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ON *M*-RINGS AND GENERAL *ZPI*-RINGS

Dedicated to Professor Kentaro Murata on his 60th birthday

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In the preceding paper [10], we have proved that a left Noetherian M-ring is a so called "general ZPI-ring" in the commutative case. Also we know that in an M-ring the multiplication of prime ideals is commutative [8]. In the present paper we define general ZPI-rings in section 1 and we study general properties of them, and as an important example of such rings we can give a left Noetherian semi-prime Asano left order. In section 2 we research the condition for a left Noetherian general ZPI-ring to be an M-ring, using minimal prime divisors of an ideal. The notation "$<$" means a proper inclusion as the preceding papers [8], [9], [10].

1. *M*-rings and general *ZPI*-rings

DEFINITION. If the multiplication of any two prime ideals of a ring R is commutative, and any ideal of R can be written as a produkt of powers of prime (considering R as a prime ideal) ideals of R, then we call R *a general ZPI-ring*. Therefore the multiplication of ideals is commutative.

In the commutative case a general ZPI-ring is necessarily Noetherian no matter whether the ring has an identity or not. But in our case the general ZPI-ring is not necessarily Noetherian as the example in [9] shows.

Proposition 1. *Let R be a left Noetherian general ZPI-ring, let P be any prime ideal of R, and let $\mathfrak{q}$ be maximal in the set of prime ideals such that $\mathfrak{q}<P$. Then for any ideal $\mathfrak{a}$ with $\mathfrak{q}<\mathfrak{a}<P$, there is an ideal $\mathfrak{b}$ such that $\mathfrak{a}=P\,\mathfrak{b}=\mathfrak{b}P$.*

Proof. Let $\mathfrak{a}=\mathfrak{p}_1\cdots\mathfrak{p}_r<P$, since R is a general ZPI-ring. Then $\mathfrak{p}_i\subseteq P$ for some $\mathfrak{p}_i$. Since $\mathfrak{q}<\mathfrak{a}\subseteq\mathfrak{p}_i$, $\mathfrak{q}<\mathfrak{p}_i\subseteq P$, so $\mathfrak{p}_i=P$. Therefore $\mathfrak{a}=P\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\mathfrak{p}_{i+1}\cdots\mathfrak{p}_r=\mathfrak{b}\,P$, where $\mathfrak{b}=\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\mathfrak{p}_{i+1}\cdots\mathfrak{p}_r$.

As in the commutative case we have

Proposition 2. *Let R be be a left Noetherian general ZPI-ring, and let P be a maximal ideal of R. Then there are no ideals between P and P^2 (including the case that $P=P^2$), more generally for any positive integer n, the only ideals* *between P and P^n are P, $P^2,\cdots,P^n$ (including the case that $P^i=P^{i+1}$ for some i, $1\leq i<n$).*

REMARK. Let R be as above. If every proper ideal $\mathfrak{a}$ of R can be written as a product of minimal prime divisors of $\mathfrak{a}$, then for any proper prime ideal $\mathfrak{p}$ of R and for any positive integer n, the only ideals between $\mathfrak{p}$ and $\mathfrak{p}^n$ are $\mathfrak{p}$, $\mathfrak{p}^2,\cdots,\mathfrak{p}^n$.

Proposition 3. *Let R be a left Noetherian general ZPI-ring, and let min-$\mathcal{P}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$ be the set of minimal prime ideals of R. Then for any subset $\{\mathfrak{p}_{i_1},\cdots,\mathfrak{p}_{i_k}\}$ of min-$\mathcal{P}$, $\mathfrak{p}_{i_1}\cap\cdots\cap\mathfrak{p}_{i_k}=\mathfrak{p}_{i_1}\cdots\mathfrak{p}_{i_k}$. Especially for the prime radical N_1 of R, $N_1=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_r=\mathfrak{p}_1\cdots\mathfrak{p}_r$.*

Proof. Since R is a general ZPI-ring, $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_i=P_1\cdots P_k$ for some prime ideals $P_1,\cdots,P_k$ of R. Then for any $\mathfrak{p}_j$ $1\leq j\leq i$ we have $P_j\equiv 0 \pmod{\mathfrak{p}_j}$ for some P_j, and so $P_j=\mathfrak{p}_j$, therefore $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_i=\mathfrak{p}_1\cdots\mathfrak{p}_i P_{i+1}\cdots P_k$. Now $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_i\supseteq\mathfrak{p}_1\cdots\mathfrak{p}_i\supseteq\mathfrak{p}_1\cdots\mathfrak{p}_i P_{i+1}\cdots P_k=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_i$, hence $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_i=\mathfrak{p}_1\cdots\mathfrak{p}_i$.

