$ of $G_{2m}$, then $F$ is holomorphic in $U$. By Lemma 1, we have $f|_{K_m} = F|_{K_m} \in H(K_m)$, which proves the lemma.

**4. Global approximation.** Let $f$ be an arbitrary function in $C^\infty(G)$ and let $\epsilon$ be any positive number. We shall construct a sequence $\{ f_m \}$ of functions holomorphic in $G$ and satisfying

$$|f_m - f_{m-1}| < 2^{-m-1}\epsilon \quad \text{on} \quad K_m$$

and

$$|f_m - f| < \sum_{\gamma=1}^{m} 2^{-\gamma}\epsilon \quad \text{on} \quad T \cap \overline{G}_{2m+3}.$$ 

We define the function $f_\epsilon = \lim f_m$. A standard argument shows that $f_\epsilon$ is holomorphic in $G$ and that $|f_\epsilon - f| < \epsilon$ on $T$.

The construction of $\{ f_m \}$ is as follows. By Lemma 2, we can find a function $f_0$ holomorphic in $G$ such that

$$|f_0 - f| < 2^{-1}\epsilon \quad \text{on} \quad K_0 = T \cap \overline{G}_3.$$ 

Suppose $f_j, j=1, \ldots, m-1$ are already defined. Let $\psi$ be a $C^\infty$ function: $G \rightarrow [0, 1]$ satisfying $\psi = 1$ in an open neighborhood $U$ of $G_{2m}$ and $\psi = 0$ in $G \setminus G_{2m+1}$. Set $g = \psi f_{m-1} + (1 - \psi)f$. Then $g$ is holomorphic in $U$. By Lemma 2, we can find a function $f_m$ holomorphic in $G$ so that

$$|f_m - g| < 2^{-m-1}\epsilon \quad \text{on} \quad K_m.$$ 

Since $g = f_{m-1}$ in $U$ and $K_m \subset U$, we have

$$|f_m - f_{m-1}| < 2^{-m-1}\epsilon \quad \text{on} \quad K_m.$$ 

Since $|g - f| = \psi |f_{m-1} - f| < \sum_{\gamma=1}^{m} 2^{-\gamma}\epsilon$ on $T \cap \overline{G}_{2m+1}$ and since $g = f$ on $T \setminus G_{2m+1}$, we have

$$|f_m - f| < (2^{-m-1} + \sum_{\gamma=1}^{m} 2^{-\gamma})\epsilon \quad \text{on} \quad T \cap \overline{G}_{2m+3}.$$ 

This completes the proof of the theorem.
Remark 1. The question arises whether the same conclusion as Theorem can be obtained under the condition that $\rho$ is $C^\infty$ strictly plurisubharmonic in $G$. (There is a simple example of $T$ such that every defining function of $T$ is not strictly plurisubharmonic in $G$ and such that $H(T, G) = C(T)$.) When $T$ is compact this condition is sufficient. This follows at once from Theorem 2 of [7] and the fact that $T$ is then $\mathcal{O}_C$-convex. We do not know whether it is true even when $T$ is not assumed to be compact.

Remark 2. It is reasonable to conjecture that the theorem will be valid even when a defining function $\rho$ of $T$ is of class $C^2$. In fact, when $T$ is a submanifold, $C^2$-differentiability of $\rho$ is sufficient to derive the approximation by functions holomorphic in a neighborhood of $T$ (c.f. Harvey-Wells [2] and Nunemacher [6]). The $C^\infty$ differentiability assumption in this paper was necessary because of the $L^2$-method we employed.

References


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