1. Introduction

Let $G/K$ be a Hermitian symmetric space where $G$ is a connected non-compact semisimple Lie group and $K \subset G$ is a maximal compact subgroup. We fix a discrete subgroup $\Gamma$ of $G$ which acts freely on $G/K$ and for which the quotient $X = \Gamma \backslash G/K$ is compact. Let $E_\tau \to G/K$ be a homogeneous $C^\infty$ vector bundle over $G/K$ induced by a finite-dimensional irreducible representation $\tau$ of $K$. Then $E_\tau$ has a holomorphic structure and one can define a presheaf by assigning to an open set $U$ in $X$ the abelian group of $\Gamma$-invariant holomorphic sections of $E_\tau$ on the inverse image (under the map $G/K \to X$) of $U$ in $G/K$. Let $\Theta_\tau \to X$ be the sheaf generated by this presheaf and let $H^q(X, \Theta_\tau)$ denote the $q$th cohomology space of $X$ with coefficients in $\Theta_\tau$. In this paper we continue the program initiated in [23] of obtaining some general vanishing theorems for the spaces $H^q(X, \Theta_\tau)$ by the application of recent representation-theoretic results. This allows for a unified viewpoint and one by which, in particular, the classical vanishing theorems of [3], [4], [5], [6], [7], [12], and [13] may be deduced.

Following Hotta and Murakami [4] we represent $H^q(X, \theta_\tau)$ as a space of automorphic forms. Then its dimension can be expressed by a formula of Matsushima and Murakami [14] in terms of certain irreducible unitary representations $\pi$ of $G$, the multiplicity of $\pi$ in $L^2(\Gamma \backslash G)$, and the $K$ intertwining number of $\pi$ with $\text{Ad}_\pi \otimes \tau$ where $\text{Ad}_\pi$ is the $q$th exterior power of the adjoint representation of $K$ on the space of holomorphic tangent vectors at the origin of $G/K$. Based on results of Kumaresan [9], Parthasarathy [17], and Vogan [21], we have been able to obtain in [23] and [24] a clearer understanding of the structure of the unitary representations $\pi$ of $G$ in the Matsushima-Murakami formula; also see Theorem 3.3 of the present paper. We apply this new knowledge in conjunction with the Matsushima-Murakami formula to deduce the main result of this paper, which is Theorem 4.3. We can deduce, in particular, results of [23] from Theorem 4.3 without assuming the linearity of $G$. Thus we drop the linearity assumption in the present paper, which was enforced in [23].
2. Unitary representations intertwining $\chi^\pm \otimes \tau_{A+\delta_n}$

In this section $G$ will denote a non-compact connected semisimple Lie group with finite center and $K \subset G$ will denote a maximal compact subgroup of $G$. However, proceeding more generally, we shall not assume that $G/K$ is Hermitian symmetric (until later). Let $g_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of $G$, where $\mathfrak{k}_0$ is the Lie algebra of $K$ and $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{k}_0$ relative to the Killing form $(\ , \ )$ of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ denote, respectively, the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$. We shall assume throughout that $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; i.e. we assume $G$ and $K$ have the same rank. This will be the case in particular when $G/K$ is Hermitian. Let $\Delta$ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{h})$, let $\Delta_1, \Delta_2$ denote the compact, non-compact roots respectively in $\Delta$, let $\Delta^+ = \Delta^+ \cap \Delta_1, \Delta^+ = \Delta^+ \cap \Delta_2$, and let $2\delta = \langle \Delta^+ \rangle$, $2\delta = \langle \Delta^+ \rangle$, $2\delta = \langle \Delta^+ \rangle$, where we write $\langle \Phi \rangle = \sum_{\alpha \in \Phi} \alpha$ for $\Phi \subset \Delta$. Let $\mathcal{L}$ denote the integral linear forms $\Lambda$ on $\mathfrak{h}$; i.e. $\Lambda \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) satisfies: 

\[
(2.1) \quad \mathcal{L} = \{\Lambda \in \mathcal{L} | (\Lambda + \delta, \alpha) \neq 0 \text{ for } \alpha \in \Delta \text{ and } (\Lambda + \delta, \alpha) > 0 \text{ for } \alpha \in \Delta^+ \}.
\]

Let $g_\alpha$ be the (one dimensional) root space of $\alpha \in \Delta$. Given $\Lambda \in \mathcal{L}$, $\Lambda + \delta$ is the highest weight with respect to $\Delta^+$ of an irreducible representation $\tau_{A+\delta_n}$ of $\mathfrak{k}$.

The Killing form of $\mathfrak{g}_0$ induces a real inner product on $\mathfrak{p}_0$ and since $\mathfrak{p}_0$ is even-dimensional (because $G$ and $K$ are of equal rank) the spin representation $\sigma$ of $\mathfrak{so}(\mathfrak{p}_0)$ has a decomposition $\sigma = \sigma^+ \oplus \sigma^-$ into two irreducible representations $\sigma^\pm$. Let

\[
(2.2) \quad \chi^\pm = \sigma^\pm \circ (\text{ad}_{\mathfrak{k}_0}) |_{\mathfrak{p}_0}
\]

where $(\text{ad}_{\mathfrak{k}_0}) |_{\mathfrak{p}_0}$ is the adjoint representation of $\mathfrak{k}_0$ on $\mathfrak{p}_0$. Then $\chi^\pm \otimes \tau_{A+\delta_n}$ always integrates to a representation of $K$ (which we shall denote by the same symbol) for $\Lambda \in \mathcal{L}$ even though $\tau_{A+\delta_n}$ may not. Let $\Omega$ denote the Casimir operator of $G$ and let $\hat{G}$ denote the equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $G$ on a Hilbert space $H_\pi$. Given $\Lambda \in \mathcal{L}$, we shall want to pin down the structure of a $(\pi, H_\pi) \in \hat{G}$ such that $\pi(\Omega) = (\Lambda, \Lambda + 2\delta)1$ and such that $\text{Hom}_K(\pi, \chi^\pm \otimes \tau_{A+\delta_n}) \neq 0$. Here $H_\pi$ also denotes the space of $K$ finite vectors in $H_\pi$ which is regarded as a $U\mathfrak{g}$ module where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$; thus $\pi(\Omega)$ is well-defined. We shall need the following additional notation. If $\theta \subset \mathfrak{g}$ is a parabolic subalgebra we shall write $\theta = \mathfrak{m} + \mathfrak{n}$ for its Levi decomposition where $\mathfrak{m}$ and $\mathfrak{n}$ denote the reductive and nilpotent parts respectively of $\theta$, $\Delta(\mathfrak{m})$ for the roots of $\mathfrak{m}$, $\theta_\pm$ for the set of non-compact roots in the nilpotent radical $\mathfrak{n}$, $M$ for the closed Lie subgroup of $G$ whose complexified Lie algebra is $\mathfrak{m}$, and we shall write $2\delta_\pm = \langle \theta_\pm \rangle$. Let $c: \mathfrak{g}_0 \to \mathfrak{g}_0$ denote the Cartan
involution for the Cartan decomposition \( g = \mathfrak{t} + \mathfrak{p} \) above. Let \( F \) be a finite-dimensional irreducible \( g \)-module and let \( \theta = \mathfrak{m} + \mathfrak{n} \supseteq \mathfrak{h} \) be a \( c \)-stable parabolic subalgebra of \( g \) such that the space \( F^\ast \) of \( \mathfrak{u} \) invariants is a one dimensional unitary \( M \) module. If \( \lambda \in \mathfrak{m}^\ast \) is the differential of \( F^\ast \) then \( \lambda(\Delta(\mathfrak{m})) = 0 \) and we shall write \( A_\theta(\lambda) \) for the unique (up to equivalence) irreducible \( g \)-module with minimal \( \mathfrak{k} \) type \( \lambda \mid \mathfrak{k} + 2\delta_{\mathfrak{k},\mathfrak{n}} \). This means that \( A_\theta(\lambda) \) is the only irreducible \( g \)-module such that (i) \( A_\theta(\lambda) \mid \mathfrak{k} \) contains the irreducible \( \mathfrak{k} \)-module with \( \mathfrak{k} \)-highest weight \( \lambda \mid \mathfrak{k} + 2\delta_{\mathfrak{k},\mathfrak{n}} \) and (ii) the \( \Delta_+ \)-highest weight of any irreducible \( \mathfrak{k} \)-submodule of \( A_\theta(\lambda) \mid \mathfrak{k} \) is of the form \( \lambda \mid \mathfrak{k} + 2\delta_{\mathfrak{k},\mathfrak{n}} + \sum_{\beta \in \Delta \setminus \Delta_+} n_\beta \beta \) where \( n_\beta \geq 0 \). For the existence and construction of the \( g \) modules \( A_\theta(\lambda) \) the reader may consult [16], [25]. One knows that the special \( \mathfrak{k} \) type \( \lambda \mid \mathfrak{k} + 2\delta_{\mathfrak{k},\mathfrak{n}} \) occurs exactly once in \( A_\theta(\lambda) \mid \mathfrak{k} \).

