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AN APPLICATION OF THE ITERATED LOOP SPACE THEORY TO COHOMOLOGY SUSPENSIONS

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0. Introduction

For a based space X, ΣX and ΩX denote respectively the reduced suspension and loop space of X. There is a natural (iterated) isomorphism

$$\phi^n\colon [\Sigma^n X, Y] \xrightarrow[\cong]{} [X, \Omega^n Y]$$

For $i\geq 1$ let $\Sigma^*\colon H^i(\Sigma X)\to \tilde{H}^{i-1}(X)$ be the suspension isomorphism and $\sigma^*\colon H^i(X)\to \tilde{H}^{i-1}(\Omega X)$ the cohomology suspension (see, for example, [16, VIII]). For an n-fold loop space $X=\Omega^n Y$, let

$$\xi_n = \phi^{-n}(1_X)\colon \Sigma^n X \to Y\,.$$

Then $\xi_n^*\colon H^i(Y)\to H^i(\Sigma^n X)$ factors as the composite

$$H^i(Y)\xrightarrow{(\sigma^*)^n} \tilde{H}^{i-n}(X)\xrightarrow[\cong]{(\Sigma^*)^{-n}} H^i(\Sigma^n X)\,.$$

So we can obtain results on $(\sigma^*)^n$ by studying ξ_n^*.

Convert ξ_n into a fibre map and denote by $G_n X$ its fibre. (It is known by Barcus and Meyer [2] that $G_1X\simeq\Sigma(X\wedge X)$.) Suspose that X is $(m-1)$-connected ($m>1$) and consider the Serre spectral sequence for the mod p cohomology of this fibration. Then Milgram [12, I] showed that there is a $(3m+n-1)$-equivalence of $\Sigma^n e_n^2 X$ into $G_n X$ (where $e_n^2X\simeq S^{n-1}|\!\times_{Z_2}(X\wedge X)$, the extended square of X [11]). Using it, he found formulas for the differentials of this spectral sequence in total degrees less than $3m+n-1$, which gives a precise description of the relationship between the cohomology of Y and that of X. Our aim is to extend this result to total degrees less than $4m+n-1$.

Throughout this paper, all spaces are assumed to be of the homotopy type of a based CW-complex. p will always denote a prime, and let $H_*(X)$ and $H^*(X)$ denote respectively the mod p homology and cohomology of X. For all X, $H_*(X)$ is assumed to be of finite type. So we have a dual pairing

$$\langle\ ,\ \rangle\colon H^i(X)\otimes H_i(X)\to Z_p\,.$$

This paper is organized as follows. In §1 we collect some results about n-fold loop spaces. In §2 we mention the result of Milgram [12, Theorem 4.6] (Theorem 3) in our terminology. With the aid of this theorem, our main result (Theorem 7) is stated in §3. Its proof is facilitated by use of two lemmas (Lemmas 8 and 9) which are also due to Milgram; we treat them in §4. §§5 and 6 are devoted to prove Theorems 3 and 7 respectively. §7 contains several remarks.

1. Results about n-fold loop spaces

F. Cohen [5, III] constructed a satisfactory theory of homology operations on n-fold loop spaces. We exhibit some of his results which we need. For more complete accounts see [5, III].

Let Y be an arbitrary space and $n \geq 1$. Then

(1.1) *In $H_*(\Omega^n Y)$ there exist operations*

$$Q^s\colon H_i(\Omega^n Y) \to H_{i+s}(\Omega^n Y) \text{ for } p=2 \text{ and } 0 \leq s \leq i+n-1\,,$$
$$Q^s\colon H_i(\Omega^n Y) \to H_{i+2(p-1)s}(\Omega^n Y) \text{ for } p>2 \text{ and } 0 \leq 2s \leq i+n-1\,,$$
$$\lambda_{n-1}\colon H_i(\Omega^n Y) \otimes H_j(\Omega^n Y) \to H_{i+j+n-1}(\Omega^n Y)$$

which are natural with respect to n-fold loop maps and satisfy the following properties:

(1.2) *$Q^s(a)=0$ if $p=2$ and $s<|a|$ or $p>2$ and $2s<|a|$ (where $|a|$ denotes the degree of a).*

(1.3) *$Q^s(a)=a*\cdots*a$ (p-fold) if $p=2$ and $s=|a|$ or $p>2$ and $2s=|a|$ (where $*$ denotes the Pontrjagin product).*

(1.4) *$Q^s(1)=0$ if $s>0$ (where $1 \in H_0(\Omega^n Y)$ is the identity element).*

(1.5) *Let $\psi\colon H_*(\Omega^n Y) \to H_*(\Omega^n Y) \otimes H_*(\Omega^n Y)$ be the coproduct induced by the diagonal map of $\Omega^n Y$. If $\psi(a)=\sum a' \otimes a''$, then*

$$\psi Q^s(a) = \sum_{i+j=s} Q^i(a') \otimes Q^j(a'')\,.$$

(1.6) *If $s>pt$, then*

$$Q^s Q^t = \sum_i (-1)^{s+i} \binom{(p-1)(i-t)-1}{pi-s} Q^{s+t-i} Q^i\,;$$

if $p>2$, $s \geq pt$ and Δ is the mod *p homology Bockstein, then*

$$Q^s \Delta Q^t = \sum_i (-1)^{s+i} \binom{(p-1)(i-t)}{pi-s} \Delta Q^{s+t-i} Q^i - \sum_i (-1)^{s+i} \binom{(p-1)(i-t)-1}{pi-s-1} Q^{s+t-i} \Delta Q^i$$

(*where* $\binom{i}{j}=i!/j!\,(i-j)!$).

(1.7) *Suppose* $p=2$ *and let* $Sq_*^r\colon H_i(\Omega^n Y)\to H_{i-r}(\Omega^n Y)$ *be the dual of the Steenrod square* Sq^r [14]. *Then*

$$Sq_*^r Q^s(a)=\begin{cases}\sum_i \binom{s-r}{r-2i} Q^{s-r+i}(Sq_*^i a) & \text{if } s<|a|+n-1\\ \sum_i \binom{s-r}{r-2i} Q^{s-r+i}(Sq_*^i a) + \sum_{\substack{i+j=r\\ i<j}} \lambda_{n-1}(Sq_*^i a,\, Sq_*^j a) & \text{if } s=|a|+n-1\,.\end{cases}$$

(1.8) *If Y is a loop space, then* $\lambda_{n-1}(a, b)=0$.

(1.9) $\lambda_0(a, b)=a*b-(-1)^{|a||b|}b*a$.

(1.10) $\lambda_{n-1}(a, b)=(-1)^{|a||b|+(|a|+|b|)(n-1)+n}\lambda_{n-1}(b, a)$; *if* $p=2$, $\lambda_{n-1}(a, a)=0$.

(1.11) $\lambda_{n-1}(1, a)=\lambda_{n-1}(a, 1)=0$.

(1.12) *If* $\psi(a)=\sum a'\otimes a''$ *and* $\psi(b)=\sum b'\otimes b''$, *then*

$$\psi\lambda_{n-1}(a, b)=\sum(-1)^{|a''||b'|+|a''|(n-1)}\lambda_{n-1}(a', b')\otimes(a''*b'') +(-1)^{|a''||b'|+|b'|(n-1)}(a'*b')\otimes\lambda_{n-1}(a'', b'')\,.$$

(1.13) $(-1)^{(|a|+n-1)(|c|+n-1)}\lambda_{n-1}(a, \lambda_{n-1}(b,c))+(-1)^{(|b|+n-1)(|a|+n-1)}\lambda_{n-1}(b, \lambda_{n-1}(c,a)) +(-1)^{(|c|+n-1)(|b|+n-1)}\lambda_{n-1}(c, \lambda_{n-1}(a, b))=0$; *if* $p=3$, $\lambda_{n-1}(a, \lambda_{n-1}(a, a))=0$.

(1.14) *Suppose* $p=2$. *Then*

$$Sq_*^r\lambda_{n-1}(a, b)=\sum_{i+j=r}\lambda_{n-1}(Sq_*^i a,\, Sq_*^j b)\,.$$

(1.15) *For* $n>1$ *let* $\sigma_*\colon \tilde{H}_i(\Omega^n Y)\to H_{i+1}(\Omega^{n-1}Y)$ *be the homology suspension. Then* $\sigma_* Q^s(a)=Q^s(\sigma_* a)$ *and* $\sigma_*\lambda_{n-1}(a, b)=\lambda_{n-2}(\sigma_* a, \sigma_* b)$.

(1.16) *If* $\Omega^{n-1}Y$, $n>1$, *is simply connected and* a', $b'\in H_{*+1}(\Omega^{n-1}Y)$ *transgress to* a, $b\in H_*(\Omega^n Y)$ *respectively in the Serre spectral sequence of the path fibration* $\Omega^n Y\to P\Omega^{n-1}Y\to\Omega^{n-1}Y$, *then* $Q^s(a')$ *and* $\lambda_{n-2}(a', b')$ *transgress to* $Q^s(a)$ *and* $\lambda_{n-1}(a, b)$ *respectively.*
(Here we have written

$$\begin{cases} Q^{|a|+n-1}(a) & \text{when } p=2\\ Q^{(|a|+n-1)/2}(a) & \text{when } p>2\end{cases}$$

instead of $\xi_{n-1}(a)$; for this notation see Theorem 1.3 of [5, III].)

Throughout the remainder of this section, X will denote an arbitrary *con-*

nected space. Let

$$\{a, b, c, \cdots\}$$

be a totally ordered Z_p-basis of homogeneous elements for $\tilde{H}_*(X)$. (This ordering has no essential influence on the following argument.) Then the *basic* λ_{n-1}*-products* are defined as follows. Define $a, b, \cdots$ to be the *basic* λ_{n-1}*-products of weight* 1. Assume inductively that the basic λ_{n-1}-products of weight j, $1 \leq j \leq k$, are defined and totally ordered among themselves. Then a *basic* λ_{n-1}*-product of weight* k is defined to be $\lambda_{n-1}(x, y)$ where

(1) x and y are basic λ_{n-1}-products with weight(x)+weight$(y)=k$;

(2) $x<y$ and if $y=\lambda_{n-1}(z, w)$ for z and w basic λ_{n-1}-products with $z<w$, then $x \geq z$;

or

(2)′ $x=y$ if $p>2$ where x is a basic λ_{n-1}-product of weight 1 and $|x|+n$ is even.

For example, the basic λ_{n-1}-products of weight 2 are

$$\lambda_{n-1}(a, b) \text{ for } a<b;$$
$$\lambda_{n-1}(a, a) \text{ for } p>2 \text{ where } |a|+n \text{ is even,}$$

and those of weight 3 are

$$\lambda_{n-1}(b, \lambda_{n-1}(a, c)),\ \lambda_{n-1}(c, \lambda_{n-1}(a, b)) \text{ for } a<b<c\,;$$
$$\lambda_{n-1}(a, \lambda_{n-1}(a, b)),\ \lambda_{n-1}(b, \lambda_{n-1}(a, b)) \text{ for } a<b\,.$$

REMARK. The notion of basic λ_{n-1}-products is derived from (1.10) and (1.13). It will be regarded as a procedure for choosing certain indecomposable elements of $H_*(\Omega^n\Sigma^n X)$.

Consider sequences of non-negative integers

$$J = \begin{cases} (s_1, \cdots, s_k) & \text{when } p=2 \\ (\varepsilon_1, s_1, \cdots, \varepsilon_k, s_k) & \text{when } p>2 \end{cases}$$

where $\varepsilon_j=0$ or 1. Define the *length* and *excess* of J by

$$l(J) = k \quad \text{and}$$

$$e(J) = \begin{cases} s_1 - \sum_{j=2}^{k} s_j & \text{when } p=2 \\ 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} (2(p-1)s_j - \varepsilon_j) & \text{when } p>2\,. \end{cases}$$

J is said to be *admissible* if

$$2s_j \geq s_{j-1} \qquad \text{when } p=2$$

$$ps_j-\varepsilon_j\geq s_{j-1} \text{ when } p>2$$

for $2\leq j\leq k$. J determines the homology operation

$$Q^J=\begin{cases}Q^{s_1}\cdots Q^{s_k} & \text{when } p=2\\ \Delta^{\varepsilon_1}Q^{s_1}\cdots\Delta^{\varepsilon_k}Q^{s_k} & \text{when } p>2\end{cases}$$

REMARK. The notion of admissibility is derived from (1.6).

For any space X let

$$\eta_n=\phi^n(1_{\Sigma^n X})\colon X\to\Omega^n\Sigma^n X.$$

It is well known that $\eta_{n*}\colon H_*(X)\to H_*(\Omega^n\Sigma^n X)$ is injective. So we may regard that $H_*(X)\subset H_*(\Omega^n\Sigma^n X)$. Then, for $a, b\in H_*(X)$, we have the following elements of $H_*(\Omega^n\Sigma^n X)$:

$$a*b,\ Q^s(a),\ \lambda_{n-1}(a, b), \text{ etc.}$$

Under the above notations and terminologies, we have

(1.17) *If $n>1$, $H_*(\Omega^n\Sigma^n X)$ is the free (associative and) commutative Z_p-algebra generated by*

$$\left\{Q^J(x)\;\middle|\;\begin{array}{l} x \text{ is a basic } \lambda_{n-1}\text{-product}; J \text{ is admissible};\\ \text{if } p=2,\ e(J)>|x| \text{ and } s_k\leq|x|+n-1;\\ \text{if } p>2,\ e(J)+\varepsilon_1>|x| \text{ and } 2s_k\leq|x|+n-1\,.\end{array}\right\}$$

and if $n=1$, $H_*(\Omega\Sigma X)$ is the free associative Z_p-algebra generated by $\{a, b, \cdots\}$.