Lemma 4. *Let R be a left Noetherian semi-prime general ZPI-ring, and let min-$\mathcal{P}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$ be the set of minimal prime ideals of R. Then for any $1\leq i<r$ and any positive integers $m_1,\cdots,m_i$ $\mathfrak{p}_1^{m_1}\cdots\mathfrak{p}_i^{m_i}\neq 0$.*

Theorem 1. *Let R be a left Noetherian semi-prime general ZPI-ring, and let min-$\mathcal{P}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$ be the set of minimal prime ideals of R. If a proper ideal $\mathfrak{a}$ of R has the form $\mathfrak{a}=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_s^{e_s}P_1^{f_1}\cdots P_t^{f_t}$ where $\mathfrak{p}_i\in$min-$\mathcal{P}$ for $i=1,\cdots,s$ and $P_j\notin$min-$\mathcal{P}$ for $j=1,\cdots,t$, then $P_1^{f_1}\cdots P_t^{f_t}\subseteq' R$, i.e. essential as a left R-module, and the set $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_s\}$ is uniquely determined by $\mathfrak{a}$.*

Proof. Let P be a prime ideal of R. By proposition 2.11 [5] and Lemma 4, P is not essential as a left R-module if and only if $P\in$min-$\mathcal{P}$. Hence $P_1^{f_1}\cdots P_t^{f_t}\subseteq' R$ as a left R-module. Let $\mathfrak{a}=\mathfrak{p}_{i_1}^{\alpha_1}\cdots\mathfrak{p}_{i_k}^{\alpha_k}Q_1\cdots Q_w$ where $\mathfrak{p}_{i_j}\in$min-$\mathcal{P}$ for $1\leq j\leq k$, $Q_i\notin$min-$\mathcal{P}$ for $1\leq i\leq w$ be another form of $\mathfrak{a}$. Assume that two set $\mathfrak{M}_1=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_s\}$, $\mathfrak{M}_2=\{\mathfrak{p}_{i_1},\cdots,\mathfrak{p}_{i_k}\}$ are distinct. If $\mathfrak{M}_1>\mathfrak{M}_2$, then $0=\mathfrak{a}\,\mathfrak{p}_{s+1}\cdots\mathfrak{p}_r=\mathfrak{p}_{i_1}^{\alpha_1}\cdots\mathfrak{p}_{i_k}^{\alpha_k}\mathfrak{p}_{s+1}\cdots\mathfrak{p}_r Q_1\cdots Q_w$ and $Q_1\cdots Q_w$ contains some regular element, hence $0=\mathfrak{p}_{i_1}^{\alpha_1}\cdots\mathfrak{p}_{i_k}^{\alpha_k}\mathfrak{p}_{s+1}\cdots\mathfrak{p}_r$ contradicting Lemma 4. Next we consider the case that $\mathfrak{M}_1\not\supseteq\mathfrak{M}_2$ and also $\mathfrak{M}_1\not\subseteq\mathfrak{M}_2$. We denote the product of minimal prime ideals belonging to the set $\mathfrak{M}_1$ by $[\mathfrak{M}_1]$, for example. Then $0=\mathfrak{a}\,[$min-$\mathcal{P}-\mathfrak{M}_1]$ since $\mathfrak{a}=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_s^{e_s}P_1^{f_1}\cdots P_t^{f_t}$. On the other hand, min-$\mathcal{P}-\mathfrak{M}_1\not\supseteq$min-$\mathcal{P}-\mathfrak{M}_2$ and $\mathfrak{a}=\mathfrak{p}_{i_1}^{\alpha_1}\cdots\mathfrak{p}_{i_k}^{\alpha_k}Q_1\cdots Q_w$, hence $0\neq\mathfrak{a}\,[$min-$\mathcal{P}-\mathfrak{M}_1]$ which is a contradiction. So we have $\mathfrak{M}_1=\mathfrak{M}_2$.

As a result of Theorem 1 we have

Proposition 5. *Let R be as above, and let min-$\mathcal{P}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$. Then*

$(\mathfrak{p}_1^{\alpha_1}\cdots\mathfrak{p}_i^{\alpha_i},\ \mathfrak{p}_{i+1}^{\alpha_{i+1}}\cdots\mathfrak{p}_j^{\alpha_j})$ *is a regular*[1] *ideal of* R, *where* $\mathfrak{p}_1,\cdots,\mathfrak{p}_j$ *are distinct minimal prime ideal of* R, $1\leq i<j\leq r$ *and* $\alpha_1,\cdots,\alpha_j$ *are any positive integers.*

Proposition 6. *Let* R *be a left Noetherian general ZPI-ring, and let* P *be a maximal ideal of* R *such that* $P^i>P^{i+1}$ *for any positive integer* i. *Then* $\bigcap_{n=1}^{\infty} P^n$ *is a prime ideal of* R.