Now let \( W \) be the Weyl group of \((g, \mathfrak{k})\) and let \( W_K \) be the subgroup of \( W \) generated by reflections corresponding to compact roots. For \( \Lambda \in \mathfrak{F}_+ \) let

\[
P(\Lambda) = \{ \alpha \in \Delta | (\Lambda + \delta, \alpha) > 0 \}
\]

be the system of positive roots corresponding to the regular element \( \Lambda + \delta \), let

\[
Q_\Lambda = \{ \alpha \in \Delta_+ | (\Lambda + \delta, \alpha) > 0 \}
\]

\[
P^\pm(\Lambda) = P(\Lambda) \cap \Delta_\pm, \quad 2\delta(\Lambda) = \langle P(\Lambda) \rangle, \quad 2\delta_\pm(\Lambda) = \langle P_\pm(\Lambda) \rangle
\]

and for \( w \in W, \tau \in W_K \) let

\[
\Phi_w(\alpha) = w(\alpha + \Lambda) \cap P(\Lambda), \quad \Phi_{w_1} = w_1(\Lambda + \delta) \cap \Delta_+^+, \quad \Phi_{\tau_1} = \tau_1(\Lambda + \delta) \cap \Delta_+^+
\]

Proposition 2.6. Let \( \tau \in W_K \) and let \( w \in W \) be such that \( \Delta_+^+ \subset wP(\Lambda) \). Then \( \Phi_{\tau_1} = \Phi_{w_1} \cup (\Phi_{\tau_1} \setminus \Phi_{\tau_1}^k) \), \( \Phi_{\tau_1} = \Phi_{w_1}^k \). Also \( \Phi_{\tau_1} = \Phi_{w_1} \).

Proof. If \( \alpha \in \Phi_{w_1}^k \) then \( \alpha \in \Delta_+^+ \subset P(\Lambda) \) and \( \tau \alpha \in \Delta_+^+ \subset w(\alpha + \Lambda) \cap P(\Lambda) \). If \( \alpha \in \Phi_{w_1} \) then \( \alpha \in P(\Lambda) \) and \( \tau \alpha \in \Delta_+^+ \subset wP(\Lambda) \). Conversely \( \alpha \in \Phi_{\tau_1} \cap \Delta_+^+ \subset \Phi_{\tau_1} \cap \Delta_+^+ \). Clearly \( \Phi_{\tau_1} = \Phi_{w_1} \).

Using Proposition 2.6 we can now state the following theorem whose proof is given in [24] (see Theorem 2.15 there).

Theorem 2.7. Let \( \Lambda \in \mathfrak{F}_+ \) in (2.1), let \( P(\Lambda) \) be the corresponding positive system in (2.3), and let \( \sigma \in W \) be the unique Weyl group element such that \( \sigma \Delta_+^+ = P(\Lambda) \).
Let \((\pi, H_\pi) \in \hat{G}\) be such that \(\pi(\Omega) = (\Lambda, \Lambda + \Delta)\) and such that \(\text{Hom}_G(\pi, X^{\otimes} \otimes \tau_{\Lambda + \delta}) = 0\). Then there is a pair \((\tau, w) \in W_K \times W\) and a \(c\)-stable parabolic subalgebra \(\theta = m + u\) of \(g\) containing a Borel subalgebra \(\mathfrak{b} + \sum_{\alpha \in \Delta^+} g_\alpha\) where \(\Delta^+ \supset \Delta^+\) such that

(i) \(H_\pi = A_\theta(\lambda)\) and the minimal \(\mathfrak{l}\) type \(\lambda\) at \(\mathfrak{b} + 2\delta_u, n\) (which characterizes \(H_\pi\)) has the form \(\lambda|_{\mathfrak{b} + 2\delta_u, n} = \Lambda + \delta + \tau^{-1}(w\delta(\Lambda) - \delta)\)

(ii) \((\tau, w)\) satisfy \(\Delta^+ \subset wP(\Lambda), \tau(\Lambda + \delta - \delta(\Lambda)) = w(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda)\), \(\Phi^\Lambda_w, \Phi^\Lambda_w - \Phi^\Lambda_{-1}\), and \(\{\alpha \in P^\Lambda_n | \tau\alpha \in -P^\Lambda_n\}\) are contained in \(\{\alpha \in P^\Lambda_n | (\Lambda + \delta - \delta(\Lambda), \alpha) = 0\}\), and \((-1)^{\varphi_w} = \pm (\Lambda + \delta - \delta(\Lambda)) = \pm (-1)^n |\theta_{u, n}|\) where \(|S|\) denotes the cardinality of a set \(S\) and \(n = \frac{1}{2} \dim \mathfrak{g}/K\) (see (2.5)); also \(\Phi^\Lambda_{-1} \subset \{\alpha \in \Delta^+_n | (\Lambda + \delta - \delta(\Lambda), \alpha) = 0\}\)

(iii) the relative Lie algebra cohomology \(H^j(m, m \cap \mathfrak{l}, C)\) (for the trivial module \(C = \text{the complex numbers}\)) is non-zero for \(j = n - |\theta_{u, n}| - |\{\alpha \in P^\Lambda_n | \omega^{-1}\tau\alpha \in -P^\Lambda_n\}|\). Hence the latter number is even.