Thus for $n\geq 1$ $H_*(\Omega^n\Sigma^n X)$ has a Z_p-basis consisting of all monomials in the above generators. Let us define the *height* of a monomial as follows:

$$\begin{aligned}&\mathrm{height}(Q^J(x))=p^{l(J)}\,\mathrm{weight}(x) \text{ and}\\ &\mathrm{height}(Q^J(x)*Q^K(y))=\mathrm{height}(Q^J(x))+\mathrm{height}(Q^K(y))\,.\end{aligned}$$

According to May [7], there is a functor C_n from spaces to spaces together with a natural transformation $\alpha_n\colon C_n\to\Omega^n\Sigma^n$ such that $\alpha_n X\colon C_nX\to\Omega^n\Sigma^n X$ is a (weak) homotopy equivalence for all X. The space C_nX has a natural filtration $\{F_kC_nX\,|\,k\geq 0\}$ (such that $F_0C_nX=\{*\}$, $F_1C_nX\simeq X$ and $F_kC_nX\subset F_{k+1}C_nX$ is a cofibration for all k). $H_*(F_kC_nX)$ may be regarded as a sub-Z_p-module of $H_*(\Omega^n\Sigma^n X)$ and then it is additively generated by the elements of height $\leq k$.

For $k, n\geq 1$ let

$$e_n^kX=F_kC_nX/F_{k-1}C_nX\,.$$

As displayed in [9], if X is $(m-1)$-connected, $m>1$, then e_n^kX is $(km-1)$-connected and therefore

(1.18) *The composite*

$$F_kC_nX \xrightarrow{\subset} C_nX \xrightarrow{\alpha_n X} \Omega^n\Sigma^n X$$

(*which we denote by* j_k) *is a* $((k+1)m-1)$*-equivalence.*

So there is an isomorphism

$$H^i(\Omega^n\Sigma^n X)\cong H^i(F_kC_nX) \text{ for } i<(k+1)m-1\,.$$

For $\alpha\in H^i(X)$ let α_* denote its dual. We regard it as an element of $H_i(\Omega^n\Sigma^n X)$. Then, for α, $\beta\in H^*(X)$, we have the following elements of $H^*(\Omega^n\Sigma^n X)$:

$$\alpha*\beta = \text{the dual of } \alpha_* * \beta_*\,,$$
$$Q^s(\alpha) = \text{the dual of } Q^s(\alpha_*)\,,$$
$$\lambda_{n-1}(\alpha, \beta) = \text{the dual of } \lambda_{n-1}(\alpha_*, \beta_*), \text{ etc.}$$

Combining the above notations and results, we obtain

Proposition 1. *Suppose that X is* $(m-1)$*-connected and let* $\{\alpha, \beta, \gamma, \cdots\}$ *be a totally ordered* Z_p*-basis for* $\tilde{H}^*(X)$. *Then a* Z_p*-basis for* $\tilde{H}^*(\Omega^n\Sigma^n X)$ *in dimensions* $<3m-1$ *is given by*

height 1: α,
height 2: $\alpha*\beta$ *for* $\alpha\leq\beta$ *where if* $\alpha=\beta$, $p>2$ *and* $|\alpha|$ *is even*;
$Q^s(\alpha)$ *for* $p=2$ *and* $|\alpha|\leq s\leq|\alpha|+n-1$;
$\lambda_{n-1}(\alpha, \beta)$ *for* $\alpha\leq\beta$ *where if* $\alpha=\beta$, $p>2$ *and* $|\alpha|+n$ *is even,*

and that in dimensions $<4m-1$ *is given by the above together with*

height 3: $\alpha*\beta*\gamma$ *for* $\alpha\leq\beta\leq\gamma$ *where if* $\alpha=\beta=\gamma$, $p>3$ *and* $|\alpha|$ *is even, and if* $\alpha=\beta\neq\gamma$ *or* $\alpha\neq\beta=\gamma$, $p>2$ *and* $|\beta|$ *is even*;
$\alpha*Q^s(\beta)$ *for* $p=2$ *and* $|\beta|\leq s\leq|\beta|+n-1$;
$\alpha*\lambda_{n-1}(\beta, \gamma)$ *for* $\beta\leq\gamma$ *where if* $\beta=\gamma$, $p>2$ *and* $|\beta|+n$ *is even*;
$\Delta^\varepsilon Q^s(\alpha)$ *for* $p=3$, $\varepsilon=0$ *or* 1 *and* $|\alpha|+\varepsilon\leq 2s\leq|\alpha|+n-1$;
$\lambda_{n-1}(\alpha, \lambda_{n-1}(\beta, \gamma))$ *for* $\alpha\geq\beta<\gamma$.

2. Review of Milgram's work

As in §0, if $X=\Omega^n Y$, we have a fibration

$$G_nX \xrightarrow{\nu_n} \Sigma^n X \xrightarrow{\xi_n} Y\,. \tag{2.1}$$

Application of the functor Ω^n yields a fibration

$$\Omega^n G_nX \xrightarrow{\Omega^n\nu_n} \Omega^n\Sigma^n X \xrightarrow{\Omega^n\xi_n} X\,. \tag{2.2}$$

Put

$$F_nX = \Omega^n G_n X\,.$$

Since $(\Omega^n\xi_n)\eta_n = 1_X$ it follows that (2.2) is fibre (weak) homotopically trivial (see [12, Lemma 4.1]). So we have

Lemma 2. *The following equivalent statements hold*:

(i) *The* mod p *cohomology Serre spectral sequence of the fibration* (2.2) *collapses.*

(ii) $(\Omega^n\nu_n)^*\colon H^*(\Omega^n\Sigma^n X) \to H^*(F_nX)$ *is surjective and its kernel coincides with the ideal generated by* $(\Omega^n\xi_n)^*(\sum_{i>0} H^i(X))$.

For the proof see [13].

Suppose again that $X=\Omega^n Y$ is $(m-1)$-connected for $m>1$. Then it follows from Proposition 1 and Lemma 2(ii) that F_nX is $(2m-1)$-connected. Let $(F_nX)_{3m-1}$ be the $(3m-1)$-skeleton of F_nX. Then the inclusion map i_{3m-1}: $(F_nX)_{3m-1}\to F_nX$ is a $(3m-1)$-equivalence. Since $F_nX=\Omega^n G_nX$, we have a map

$$\phi^{-n}(i_{3m-1})\colon \Sigma^n(F_nX)_{3m-1} \to G_nX\,.$$

Consider the commutative diagram

$$\begin{array}{ccc} \pi_i((F_nX)_{3m-1}) & \xrightarrow{i_{3m-1\#}} & \pi_i(F_nX) \\ \Big\downarrow{\scriptstyle \Sigma^n} & & \cong\Big\downarrow{\scriptstyle \phi^{-n}} \\ \pi_{i+n}(\Sigma^n(F_nX)_{3m-1}) & \xrightarrow{\phi^{-n}(i_{3m-1})_\#} & \pi_{i+n}(G_nX) \end{array}$$

where Σ^n is the n-fold suspension homomorphism. By the Freudenthal suspension theorem, Σ^n is an isomorphism for $i<4m-1$ and an epimorphism for $i=4m-1$. Therefore $\phi^{-n}(i_{3m-1})$ is a $(3m+n-1)$-equivalence. So there is an isomorphism

$$H^i(G_nX)\cong H^i(\Sigma^n(F_nX)_{3m-1}) \text{ for } i<3m+n-1\,.$$

Through this isomorphism we shall identify them. Then, for $\omega\in H^i(F_nX)$ with $i<3m-1$, we have an element $\sigma^n(\omega)\in H^{i+n}(G_nX)$ (hereafter we often write $\sigma^n(\ \ )$ for $(\Sigma^*)^{-n}(\ \ )$).

Let us compute $H^*((F_nX)_{3m-1})$ by using the Serre exact sequence of the fibration (2.2); it is valid for dimensions $\leq 3m-1$. Moreover, the transgression τ is trivial, by (i) of Lemma 2. Thus we have a short exact sequence

$$0 \to H^i(X) \xrightarrow{(\Omega^n\xi_n)^*} H^i(\Omega^n\Sigma^n X) \xrightarrow{(\Omega^n\nu_n)^*} H^i(F_nX) \to 0$$

for $i<3m-1$. For $\chi\in H^i(\Omega^n\Sigma^n X)$ we denote by $[\chi]$ the image of χ under

$(\Omega^n\nu_n)^*$:

$$(\Omega^n\nu_n)^*(\chi) = [\chi] .$$

Then from the former part of Proposition 1 it follows that

(2.3) *Suppose that* $X=\Omega^n Y$ *is* $(m-1)$*-connected* $(m>1)$ *and let* $\{\alpha, \beta, \cdots\}$ *be a totally ordered* Z_p*-basis for* $\tilde{H}^*(X)$. *Then a* Z_p*-basis for* $\tilde{H}^*(G_nX)$ *in dimensions* $<3m+n-1$ *is given by*:

(1) $\sigma^n[\alpha*\beta]$ *for* $\alpha\leq\beta$ *where if* $\alpha=\beta$, $p>2$ *and* $|\alpha|$ *is even*;

(2) $\sigma^n[Q^s(\alpha)]$ *for* $p=2$ *where* $|\alpha|\leq s\leq|\alpha|+n-1$;

(3) $\sigma^n[\lambda_{n-1}(\alpha, \beta)]$ *for* $\alpha\leq\beta$ *where if* $\alpha=\beta$, $p>2$ *and* $|\alpha|+n$ *is even.*

Notice that the elements α and β appearing in (2.3) have dimension $<2m-1$. We now recall the following fact (see (3.1) of [16, VIII]):

(2.4) *If* $X=\Omega^n Y$ *is* $(m-1)$*-connected, then*

$$(\sigma^*)^n\colon H^{i+n}(Y)\to H^i(X) \text{ or}$$
$$\xi_n^*\colon H^{i+n}(Y)\to H^{i+n}(\Sigma^n X)$$

is an isomorphism for $i\leq 2m-1$.

For $\alpha\in H^i(X)$ we denote by ${}^n\tilde{\alpha}$ an element of $H^{i+n}(Y)$ such that

$$(\sigma^*)^n({}^n\tilde{\alpha}) = \alpha \quad\text{or}\quad \xi_n^*({}^n\tilde{\alpha}) = \sigma^n(\alpha) .$$

Thus, for each $\alpha\in H^i(X)$ with $i\leq 2m-1$, ${}^n\tilde{\alpha}$ exists uniquely.

Consider the fibration (2.1). Since Y and G_nX are $(m+n-1)$- and $(2m+n-1)$-connected respectively, its Serre exact sequence

$$\text{(2.5)}\qquad \cdots\to H^i(Y)\xrightarrow{\xi_n^*} H^i(\Sigma^n X)\xrightarrow{\nu_n^*} H^i(G_nX)\xrightarrow{\tau} H^{i+1}(Y)\to\cdots$$

is valid for $i\leq 3m+2n-1$.

Theorem 3 (Milgram). *Under the above situation, the following formulas hold up to non-zero constants*:

(1) $\nu_n^*(\sigma^n(\alpha\cup\beta))=\sigma^n[\alpha*\beta]$ (*where* $\cup$ *denotes the cup product*) *and so* $\tau(\sigma^n[\alpha*\beta])=0$;

(2) *If* $p=2$, $\tau(\sigma^n[Q^s(\alpha)])=Sq^{s+1}({}^n\tilde{\alpha})$;

(3) $\tau(\sigma^n[\lambda_{n-1}(\alpha, \beta)])={}^n\tilde{\alpha}\cup{}^n\tilde{\beta}$.

REMARK. In (1) $\alpha\cup\beta$ is always non-zero; see the Remark below Lemma 5.

For the proof see §5. Assuming this Theorem for a while, we proceed with our argument.