Proof. Set $\bigcap_{n=1}^{\infty} P^n=\mathfrak{a}$. Let A, B be ideals of R such that $AB\equiv 0 \pmod{\mathfrak{a}}$ $A\not\equiv 0$ and $B\not\equiv 0 \pmod{\mathfrak{a}}$. Therefore there is a maximal $i\geq 0$ such that $A\subseteq P^i$ and so $A\not\subseteq P^{i+1}$. Similarly there is a maximal $j\geq 0$ such that $B\subseteq P^j$ and so $B\not\subseteq P^{j+1}$, where $P^0=R$. Then $P^{i+1}<(A,\ P^{i+1})\subseteq P^i$, therefore $(A,\ P^{i+1})=P^i$ by Proposition 2, and similarly $(B,\ P^{j+1})=P^j$. Hence $P^{i+j}=(A,\ P^{i+1})\ (B,\ P^{j+1})\subseteq P^{i+j+1}$, thus $P^{i+j}=P^{i+j+1}$ contradicting the assumptions.

REMARK. Let R be as above. Let $\mathfrak{p}$ be any proper prime ideal such that for any positive integer i $\mathfrak{p}^i>\mathfrak{p}^{i+1}$. If every proper ideal of R can be written as a product of minimal prime divisors, then $\bigcap_{n=1}^{\infty}\mathfrak{p}^n$ is a prime ideal of R.

Theorem 2. *Let* R *be a Noetherian (left and right) prime ring with an identity. If* R *satisfies the following*

1) *R is a general ZPI-ring;*
2) *every non-zero proper prime ideal of R is maximal;*
3) *every ideal of R is projective both as a left and as a right R-module,*

the R *is an M-ring.*

Proof. We shall prove the existence of an ideal C with $A=BC=CB$ for ideals A, B such that $O<A<B<R$. Let $A=P_1^{e_1}\cdots P_\alpha^{e_\alpha}<B=Q_1^{f_1}\cdots Q_\beta^{f_\beta}$ where $P_1,\cdots,P_\alpha,\ Q_1,\cdots,Q_\beta$ are prime ideals of R and $e_k>0$ for $k=1,\cdots,\alpha, f_j>0$ for $j=1,\cdots,\beta$, so for every Q_k there is some P_k with $P_k=Q_k$ for $k=1,\cdots,\beta$. Hence $A=Q_1^{e_1}\cdots Q_\beta^{e_\beta}P_{\beta+1}^{e_{\beta+1}}\cdots P_\alpha^{e_\alpha}<B=Q_1^{f_1}\cdots Q_\beta^{f_\beta}$. Now by Proposition 2.2 [3], each maximal ideal of R is either idempotent or invertible. Let $Q_1,\cdots,Q_j$ be the set of idempotent maximal ideals in the set of maximal ideals $Q_1,\cdots,Q_j,\cdots,Q_\beta$ (including the case that $\{Q_1,\cdots,Q_j\}$ is empty). Then $A=Q_1\cdots Q_j\ Q_{j+1}^{e_{j+1}}\cdots Q_\beta^{e_\beta}\ P_{\beta+1}^{e_{\beta+1}}\cdots P_\alpha^{e_\alpha}<B=Q_1\cdots Q_j\ Q_{j+1}^{f_{j+1}}\cdots Q_\beta^{f_\beta}$, where $Q_{j+1},\cdots,Q_\beta$ are invertible ideals of R. If $e_{j+1}<f_{j+1}$ for example, multiplying $(Q_{j+1}^{-1})^{e_{j+1}}$ on each side, we have $Q_1\cdots Q_j\ Q_{j+2}^{e_{j+2}}\cdots Q_\beta^{e_\beta}\ P_{\beta+1}^{e_{\beta+1}}\cdots P_\alpha^{e_\alpha}<Q_1\cdots Q_j\ Q_{j+1}^{f_{j+1}-e_{j+1}}\cdots Q_\beta^{f_\beta}\equiv 0 \pmod{Q_{j+1}}$, which is a contradiction. Therefore $e_{j+1}\geq f_{j+1},\cdots,e_\beta\geq f_\beta$. Thus $A=B\ Q_{j+1}^{e_{j+1}-f_{j+1}}\cdots Q_\beta^{e_\beta-f_\beta}\ P_{\beta+1}^{e_{\beta+1}}\cdots P_\alpha^{e_\alpha}$, hence R is an M-ring.