**Remarks.**

(i) If \(F\) is the finite-dimensional irreducible \(g\) module with \(P(\Lambda)\)-highest weight \(\Lambda + \delta - \delta(\Lambda)\) then \(H_\pi\) in Theorem 2.7 satisfies

\[
\text{Hom}_G(H_{\pi, \Lambda} \wedge \mathfrak{l} \otimes F) = H^i(\mathfrak{l}, \mathfrak{g}, \mathfrak{t} \otimes \mathfrak{r}_\Lambda) = H^i|_{\theta_{u, n}}(m, m \cap \mathfrak{l}, C) \quad \text{for } i \geq 0
\]

(ii) \(\Lambda + \delta_u + \tau^{-1}(w\delta(\Lambda) - \delta)\) is the only \(\mathfrak{l}\) type which occurs both in \(\pi|_K\) and in \(\mathfrak{r}_\Lambda \otimes \tau_{\Lambda + \delta_u}\)

(iii) If \(\sigma_1 \in W\) is the unique Weyl group element such that \(\sigma_1 \Delta^+ = P(\Lambda)\) then \(\sigma_1 \lambda|_{\mathfrak{b} + 2\delta_u, n} = \Lambda + \delta - \delta(\Lambda)\) (see [24]).

(iv) The proof of Theorem 2.7 leans heavily on the recent unpublished results of D. Vogan [21]. Vogan's results depend in part on the important theorem of S. Kumaresan [9] which specifies the structure of an irreducible \(\mathfrak{l}\) component of \(\Lambda \otimes \mathfrak{r}\) that can occur in an irreducible unitary \(g\) module \(H_\pi\) when \(\pi(\Omega) = 0\).

(v) \(\Phi_\pi = \Delta^+_n - Q_\Lambda\).

**3. Unitary representations intertwining \(\text{Ad}_g \otimes \tau_\Lambda\)**

We now assume that for \(G, K\) in section 2, the quotient \(G/K\) admits a \(G\) invariant complex structure; i.e. \(G/K\) is a Hermitian symmetric domain. We choose the positive system \(\Delta^+\) above to be compatible with the complex structures on \(G/K\). This means that

\[
\mathfrak{p}^\pm = \sum_{\alpha \in \Delta^+_n} g_\alpha
\]

where \(\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-\) is the splitting of \(\mathfrak{p}\) into the spaces of holomorphic and anti-
holomorphic tangent vectors \( p^+ \), \( p^- \) respectively at the origin in \( G/K \). The spaces \( p^\pm \) are \( K \) and \( \mathfrak{g} \) stable abelian subalgebras of \( \mathfrak{g} \). The condition of the compatibility of \( \Delta^+ \) with a \( G \) invariant complex structure is equivalent to the following: every \( \alpha \in \Delta^+ \) is totally positive; i.e. for each \( \alpha \) in \( \Delta^+ \) we have \( \alpha + \beta \in \Delta^+ \) for any \( \beta \in \Delta \) such that \( \alpha + \beta \in \Delta \). If \( \mu \in \mathfrak{h}^* \) is integral and \( \Delta^+ \) dominant we write \( (\tau_\mu, V_\mu) \) for the corresponding irreducible of representation of \( \mathfrak{g} \) (or of \( K \) if \( (\tau_\mu, V_\mu) \in \hat{K} \)). Let \( L^\pm \) denote the representation space of \( \chi^\pm \). Then we have

\[
\sum_{(-1)^j=\pm 1} \oplus \Lambda^{\ast-j} \otimes V_{\lambda_j} \cong L^\pm \otimes V_{\lambda_j}^c
\]
as \( K \) modules. Here note that \( \dim V_{\lambda_j} = 1 \) by Weyl's formula since \( (\delta_\alpha, \alpha) = 0 \) for \( \alpha \in \Delta^+ \) in the Hermitian symmetric case. Again \( n = \frac{1}{2} \dim \mathfrak{g} G/K = \dim \mathfrak{c} G/K = |\Delta^+| \). We now prove the following Hermitian analogue of Theorem 2.7.

**Theorem 3.3.** Let \( \Lambda, P(\Lambda) \sigma \) be as in Theorem 2.7 where \( \Lambda \) is the \( \Delta^+ \)-highest weight of \( (\tau_\Lambda, V_\Lambda) \in \hat{K} \). Let \( (\pi, H_\sigma) \in \hat{G} \) be such that \( \pi(\Omega) = (\Lambda, \Lambda + 2\delta) \) and such that \( \text{Hom}_K(H_\sigma, \Lambda^\pm \otimes V_\Lambda) = 0 \) where \( q^\pm \) is fixed. Then there is a pair \( (\pi, \sigma) \) and a \( c \) stable parabolic subalgebra \( \theta = \pi^+ \Sigma \alpha^+ \) where \( \Lambda^\pm \Delta^+ \) such that \( H_\sigma(\pi, \sigma), \theta \) satisfy conditions (i), (ii), (iii) of Theorem 2.7 where in (ii) \( \pm \) is chosen according as \( (-1)^{q^-} = \pm 1 \).

If \( A_{\Lambda, \tau, w} = \{ \alpha \in P(\Lambda) \mid w^{-1} \tau \alpha \in -P(\Lambda) \} \) (see Proposition 2.6), then \( q \) satisfies \( q = n - (n - q) \) and using (3.2) we have for \( (-1)^{q^-} = \pm 1 \) the \( K \) module inclusion \( \Lambda^q \otimes V_\Lambda \subset L^\pm \otimes V_{\delta_\alpha} \otimes V_\Lambda = L^\pm \otimes V_{\Lambda + \delta_\alpha} \) so that \( \text{Hom}_K(H_\sigma, L^\pm \otimes V_{\Lambda + \delta_\alpha}) = 0 \) since \( H_\sigma \otimes K \) and \( \Lambda^q \otimes V_\Lambda \) contain a common \( K \) type \( V_\pi \). Thus Theorem 2.7 applies. The \( \Delta^+ \)-highest weight \( \mu \) satisfies \( \mu = \Lambda + \langle Q \rangle \) where \( Q \subset \Delta^+ \) such that \( |Q| = q \).

Let \( Q = \Delta^+ - Q_1 \) so that \( \mu = \Lambda + 2\delta_\alpha - \langle Q \rangle \). Define \( Q_3 = (Q_\Lambda - Q_2) \cup -Q_2 \cap Q_\Lambda \subset P(\Lambda) = Q_\Lambda \cup Q_\Lambda \) where \( Q_\Lambda = \Delta^+ - Q_\Lambda \). Then one easily checks that

\[
|Q_3| = |Q_\Lambda| - 2|Q_2 \cap Q_\Lambda| + |Q_\Lambda|
\]
and

\[
\langle Q_3 \rangle = \langle Q_\Lambda \rangle - \langle Q_2 \rangle.
\]

Let \( Q_4 = P(\Lambda) - Q_3 \). One has \( \delta_\alpha + \delta_{\Lambda} = \langle Q_4 \rangle \) so that using (3.4) \( \mu = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_2 \rangle = \langle Q_4 \rangle \).