In the exact sequence (2.5), for $\omega\in H^i((F_nX)_{3m-1})$ with $\tau(\sigma^n(\omega))=0$, we denote by $\{\omega\}$ an element of $H^i(X)$ such that

$$\nu_n^*(\sigma^n\{\omega\}) = \sigma^n(\omega)$$

in $H^{i+n}(G_nX)$. (2.5) gives rise to a short exact sequence

$$0 \to \mathrm{Cok}\,\tau \xrightarrow{\xi_n^*} H^i(\Sigma^n X) \xrightarrow{\nu_n^*} \mathrm{Ker}\,\tau \to 0$$

for $i<3m+n-1$. By Theorem 3, the additive structures of Im τ and Ker τ can be easily described. Thus we have

Corollary 4. *Let*

$$\sum_{i<2m-1} \tilde{H}^i(X) = Z_p\{\alpha, \beta, \cdots\} .$$

Then $\tilde{H}^*(X)$ in dimensions $<3m-1$ has a Z_p-basis consisting of elements of the following four kinds:

(1) θ *where* $\sigma^n(\theta) \in \mathrm{Im}\, \xi_n^*$;

(2) $\alpha \cup \beta$ *for* $\alpha \leq \beta$ *where if* $\alpha = \beta$, $p>2$ *and* $|\alpha|$ *is even*;

(3) $\{Q^s(\alpha)\}$ *for* $p=2$ *and* $|\alpha| \leq s \leq |\alpha|+n-1$ *where* $Sq^{s+1}({}^n\tilde{\alpha})=0$;

(4) $\{\lambda_{n-1}(\alpha, \beta)\}$ *for* $\alpha \leq \beta$ *where if* $\alpha=\beta$, $p>2$ *and* $|\alpha|+n$ *is even, and* ${}^n\tilde{\alpha} \cup {}^n\tilde{\beta}=0$.

NOTATION. From now on, we use the letters α, β, γ to denote elements of $\tilde{H}^*(X)$ of dimension $\leq 2m-1$ and the letter θ to denote an element of $\tilde{H}^*(X)$ of dimension $<3m-1$ for which ${}^n\tilde{\theta}$ exists, unless otherwise stated. Of course, the θ includes α.

Since the fibration (2.2) is fibre (weak) homotopically trivial, we may assume that there is a fibration

$$X \xrightarrow{\eta_n} \Omega^n \Sigma^n X \to F_n X .$$

Consider the following commutative diagram

$$\begin{array}{ccccc} F_1C_nX & \to & F_2C_nX & \to & e_n^2X \\ \simeq \downarrow & & \downarrow j_2 & & \vdots\, j_2' \\ X & \xrightarrow{\eta_n} & \Omega^n\Sigma^n X & \to & F_nX \end{array}$$

where the upper row is a cofibration. Then it follows from (1.18) that the induced map j_2': $e_n^2X \to F_nX$ is a $(3m-1)$-equivalence. Since e_n^2X is homotopy equivalent to $S^{n-1}|\!\!\times_{Z_2}(X\wedge X)$ (see Proposition 2.6 and Remark 4.10 of [8]), we can use $S^{n-1}|\!\!\times_{Z_2}(X\wedge X)$ instead of $(F_nX)_{3m-1}$ in the argument of this section, which is just the argument of Milgram [12, I].

3. The main theorem

We now take the $(4m-1)$-skeleton $(F_nX)_{4m-1}$ of F_nX. Since the inclusion map $i_{4m-1}\colon (F_nX)_{4m-1}\to F_nX$ is a $(4m-1)$-equivalence, by the same argument as in §2, the map

$$\rho_n = \phi^{-n}(i_{4m-1})\colon \Sigma^n(F_nX)_{4m-1} \to G_nX$$

is a $(4m+n-1)$-equivalence. (Note that this equivalence is natural in X; see the diagram (5.1).) So there is an isomorphism

$$H^i(G_nX)\cong H^i(\Sigma^n(F_nX)_{4m-1}) \text{ for } i<4m+n-1\,.$$

(It follows from (2.4) that this isomorphism holds for $i=4m+n-1$.) Similarly we shall identify them.

Let us compute $H^*((F_nX)_{4m-1})$ by using the Serre spectral sequence $\{E_r, d_r\}$ of the fibration (2.2); that is,

$$E_2^{i,j} = H^i(F_nX)\otimes H^j(X) \text{ and } E_\infty^{*,*} = \mathrm{Gr}\, H^*(\Omega^n\Sigma^nX)\,.$$

By (i) of Lemma 2, $E_r^{*,*}=E_\infty^{*,*}$ for all $r\geq 2$. It follows from (2.3) that $E_2^{i,j}$ for $i+j<4m-1$ with $i, j>0$ has a Z_p-basis consisting of elements

$$[\beta*\gamma]\otimes\alpha,\ [Q^s(\beta)]\otimes\alpha\ (p=2) \text{ and } [\lambda_{n-1}(\beta, \gamma)]\otimes\alpha\,.$$

For $\alpha\in H^i(X)$ let $\bar\alpha\in H^i(\Omega^n\Sigma^nX)$ denote the dual of $\alpha_*\in H_i(\Omega^n\Sigma^nX)$; then $\eta_n^*(\bar\alpha)=\alpha$. By the multiplicative properties of the cohomology spectral sequence, $[\beta*\gamma]\otimes\alpha$, $[Q^s(\beta)]\otimes\alpha$ $(p=2)$, $[\lambda_{n-1}(\beta, \gamma)]\otimes\alpha\in E_\infty^{*,*}$ are represented by $\bar\alpha\cup(\beta*\gamma)$, $\bar\alpha\cup Q^s(\beta)$ $(p=2)$, $\bar\alpha\cup\lambda_{n-1}(\beta, \gamma)\in H^*(\Omega^n\Sigma^nX)$ respectively.

Lemma 5. *In* $\sum_{i<4m-1}\tilde H^i(\Omega^n\Sigma^nX)$ *the following relations hold*:

(i) (1) *If* α, β, γ *are distinct*,

$$\bar\alpha\cup(\beta*\gamma) = (-1)^{|\alpha||\beta|}\beta*(\alpha\cup\gamma)+(-1)^{|\alpha||\gamma|+|\beta||\gamma|}\gamma*(\alpha\cup\beta)+\alpha*\beta*\gamma;$$

(2) *If* $\alpha\neq\beta$,

$$\bar\alpha\cup(\beta*\beta) = \beta*(\alpha\cup\beta)+\alpha*\beta*\beta \text{ and}$$
$$\bar\beta\cup(\alpha*\beta) = 2\alpha*(\beta\cup\beta)+\beta*(\alpha\cup\beta)+2\alpha*\beta*\beta;$$

(3) $$\bar\alpha\cup(\alpha*\alpha) = \alpha*(\alpha\cup\alpha)+3\alpha*\alpha*\alpha\,.$$

(ii) *If* $p=2$,

$$\bar\alpha\cup Q^s(\beta) = \alpha*Q^s(\beta)\,.$$

(iii) (1) *If* α, β, γ *are distinct*,

$$\bar\alpha\cup\lambda_{n-1}(\beta, \gamma) = (-1)^{|\alpha||\beta|+|\alpha|(n-1)}\lambda_{n-1}(\beta, \alpha\cup\gamma)$$
$$+(-1)^{|\alpha||\gamma|+|\beta||\gamma|+(|\alpha|+|\beta|+|\gamma|)(n-1)+n}\lambda_{n-1}(\gamma, \alpha\cup\beta)+\alpha*\lambda_{n-1}(\beta, \gamma);$$

(2) *If* $\alpha\neq\beta$,

$$\bar\alpha\cup\lambda_{n-1}(\beta, \beta) = (-1)^{|\alpha|}\lambda_{n-1}(\beta, \alpha\cup\beta)+\alpha*\lambda_{n-1}(\beta, \beta) \text{ and}$$

$$\bar{\beta}\cup\lambda_{n-1}(\alpha,\beta)=(-1)^{|\alpha||\beta|+|\beta|(n-1)}\lambda_{n-1}(\alpha,\beta\cup\beta)$$
$$+(-1)^{|\beta|+|\alpha|(n-1)+n}\lambda_{n-1}(\beta,\alpha\cup\beta)+\beta*\lambda_{n-1}(\alpha,\beta);$$
(3) $\bar{\alpha}\cup\lambda_{n-1}(\alpha,\alpha)=(-1)^{|\alpha|}\lambda_{n-1}(\alpha,\alpha\cup\alpha)+\alpha*\lambda_{n-1}(\alpha,\alpha)$.

REMARK. Note that for $\alpha,\beta\in H^*(X)$, $\alpha\cup\beta\neq 0$ if $\alpha\neq\beta$, and $\alpha\cup\alpha\neq 0$ if $p>2$. In fact, since $X=\Omega^n Y$ is a connected H-space, $H^*(X)$ becomes a connected, associative and commutative Hopf algebra of finite type over Z_p; hence the Borel structure theorem (see (8.12) of [16, III]) implies the result.

Proof. Since $|\alpha|$, $|\beta|$, $|\gamma|\leq 2m-1$, α,β,γ are primitive. So

$$\begin{aligned}\psi(\alpha_* *(\beta\cup\gamma)_*)&=\psi(\alpha_*)*\psi((\beta\cup\gamma)_*)\\&=(\alpha_*\otimes 1+1\otimes\alpha_*)*((\beta\cup\gamma)_*\otimes 1+\beta_*\otimes\gamma_*\\&\quad+(-1)^{|\beta||\gamma|}\gamma_*\otimes\beta_*+1\otimes(\beta\cup\gamma)_*)\\&=(\alpha_* *(\beta\cup\gamma)_*)\otimes 1+(\alpha_* *\beta_*)\otimes\gamma_*\\&\quad+(-1)^{|\beta||\gamma|}(\alpha_* *\gamma_*)\otimes\beta_*+\alpha_*\otimes(\beta\cup\gamma)_*\\&\quad+(-1)^{|\alpha||\beta|+|\alpha||\gamma|}(\beta\cup\gamma)_*\otimes\alpha_*+(-1)^{|\alpha||\beta|}\beta_*\otimes(\alpha_* *\gamma_*)\\&\quad+(-1)^{|\alpha||\gamma|+|\beta||\gamma|}\gamma_*\otimes(\alpha_* *\beta_*)+1\otimes(\alpha_* *(\beta\cup\gamma)_*)\end{aligned}$$

and

$$\begin{aligned}\psi(\alpha_* *\beta_* *\gamma_*)&=\psi(\alpha_*)*\psi(\beta_*)*\psi(\gamma_*)\\&=(\alpha_* *\beta_* *\gamma_*)\otimes 1+(\alpha_* *\beta_*)\otimes\gamma_*+(-1)^{|\beta||\gamma|}(\alpha_* *\gamma_*)\otimes\beta_*\\&\quad+\alpha_*\otimes(\beta_* *\gamma_*)+(-1)^{|\alpha||\beta|+|\alpha||\gamma|}(\beta_* *\gamma_*)\otimes\alpha_*\\&\quad+(-1)^{|\alpha||\beta|}\beta_*\otimes(\alpha_* *\gamma_*)+(-1)^{|\alpha||\gamma|+|\beta||\gamma|}\gamma_*\otimes(\alpha_* *\beta_*)\\&\quad+1\otimes(\alpha_* *\beta_* *\gamma_*).\end{aligned}$$

Thus if $\chi=\beta_* *(\alpha\cup\gamma)_*$, $\gamma_* *(\alpha\cup\beta)_*$ or $\alpha_* *\beta_* *\gamma_*$, $\psi(\chi)$ contains the term $\alpha_*\otimes(\beta_* *\gamma_*)$ whose coefficient is $(-1)^{|\alpha||\beta|}$, $(-1)^{|\alpha||\gamma|+|\beta||\gamma|}$ or 1 respectively. This implies (1) of (i), for if χ is other base, $\psi(\chi)$ does not contain it. Similar calculations yield (2) and (3) of (i).

(ii) and (iii) are proved similarly by using (1.4), (1.5), (1.10), (1.11) and (1.12).