1) We call an R-ideal *a regular ideal.*

REMARK. If R is a Noetherian semi-prime ring with an identity, then we may replace the condition 2) by the following:

2') *the proper prime ideals of R are either comaximal minimal prime ideals or maximal prime ideals of R.*

The theorem is valid also in this case, because $R=R_1\oplus\cdots\oplus R_i\oplus\cdots\oplus R_n$ where $R_i\cong R/\mathfrak{p}_i$ for every i and $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_n\}=\text{min-}\mathcal{P}$, so every R_i is a Noetherian general ZPI-ring satisfying the condition 2).

Theorem 3. *Let R be a left Noetherian semi-prime Asano left order. Then R is a general ZPI-ring and also an M-ring, and the proper prime ideals of R are either comaximal idempotent minimal prime ideals or maximal prime ideals of R. Every proper ideal $\mathfrak{a}$ of R has the form $\mathfrak{a}=\mathfrak{p}_1\cdots\mathfrak{p}_i P_1^{e_1}\cdots P_m^{e_m}$ where $\mathfrak{p}_k\in\text{min-}\mathcal{P}$ for $1\leq k\leq i$ and $P_1,\cdots,P_m$ are maximal prime ideals of R which are regular.*

Proof. Let $Q=Q_1\oplus\cdots\oplus Q_i\oplus\cdots\oplus Q_n$ be the left quotient ring of R which is semisimple Artinian, where $Q_1,\cdots,Q_n$ are simple Artinian rings. Now we can deduce that $R=R_1\oplus\cdots\oplus R_i\oplus\cdots\oplus R_n$ where R_i is a left Noetherian Asano left order of Q_i for $1\leq i\leq n$. Each proper prime ideal of R has either the form $\mathfrak{p}_i=R_1\oplus\cdots\oplus R_{i-1}\oplus R_{i+1}\oplus\cdots\oplus R_n$ or the form $P_i=R_1\oplus\cdots\oplus R_{i-1}\oplus\mathfrak{p}_{(i)}\oplus R_{i+1}\oplus\cdots\oplus R_n$ where $\mathfrak{p}_{(i)}$ is a maximal prime ideal of R_i for $1\leq i\leq n$. Every proper ideal $\mathfrak{a}$ of R has the form $\mathfrak{a}=\mathfrak{a}_1\oplus\cdots\oplus\mathfrak{a}_i\oplus\cdots\oplus\mathfrak{a}_n$ where $\mathfrak{a}_i$ is an ideal of R_i for $1\leq i\leq n$. In order to make the proof concise we assume that $\mathfrak{a}_1=\cdots=\mathfrak{a}_{i-1}=0$ (including the case that $\{\mathfrak{a}_1,\cdots,\mathfrak{a}_{i-1}\}$ is empty) and $\mathfrak{a}_i=\mathfrak{p}_{(i)1}^{e_{i1}}\cdots\mathfrak{p}_{(i)\alpha}^{e_{i\alpha}},\cdots,\mathfrak{a}_n=\mathfrak{p}_{(n)1}^{e_{n1}}\cdots\mathfrak{p}_{(n)\lambda}^{e_{n\lambda}}$. Then $\mathfrak{a}=\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}P_{i1}^{e_{i1}}\cdots P_{i\alpha}^{e_{i\alpha}}\cdots P_{n1}^{e_{n1}}\cdots P_{n\lambda}^{e_{n\lambda}}$ where $P_{ij}=R_i\oplus\cdots\oplus R_{i-1}\oplus\mathfrak{p}_{(i)j}\oplus P_{i+1}\oplus\cdots\oplus R_n$, thus R is a general ZPI-ring. Then it is easy to see that R is an M-ring.