On the other hand by remark (ii) above \( \Lambda + \delta_\alpha + \tau^{-1}(\omega \delta(\Lambda) - \delta_\lambda) \) is the only \( \mathfrak{g} \) type occurring both in \( \pi|K \) and \( \chi^\pm \otimes \tau_{\Lambda + \delta_\alpha} \) which means that \( \mu = \Lambda + \delta_\alpha + \tau^{-1}(\omega \delta(\Lambda) - \delta_\lambda) = \Lambda + \delta_\alpha + \delta_{\Lambda} - \langle Q_4 \rangle \) and hence \( \tau^{-1}(\omega \delta(\Lambda) - \delta_\lambda) = \delta_{\Lambda} - \langle Q_4 \rangle \). Therefore \( \langle Q_2 \cup \Phi^{\pm}_1 \rangle \) (see (2.5)) = \( \langle Q_4 \rangle + \langle \Phi^{\pm}_1 \rangle = \langle Q_2 \rangle + \delta_{\Lambda} - \tau^{-1} \delta_{\lambda} = \delta_{\Lambda} - \langle Q_\Lambda \rangle - \langle Q_\Lambda \rangle = \delta_{\Lambda} - \tau^{-1} \omega \delta(\Lambda) = \langle Q_\Lambda \rangle \). Thus by (5.10.2) of Kostant [8] \( Q_4 \cup \Phi^{\pm}_1 = \Phi^{\pm}_1 \). Then \( Q_4 = \Phi^{\pm}_1 \).
A_{\Lambda,\tau,w} (by Proposition 2.6) and since $Q_{4}=P_{n}^{(\Lambda)}-Q_{3}$, $Q_{3}=\Delta^{+}-Q_{1}$ we get $|A_{\Lambda,\tau,w}|=n-|Q_{3}|=n-|Q_{2}|+2|Q_{2}\cap Q_{\Lambda}|=|Q_{\Lambda}|$ (by (3.4)) $=|Q_{1}|+2|Q_{2}\cap Q_{\Lambda}|=|Q_{\Lambda}|$. But by definition of $Q_{3}$ we have $Q_{2}\cap Q_{\Lambda}=Q_{\Lambda}-Q_{3}$, $Q_{\Lambda}=Q_{\Lambda}\cap A_{\Lambda,\tau,w}$ and hence $|A_{\Lambda,\tau,w}|=q+2|Q_{\Lambda}\cap A_{\Lambda,\tau,w}|=|Q_{\Lambda}|$. This proves Theorem 3.3.

In the statement of Theorem 3.3 no conditions are imposed on $\Lambda \in \mathbb{E}'$. However suppose for example that we impose the following condition: we assume every $\alpha \in P_{n}^{(\Lambda)}$ is totally positive. Then we have the following refinement of Theorem 3.3.

**Corollary 3.5.** Let $(\tau, V_{\Lambda}), P^{(\Lambda)}, \sigma, (\pi, H_{\Lambda})$ be as in Theorem 3.3 with $q$ fixed. Suppose in addition that $P^{(\Lambda)}$ is compatible with a $G$ invariant complex structure on $G/K$; i.e. assume every non-compact root in $P^{(\Lambda)}$ is totally positive. Then there is a Weyl group element $w$ and a c stable parabolic subalgebra $\theta=\mathfrak{m}+\mathfrak{u}$ satisfying the conditions of Theorem 2.7 where in (i), (ii), (iii) $\tau \in W_{K}$ may be assumed equal to the identity element (thus for example $H_{\Lambda}$ is characterized by the minimal $\mathfrak{f}$ type $\Lambda+\delta_{\mathfrak{u}}+w\delta^{(\Lambda)}-\delta_{\mathfrak{k}}$ and $j=n-|\theta_{\mathfrak{u},w}|=\Phi_{\mathfrak{u}}^{(\Lambda)}|$) and in (ii) $\pm$ is chosen according as $(-1)^{n-\eta}=\pm 1$. $q$ satisfies $q=|\Phi_{\mathfrak{w}}^{(\Lambda)}|-2|Q_{\Lambda}\cap \Phi_{\mathfrak{w}}^{(\Lambda)}|+|Q_{\Lambda}|$.

**Proof.** Choose $(\tau, \omega), \theta=m+\mathfrak{u}$ as in Theorem 2.7 or Theorem 3.3. Since every non-compact root in $P^{(\Lambda)}$ is totally positive and since $\tau_{\w}^{W}$ we have $\tau P^{(\Lambda)}=P^{(\Lambda)}$. This implies that

$$A_{\Lambda,\tau,w}=\tau^{-1}\Phi_{\mathfrak{w}}^{(\Lambda)}$$

Also one has $\tau Q_{\Lambda}=Q_{\Lambda}$ and hence by (3.6)

$$\tau(Q_{\Lambda}\cap A_{\Lambda,\tau,w})=Q_{\Lambda}\cap \Phi_{\mathfrak{w}}^{(\Lambda)}.$$
Proposition 3.8. Suppose in Theorem 3.3 the parabolic subalgebra \( \theta = m + u \) is \( g \) itself. Then \( \Lambda = \delta^{(\Lambda)} - \delta \) and \( q = n - |Q_\Lambda| \).

Proof. \( \theta = g \) means that \( u = 0 \), \( m = g \). Then \( \theta_{u,n} = \phi \) and \( \Delta(m) = \Delta \).

Proposition 3.9. Let \( \Lambda \in \mathcal{P}_\theta \) be such that every non-compact root in \( P(\Lambda) \) is totally positive. Let

\[
\mathfrak{p}^{(\Lambda)^+} = \sum_{\alpha \in P_+^{(\Lambda)}} g_\alpha
\]

be the \( \mathfrak{g} \) module of holomorphic tangent vectors for the corresponding \( G \) invariant complex structure on \( G/K \) compatible with \( P(\Lambda) \); cf. (3.1). Suppose \( w \in W \) is a Weyl group element such that \( \xi w \in \mathfrak{p}^{(\Lambda)^+} \). Then we have a \( \mathfrak{g} \) module inclusion

\[
V_{\delta_n^{(\Lambda)} + w\delta^{(\Lambda)} - \delta} \subset \wedge \cap n - |q^{(\Lambda)}_w| \mathfrak{p}^{(\Lambda)^+}.
\]
Corollary 3.14. Let $\Lambda$, $P(\Lambda)$, and $w$ be as in Proposition 3.9. Then we have the $k$ module inclusion $V_{\Lambda+\delta_n+\omega\delta(\Lambda)-\delta_k} \subset V_{\Lambda+\delta-\delta(\Lambda)\wedge} V_{\delta(\Lambda)+w\delta(\Lambda)-\delta_k} \subset V_{\Lambda+\delta-\delta(\Lambda)\wedge} \wedge^t\mathfrak{p}(\Lambda)^+$ where $t=n-|\Phi(\Lambda)|$.

Proof. $\Lambda+\delta_n+\omega\delta(\Lambda)-\delta_k = \Lambda+\delta_n-\delta(\Lambda)+\omega\delta(\Lambda)-\delta_k$

$= \Lambda+\delta-\delta(\Lambda)+\delta(\Lambda)+\omega\delta(\Lambda)-\delta_k$

Corollary 3.15. Let $(\tau_\Lambda, V_\Lambda) \in \hat{G}$ where $\Lambda \in \mathcal{F}_0$ and every non-compact root in $P(\Lambda)$ is totally positive. Let $(\pi, H_\pi) \in \mathcal{G}$ be such that $\pi(\Omega)=(\Lambda, \Lambda+2\delta)$. Let $\mu=\Lambda+\delta_n+\omega\delta(\Lambda)-\delta_k$ be the minimal $\mathfrak{g}$ type of $H_\mu$ given by Corollary 3.5. Then relative to the positive system $P(\Lambda)=P(\Lambda)^+ \cup -P(\Lambda)^+ = \Delta^+ \cup -P(\Lambda)^+$, $H_\mu$ is a highest weight $\mathfrak{g}$ module with highest weight $\mu$.