It follows from Lemma 2 (ii) and Lemma 5 (i) (1) that if α,β,γ are distinct,

$$\begin{aligned}0&=[\bar{\alpha}\cup(\beta*\gamma)]=(-1)^{|\alpha||\beta|}[\beta*(\alpha\cup\gamma)]\\&\quad+(-1)^{|\alpha||\gamma|+|\beta||\gamma|}[\gamma*(\alpha\cup\beta)]+[\alpha*\beta*\gamma],\\0&=[\bar{\beta}\cup(\alpha*\gamma)]=(-1)^{|\alpha||\beta|}[\alpha*(\beta\cup\gamma)]\\&\quad+(-1)^{|\alpha||\beta|+|\alpha||\gamma|+|\beta||\gamma|}[\gamma*(\alpha\cup\beta)]+(-1)^{|\alpha||\beta|}[\alpha*\beta*\gamma],\\0&=[\bar{\gamma}\cup(\alpha*\beta)]=(-1)^{|\alpha||\gamma|+|\beta||\gamma|}[\alpha*(\beta\cup\gamma)]\\&\quad+(-1)^{|\alpha||\beta|+|\alpha||\gamma|+|\beta||\gamma|}[\beta*(\alpha\cup\gamma)]+(-1)^{|\alpha||\gamma|+|\beta||\gamma|}[\alpha*\beta*\gamma]\end{aligned}$$

in $H^*(F_nX)$. Hence

$$(3.1)\qquad \begin{aligned}&[\beta*(\alpha\cup\gamma)]=(-1)^{|\alpha||\beta|}[\alpha*(\beta\cup\gamma)]\,,\\&[\gamma*(\alpha\cup\beta)]=(-1)^{|\alpha||\gamma|+|\beta||\gamma|}[\alpha*(\beta\cup\gamma)]\quad\textit{and}\\&[\alpha*\beta*\gamma]=-2[\alpha*(\beta\cup\gamma)]\,.\end{aligned}$$

Similarly from (2) of Lemma 5 (i) it follows that if $\alpha\neq\beta$,

$$(3.2)\qquad \begin{aligned}&[\beta*(\alpha\cup\beta)]=2[\alpha*(\beta\cup\beta)]\quad\textit{and}\\&[\alpha*\beta*\beta]=-2[\alpha*(\beta\cup\beta)]\,.\end{aligned}$$

From (3) of Lemma 5 (i) it follows that

$$(3.3)\qquad [\alpha*(\alpha\cup\alpha)]=-3[\alpha*\alpha*\alpha]\,.$$

From (ii) of Lemma 5 it follows that

$$(3.4)\qquad [\alpha*Q^s(\beta)]=0\,.$$

From (1) of Lemma 5 (iii) it follows that if α, β, γ are distinct,

$$(3.5)\qquad \begin{aligned}&[\alpha*\lambda_{n-1}(\beta,\gamma)]=(-1)^{|\alpha||\beta|+|\alpha|(n-1)+1}[\lambda_{n-1}(\beta,\alpha\cup\gamma)]\\&\quad+(-1)^{|\alpha||\gamma|+|\beta||\gamma|+(|\alpha|+|\beta|+|\gamma|+1)(n-1)}[\lambda_{n-1}(\gamma,\alpha\cup\beta)]\,,\\&[\beta*\lambda_{n-1}(\alpha,\gamma)]=(-1)^{|\alpha||\beta|+|\beta|(n-1)+1}[\lambda_{n-1}(\alpha,\beta\cup\gamma)]\\&\quad+(-1)^{|\alpha||\beta|+|\alpha||\gamma|+|\beta||\gamma|+(|\alpha|+|\beta|+|\gamma|)(n-1)}[\lambda_{n-1}(\gamma,\alpha\cup\beta)]\,,\\&[\gamma*\lambda_{n-1}(\alpha,\beta)]=(-1)^{|\alpha||\gamma|+|\beta||\gamma|+|\gamma|(n-1)+1}[\lambda_{n-1}(\alpha,\beta\cup\gamma)]\\&\quad+(-1)^{|\alpha||\beta|+|\alpha||\gamma|+|\beta||\gamma|+(|\alpha|+|\beta|+|\gamma|+1)(n-1)}[\lambda_{n-1}(\beta,\alpha\cup\gamma)]\,.\end{aligned}$$

From (2) of Lemma 5 (iii) it follows that if $\alpha\neq\beta$,

$$(3.6)\qquad \begin{aligned}&[\alpha*\lambda_{n-1}(\beta,\beta)]=(-1)^{|\alpha||\beta|+|\alpha|(n-1)+1}[\lambda_{n-1}(\beta,\alpha\cup\beta)]\quad\textit{and}\\&[\beta*\lambda_{n-1}(\alpha,\beta)]=(-1)^{|\alpha||\beta|+|\beta|(n-1)+1}[\lambda_{n-1}(\alpha,\beta\cup\beta)]\\&\quad+(-1)^{|\beta|+(|\alpha|+1)(n-1)}[\lambda_{n-1}(\beta,\alpha\cup\beta)]\,.\end{aligned}$$

From (3) of Lemma 5 (iii) it follows that

$$(3.7)\qquad [\alpha*\lambda_{n-1}(\alpha,\alpha)]=(-1)^{|\alpha|+1}[\lambda_{n-1}(\alpha,\alpha\cup\alpha)]\,.$$

Combining Proposition 1, Lemma 2, Corollary 4 and relations (3.1)–(3.7), we obtain

Proposition 6. *Suppose that $X=\Omega^nY$ is $(m-1)$-connected $(m>1)$. Then a Z_p-basis for $\tilde{H}^*(G_nX)$ in dimensions $<4m+n-1$ is given by:*

(1) $\sigma^n[\alpha*\theta]$ *for* $\alpha\leq\theta$ *where if* $\alpha=\theta$, $p>2$ *and* $|\alpha|$ *is even;*

(2) $\sigma^n[\alpha*(\beta\cup\gamma)]$ *for* $\alpha\leq\beta\leq\gamma$ *where if* $\alpha=\beta=\gamma$, $p>3$ *and* $|\alpha|$ *is even, and if* $\alpha=\beta\neq\gamma$ *or* $\alpha\neq\beta=\gamma$, $p>2$ *and* $|\beta|$ *is even;*

(3) $\sigma^n[\alpha * \{Q^s(\beta)\}]$ *for* $p=2$ *and* $|\beta| \leq s \leq |\beta|+n-1$ *where* $Sq^{s+1}({}^n\tilde{\beta})=0$;
(4) $\sigma^n[\alpha * \{\lambda_{n-1}(\beta, \gamma)\}]$ *for* $\beta \leq \gamma$ *where if* $\beta=\gamma$, $p>2$ *and* $|\beta|+n$ *is even, and* ${}^n\tilde{\beta} \cup {}^n\tilde{\gamma}=0$;
(5) $\sigma^n[Q^s(\alpha)]$ *for* $p=2$ *and* $|\alpha| \leq s \leq |\alpha|+n-1$;
(6) $\sigma^n[\lambda_{n-1}(\alpha, \theta)]$ *for* $\alpha \leq \theta$ *where if* $\alpha=\theta$, $p>2$ *and* $|\alpha|+n$ *is even*;
(7) $\sigma^n[\lambda_{n-1}(\alpha, \beta \cup \gamma)]$ *for* $\beta \leq \gamma$ *where if* $\beta=\gamma$, $p>2$ *and* $|\beta|$ *is even*;
(8) $\sigma^n[\lambda_{n-1}(\alpha, \{Q^s(\beta)\})]$ *for* $p=2$ *and* $|\beta| \leq s \leq |\beta|+n-1$ *where* $Sq^{s+1}({}^n\tilde{\beta})=0$;
(9) $\sigma^n[\lambda_{n-1}(\alpha, \{\lambda_{n-1}(\beta, \gamma)\})]$ *for* $\beta \leq \gamma$ *where if* $\beta=\gamma$, $p>2$ *and* $|\beta|+n$ *is even, and* ${}^n\tilde{\beta} \cup {}^n\tilde{\gamma}=0$;
(10) $\sigma^n[Q^s(\alpha)]$ *for* $p=3$ *and* $|\alpha| \leq 2s \leq |\alpha|+n-1$;
(11) $\sigma^n[\Delta Q^s(\alpha)]$ *for* $p=3$ *and* $|\alpha| < 2s \leq |\alpha|+n-1$;
(12) $\sigma^n[\lambda_{n-1}(\alpha, \lambda_{n-1}(\beta, \gamma))]$ *for* $\alpha \geq \beta < \gamma$.

Consider the mod p cohomology spectral sequence $\{E_r, d_r\}$ of the fibration (2.1) in total degrees $<4m+n-1$; that is,

(3.8)
$$E_2^{i,j} = H^i(Y) \otimes H^j(G_n X),\ d_r\colon E_r^{i,j} \to E_r^{i+r, j-r+1} \text{ and}$$
$$E_\infty^{*,*} = \operatorname{Gr} H^*(\Sigma^n X)\,.$$

Then our main result is

Theorem 7. *Under the above situation, the following formulas hold up to non-zero constants*:
(1) $\nu_n^*(\sigma^n(\alpha \cup \theta))=\sigma^n[\alpha * \theta]$;
(2) $\nu_n^*(\sigma^n(\alpha \cup \beta \cup \gamma))=\sigma^n[\alpha * (\beta \cup \gamma)]$;
(3) *If* $p=2$ *and* $Sq^{s+1}({}^n\tilde{\beta})=0$, $\nu_n^*(\sigma^n(\alpha \cup \{Q^s(\beta)\}))=\sigma^n[\alpha * \{Q^s(\beta)\}]$;
(4) *If* ${}^n\tilde{\beta} \cup {}^n\tilde{\gamma}=0$, $\nu_n^*(\sigma^n(\alpha \cup \{\lambda_{n-1}(\beta, \gamma)\}))=\sigma^n[\alpha * \{\lambda_{n-1}(\beta, \gamma)\}]$;
(5) *If* $p=2$, $\tau(\sigma^n[Q^s(\alpha)])=Sq^{s+1}({}^n\tilde{\alpha})$;
(6) $\tau(\sigma^n[\lambda_{n-1}(\alpha, \theta)])={}^n\tilde{\alpha} \cup {}^n\tilde{\theta}$;
(7) $d_{|\alpha|+n}(1 \otimes \sigma^n[\lambda_{n-1}(\alpha, \beta \cup \gamma)])={}^n\tilde{\alpha} \otimes \sigma^n[\beta * \gamma]$;
(8) *If* $p=2$ *and* $Sq^{s+1}({}^n\tilde{\beta})=0$, $d_{|\alpha|+n}(1 \otimes \sigma^n[\lambda_{n-1}(\alpha, \{Q^s(\beta)\})])={}^n\tilde{\alpha} \otimes \sigma^n[Q^s(\beta)]$;
(9) (a) *If* ${}^n\tilde{\beta} \cup {}^n\tilde{\gamma}=0$, $d_{|\alpha|+n}(1 \otimes \sigma^n[\lambda_{n-1}(\alpha, \{\lambda_{n-1}(\beta, \gamma)\})])={}^n\tilde{\alpha} \otimes \sigma^n[\lambda_{n-1}(\beta, \gamma)]$;

(b) *If* ${}^n\tilde{\alpha} \cup {}^n\tilde{\beta}={}^n\tilde{\beta} \cup {}^n\tilde{\gamma}=0$, ((a) *holds and*) $\tau(\sigma^n[\lambda_{n-1}(\alpha, \{\lambda_{n-1}(\beta, \gamma)\})]+c' \cdot \sigma^n[\lambda_{n-1}(\gamma, \{\lambda_{n-1}(\alpha, \beta)\})])=\langle {}^n\tilde{\alpha}, {}^n\tilde{\beta}, {}^n\tilde{\gamma} \rangle$ (*where* c' *is a non-zero constant and* $\langle\ ,\ ,\ \rangle$ *denotes the Massey product* [15]);

(c) *If* ${}^n\tilde{\alpha} \cup {}^n\tilde{\beta}={}^n\tilde{\beta} \cup {}^n\tilde{\gamma}={}^n\tilde{\gamma} \cup {}^n\tilde{\alpha}=0$, ((a), (b) *hold and*) $\tau(\sigma^n[\lambda_{n-1}(\beta, \{\lambda_{n-1}(\gamma, \alpha)\})]+c'' \cdot \sigma^n[\lambda_{n-1}(\alpha, \{\lambda_{n-1}(\beta, \gamma)\})])=\langle {}^n\tilde{\beta}, {}^n\tilde{\gamma}, {}^n\tilde{\alpha} \rangle$ (*where* c'' *is a non-zero constant*);

(10) *If* $p=3$, $\tau(\sigma^n[Q^s(\alpha)])=\Delta^* \mathfrak{P}^s({}^n\tilde{\alpha})$ (*where* Δ^* *is the* mod 3 *cohomology Bockstein and* $\mathfrak{P}^s$ *is the Steenrod 3rd power* [14]);

(11) *If* $p=3$, $\tau(\sigma^n[\Delta Q^s(\alpha)])=\mathfrak{P}^s({}^n\tilde{\alpha})$;
(12) $d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\alpha, \lambda_{n-1}(\beta, \gamma))])={}^n\tilde{\alpha}\otimes\sigma^n[\lambda_{n-1}(\beta, \gamma)]$.

The proof is postponed until §6.

REMARK. In the proof of Theorem 7, for the convenience of argument only, we will take as a Z_p-basis for $\tilde{H}^*(G_nX)$ the set of elements given in Proposition 6. However, Theorem 7 is valid, independently of the choice of Z_p-basis for $\tilde{H}^*(G_nX)$ and of the ordering of Z_p-basis for $\tilde{H}^*(X)$. This assertion will be discussed in §6.

4. Lemmas

In §3 we have shown that

(4.1) *If* $X=\Omega^n Y$ *is* $(m-1)$*-connected, there is a* $(4m+n-1)$*-equivalence* ρ_n: $\Sigma^n(F_nX)_{4m-1}\to G_nX$.