By Proposition 6 we have

Corollary 4. *Let R be a left Noetherian semi-prime Asano left order and let P be a regular prime ideal of R, then $\bigcap_{n=1}^{\infty} P^n=\mathfrak{p}$ is a minimal prime ideal of R.*

2. Minimal prime divisors of ideals

Let $\mathfrak{a}$ be a proper ideal of R. *A minimal prime divisor of $\mathfrak{a}$ is a prime ideal $\mathfrak{p}$ with $\mathfrak{a}\subseteq\mathfrak{p}$ such that there are no prime ideals $\mathfrak{p}'$ with $\mathfrak{a}\subseteq\mathfrak{p}'<\mathfrak{p}$.* We denote the set of minimal prime divisors of $\mathfrak{a}$ by $\text{min-}\mathcal{P}_{\mathfrak{a}}$. The set $\text{min-}\mathcal{P}$ of minimal prime ideals of R is $\text{min-}\mathcal{P}_0$. As a consequence of Theorem 3 [10] and Proposition 1 [8], we have

Proposition 7. *Let R be a left Noetherian general ZPI-ring. Moreover if R is an M-ring, then*

(*) i) *For any prime ideal* $\mathfrak{p}$, $\mathfrak{q}$ *with* $\mathfrak{p}<\mathfrak{q}$, $\mathfrak{p}=\mathfrak{p}\,\mathfrak{q}=\mathfrak{q}\,\mathfrak{p}$.
ii) *Let* $\mathfrak{a}$ *be any proper ideal of* R, *and let* $min\text{-}\mathcal{P}_{\mathfrak{a}}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$. *Then* $\mathfrak{a}=\mathfrak{p}_1^{f_1}\cdots\mathfrak{p}_r^{f_r}$ *for some positive integers* $f_1,\cdots,f_r$.

REMARK. Let R be a left Noetherian general *ZPI*-ring. Then i) of the above condition (*) is equivalent to the following:

i′) *For any prime ideal* $\mathfrak{p}$ *and any ideal* $\mathfrak{b}$ *properly containing* $\mathfrak{p}$, $\mathfrak{p}=\mathfrak{b}\,\mathfrak{p}=\mathfrak{p}\,\mathfrak{b}$.

Next we consider the converse of this apparent proposition.

Proposition 8. *Let R be a left Noetherian general ZPI-ring which satisfies the condition (*) in Proposition 7 and let $\mathfrak{a}$ be a proper ideal of R. Then for any minimal prime divisor $\mathfrak{p}$ of $\mathfrak{a}$, either $\mathfrak{p}^i=\mathfrak{p}^{i+1}$ for some positive integer i or else there is some positive integer j such that $\mathfrak{p}^j\not\supseteq\mathfrak{a}$.*

Proof. We assume that for any positive integer i $\mathfrak{p}^i>\mathfrak{p}^{i+1}$, and we shall show that $\mathfrak{p}^j\not\supseteq\mathfrak{a}$ for some positive integer j. If $\mathfrak{p}^i>\mathfrak{p}^{i+1}$ for any positive integer i and moreover $\mathfrak{p}^k\supseteq\mathfrak{a}$ for any positive integer k, then $\mathfrak{a}\subseteq\bigcap_{n=1}^{\infty}\mathfrak{p}^n=\mathfrak{n}<\mathfrak{p}$ where $\mathfrak{n}$ is a prime ideal by the remark of Proposition 6, a contradiction.

Proposition 9. *Let R be a left Noetherian general ZPI-ring which satisfies the condition (*), let $\mathfrak{a}$ be a proper ideal of R, and let $min\text{-}\mathcal{P}_{\mathfrak{a}}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$. Then for any $i\neq j$ and any positive integer e_i, e_j, $(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})$ is an idempotent ideal of R.*

Proof. First we prove that $\mathfrak{p}_i^{e_i}(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=\mathfrak{p}_i^{e_i}$, and similarly $\mathfrak{p}_j^{e_j}(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=\mathfrak{p}_j^{e_j}$. Since $\mathfrak{p}_j^{e_j}\not\equiv 0\ (\mathrm{mod}\ \mathfrak{p}_i^{e_i})$, $\mathfrak{p}_i^{e_i}<(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=P_1^{f_1}\cdots P_s^{f_s}$ where $P_1,\cdots,P_s$ are minimal prime divisors of $(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})$. Now we know that $\mathfrak{p}_i\equiv 0\ (\mathrm{mod}\ P_k)$ for every P_k, $1\leq k\leq s$. If $\mathfrak{p}_i=P_k$ for some P_k, then $(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=P_1^{f_1}\cdots P_{k-1}^{f_{k-1}}\mathfrak{p}_i^{f_k}P_{k+1}^{f_{k+1}}\cdots P_s^{f_s}\equiv 0$ $(\mathrm{mod}\ \mathfrak{p}_i)$, hence $\mathfrak{p}_j\equiv 0\ (\mathrm{mod}\ \mathfrak{p}_i)$, a contradiction. Therefore $\mathfrak{p}_i<P_k$ for $1\leq k\leq s$, hence $\mathfrak{p}_i^{e_i}(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=\mathfrak{p}_i^{e_i}P_1^{f_1}\cdots P_s^{f_s}=\mathfrak{p}_i^{e_i}$. Then $(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})^2=(\mathfrak{p}_i^{e_i}(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j}),\mathfrak{p}_j^{e_j}(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j}))$ $=(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})$.