Proof. We have $\mathfrak{g}$ module inclusions $V_\mu \subset H_\mu$ and (by Corollary 3.14) $V_\mu \subset V_{\Lambda+\delta-\delta(\Lambda)\wedge} \wedge^t\mathfrak{p}(\Lambda)^+$ where $t=n-|\Phi(\Lambda)|$ and where $\Lambda+\delta-\delta(\Lambda)$ is $P(\Lambda)^+$ dominant. Since $|\Lambda+\delta-\delta(\Lambda)|+\delta(\Lambda)|^2-|\delta(\Lambda)|^2=|\Lambda+\delta|^2-|\delta|^2=\pi(\Omega)$ Corollary 3.15 follows from Lemma 3.7 of [6] or from the proof of Lemma 2 of [4].

The fact that any $(\pi, H_\pi) \in \hat{G}$ as in Corollary 3.15 has to be a $P(\Lambda)^+$-highest weight $\mathfrak{g}$ module is also proved in [23] (see the proof of Lemma 2.4 there) by different means.

4. Vanishing theorems

In this section we again assume, as in section 3, that $G/K$ is a Hermitian symmetric domain and that the positive system $\Delta^+$ is compatible with the $G$ invariant complex structure on $G/K$. We fix a discrete subgroup $\Gamma$ of $G$ which acts freely on $G/K$ and for which the quotient $X=\Gamma\backslash G/K$ is compact. Let $\tau=\tau_\Lambda \in \hat{K}$ be a fixed finite-dimensional irreducible representation of $K$ acting on a complex vector space $V_\Lambda$ where $\Lambda \in \mathcal{F}_0$ is the $\Delta^+$-highest weight of $\tau$. The induced $C^\infty$ vector bundle $E_\tau \rightarrow G/K$ has a holomorphic structure. To prove this one usually assumes that $G$ is a real form of a complex Lie group $G^\mathfrak{c}$ (i.e. $G$ is linear). Since we are not imposing the latter assumption on $G$ we appeal to the more general criteria of [19], [20] for the existence of holomorphic structures on homogeneous bundles. The induced sheaf $\theta_\tau \rightarrow X$ of abelian groups over $X$ given in the introduction will also be denoted by $\theta_\Lambda$. Let $\text{Ad}_\Lambda^\mathfrak{c}$ denote the adjoint representation of $K$ on $\wedge^t\mathfrak{p}^+$. Then as in [4] the sheaf cohomology $H^q(X, \theta_\Lambda)$ can be identified with the space $A(\text{Ad}_\Lambda^\mathfrak{c} \otimes \tau_\Lambda, (\Lambda, \Lambda+2\delta), \Gamma)$ of automorphic forms of type $(\text{Ad}_\Lambda^\mathfrak{c} \otimes \tau_\Lambda, (\Lambda, \Lambda+2\delta), \Gamma)$; i.e.
(4.1) \[ H^q(X, \theta_\Lambda) = \{ f: G \to \wedge^q V_\Lambda | f \text{ is } C^\infty, \ f(\gamma a) = f(a), \] 
\[ f(ak^{-1}) = (\text{Ad}_a^* \otimes \tau_\Lambda)(k)f(a) \text{ for } (\gamma, a, k) \text{ in } \Gamma \times G \times K \text{ and} \] 
\[ \Omega f = (\Lambda, \Lambda + 2\delta)f \}. \]

By the formula of Matsushima-Murakami [14] we therefore have

(4.2) \[ \dim H^q(X, \theta_\Lambda) = \sum_{\pi(\Omega) \in G} m_\pi(\Gamma) \dim \text{Hom}_G(H_{\pi}, \wedge^q V_\Lambda) \]

where \( m_\pi(\Gamma) \) is the multiplicity of \( \pi \) in the right regular representation of \( G \) on \( L^2(\Gamma \backslash G) \).

Using (4.2) we immediately deduce from Theorem 3.3 the following main theorem.

**Theorem 4.3.** Let \( \Lambda \in \mathcal{T}_\delta \) in (2.1) be the \( \Delta_\Lambda \)-highest weight of \((\tau_\Lambda, V_\Lambda) \in K \).

Let \( \sigma \in W \) be the unique Weyl group element such that \( \sigma \Delta_\Lambda = P(\Lambda) \) where \( P(\Lambda) \) is the system of positive roots in (2.3). Suppose that \( H^q(\Gamma \backslash G, \theta_\Lambda) \neq 0 \). Then there is a pair \((\tau, \omega) \) in \( W_X \times W \) and a stable parabolic subalgebra \( \theta = \mathfrak{m} + \mathfrak{u} \) of \( \mathfrak{g} \) containing the Borel subalgebra \( \mathfrak{h} + \sum \mathfrak{g}_\alpha \) for some positive system \( \Delta_\alpha \supset \Delta_\Lambda \) (cf. earlier notation) such that

(i) \( q = |A_{\Lambda, \tau, \omega}| - 2|Q_\Lambda \cap A_{\Lambda, \tau, \omega}| + |Q_\Lambda| \) where \( A_{\Lambda, \tau, \omega} = \{ \alpha \in P(\Lambda) | \omega^{-1} \tau \alpha \in -P(\Lambda) \} \) and where \( Q_\Lambda \) is given by (2.4)

(ii) \( \Delta_\alpha \subset wP(\Lambda) \) (so that by Proposition 2.6 \( A_{\Lambda, \tau, \omega} = \Phi(\Lambda)_w - \Phi(\Lambda)_{w^{-1}} \), \( \tau(\Lambda + \delta - \delta(\Lambda)) = w(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda) \), and \( A_{\Lambda, \tau, \omega}, \Phi(\Lambda)_w, \Phi(\Lambda)_{w^{-1}} \) are all contained in \( \{ \alpha \in P(\Lambda) | \tau(\Lambda + \delta - \delta(\Lambda)), \alpha) = 0 \}; \Phi(\Lambda)_{w^{-1}} \subset \{ \alpha \in \Delta_\Lambda \} |(\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \}; \) see notation of (2.5)

(iii) the relative Lie algebra cohomology \( H^J(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{f}, \mathcal{C}) \neq 0 \) for \( j = n - |\theta_{\Lambda, \omega}| - |A_{\Lambda, \tau, \omega}| \) (hence the latter is an even number) where, as above, \( \theta_{\Lambda, \omega} \) is the set of non-compact roots in the nilradical \( \mathfrak{u} \) of \( \theta \) and \( n = \frac{1}{2} \dim_G \Gamma/G \).

(iv) For \( (-1)^{n-q} = \pm 1 \) we have \( (-1)^{|\Phi(\Lambda)|} = \pm (-1)^{|\Phi(\Lambda)|} = \pm (-1)^{n+|\theta_{\Lambda, \omega}|} \).