For $n>k\geq 1$ consider the following diagram

$$\text{(4.2)}\qquad \begin{tikzcd} G_{n-k}\Omega^{n-k}\Omega^k Y \arrow[r, "\nu_{n-k}"] \arrow[d, dotted, "\tilde{\eta}'_k"] & \Sigma^{n-k}\Omega^{n-k}\Omega^k Y \arrow[r, "\xi_{n-k}"] \arrow[d, "\eta_k"] & \Omega^k Y \arrow[d, "="] \\ \Omega^k G_n\Omega^n Y \arrow[r, "\Omega^k\nu_n"] & \Omega^k\Sigma^n\Omega^n Y \arrow[r, "\Omega^k\xi_n"] & \Omega^k Y \end{tikzcd}$$

where the rows are fibrations. Commutativity of the right-hand square yields a map $\tilde{\eta}'_k$: $G_{n-k}\Omega^n Y\to\Omega^k G_n\Omega^n Y$. Application of the functor Ω^{n-k} to the diagram (4.2) yields a commutative diagram

$$\text{(4.3)}\qquad \begin{tikzcd} F_{n-k}\Omega^n Y \arrow[r, "\Omega^{n-k}\nu_{n-k}"] \arrow[d, "\Omega^{n-k}\tilde{\eta}'_k"] & \Omega^{n-k}\Sigma^{n-k}\Omega^n Y \arrow[r, "\Omega^{n-k}\xi_{n-k}"] \arrow[d, "\Omega^{n-k}\eta_k"] & \Omega^n Y \arrow[d, "="] \\ F_n\Omega^n Y \arrow[r, "\Omega^n\nu_n"] & \Omega^n\Sigma^n\Omega^n Y \arrow[r, "\Omega^n\xi_n"] & \Omega^n Y \end{tikzcd}$$

Let η'_k: $(F_{n-k}\Omega^n Y)_{4m-1}\to(F_n\Omega^n Y)_{4m-1}$ be the restriction to the $(4m-1)$-skeleton (of a cellular approximation) of the map $\Omega^{n-k}\tilde{\eta}'_k$. Then there is a commutative diagram

$$\begin{tikzcd} \Sigma^{n-k}(F_{n-k}\Omega^n Y)_{4m-1} \arrow[r, "\Sigma^{n-k}\eta'_k"] \arrow[d, "\rho_{n-k}"] & \Sigma^{n-k}(F_n\Omega^n Y)_{4m-1} \arrow[d, "\phi^k(\rho_n)"] \\ G_{n-k}\Omega^n Y \arrow[r, "\tilde{\eta}'_k"] & \Omega^k G_n\Omega^n Y \end{tikzcd}$$

and by (4.1), both ρ_{n-k} and $\phi^k(\rho_n)$ are $(4m+n-k-1)$-equivalences. Thus

$$H^i(G_n\Omega^n Y)\xrightarrow{(\sigma^*)^k}H^{i-k}(\Omega^k G_n\Omega^n Y)\xrightarrow{(\tilde{\eta}'_k)^*}H^{i-k}(G_{n-k}\Omega^n Y)$$

may be identified with the composite

$$H^i(\Sigma^n(F_n\Omega^n Y)_{4m-1}) \xrightarrow[\cong]{(\Sigma^*)^k} H^{i-k}(\Sigma^{n-k}(F_n\Omega^n Y)_{4m-1})$$
$$\xrightarrow{(\Sigma^{n-k}\eta_k')^*} H^{i-k}(\Sigma^{n-k}(F_{n-k}\Omega^n Y)_{4m-1})$$

for $i<4m+n-1$. Then we have

Lemma 8. *For any α, $\beta\in H^*(\Omega^n Y)$ the following relations hold:*

(1) $(\tilde{\eta}_k')^*(\sigma^*)^k(\sigma^n[\alpha*\beta]) = \sigma^{n-k}[\alpha*\beta]$;

(2) $(\tilde{\eta}_k')^*(\sigma^*)^k(\sigma^n[Q^s(\alpha)]) = \begin{cases} \sigma^{n-k}[Q^s(\alpha)] & \text{if } p=2 \text{ and } s\leq|\alpha|+n-k-1 \text{ or} \\ & p>2 \text{ and } 2s\leq|\alpha|+n-k-1 \\ 0 & \text{otherwise} \end{cases}$

(3) $(\tilde{\eta}_k')^*(\sigma^*)^k(\sigma^n[\lambda_{n-1}(\alpha, \beta)]) = 0$.

Proof. By (1.1) and (1.8), $(\Omega^{n-k}\eta_k)^*\colon H^i(\Omega^n\Sigma^n\Omega^n Y) \to H^i(\Omega^{n-k}\Sigma^{n-k}\Omega^n Y)$ satisfies:

$$(\Omega^{n-k}\eta_k)^*(\alpha*\beta) = \alpha*\beta;$$

$$(\Omega^{n-k}\eta_k)^*(Q^s(\alpha)) = \begin{cases} Q^s(\alpha) & \text{if } p=2 \text{ and } s\leq|\alpha|+n-k-1 \text{ or } p>2 \text{ and} \\ & 2s\leq|\alpha|+n-k-1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\Omega^{n-k}\eta_k)^*(\lambda_{n-1}(\alpha, \beta)) = 0\,.$$

So the result follows from (4.3) and the definition of η_k'.

For $n>k\geq 1$ consider the following diagram

$$\text{(4.4)}\qquad \begin{array}{ccccc} G_n\Omega^n Y & \xrightarrow{\nu_n} & \Sigma^n\Omega^n Y & \xrightarrow{\xi_n} & Y \\ \vdots\downarrow \bar{\xi}_k' & & \downarrow \Sigma^{n-k}\xi_k & & \downarrow = \\ G_{n-k}\Omega^{n-k} Y & \xrightarrow{\nu_{n-k}} & \Sigma^{n-k}\Omega^{n-k} Y & \xrightarrow{\xi_{n-k}} & Y \end{array}$$

where the rows are fibrations. Commutativity of the right-hand square yields a map $\bar{\xi}_k'\colon G_n\Omega^n Y \to G_{n-k}\Omega^{n-k}Y$. Application of Ω^n to (4.4) yields a commutative diagram

$$\text{(4.5)}\qquad \begin{array}{ccccc} F_n\Omega^n Y & \xrightarrow{\Omega^n\nu_n} & \Omega^n\Sigma^n\Omega^n Y & \xrightarrow{\Omega^n\xi_n} & \Omega^n Y \\ \downarrow \Omega^n\bar{\xi}_k' & & \downarrow \Omega^n\Sigma^{n-k}\xi_k & & \downarrow = \\ \Omega^k F_{n-k}\Omega^{n-k} Y & \xrightarrow{\Omega^n\nu_{n-k}} & \Omega^n\Sigma^{n-k}\Omega^{n-k} Y & \xrightarrow{\Omega^n\xi_{n-k}} & \Omega^n Y\,. \end{array}$$

Let $\xi_k'\colon \Sigma^k(F_n\Omega^n Y)_{4m-1} \to (F_{n-k}\Omega^{n-k}Y)_{4m-1}$ be the restriction to the $(4m-1)$-

skeleton of the map $\phi^{-k}(\Omega^n\xi'_k)\colon \Sigma^k F_n\Omega^n Y \to F_{n-k}\Omega^{n-k}Y$. Then there is a commutative diagram

$$\begin{tikzcd}
\Sigma^n(F_n\Omega^n Y)_{4m-1} \arrow[r, "\Sigma^{n-k}\xi'_k"] \arrow[d, "\rho_n"] & \Sigma^{n-k}(F_{n-k}\Omega^{n-k}Y)_{4m-1} \arrow[r, "\subset"] & \Sigma^{n-k}(F_{n-k}\Omega^{n-k}Y)_{4m+4k-1} \arrow[d, "\rho_{n-k}"] \\
G_n\Omega^n Y \arrow[rr, "\tilde{\xi}'_k"] & & G_{n-k}\Omega^{n-k}Y
\end{tikzcd}$$

and by (4.1), ρ_n and ρ_{n-k} are $(4m+n-1)$- and $(4m+n+3k-1)$-equivalences respectively. Thus

$$H^i(G_{n-k}\Omega^{n-k}Y) \xrightarrow{(\tilde{\xi}'_k)^*} H^i(G_n\Omega^n Y)$$

may be identified with the composite

$$H^i(\Sigma^{n-k}(F_{n-k}\Omega^{n-k}Y)_{4m+4k-1}) \longrightarrow H^i(\Sigma^{n-k}(F_{n-k}\Omega^{n-k}Y)_{4m-1}) \xrightarrow{(\Sigma^{n-k}\xi'_k)^*} H^i(\Sigma^n(F_n\Omega^n Y)_{4m-1})$$

for $i<4m+n-1$.

Lemma 9. *For any ${}^k\tilde{\alpha}$, ${}^k\tilde{\beta}\in H^*(\Omega^{n-k}Y)$ the following relations hold:*

(1) $(\tilde{\xi}'_k)^*(\sigma^{n-k}[{}^k\tilde{\alpha}*{}^k\tilde{\beta}])=0$;

(2) $(\tilde{\xi}'_k)^*(\sigma^{n-k}[Q^s({}^k\tilde{\alpha})])=\sigma^n[Q^s(\alpha)]$;

(3) $(\tilde{\xi}'_k)^*(\sigma^{n-k}[\lambda_{n-k-1}({}^k\tilde{\alpha},\ {}^k\tilde{\beta})])=\sigma^n[\lambda_{n-1}(\alpha,\ \beta)]$,

where α (resp. β) is the image of ${}^k\tilde{\alpha}$ (resp. ${}^k\tilde{\beta}$) under $(\sigma^*)^k\colon\ H^i(\Omega^{n-k}Y)\to \tilde{H}^{i-k}(\Omega^n Y)$.

Proof. Recall (e.g. from §3 of [16, VIII]) that

(4.6) *For any Y, $\sigma^*\colon H^i(Y)\to\tilde{H}^{i-1}(\Omega Y)$ maps every decomposable element into zero.*

By this fact and (1.15), $\xi_k^*\colon H^i(\Omega^{n-k}Y)\to H^i(\Sigma^k\Omega^n Y)$ satisfies:

$$\xi_k^*({}^k\tilde{\alpha}*{}^k\tilde{\beta})=0;$$
$$\xi_k^*(Q^s({}^k\tilde{\alpha}))=\sigma^k(Q^s(\alpha));$$
$$\xi_k^*(\lambda_{n-k-1}({}^k\tilde{\alpha},\ {}^k\tilde{\beta}))=\sigma^k(\lambda_{n-1}(\alpha,\ \beta)).$$

So the result follows from (4.5) and the definition of ξ'_k.

5. Proof of Theorem 3

Milgram [12, I] did not give a detailed proof of Theorem 3. Here we present it for later convenience.

If Y' and Y'' are $(m'+n-1)$- and $(m''+n-1)$-connected respectively, where $m''\geq m'>1$, and if $g\colon Y'\to Y''$ is a map, there is a commutative diagram of fibrations

$$
(5.1)\qquad
\begin{tikzcd}
\left(\Sigma^n(F_nX')_{4m'-1}\right. \arrow[r, "\rho_n"] \arrow[d, dotted] & \left.G_nX'\right) \arrow[r, "\nu_n"] \arrow[d, dotted, "G_nf"] & \Sigma^nX' \arrow[r, "\xi_n"] \arrow[d, "\Sigma^nf"] & Y' \arrow[d, "g"] \\
\left(\Sigma^n(F_nX'')_{4m''-1}\right. \arrow[r, "\rho_n"] & \left.G_nX''\right) \arrow[r, "\nu_n"] & \Sigma^nX'' \arrow[r, "\xi_n"] & Y''
\end{tikzcd}
$$

where $X'=\Omega^nY'$, $X''=\Omega^nY''$ and $f=\Omega^ng$. Then the naturality of the Serre exact sequence yields a commutative diagram of exact sequences

$$
(5.2)\qquad
\begin{tikzcd}
0 \arrow[r] & Cok\ \xi_n^* \arrow[r, "\nu_n^*"] \arrow[d, "(\Sigma^nf)^*"] & H^i(G_nX'') \arrow[r, "\tau"] \arrow[d, "(G_nf)^*"] & Ker\ \xi_n^* \arrow[r] \arrow[d, "g^*"] & 0 \\
0 \arrow[r] & Cok\ \xi_n^* \arrow[r, "\nu_n^*"] & H^i(G_nX') \arrow[r, "\tau"] & Ker\ \xi_n^* \arrow[r] & 0
\end{tikzcd}
$$

for $i<3m'+n-1$.

Let $K(Z_p,\ i)$ be an Eilenberg-MacLane space of type $(Z_p,\ i)$ and let $\iota_i\in H^i(K(Z_p,\ i))$ be its fundamental class.

Proof of (1).

In the diagram (5.2), set $g=({}^n\tilde{\alpha},\ {}^n\tilde{\beta})\colon Y\to K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n)$; then we see that to show (1) it suffices to prove

$$(1)'\qquad \nu_n^*(\sigma^n(\iota_{|\alpha|}\times\iota_{|\beta|}))=\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\iota_{|\beta|})]$$

in the case $Y=K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n)$.