Lemma. *Under the same assumptions as above, for any $i\neq j$ and any positive integer e_i, e_j, $\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}=\mathfrak{p}_i^{e_i}\mathfrak{p}_j^{e_j}$.*

Proof. First we prove that $\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}=(\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j})(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})$. For some positive integer ρ $\mathfrak{a}^{\rho}\subseteq\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}=P_1^{f_1}\cdots P_s^{f_s}\equiv 0\ (\mathrm{mod}\ \mathfrak{p}_i)$, where $P_1,\cdots,P_s$ are minimal prime divisors of $\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}$. Therefore $\mathfrak{a}\subseteq P_1\equiv 0\ (\mathrm{mod}\ \mathfrak{p}_i)$ for some P_1, so $P_1=\mathfrak{p}_i$. Similarly for some P_2, $\mathfrak{a}\subseteq P_2\equiv 0\ (\mathrm{mod}\ \mathfrak{p}_j)$, so $P_2=\mathfrak{p}_j$, and $\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}=\mathfrak{p}_i^{f_1}\mathfrak{p}_j^{f_2}P_3^{f_3}\cdots P_s^{f_s}$. Let $(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=Q_1\cdots Q_t$ where $Q_1,\cdots,Q_t$ are minimal prime divisors of $(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})$. For every Q_k, $\mathfrak{p}_i\equiv 0\ (\mathrm{mod}\ Q_k)$ and $\mathfrak{p}_j\equiv 0\ (\mathrm{mod}\ Q_k)$, hence $\mathfrak{p}_i<Q_k$ and $\mathfrak{p}_j<Q_k$ for every Q_k, $1\leq k\leq t$. From the above arguments $(\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j})(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=$

$\mathfrak{p}_1^{f_1}\mathfrak{p}_j^{f_2}P_3^{f_3}\cdots P_s^{f_s}Q_1\cdots Q_t=\mathfrak{p}_i^{f_1}\mathfrak{p}_j^{f_2}P_3^{f_3}\cdots P_s^{f_s}=\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}$ by the condition (*). Hence $\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}=(\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j})\,(\mathfrak{p}_i^{e_i},\mathfrak{p}_j^{e_j})=((\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j})\mathfrak{p}_i^{e_i},(\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j})\mathfrak{p}_j^{e_j})\subseteq(\mathfrak{p}_j^{e_j}\mathfrak{p}_i^{e_i},\mathfrak{p}_i^{e_i}\mathfrak{p}_j^{e_j})=\mathfrak{p}_i^{e_i}\mathfrak{p}_j^{e_j}$. The other inclusion is obvious, so $\mathfrak{p}_i^{e_i}\cap\mathfrak{p}_j^{e_j}=\mathfrak{p}_i^{e_i}\mathfrak{p}_j^{e_j}$.

Now by the induction we have

Theorem 5. *Let R be a left Noetherian general ZPI-ring which satisfies the condition (*), let $\mathfrak{a}$ be a proper ideal of R, and let min-$\mathcal{P}_{\mathfrak{a}}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$. Then for any subset $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_k\}$ of min-$\mathcal{P}_{\mathfrak{a}}$ and for any positive integers e_i, $i=1,\cdots,k$, $\mathfrak{p}_1^{e_1}\cap\cdots\cap\mathfrak{p}_k^{e_k}=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_k^{e_k}$.*

Theorem 6. *Let R be a left Noetherian general ZPI-ring which satisfies the condition (*), let $\mathfrak{a}$ be a proper ideal of R, and let $\mathfrak{a}=\mathfrak{p}_1^{x_1}\cdots\mathfrak{p}_r^{x_r}$ where min-$\mathcal{P}_{\mathfrak{a}}=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_r\}$ and $x_i>0$ for $1\leq i\leq r$. Let $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_k\}$ be the subset of min-$\mathcal{P}_{\mathfrak{a}}$ every $\mathfrak{p}_i$ of which has a maximal index α_i such that $\mathfrak{p}_1^{\alpha_i}\supseteq\mathfrak{a}$ and so $\mathfrak{p}_i^{\alpha_i+1}\not\supseteq\mathfrak{a}$, and assume that for $\mathfrak{p}_{k+1},\cdots,\mathfrak{p}_r$ there are no maximal β_j among indices β_j such that $\mathfrak{p}_j^{\beta_j}\supseteq\mathfrak{a}$ (including the case that one of the sets $\{1,\cdots,k\}$, $\{k+1,\cdots,r\}$ is empty). Then $\mathfrak{a}$ has the form $\mathfrak{a}=\mathfrak{p}_1^{\beta_1}\cdots\mathfrak{p}_k^{\beta_k}\mathfrak{p}_{k+1}^{y_{k+1}}\cdots\mathfrak{p}_r^{y_r}$, where β_i is any positive integer such that $x_i\leq\beta_i\leq\alpha_i$ for $1\leq i\leq k$ and y_j is any positive integer with $x_j\leq y_j$ for $k<j\leq r$.*