As has been noted \( \Phi(\Lambda) = \Delta_\Lambda - Q_\Lambda \), and if \( \sigma_1 \in W \) is the unique Weyl group element such that \( \sigma_1 \Delta_\Lambda = P(\Lambda) \) then \( (\Lambda + \delta - \delta(\Lambda), \sigma_1(\Delta(\mathfrak{m}))) = 0 \) where \( \Delta(\mathfrak{m}) \) is the set of roots for the reductive part \( \mathfrak{m} \) of \( \theta \). From Corollary 3.4 we obtain

**Corollary 4.4.** Let \( \Lambda \in \mathcal{T}_\delta \) in Theorem 4.3 satisfy the condition that every non-compact root in \( P(\Lambda) \) is totally positive. Then if \( H^q(\Gamma \backslash G, \theta_\Lambda) \neq 0 \) we can choose \( \omega \in W \) satisfying \( \Delta_\alpha \subset wP(\Lambda) \) and a stable parabolic subalgebra \( \theta = \mathfrak{m} + \mathfrak{u} \subset \mathfrak{h} + \sum \mathfrak{g}_\alpha \) such that

(i) \( q = |\Phi(\Lambda)| - 2|Q_\Lambda \cap \Phi(\Lambda)| + |Q_\Lambda| \)

(ii) \( H^{n-q}(-2|\theta_{\Lambda, \omega}| - |\Phi(\Lambda)|) (\mathfrak{m}, \mathfrak{m} \cap \mathfrak{f}, \mathcal{C}) \neq 0 \)

(iii) \( \Phi(\Lambda) \subset \{ \alpha \in P(\Lambda) | (\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \} \).
Statement (iv) of Theorem 4.3 holds.

Consider for example the special case when $\Lambda$ is actually $\Delta^+$-dominant. Then $P^{(\Lambda)}=\Delta^+$ so that $\Lambda$ indeed satisfies Corollary 4.4. Also in this case $Q_{\Lambda}=\Delta^+_1$ so that $Q_{\Lambda} \cap \Phi^{(\Lambda)}_\theta = \Phi^{(\Lambda)}_\theta$. Thus by (i) of Corollary 4.4 $H^q=0$ for $q=|\Phi^{(\Lambda)}_\theta|-2|\Phi^{(\Lambda)}_{\Lambda}|+n=n-|\Phi^{(\Lambda)}_{\Lambda}|$ and hence by (ii) $H^{q-|\theta,\Phi^{(\Lambda)}_\theta|}(m, m \cap \mathfrak{k}, C) \neq 0$. Thus we have proved the following conjecture of R. Parthasarathy.

**Corollary 4.5.** Suppose the $\Delta^+_1$-highest weight $\Lambda$ of $\tau$ is actually $\Delta^+$-dominant. Then if $H^q(\Gamma \backslash \Gamma, \theta_{\Lambda})=0$ so is $H^{q-|\theta,\Phi^{(\Lambda)}_{\Lambda}|}(m, m \cap \mathfrak{k}, C)$ for some $c$ stable parabolic subalgebra $\theta=m+\mathfrak{n}$ of $\mathfrak{g}$.

Our argument shows moreover that in Corollary 4.5 $q=n-|\mathfrak{w}(\Delta^+) \cap \Delta^+|$ for some $\mathfrak{w} \in W$ with $\Delta^+_1 \subseteq \mathfrak{w} \Delta^+$, $\mathfrak{w}(\Delta^+) \cap \Delta^+ = \{ \alpha \in \Delta^+_1 | (\Lambda, \alpha)=0 \} ; \mathfrak{w} \Lambda=\Lambda$. Let $l(\mathfrak{w})=|\mathfrak{w}(\Delta^+) \cap \Delta^+|$ (=length of $\mathfrak{w}$) and let

$$n_{\Lambda} = | \{ \alpha \in \Delta^+_1 | (\Lambda, \alpha)>0 \} |.$$

Then $| \{ \alpha \in \Delta^+_1 | (\Lambda, \alpha)=0 \} | = n-n_{\Lambda}$ so that by (b.) $l(\mathfrak{w}) \leq n-n_{\Lambda}$ and by (a.) $q=n-l(\mathfrak{w}) \geq n_{\Lambda}$. That is

**Corollary 4.7** (Hotta-Murakami [4]). Suppose $\Lambda$ is $\Delta^+$-dominant. Then $H^q(\Gamma \backslash \Gamma, \theta_{\Lambda})=0$ for $q<n_{\Lambda}$ in (4.6). More generally for $H^q(\Gamma \backslash \Gamma, \theta_{\Lambda})=0$ $q=n-l(\mathfrak{w})$ for some $\mathfrak{w} \in W$ satisfying $\mathfrak{w}(\Delta^+) \cap \Delta^+ = \{ \alpha \in \Delta^+_1 | (\Lambda, \alpha)=0 \} , \mathfrak{w} \Lambda=\Lambda$.

We define

$$R= R(Q) = \min \{|\theta_{\mathfrak{u}, \mathfrak{n}}| \theta=\mathfrak{c} \text{ stable parabolic subalgebra of } \mathfrak{g}, \theta=\mathfrak{g}\} .$$

Again note that for $\theta=\mathfrak{g}$ $\mathfrak{u}=0$ and hence $|\theta_{\mathfrak{u}, \mathfrak{n}}|=\dim \mathfrak{u} \cap \mathfrak{p}=0$. The values $R(G)$ have been computed by Vogan for general symmetric spaces. Specializing his results to the Hermitian case we have the following table for the irreducible Hermitian symmetric spaces.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$R(G)$</th>
<th>$\frac{1}{2} \dim_{\mathbb{R}} G/K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(n, m), n \geq m$</td>
<td>$m$</td>
<td>$nm$</td>
</tr>
<tr>
<td>$Sp(n, R)$</td>
<td>$n$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$SO(n, 2), n \geq 2$</td>
<td>$2$</td>
<td>$n$</td>
</tr>
<tr>
<td>$SO^*(2n), n &gt; 3$</td>
<td>$n-1$</td>
<td>$\frac{n-1}{2}$</td>
</tr>
<tr>
<td>real form of $E_6$</td>
<td>$8$</td>
<td>$16$</td>
</tr>
<tr>
<td>real form of $E_7$</td>
<td>$11$</td>
<td>$17$</td>
</tr>
</tbody>
</table>
In Theorem 4.3 $H^j(m, m \cap \mathfrak{t}, \mathfrak{c}) \neq 0$ for $j = n - |\theta_{v,s}| - |A_{\lambda,-w}|$ by (iii); hence $j \geq 0$. That is $|A_{\lambda,-w}| \leq n - |\theta_{v,s}|$ and if $\theta \neq q$ $|A_{\lambda,-w}| \leq n - R(G)$. Thus applying Proposition 3.8 we get

**Proposition 4.10.** Suppose in Theorem 4.3 that either $\Lambda = \delta^{(\Lambda)} - \delta$ or $q \neq n - |Q_{\Lambda}|$. Then $A_{\lambda,-w}$ there satisfies $|A_{\lambda,-w}| \leq n - R(G)$. Similarly $w$ in Corollary 4.4 satisfies $|\Phi^{(\Lambda)}_w| \leq n - R(G)$.

Note that, in general, by Theorem 4.3 we always have $|A_{\lambda,-w}|, |\Phi^{(\Lambda)}_w| \leq |\{\alpha \in P_{\Lambda}^{(\Lambda)}| (\Delta + \delta - \delta^{(\Lambda)}, \alpha) = 0\}|$. In Corollary 4.7 $q = n - l(w)$ for $H^q \neq 0$. By Proposition 4.10. $l(w) \leq n - R(G)$ if either $\Lambda \neq 0$ or $q \neq 0$; i.e. $q = n - l(w) > R(G)$ which establishes

**Corollary 4.11.** Suppose $\Lambda$ is $\Delta^+$-dominant. If $\Lambda \neq 0$ then $H^q(\Gamma \backslash G/K, \theta_{\Lambda}) = 0$ for $0 \leq q < R(G)$. If $\Lambda = 0$ then $H^q(\Gamma \backslash G/K, \theta_{\Lambda}) = 0$ for $1 \leq q < R(G)$.