Suppose $n>1$ and consider the diagram (5.2) for the case $g=\pi_1\colon K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n)\to K(Z_p,\ |\alpha|+n)$, the projection to the first factor. Then

$$
\begin{aligned}
&H^{|\alpha|+|\beta|+n}(G_n(K((Z_p|\alpha|)\times K(Z_p,\ |\beta|)))\\
&\quad =Z_p\{\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\iota_{|\beta|})]\}\ \text{modulo Im}\ (G_n\pi_1)^*\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&Cok[\xi_n^*\colon H^{|\alpha|+|\beta|+n}(K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n))\\
&\quad \to H^{|\alpha|+|\beta|+n}(\Sigma^n(K(Z_p,\ |\alpha|)\times K(Z_p,\ |\beta|)))]\\
&\quad =Z_p\{\sigma^n(\iota_{|\alpha|}\times\iota_{|\beta|})\}\ \text{modulo Im}\ (\Sigma^n\pi_1)^*
\end{aligned}
$$

and

$$
\begin{aligned}
&Ker[\xi_n^*\colon H^{|\alpha|+|\beta|+n+1}(K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n))\\
&\quad \to H^{|\alpha|+|\beta|+n+1}(\Sigma^n(K(Z_p,\ |\alpha|)\times K(Z_p,\ |\beta|)))]\\
&\quad =0\ \text{modulo Im}\ \pi_1^*\,.
\end{aligned}
$$

For $\sigma^n(\omega)\in\text{Im}\,(G_n\pi_1)^*$ let $\sigma^n(\varpi)\in H^*(G_nK(Z_p,\ |\alpha|))$ be such that $(G_n\pi_1)^*(\sigma^n(\varpi))=\sigma^n(\omega)$. Then the behavior of $\sigma^n(\omega)$ in the lower sequence of (5.2) depends on that of $\sigma^n(\varpi)$ in the upper sequence of (5.2). So the above observation implies (1)$'$ for $n>1$.

It remains to prove the case $n=1$. Consider the diagram (4.2) for the case that $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)$ and $k=n-1$; then there is a commutative diagram

$$\text{(5.3)} \qquad \begin{tikzcd} H^i(\Sigma^n X) \arrow[r, "\nu_n^*"] \arrow[d, "(\sigma^*)^{n-1}"] \arrow[dd, bend right=90, "(\Sigma^*)^{n-1}"', "\cong"] & H^i(G_n X) \arrow[d, "(\sigma^*)^{n-1}"] \\ H^{i-n+1}(\Omega^{n-1}\Sigma^n X) \arrow[r, "(\Omega^{n-1}\nu_n)^*"] \arrow[d, "\eta_{n-1}^*"] & H^{i-n+1}(\Omega^{n-1}G_n X) \arrow[d, "(\tilde{\eta}'_{n-1})^*"] \\ H^{i-n+1}(\Sigma X) \arrow[r, "\nu_1^*"] & H^{i-n+1}(G_1 X) \end{tikzcd}$$

where $X=K(Z_p, |\alpha|)\times K(Z_p, |\beta|)$, and by (1) of Lemma 8,

$$\begin{aligned} \nu_1^*(\sigma(\iota_{|\alpha|}\times\iota_{|\beta|})) &= \nu_1^*(\Sigma^*)^{n-1}(\sigma^n(\iota_{|\alpha|}\times\iota_{|\beta|})) \\ &= (\tilde{\eta}'_{n-1})^*(\sigma^*)^{n-1}\nu_n^*(\sigma^n(\iota_{|\alpha|}\times\iota_{|\beta|})) \\ &= (\tilde{\eta}'_{n-1})^*(\sigma^*)^{n-1}(\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\iota_{|\beta|})]) \\ &= \sigma[(\iota_{|\alpha|}\times 1)*(1\times\iota_{|\beta|})]\,. \end{aligned}$$

Proof of (2).

In the diagram (5.2), set $g={}^n\tilde{\alpha}\colon Y\to K(Z_2, |\alpha|+n)$; then we see that to show (2) it suffices to prove

$$\text{(2)}' \qquad \tau(\sigma^n[Q^s(\iota_{|\alpha|})]) = Sq^{s+1}(\iota_{|\alpha|+n})$$

in the case $Y=K(Z_2, |\alpha|+n)$.

Consider the lower sequence of (5.2) for the case that $Y'=K(Z_2, s+1)$ and $n=1$. Then

$$H^{2s+1}(G_1K(Z_2, s)) = Z_2\{\sigma[Q^s(\iota_s)]\}\,.$$

On the other hand,

$$Cok[\xi_1^*\colon H^{2s+1}(K(Z_2, s+1))\to H^{2s+1}(\Sigma K(Z_2, s))] = 0$$

and

$$\begin{aligned} &Ker[\xi_1^*\colon H^{2s+2}(K(Z_2, s+1))\to H^{2s+2}(\Sigma K(Z_2, s))] \\ &\quad = Z_2\{Sq^{s+1}(\iota_{s+1})\}\,. \end{aligned}$$

So we have

$$\tau(\sigma[Q^s(\iota_s)]) = Sq^{s+1}(\iota_{s+1})\,.$$

Consider the diagram (4.4) for the case that $Y=K(Z_2, s+1)$, $n=-|\alpha|+s+1$ and $k=n-1=-|\alpha|+s$; then there is a commutative diagram

$$(5.4)\qquad \begin{array}{ccc} H^i(G_1\Omega Y) & \xrightarrow{\tau} & H^{i+1}(Y) \\ \Big\downarrow {\scriptstyle \xi'^*_{n-1}} & & \Big\downarrow {\scriptstyle =} \\ H^i(G_nX) & \xrightarrow{\tau} & H^{i+1}(Y) \end{array}$$

where $X=K(Z_2, |\alpha|)$ and $Y=K(Z_2, s)$, and by (2) of Lemma 9,

$$\begin{aligned} \tau(\sigma^{-|\alpha|+s+1}[Q^s(\iota_{|\alpha|})]) &= \tau(\xi'_{-|\alpha|+s})^*(\sigma[Q^s(\iota_s)]) \\ &= \tau(\sigma[Q^s(\iota_s)]) \\ &= Sq^{s+1}(\iota_{s+1})\,. \end{aligned}$$

Consider the diagram (4.2) for the case that $Y=K(Z_2, |\alpha|+n)$ and $k=|\alpha|+n-s-1$; then there is a commutative diagram

$$(5.5)\qquad \begin{array}{ccc} H^{i+k}(G_nX) & \xrightarrow{\tau} & H^{i+k+1}(Y) \\ \Big\downarrow {\scriptstyle (\sigma^*)^k} & & \Big\downarrow {\scriptstyle (\sigma^*)^k} \\ H^i(\Omega^kG_nX) & \xrightarrow{\tau} & H^{i+1}(\Omega^kY) \\ \Big\downarrow {\scriptstyle \tilde{\eta}'^*_k} & & \Big\downarrow {\scriptstyle =} \\ H^i(G_{n-k}X) & \xrightarrow{\tau} & H^{i+1}(\Omega^kY) \end{array}$$

where $X=K(Z_2, |\alpha|)$ and $\Omega^kY=K(Z_2, s+1)$, and by (2) of Lemma 8,

$$\begin{aligned} &(\sigma^*)^{|\alpha|+n-s-1}\tau(\sigma^n[Q^s(\iota_{|\alpha|})]) \\ &= \tau(\sigma^*)^{|\alpha|+n-s-1}(\sigma^n[Q^s(\iota_{|\alpha|})]) \\ &= \tau(\tilde{\eta}'_{|\alpha|+n-s-1})^*(\sigma^*)^{|\alpha|+n-s-1}(\sigma^n[Q^s(\iota_{|\alpha|})]) \\ &= \tau(\sigma^{-|\alpha|+s+1}[Q^s(\iota_{|\alpha|})]) \\ &= Sq^{s+1}(\iota_{s+1}) \\ &= (\sigma^*)^{|\alpha|+n-s-1}(Sq^{s+1}(\iota_{|\alpha|+n}))\,. \end{aligned}$$

Since $(\sigma^*)^{|\alpha|+n-s-1}\colon H^{|\alpha|+n+s+1}(K(Z_2, |\alpha|+n))\to H^{2s+2}(K(Z_2, s+1))$ is monomorphic (see [4]), (2)$'$ follows.

Proof of (3).

In the diagram (5.2), set $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta})\colon Y\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)$; then we see that to show (3) it suffices to prove

$$(3)'\qquad \tau(\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1, 1\times\iota_{|\beta|})]) = \iota_{|\alpha|+n}\times\iota_{|\beta|+n}$$

in the case $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)$.

Consider the diagram (5.2) for the case that $g=\pi_1\colon K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\to K(Z_p, |\alpha|+n)$ and $n=1$. Then

$$H^{|\alpha|+|\beta|+2n-1}(G_1(K(Z_p, |\alpha|+n-1)\times K(Z_p, |\beta|+n-1)))$$

$$= Z_p\{\sigma[(\iota_{|\alpha|+n-1}\times 1)*(1\times \iota_{|\beta|+n-1})],$$
$$\sigma[\lambda_0(\iota_{|\alpha|+n-1}\times 1,\ 1\times \iota_{|\beta|+n-1})]\} \text{ modulo Im } (G_1\pi_1)^*.$$

On the other hand,

$$Cok[\xi_1^*\colon H^{|\alpha|+|\beta|+2n-1}(K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n))\to$$
$$H^{|\alpha|+|\beta|+2n-1}(\Sigma(K(Z_p,\ |\alpha|+n-1)\times K(Z_p,\ |\beta|+n-1)))]$$
$$= Z_p\{\sigma(\iota_{|\alpha|+n-1}\times \iota_{|\beta|+n-1})\} \text{ modulo Im } (\Sigma\pi_1)^*$$

and

$$Ker[\xi_1^*\colon H^{\alpha|+|\beta|+2n}(K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n))\to$$
$$H^{|\alpha|+|\beta|+2n}(\Sigma(K(Z_p,\ |\alpha|+n-1)\times K(Z_p,\ |\beta|+n-1)))]$$
$$= Z_p\{\iota_{|\alpha|+n}\times \iota_{|\beta|+n}\} \text{ modulo Im } \pi_1^*.$$

In view of the formula (1), we find that

$$\tau(\sigma[\lambda_0(\iota_{|\alpha|+n-1}\times 1,\ 1\times \iota_{|\beta|+n-1})]) = \iota_{|\alpha|+n}\times \iota_{|\beta|+n}.$$

Suppose $n>1$ and consider the diagram (4.4) for the case that $Y=K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n)$ and $k=n-1$; then we have the commutative diagram (5.4) (where $X=K(Z_p,\ |\alpha|)\times K(Z_p,\ |\beta|)$ and $\Omega Y=K(Z_p,\ |\alpha|+n-1)\times K(Z_p,\ |\beta|+n-1)$), and by (3) of Lemma 9,

$$\begin{aligned}
&\tau(\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times \iota_{|\beta|})]) \\
&\quad= \tau(\xi'_{n-1})^*(\sigma[\lambda_0(\iota_{|\alpha|+n-1}\times 1,\ 1\times \iota_{|\beta|+n-1})]) \\
&\quad= \tau(\sigma[\lambda_0(\iota_{|\alpha|+n-1}\times 1,\ 1\times \iota_{|\beta|+n-1})]) \\
&\quad= \iota_{|\alpha|+n}\times \iota_{|\beta|+n}.
\end{aligned}$$

6. Proof of Theorem 7

We begin by introducing some notations.

For $i\leq j$ let $L(Z_2,\ i;\ j)$ denote the mapping fibre of

$$Sq^i(\iota_j)\colon K(Z_2,\ j)\to K(Z_2,\ i+j),$$

and for $i>j$ let

$$L(Z_2,\ i;\ j) = \Omega^{i-j}L(Z_2,\ i;\ i).$$

Then for any $(i,\ j)$ there is a fibration

$$K(Z_2,\ i+j-1)\xrightarrow{\varepsilon_L} L(Z_2,\ i;\ j)\xrightarrow{\zeta_L} K(Z_2,\ j)$$

which is induced by $Sq^i(\iota_j)$. Put $\iota^j=\zeta_L^*(\iota_j)$. Since Sq^i is stable, it follows that $\Omega^nL(Z_2,\ i;\ j+n)\simeq L(Z_2,\ i;\ j)$ for all $i,\ j$ and n, i.e., $L(Z_2,\ i;\ j)$ is an infinite loop space.

Suppose $i>j$. Then $Sq^i(\iota_j)=0$ and therefore

$$(6.1) \qquad L(Z_2, i; j) \simeq K(Z_2, j) \times K(Z_2, i+j-1).$$

Let $\kappa^{i;j} \in H^{i+j-1}(L(Z_2, i; j))$ be the element such that

$$\varepsilon_L^*(\kappa^{i;j}) = \iota_{i+j-1}.$$

We now take integers i, j and n so that (2) of Theorem 3 is applicable to $\sigma^n[Q^{i-1}(\iota^{j-n})] \in H^{i+j-1}(G_n L(Z_2, i; j-n))$, where $Y=L(Z_2, i; j)$. Then $\tau(\sigma^n[Q^{i-1}(\iota^{j-n})]) = Sq^i(\iota^j)$, which is equal to zero by the definition of $L(Z_2, i; j)$. So $\sigma^n[Q^{i-1}(\iota^{j-n})]$ lies in the image of ν_n^*. In view of (6.1), we find that

$$(6.2) \qquad \nu_n^*(\sigma^n(\kappa^{i;j-n})) = \sigma^n[Q^{i-1}(\iota^{j-n})].$$

For $i \leq j$ let $M(Z_p; i, j)$ denote the mapping fibre of

$$\iota_i \times \iota_j \colon K(Z_p, i) \times K(Z_p, j) \to K(Z_p, i+j).$$

Then there is a fibration

$$K(Z_p, i+j-1) \xrightarrow{\varepsilon_M} M(Z_p; i, j) \xrightarrow{\zeta_M} K(Z_p, i) \times K(Z_p, j).$$

Application of Ω^n yields a fibration

$$K(Z_p, i+j-n-1) \xrightarrow{\varepsilon_M} \Omega^n M(Z_p; i, j) \xrightarrow{\zeta_M} K(Z_p, i-n) \times K(Z_p, j-n)$$

which is induced by $(\sigma^*)^n(\iota_i \times \iota_j)$ for $n \geq 0$. Put $\iota^{i-n} = \zeta_M^*(\iota_{i-n} \times 1)$ and $\iota^{j-n} = \zeta_M^*(1 \times \iota_{j-n})$.