Proof. By Theorem 5 $\mathfrak{a}=\mathfrak{p}_1^{x_1}\cap\cdots\cap\mathfrak{p}_r^{x_r}\supseteq\mathfrak{p}_1^{\beta_1}\cap\cdots\cap\mathfrak{p}_{k+1}^{y_{k+1}}\cap\cdots\cap\mathfrak{p}_r^{y_r}$, since $x_i\leq\beta_i\leq\alpha_i$ for $1\leq i\leq k$ and $x_j\leq y_j$ for $k<j\leq r$. Conversely $\mathfrak{a}\subseteq\mathfrak{p}_i^{\beta_i}$ for $1\leq i\leq k$ since $\beta_i\leq\alpha_i$, and also $\mathfrak{a}\subseteq\mathfrak{p}_{k+1}^{y_{k+1}}\cap\cdots\cap\mathfrak{p}_r^{y_r}$ for any $y_j\geq x_j$, $k<j\leq r$; hence $\mathfrak{a}\subseteq\mathfrak{p}_1^{\beta_1}\cap\cdots\cap\mathfrak{p}_k^{\beta_k}\cap\mathfrak{p}_{k+1}^{y_{k+1}}\cap\cdots\cap\mathfrak{p}_r^{y_r}$. Thus $\mathfrak{a}=\mathfrak{p}_1^{\beta_1}\cap\cdots\cap\mathfrak{p}_k^{\beta_k}\cap\mathfrak{p}_{k+1}^{y_{k+1}}\cap\cdots\cap\mathfrak{p}_r^{y_r}=\mathfrak{p}_1^{\beta_1}\cdots\mathfrak{p}_k^{\beta_k}\mathfrak{p}_{k+1}^{y_{k+1}}\cdots\mathfrak{p}_r^{y_r}$ by Theorem 5.

The following definition of primary ideal is defined in [2]. Let $\mathfrak{a}$ be an ideal of R. If for ideals A, B $AB\equiv 0\ (\mathrm{mod}\ \mathfrak{a})$ implies $A\equiv 0\ (\mathrm{mod}\ \mathfrak{a})$ or $B^\rho\equiv 0\ (\mathrm{mod}\ \mathfrak{a})$ for some positive integer ρ, then $\mathfrak{a}$ is called *r-primary*. And a *l-primary* ideal is defined similarly. A 1- and r-primary ideal is called *a primary ideal.*

Theorem 7. *Let R be as above. Then for every proper prime ideal $\mathfrak{p}$ of R $\mathfrak{p}^e$ is a primary ideal for any positive integer e.*

Proof. Let $AB\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$ for ideals A, B. We may assume that $A\not\supseteq\mathfrak{a}$ and $B\not\supseteq\mathfrak{a}$ where we set $\mathfrak{p}^e=\mathfrak{a}$. We set anew $A_1=(A,\mathfrak{a})$, $B_1=(B,\mathfrak{a})$. Then $A_1\,B_1\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$; and $A\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$ if and only if $A_1\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$, etc.. Therefore it is sufficient to prove that for ideals $A>\mathfrak{a}$, and $B>\mathfrak{a}$, if $AB\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$, then $A\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$ or $B^n\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$ for some positive integer n. Hence we prove that for ideals A, B such that $\mathfrak{a}<A$, $\mathfrak{a}<B$, if $AB\equiv 0$ $(\mathrm{mod}\ \mathfrak{p}^e)$ and for any positive integer m $B^m\not\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$, then $A\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$. Let min-$\mathcal{P}_A=\{P_1,\cdots,P_t\}$, and let $A=P_1^{\delta_1}P_2^{\delta_2}\cdots P_t^{\delta_t}$ for some positive integers $\delta_1,\cdots,\delta_t$. Since $AB\equiv 0\ (\mathrm{mod}\ \mathfrak{p}^e)$, however $B\not\equiv 0\ (\mathrm{mod}\ \mathfrak{p})$, hence $A\equiv 0$ (mod $\mathfrak{p}$). Therefore $\mathfrak{a}<A\subseteq P_1\equiv 0\ (\mathrm{mod}\ \mathfrak{p})$ for some P_1, hence $P_1=\mathfrak{p}$ since $\mathfrak{p}$ is a