In particular we see that since for $G$ in Table 4.9 rank of $G/K \leq R(G)$ the following weaker version of Corollary 4.11 holds.

**Corollary 4.12.** If $G/K$ is irreducible then $H^q(\Gamma \backslash G/K, \theta_{\Lambda}) = 0$ for $0 \leq q < \text{rank } G/K, \Lambda$ $\Delta^+$-dominant, $\Lambda \neq 0$. The $(0, q)$ Betti number of $\Gamma \backslash G/K$ vanishes for $1 \leq q < \text{rank } G/K$.

Corollary 4.12 is of course well-known; see Theorem 4.2 of [6] and Theorem 4 of [4]. In the case where $G/K$ is irreducible a slight improvement of Corollary 4.11 is given by Theorem 3.5 of [23]. Another extreme case is the case $Q_{\Lambda} = \emptyset$; i.e. $(\Lambda + \delta, \alpha) < 0$ for $\alpha \in \Delta_+$, $P^{(\Lambda)} = \Delta_+^* = \Delta_+^* - \Delta_+^*$. If $H^q \neq 0$ then from Corollary 4.4 $q = |\Phi^{(\Lambda)}_w|$ for some $w \in W$ such that $\Delta_+^* \subseteq w \Delta_+^*$, $\Phi^{(\Lambda)}_w \subseteq \{\alpha \in -\Delta_+^* \mid (\Lambda + 2\delta_n, \alpha) = 0\}$ and (by (ii) of Corollary 4.4) $H^{n-q-10n-1}(m, m \cap \mathfrak{t}, \mathfrak{c}) \neq 0$ for some $c$ stable parabolic $\theta = m + u$. By Proposition 3.8 $\theta \neq q$ unless $\Lambda = -2\delta_n$ or $q = n$. Barring the latter two cases we have $|\Phi^{(\Lambda)}_w| \leq n - R(G)$ by Proposition 4.10 so that $q = n - R(G)$. This gives

**Corollary 4.13.** Suppose $(\Lambda + \delta, \alpha) < 0$ for $\alpha$ in $\Delta_+^*$. If $\Lambda \neq -2\delta_n$ then $H^q(\Gamma \backslash G/K, \theta_{\Lambda}) = 0$ for $q > n - R(G)$. If $\Lambda = -2\delta_n$ then $H^q(\Gamma \backslash G/K, \theta_{\Lambda}) = 0$ for $n - R(G) < q < n$. In any case we always have $H^q(\Gamma \backslash G/K, \theta_{\Lambda}) = 0$ for $q > |\{\alpha \in -\Delta_+^* \mid (\Lambda + 2\delta_n, \alpha) = 0\}|$.

The last statement of Corollary 4.13 is statement (i) of Theorem 3.12 of [23]. However in [23] $G$ is assumed to be linear. We now indicate how the main result of [23] (Theorem 2.3) can be deduced with the aid of Corollary 3.5; see Theorem 4.16.

**Proposition 4.14** Let $\Lambda \in \mathcal{P}_0$ and let $w \in W$ be a Weyl group element which
satisfies $\Delta^+_s \subset w\alpha(\Lambda, \Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda)$, and $\Phi_\alpha(\Lambda, \Lambda + \delta - \delta(\Lambda), \alpha) = 0$. (cf. (ii) of Theorem 4.3) Then $\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda)$ is a regular element (i.e. $(\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda), \alpha) = 0$ for every $\alpha$ in $\Delta$) and the corresponding positive system

\begin{equation}
(4.15) \quad P' = \{ \alpha \in \Delta \mid (\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda), \alpha) > 0 \} \text{ coincides with } w\alpha(\Lambda).
\end{equation}

Also $P'_\alpha(\Lambda) - \Phi_\alpha(\Lambda) = P' \cap P'_\alpha(\Lambda)$.

Proof. For $\alpha \in \Delta^+_s$ $(\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda), \alpha) = 0$. Suppose $\alpha \in \mathcal{P}_s(\Lambda)$. If $(\Lambda + \delta - \delta(\Lambda), \alpha) = 0$ then $(\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda), \alpha) = 0$. Assume $(\Lambda + \delta - \delta(\Lambda), \alpha) > 0$. Then $\alpha \in \Phi_\alpha(\Lambda)$ since by hypothesis $\Phi_\alpha(\Lambda) \subset \{ \alpha \in \mathcal{P}_s(\Lambda) \mid (\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \}$. Thus we must have $w\alpha \in \mathcal{P}_s(\Lambda)$. Since $\Lambda + \delta - \delta(\Lambda)$ is $\mathcal{P}_s(\Lambda)$-dominant $(\Lambda + \delta - \delta(\Lambda), \alpha) + (w\alpha, \alpha) > 0$. Then we have shown $(\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda), \alpha) = 0$ for $\alpha \in \mathcal{P}_s(\Lambda)$ which proves $\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda)$ is regular. Let $\alpha \in \mathcal{P}_s(\Lambda)$ be arbitrary. Then $(\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda), w\alpha) = (w^{-1}(\Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda)), \alpha) = (\Lambda + \delta, \alpha)$ (since $w^{-1}(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda)$ which is positive. That is $w\alpha \in \mathcal{P}_s(\Lambda)$ implies $\mathcal{P}_s(\Lambda) \cap \mathcal{P}_s(\Lambda) = \mathcal{P}_s(\Lambda) - \mathcal{P}_s(\Lambda)$ since $\mathcal{P}_s(\Lambda) \cup \mathcal{P}_s(\Lambda) = \mathcal{P}_s(\Lambda)$. Also $\mathcal{P}_s(\Lambda) = w(\mathcal{P}_s(\Lambda)) \cap \mathcal{P}_s(\Lambda)$ and since $\Phi_\alpha(\Lambda) \subset \mathcal{P}_s(\Lambda)$ the last equation further implies $\mathcal{P}_s(\Lambda) - \Phi_\alpha(\Lambda) = \mathcal{P}_s(\Lambda) - \mathcal{P}_s(\Lambda)$ since $\mathcal{P}_s(\Lambda) \cap \mathcal{P}_s(\Lambda) = \mathcal{P}_s(\Lambda)$.

Remark. In Proposition 4.14 (and hence in Theorem 4.3) the condition $\Phi_\alpha(\Lambda) \subset \{ \alpha \in \mathcal{P}_s(\Lambda) \mid (\Lambda + \delta - \delta(\Lambda), \alpha) = 0 \}$ is automatically satisfied. Indeed for $\alpha \in \Phi_\alpha(\Lambda) \subset \mathcal{P}_s(\Lambda)$ $0 \leq (\Lambda + \delta - \delta(\Lambda), \alpha) = (w^{-1}(\Lambda + \delta - \delta(\Lambda)), w^{-1}\alpha) = (\Lambda + \delta, \alpha)$ (since $w^{-1}(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda)$ which is positive. Thus we have shown $\mathcal{P}_s(\Lambda) - \Phi_\alpha(\Lambda) = \mathcal{P}_s(\Lambda) - \mathcal{P}_s(\Lambda)$ since $\mathcal{P}_s(\Lambda) \cap \mathcal{P}_s(\Lambda) = \mathcal{P}_s(\Lambda)$.