Suppose $n \geq 1$. Then $(\sigma^*)^n(\iota_i \times \iota_j)=0$ by (4.6), and therefore

$$(6.3) \qquad \Omega^n M(Z_p; i, j) \simeq K(Z_p, i-n) \times K(Z_p, j-n) \times K(Z_p, i+j-n-1).$$

Let $\lambda^{n; i-n, j-n} \in H^{i+j-n-1}(\Omega^n M(Z_p; i, j))$ be the element such that

$$\varepsilon_M^*(\lambda^{n; i-n, j-n}) = \iota_{i+j-n-1}.$$

We now take integers i, j and n so that (3) of Theorem 3 is applicable to $\sigma^n[\lambda_{n-1}(\iota^{i-n}, \iota^{j-n})] \in H^{i+j-1}(G_n \Omega^n M(Z_p; i, j))$, where $Y=M(Z_p; i, j)$. Then $\tau(\sigma^n[\lambda_{n-1}(\iota^{i-n}, \iota^{j-n})]) = \iota^i \cup \iota^j$, which is equal to zero by the definition of $M(Z_p; i, j)$. So $\sigma^n[\lambda_{n-1}(\iota^{i-n}, \iota^{j-n})]$ lies in the image of ν_n^*. In view of (6.3), we find that

$$(6.4) \qquad \nu_n^*(\sigma^n(\lambda^{n; i-n, j-n})) = \sigma^n[\lambda_{n-1}(\iota^{i-n}, \iota^{j-n})] \text{ (\textit{up to a non-zero constant})}.$$

Let $X=\Omega^n Y$ and suppose that an element $\alpha \in H^*(X)$ such that $Sq^{s+1}({}^n\tilde{\alpha})=0$ is given. Consider the following diagram

$$
\begin{tikzcd}
 & & Y \arrow[d, "{}^n\tilde{\alpha}"] \arrow[dll, dashed, "{}^n\tilde{\alpha}\wedge"'] & \\
L(Z_2, s+1;\, |{}^n\tilde{\alpha}|) \arrow[rr, "\zeta_L"] & & K(Z_2, |{}^n\tilde{\alpha}|) \arrow[r, "Sq^{s+1}(\iota_{|{}^n\tilde{\alpha}|})"] & K(Z_2, |{}^n\tilde{\alpha}|+s+1)
\end{tikzcd}
$$

where the row is a fibration. By hypothesis there is a lifting ${}^n\tilde{\alpha}\wedge$ of ${}^n\tilde{\alpha}$. Then we have the commutative diagram (5.1) for the case $g={}^n\tilde{\alpha}\wedge$, and from naturality and (6.2) it follows that

$$(\Omega^{n\,n}\tilde{\alpha}\wedge)^*(\kappa^{s+1;\,|\alpha|}) = \{Q^s(\alpha)\}\,. \tag{6.5}$$

Suppose that elements $\alpha, \beta \in H^*(X)$ such that ${}^n\tilde{\alpha} \cup {}^n\tilde{\beta}=0$ are given. Consider the following diagram

$$
\begin{tikzcd}
 & & Y \arrow[d, "({}^n\tilde{\alpha},\, {}^n\tilde{\beta})"] \arrow[dll, dashed, "({}^n\tilde{\alpha},\, {}^n\tilde{\beta})\wedge"'] & \\
M(Z_p;\, |{}^n\tilde{\alpha}|,\, |{}^n\tilde{\beta}|) \arrow[rr, "\zeta_M"] & & K(Z_p, |{}^n\tilde{\alpha}|)\times K(Z_p, |{}^n\tilde{\beta}|) \arrow[r, "\iota_{|{}^n\tilde{\alpha}|}\times \iota_{|{}^n\tilde{\beta}|}"] & K(Z_p, |{}^n\tilde{\alpha}|+|{}^n\tilde{\beta}|)
\end{tikzcd}
$$

where the row is a fibration. By hypothesis there is a lifting $({}^n\tilde{\alpha}, {}^n\tilde{\beta})\wedge$ of $({}^n\tilde{\alpha}, {}^n\tilde{\beta})$. Then we have the commutative diagram (5.1) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta})\wedge$, and from naturality and (6.4) it follows that

$$(\Omega^n({}^n\tilde{\alpha}, {}^n\tilde{\beta})\wedge)^*(\lambda^{n;\,|\alpha|,|\beta|}) = \{\lambda_{n-1}(\alpha, \beta)\}\,. \tag{6.6}$$

We enter into the proof of Theorem 7.

Let $\{E_r, d_r\}$ be the spectral sequence (3.8). It follows from (2.3) that $E_2^{i,j}$ for $i+j<4m+n-1$ with $i, j>0$ (explicitly speaking, $i\geq m+n$ and $j\geq 2m+n$) has a Z_p-basis consisting of elements

$${}^n\tilde{\alpha}\otimes\sigma^n[\beta*\gamma],\ {}^n\tilde{\alpha}\otimes\sigma^n[Q^s(\beta)]\ (p=2) \text{ and } {}^n\tilde{\alpha}\otimes\sigma^n[\lambda_{n-1}(\beta, \gamma)]\,.$$

By Corollary 4 and the multiplicative properties of the cohomology spectral sequence, if these elements survive to E_∞, they represent the following elements of $H^*(\Sigma^n X)$:

$$\sigma^n(\alpha)\cup\sigma^n(\beta\cup\gamma),\ \sigma^n(\alpha)\cup\sigma^n(\{Q^s(\beta)\})\ (p=2) \text{ and}$$
$$\sigma^n(\alpha)\cup\sigma^n(\{\lambda_{n-1}(\beta, \gamma)\})\,.$$

But all cup products in $H^*(\Sigma^n X)$ vanish (e.g., see (7.8*) of [16, III]). This implies that

(6.7) *$E_2^{i,j}$ for $i+j<4m+n-1$ with $i, j>0$ is divided into two parts: one part consists of elements which kill certain elements of $E_2^{i+j+1,0}$ (following the formulas of Theorem* 3) *and the other part consists of elements which are killed by some elements of $E_2^{0,i+j-1}$.*

Consider now the diagram (5.1) and let $\{'E_r, 'd_r\}$ and $\{''E_r, ''d_r\}$ be the mod p cohomology spectral sequences of the upper and lower fibrations, respectively. Then the naturality of the Serre spectral sequence yields a homomorphism of spectral sequences

(6.8) $\bar{g}$: *$''E \to 'E$, which is a system of maps* $\{g_r^{i,j}\}$, $g_r^{i,j}: ''E_r^{i,j} \to 'E_r^{i,j}$, such that $'d_r g_r = g_r'' d_r$, g_{r+1} is induced by g_r and the diagram

$$\begin{array}{ccc} ''E_2^{i,j} & \xrightarrow{\qquad g_2^{i,j} \qquad} & 'E_2^{i,j} \\ \| & & \| \\ H^i(Y'')\otimes H^j(G_nX'') & \xrightarrow{g^*\otimes G_nf^*} & H^i(Y')\otimes H^j(G_nX') \end{array}$$

commutes.

Proof of (1).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\theta}): Y\to K(Z_p, |\alpha|+n)\times K(Z_p, |\theta|+n)$. Then we see that to show (1) it suffices to prove

$$(1)' \qquad \nu_n^*(\sigma^n(\iota_{|\alpha|}\times\iota_{|\theta|})) = \sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\iota_{|\theta|})]$$

in the case $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\theta|+n)$.

The rest of the argument is the same as that in the proof of (1) of Theorem 3, except that one uses the spectral sequence in place of the exact sequence.

Proof of (2).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta}, {}^n\tilde{\gamma}): Y\to K(Z_p, |\alpha|+n)\times K(Z_p |\beta|+n)\times K(Z_p, |\gamma|+n)$. Then we see that to show (2) it suffices to prove

$$(2)' \qquad \nu_n^*(\sigma^n(\iota_{|\alpha|}\times\iota_{|\beta|}\times\iota_{|\gamma|})) = \sigma^n[(\iota_{|\alpha|}\times 1\times 1)*(1\times\iota_{|\beta|}\times\iota_{|\gamma|})]$$

in the case $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)$.

We use the homomorphisms (6.8) for the cases that $g=(\pi_1, \pi_2): K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)$, $g=(\pi_1, \pi_3)$ and $g=(\pi_2, \pi_3)$. Suppose $n>1$ and consider $\{'E_r, 'd_r\}$ modulo $\mathrm{Im}\ \overline{(\pi_1, \pi_2)}+\mathrm{Im}\ \overline{(\pi_1, \pi_3)}+\mathrm{Im}\ \overline{(\pi_2, \pi_3)}$; then for $i+j=|\alpha|+|\beta|+|\gamma|+n$,

$$'E_2^{i,j} = \begin{cases} Z_p\{\sigma^n[(\iota_{|\alpha|}\times 1\times 1)*(1\times\iota_{|\beta|}\times\iota_{|\gamma|})]\} & (i=0) \\ 0 & (i>0) \end{cases}$$

(recall the relation (3.1)). On the other hand,

$$\begin{aligned} &H^{|\alpha|+|\beta|+|\gamma|+n}(\Sigma^n(K(Z_p, |\alpha|)\times K(Z_p, |\beta|)\times K(Z_p, |\gamma|))) \\ &\quad = Z_p\{\sigma^n(\iota_{|\alpha|}\times\iota_{|\beta|}\times\iota_{|\gamma|})\} \text{ modulo } \mathrm{Im}\ (\Sigma^n(\pi_1, \pi_2))^* \\ &\qquad +\mathrm{Im}\ (\Sigma^n(\pi_1, \pi_3))^*+\mathrm{Im}\ (\Sigma^n(\pi_2, \pi_3))^* . \end{aligned}$$

This observation implies (2)′ for $n>1$.

It remains to prove the case $n=1$. But the argument here is analogous to that in the proof of (1) of Theorem 3.

Proof of (3).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta}^\wedge)\colon Y\to K(Z_2, |\alpha|+n)\times L(Z_2, s+1; |\beta|+n)$. Then by (6.5) we see that to show (3) it suffices to prove

$$(3)' \qquad \nu_n^*(\sigma^n(\iota_{|\alpha|}\times\kappa^{s+1;\,|\beta|}))=\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\kappa^{s+1;\,|\beta|})]$$

in the case $Y=K(Z_2, |\alpha|+n)\times L(Z_2, s+1; |\beta|+n)$.

We use the homomorphism (6.8) for the case $g=1\times\zeta_L\colon K(Z_2, |\alpha|+n)\times L(Z_2, s+1; |\beta|+n)\to K(Z_2, |\alpha|+n)\times K(Z_2, |\beta|+n)$. Suppose $n>1$ and consider $\{'E_r, 'd_r\}$ modulo Im $\overline{1\times\zeta_L}$; then for $i+j=|\alpha|+|\beta|+n+s$,

$$'E_2^{i,j}=\begin{cases}Z_2\{\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\kappa^{s+1;\,|\beta|})]\} & (i=0)\\ 0 & (i>0)\,.\end{cases}$$

On the other hand,

$$\begin{aligned}&H^{|\alpha|+|\beta|+n+s}(\Sigma^n(K(Z_2, |\alpha|)\times L(Z_2, s+1; |\beta|)))\\ &\quad =Z_2\{\sigma^n(\iota_{|\alpha|}\times\kappa^{s+1;\,|\beta|})\} \text{ modulo Im } (\Sigma^n(1\times\zeta_L))^*\,.\end{aligned}$$

This observation implies (3)′ for $n>1$.

The proof for the case $n=1$ is analogous to that in the proof of (1) of Theorem 3.

Proof of (4).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, ({}^n\tilde{\beta}, {}^n\tilde{\gamma})^\wedge)\colon Y\to K(Z_p, |\alpha|+n)\times M(Z_p; |\beta|+n, |\gamma|+n)$. Then by (6.6) we see that to show (4) it suffices to prove

$$(4)' \qquad \nu_n^*(\sigma^n(\iota_{|\alpha|}\times\lambda^{n;\,|\beta|,|\gamma|}))=\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\lambda^{n;\,|\beta|,|\gamma|})]$$

in the case $Y=K(Z_p, |\alpha|+n)\times M(Z_p; |\beta|+n, |\gamma|+n)$.