minimal prime divisor of $\mathfrak{a}$; so $A=\mathfrak{p}^{\delta_1}P_2^{\delta_2}\cdots P_t^{\delta_t}$, i.e. $\mathfrak{p}$ is a minimal prime divisor of A. Let min-$\mathcal{P}_B=\{\mathfrak{q}_1,\cdots,\mathfrak{q}_k\}$. Since $\mathfrak{a}=\mathfrak{p}^e<B=\mathfrak{q}_1^{\nu_1}\cdots\mathfrak{q}_k^{\nu_k}$ for some positive integers $\nu_1,\cdots,\nu_k$, $\mathfrak{p}<\mathfrak{q}_i$ for every q_i and since $\mathfrak{p}$ is a factor of A $AB=A$ by the condition (*), i.e. $A\equiv 0 \pmod{\mathfrak{p}^e}$.

Theorem 8. *Let R be a left Noetherian general ZPI-ring which satisfies the condition* (*). *Then R is an M-ring.*

Proof. Let $0<A<B<R$ be ideals of R, let min-$\mathcal{P}_A=\{P_1,\cdots,P_a\}$, min-$\mathcal{P}_B=\{Q_1,\cdots,Q_b\}$, and let $A=P_1^{\alpha_1}\cdots P_a^{\alpha_a}$, $B=Q_1^{\beta_1}\cdots Q_b^{\beta_b}$ where $\beta_1,\cdots,\beta_b$ are positive integers and as for $\alpha_1,\cdots,\alpha_a$ by Theorem 6 we can choose them as large as possible. Then for every Q_i, there is some P_j such that $P_j\subseteq Q_i$. If $P_j<Q_i$ for every $Q_1,\cdots,Q_b$, then $A=A\,B=B\,A$, so there is nothing to prove. If there are some Q_i such that $P_j=Q_i$, we may assume for convenience sake that $P_i=Q_i$ for $1\leq i\leq m$ and for every Q_j $(m<j\leq b)$ there are some P_k with $P_k<Q_j$. Furthermore, as to $P_1,\cdots,P_m$, let $P_1,\cdots,P_s$ be minimal prime divisors of A which have maximal indices such that $P_j^{\alpha_j}\supseteq A$ for $1\leq j\leq s$, and let $P_{s+1},\cdots,P_m$ be those which do not have such indices as above. On prime ideals P_j, $1\leq j\leq s$, $A\subseteq P_j^{\alpha_j}$. and $A<B\subseteq Q_j^{\beta_j}=P_j^{\beta_j}$, so $A\subseteq P_j^{\beta_j}$, hence $\beta_j\leq\alpha_j$ for $1\leq j\leq s$ by Theorem 6. On prime ideals $P_{s+1},\cdots,P_m$ we may assume that $\beta_i\leq\alpha_i$ for $s<i\leq m$, by Theorem 6. Therefore $A=P_1^{\alpha_1-\beta_1}\cdots P_m^{\alpha_m-\beta_m}P_1^{\beta_1}\cdots P_m^{\beta_m}P_{m+1}^{\alpha_{m+1}}\cdots P_a^{\alpha_a}=P_1^{\alpha_1-\beta_1}\cdots P_m^{\alpha_m-\beta_m}P_1^{\beta_1}\cdots P_m^{\beta_m}$ $(Q_{m+1}^{\alpha_{m+1}}\cdots Q_b^{\beta_b})P_{m+1}^{\alpha_{m+1}}\cdots P_a^{\alpha_a}=B\,C$, say. Hence R is an M-ring.

We summarize

Theorem 9. *Let R be a left Noetherian general ZPI-ring. Then R is an M-ring if, and only if,*

1) *For any prime ideals $\mathfrak{p}$, $\mathfrak{q}$ of R such that $\mathfrak{p}<\mathfrak{q}$, $\mathfrak{p}=\mathfrak{p}\,\mathfrak{q}$, and*
2) *Any proper ideal $\mathfrak{a}$ of R can be written as a product of powers of minimal prime divisors of $\mathfrak{a}$.*

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