Theorem 4.16. Assume that $G$ is linear and its complexification $G^c$ is simply connected. (In particular if $\Lambda \in \mathfrak{h}^*$ is $\Delta^+_s$-dominant integral the irreducible finite-dimensional representation of $\mathfrak{k}$ defined by $\Lambda$ integrates to a representation of $K$.) Let $\Lambda \in \mathfrak{g}^*$ be such that every non-compact root in $P(\Lambda)$ is totally positive. If $H^q(\text{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}), \mathfrak{g}) = 0$ then there is a parabolic subalgebra $\mathfrak{g} / \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{h}^* = \sum \mathfrak{m}_{\alpha}$ such that $q = 2|\mathfrak{m}_\alpha|$. Also $(\Lambda + \delta - \delta(\Lambda), \Delta(\mathfrak{m}_\alpha)) = 0$.

Proof. If $H^q(\text{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}), \mathfrak{g}) = 0$ then by (4.2) $\text{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) = 0$. Let $\Lambda \in \mathfrak{g}^*$ such that $(\Lambda, \Lambda + 2\delta) = 0$. Let $\Lambda \in \mathfrak{g}^*$ such that $(\Lambda, \Lambda + 2\delta) = 0$. By Corollary 3.5 $H_\alpha$ has minimal $\mathfrak{h}$ type $\mu = \Lambda + \delta - \delta(\Lambda) - \delta(\Lambda)$ for some Weyl group element $w$ such that $\Delta^+_s \subset w\alpha(\Lambda)$ and $q = |\Phi_\alpha(\Lambda)| - 2|\mathfrak{m}_\alpha| - |\mathfrak{m}_\alpha| + |\mathfrak{m}_\alpha|$. Also $w(\Lambda + \delta - \delta(\Lambda)) = \Lambda + \delta - \delta(\Lambda)$. By Corollary 3.15 $H_\alpha$ is a highest weight $\mathfrak{g}$ module with highest weight $\mu$ relative to the positive system $P(\Lambda) = P(\Lambda) \cup P(\Lambda) = \mathfrak{g}^* \cup \mathfrak{g}^*$. Also $\mu + \delta(\Lambda) = \Lambda + \delta - \delta(\Lambda) + w\delta(\Lambda)$ is regular by Proposition 4.14 (see remark following Proposition 4.14). Thus since $G$ is assumed to be linear we can apply Parthasarathy's Theorem A of [17] to conclude the following:
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\(\mu = \Lambda_0 + \langle \theta_{u_1}, r \rangle\) for some parabolic subalgebra \(\theta_1 = m_1 + u_1\) of \(g\) where \(\theta_1 \supset \mathfrak{h} + \sum_{\alpha \in P_{\Lambda}} \mathfrak{g}_\alpha\) and where \(\Lambda_0 \in \mathfrak{h}^*\) is \(P(\Lambda)\)-dominant integral, and \((\Lambda_0, \Delta(m_1)) = 0\).

Moreover by (3.49) of [17] \(\theta_{u_1} = \mathcal{P}' \cap P_{\Lambda}(\mathcal{A})\) where \(\mathcal{P}'\) is the positive system defined by the regular element \(\mu + \delta_\mathcal{A} - \delta_{\mathcal{A}}\). Hence by Proposition 4.14 \(\theta_{u_1} = P_{\Lambda}(\mathcal{A}) - \Phi_{\Lambda}(\mathcal{A})\). Then \(\Lambda + \delta_\mathcal{A} + w\delta^{(\Lambda)} - \delta_\mathcal{A} = \mu = \Lambda_0 + \langle \theta_{u_1}, r \rangle = \Lambda_0 + \langle P_{\Lambda}(\mathcal{A}) - \Phi_{\Lambda}(\mathcal{A}) \rangle = \Lambda_0 + \delta_\mathcal{A} + w\delta^{(\Lambda)} - \delta_\mathcal{A}\) (by (3.11)) \(\Rightarrow \Lambda_0 = \Lambda + \delta_\mathcal{A} - \delta_{\mathcal{A}} = \Lambda + \delta - \delta^{(\Lambda)} \Rightarrow (\Lambda + \delta - \delta^{(\Lambda)}, \Delta(m_1)) = 0\). We also have \(| \theta_{u_1} | = n - | \Phi_{\Lambda}(\mathcal{A}) |\) so that \(q = | \Phi_{\Lambda}(\mathcal{A}) | - 2 | \mathcal{Q}_{\Lambda} \cap \Phi_{\Lambda}(\mathcal{A}) | + | \mathcal{Q}_{\Lambda} | = n - | \theta_{u_1} | - 2 | \mathcal{Q}_{\Lambda} - \theta_{u_1} | + | \mathcal{Q}_{\Lambda} | = n - | \theta_{u_1} | - 2 | \mathcal{Q}_{\Lambda} - \theta_{u_1} | + | \mathcal{Q}_{\Lambda} | + | \mathcal{Q}_{\Lambda} | = 2 | \mathcal{Q}_{\Lambda} \cap \theta_{u_1} | - | \theta_{u_1} | + | \Delta^{(\Lambda)} - \mathcal{Q}_{\Lambda} |\).

Remark. If additional information on the Weyl group element \(\sigma_1\) above (where \(\sigma_1 \Delta^{(\Lambda)} = P(\Lambda)\)) were available the preceding proof might not require the appeal to Theorem A of [17]. For example if it were known that \(\langle P_{\Lambda}(\mathcal{A}) - \sigma_1 \Delta(m) \rangle = \delta_{\mathcal{A}} + w\delta^{(\Lambda)} - \delta_\mathcal{A} = 0\) then Theorem 4.16 would follow (even for \(G\) non-linear) by taking \(\theta_1 = \sigma_1 \theta\). However \(\mathfrak{g}\) is true only when certain additional restrictions on \(\Lambda\) are imposed.

Another classical vanishing theorem for the spaces \(H^q(\Gamma \backslash G/K, \theta_\Lambda)\) is the following one of Hotta and Parthasarathy; see Proposition 1 of [5].

Theorem 4.17. Let \(\Lambda \in \mathfrak{g}_\mathcal{K}\) be the \(\Delta^{(\Lambda)}\)-highest weight of \((\tau_\Lambda, V_\Lambda) \in \tilde{\mathcal{K}}\). Suppose that \((\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\) for every \(\alpha \in P_{\Lambda}(\mathcal{A})\). Then \(H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0\) for \(q \neq | \mathcal{Q}_{\Lambda} |\).

Here \(G\) is not assumed to be linear. Theorem 4.17 follows from a trivial application of Theorem 4.3. Namely if \(H^q(\Gamma \backslash G/K, \theta_\Lambda) = 0\) then \(q = | A_{\Lambda, r, \alpha} | - 2 | \mathcal{Q}_{\Lambda} \cap A_{\Lambda, r, \alpha} | + | \mathcal{Q}_{\Lambda} |\) where \(A_{\Lambda, r, \alpha} = \{ \alpha | P_{\Lambda}(\mathcal{A}) | (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0\}\). But \((\Lambda + \delta - \delta^{(\Lambda)}, \alpha) > 0\) for \(\alpha \in P_{\Lambda}(\mathcal{A})\) by hypothesis so \(A_{\Lambda, r, \alpha} = \emptyset\). Thus \(q = | \mathcal{Q}_{\Lambda} |\).

References