We use the homomorphism (6.8) for the case $g=1\times\zeta_M\colon K(Z_p, |\alpha|+n)\times M(Z_p; |\beta|+n, |\gamma|+n)\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)$. Suppose $n>1$ and consider $\{'E_r, 'd_r\}$ modulo Im $\overline{1\times\zeta_M}$; then for $i+j=|\alpha|+|\beta|+|\gamma|+2n-1$,

$$'E_2^{i,j}=\begin{cases}Z_p\{\sigma^n[(\iota_{|\alpha|}\times 1)*(1\times\lambda^{n;\,|\beta|,|\gamma|})]\} & (i=0)\\ 0 & (i>0)\,.\end{cases}$$

On the other hand,

$$\begin{aligned}&H^{|\alpha|+|\beta|+|\gamma|+2n-1}(\Sigma^n(K(Z_p, |\alpha|)\times\Omega^n M(Z_p; |\beta|+n, |\gamma|+n)))\\ &\quad =Z_p\{\sigma^n(\iota_{|\alpha|}\times\lambda^{n;\,|\beta|,|\gamma|})\} \text{ modulo Im } (\Sigma^n(1\times\zeta_M))^*\,.\end{aligned}$$

This observation implies (4)$'$ for $n>1$.

The proof for the case $n=1$ is analogous.

Proof of (5).

This proof is the same as that of (2) of Theorem 3.

Proof of (6).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\theta})$: $Y\to K(Z_p, |\alpha|+n)\times K(Z_p, |\theta|+n)$. Then we see that to show (6) it suffices to prove

$$(6)' \qquad \tau(\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota_{|\theta|})]) = \iota_{|\alpha|+n}\times\iota_{|\theta|+n}$$

in the case $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\theta|+n)$.

The rest of the argument is the same as that in the proof of (3) of Theorem 3, except that one uses the spectral sequence in place of the exact sequence.

Proof of (7).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta}, {}^n\tilde{\gamma})$: $Y\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)$. Then we see that to show (7) it suffices to prove

$$(7)' \qquad \begin{aligned} &d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1,\ 1\times\iota_{|\beta|}\times\iota_{|\gamma|})]) \\ &\quad = (\iota_{|\alpha|+n}\times 1\times 1)\otimes\sigma^n[(1\times\iota_{|\beta|}\times 1)*(1\times 1\times\iota_{|\gamma|})] \end{aligned}$$

in the case $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)$.

We use the homomorphisms (6.8) for the cases $g=(\pi_1, \pi_2)$: $K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)$, $g=(\pi_1, \pi_3)$ and $g=(\pi_2, \pi_3)$. Then, in ${}'E_2^{i,j}$ for $i+j=|\alpha|+|\beta|+|\gamma|+2n$ with $i, j>0$, there are elements

$$\begin{aligned} &(\iota_{|\alpha|+n}\times 1\times 1)\otimes\sigma^n[(1\times\iota_{|\beta|}\times 1)*(1\times 1\times\iota_{|\gamma|})]\,, \\ &(1\times\iota_{|\beta|+n}\times 1)\otimes\sigma^n[(\iota_{|\alpha|}\times 1\times 1)*(1\times 1\times\iota_{|\gamma|})] \text{ and} \\ &(1\times 1\times\iota_{|\gamma|+n})\otimes\sigma^n[(\iota_{|\alpha|}\times 1\times 1)*(1\times\iota_{|\beta|}\times 1)]\,. \end{aligned}$$

By (6.7) and (1) of Theorem 3, these elements must be killed by some elements of ${}'E_2^{0,|\alpha|+|\beta|+|\gamma|+2n-1}$. The elements which may kill them are

$$\begin{aligned} &1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1,\ 1\times\iota_{|\beta|}\times\iota_{|\gamma|})]\,, \\ &1\otimes\sigma^n[\lambda_{n-1}(1\times\iota_{|\beta|}\times 1,\ \iota_{|\alpha|}\times 1\times\iota_{|\gamma|})] \text{ and} \\ &1\otimes\sigma^n[\lambda_{n-1}(1\times 1\times\iota_{|\gamma|},\ \iota_{|\alpha|}\times\iota_{|\beta|}\times 1)]\,, \end{aligned}$$

since the behavior of other elements in ${}'E_r$ has been determined by the formula (2) and the naturality arguments (with respect to the maps (π_1, π_2), (π_1, π_3) and (π_2, π_3)). So (7)$'$ follows.

Proof of (8).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta}\wedge)\colon Y\to K(Z_2, |\alpha|+n)\times L(Z_2, s+1; |\beta|+n)$. Then by (6.5) we see that to show (8) it suffices to prove

$$(8)' \qquad d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1, 1\times\kappa^{s+1;\,|\beta|})]) = (\iota_{|\alpha|+n}\times 1)\otimes\sigma^n[\underset{\sim}{Q}^s(1\times\iota^{|\beta|})]$$

in the case $Y=K(Z_2, |\alpha|+n)\times L(Z_2, s+1; |\beta|+n)$.

We use the homomorphism (6.8) for the case $g=1\times\zeta_L\colon K(Z_2, |\alpha|+n)\times L(Z_2, s+1; |\beta|+n)\to K(Z_2, |\alpha|+n)\times K(Z_2, |\beta|+n)$. Then in ${}'E_2^{|\alpha|+n,\,|\beta|+n+s}$ there is an element

$$(\iota_{|\alpha|+n}\times 1)\otimes\sigma^n[\underset{\sim}{Q}^s(1\times\iota^{|\beta|})]\,.$$

By (6.7), (2) of Theorem 3 and the definition of $L(Z_2, s+1; |\beta|+n)$, this element must be killed by some element of ${}'E_2^{0,\,|\alpha|+|\beta|+2n+s-1}$. The element which may kill it is

$$1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1, 1\times\kappa^{s+1;\,|\beta|})]\,,$$

since the behavior of other elements in ${}'E_r$ has been determined by the formula (3) and the naturality argument. So (8)$'$ follows.

Proof of (12).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha}, {}^n\tilde{\beta}, {}^n\tilde{\gamma})\colon Y\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)$. Then we see that to show (12) it suffices to prove

$$(12)' \quad d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1, \lambda_{n-1}(1\times\iota_{|\beta|}\times 1, 1\times 1\times\iota_{|\gamma|}))]) = (\iota_{|\alpha|+n}\times 1\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota_{|\beta|}\times 1, 1\times 1\times\iota_{|\gamma|})]$$

in the case $Y=K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)$. (Here we suppose that $\beta<\alpha<\gamma$.)

We use the homomorphisms (6.8) for the cases $g=(\pi_1, \pi_2)\colon K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)\times K(Z_p, |\gamma|+n)\to K(Z_p, |\alpha|+n)\times K(Z_p, |\beta|+n)$, $g=(\pi_1, \pi_3)$ and $g=(\pi_2, \pi_3)$. Then, in ${}'E_2^{i,j}$ for $i+j=|\alpha|+|\beta|+|\gamma|+3n-1$ with $i, j>0$, there are elements

$$(\iota_{|\alpha|+n}\times 1\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota_{|\beta|}\times 1, 1\times 1\times\iota_{|\gamma|})]\,,$$
$$(1\times\iota_{|\beta|+n}\times 1)\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1, 1\times 1\times\iota_{|\gamma|})] \text{ and}$$
$$(1\times 1\times\iota_{|\gamma|+n})\otimes\sigma^n[\lambda_{n-1}(1\times\iota_{|\beta|}\times 1, \iota_{|\alpha|}\times 1\times 1)]\,.$$

On the other hand, in ${}'E_2^{0,\,|\alpha|+|\beta|+|\gamma|+3n-2}$ there are elements

$$1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1, \lambda_{n-1}(1\times\iota_{|\beta|}\times 1, 1\times 1\times\iota_{|\gamma|}))] \text{ and}$$

$$1\otimes\sigma^n[\lambda_{n-1}(1\times 1\times\iota_{|\gamma|},\ \lambda_{n-1}(1\times\iota_{|\beta|}\times 1,\ \iota_{|\alpha|}\times 1\times 1))]\ .$$

Furthermore, in $'E_2^{|\alpha|+|\beta|+|\gamma|+3n,0}$ there is an element

$$(\iota_{|\alpha|+n}\times\iota_{|\beta|+n}\times\iota_{|\gamma|+n})\otimes 1$$

which must be in the image of $'d_r$ (for some r), by (4.6). In view of (6.7) and (3) of Theorem 3, we may conclude that

$$\begin{aligned}
&'d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1,\ \lambda_{n-1}(1\times\iota_{|\beta|}\times 1,\ 1\times 1\times\iota_{|\gamma|}))])\\
&\quad=(\iota_{|\alpha|+n}\times 1\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota_{|\beta|}\times 1,\ 1\times 1\times\iota_{|\gamma|})]\ ,\\
&'d_{|\gamma|+n}(1\otimes\sigma^n[\lambda_{n-1}(1\times 1\times\iota_{|\gamma|},\ \lambda_{n-1}(1\times\iota_{|\beta|}\times 1,\ \iota_{|\alpha|}\times 1\times 1))])\\
&\quad=(1\times 1\times\iota_{|\gamma|+n})\otimes\sigma^n[\lambda_{n-1}(1\times\iota_{|\beta|}\times 1,\ \iota_{|\alpha|}\times 1\times 1)]\ \text{and}\\
&'d_{|\alpha|+|\gamma|+2n}((1\times\iota_{|\beta|+n}\times 1)\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1\times 1,\ 1\times 1\times\iota_{|\gamma|})])\\
&\quad=\pm(\iota_{|\alpha|+n}\times\iota_{|\beta|+n}\times\iota_{|\gamma|+n})\otimes 1\ ,
\end{aligned}$$

since the behavior of other elements in $'E_r$ has been determined by the formulas (2) and (7) and the naturality arguments.

REMARK. It follows from (1.10) and (1.13) that any two of

$$\begin{aligned}
&\lambda_{n-1}(\iota_i\times 1\times 1,\ \lambda_{n-1}(1\times\iota_j\times 1,\ 1\times 1\times\iota_k))\\
&\quad(=\pm\lambda_{n-1}(\iota_i\times 1\times 1,\ \lambda_{n-1}(1\times 1\times\iota_k,\ 1\times\iota_j\times 1)))\ ,\\
&\lambda_{n-1}(1\times\iota_j\times 1,\ \lambda_{n-1}(1\times 1\times\iota_k,\ \iota_i\times 1\times 1))\ \text{and}\\
&\lambda_{n-1}(1\times 1\times\iota_k,\ \lambda_{n-1}(\iota_i\times 1\times 1,\ 1\times\iota_j\times 1))
\end{aligned}$$

constitute a part of a Z_p-basis for $\tilde{H}^*(F_n(K(Z_p,i)\times K(Z_p,j)\times K(Z_p,k)))$. Taking this into consideration, we abandon the idea of fixing a Z_p-basis for $\tilde{H}^*(G_nX)$ and assert that (12) always holds. The reader should refer to the Remark below the proof of (9) (a).

Proof of (9) (a).

Consider the homomorphism (6.8) for the case $g=({}^n\tilde{\alpha},\ ({}^n\tilde{\beta},\ {}^n\tilde{\gamma})^\wedge)\colon Y\to K(Z_p,\ |\alpha|+n)\times M(Z_p;\ |\beta|+n,\ |\gamma|+n)$. Then by (6.6) we see that to show (9) (a) it suffices to prove

(9) (a)′ $$\begin{aligned}&d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\lambda^{n;\,|\beta|,|\gamma|})])\\&\quad=(\iota_{|\alpha|+n}\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\beta|},\ 1\times\iota^{|\gamma|})]\end{aligned}$$

in the case $Y=K(Z_p,\ |\alpha|+n)\times M(Z_p;\ |\beta|+n,\ |\gamma|+n)$.

We use the homomorphism (6.8) for the case $g=1\times\zeta_M\colon K(Z_p,\ |\alpha|+n)\times M(Z_p;\ |\beta|+n,\ |\gamma|+n)\to K(Z_p,\ |\alpha|+n)\times K(Z_p,\ |\beta|+n)\times K(Z_p,\ |\gamma|+n)$. Then, in $'E_2^{i,j}$ for $i+j=|\alpha|+|\beta|+|\gamma|+3n-1$ with $i,j>0$, there are elements

$$(\iota_{|\alpha|+n}\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\beta|},\ 1\times\iota^{|\gamma|})]\ ,$$

$$(1\times\iota^{|\beta|+n})\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\gamma|})] \text{ and}$$
$$(1\times\iota^{|\gamma|+n})\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\beta|})]\,.$$

On the other hand, in $'E_2^{0,|\alpha|+|\beta|+|\gamma|+3n-2}$ there are elements

$$1\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\beta|},\ \lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\gamma|}))]\,,$$
$$1\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\gamma|},\ \lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\beta|}))] \text{ and}$$
$$1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\lambda^{n;\,|\beta|,|\gamma|})]\,.$$

It follows from the naturality argument (cf. the proof of (12)) that

$$\begin{aligned}
&'d_{|\beta|+n}(1\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\beta|},\ \lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\gamma|}))])\\
&\quad=(1\times\iota^{|\beta|+n})\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\gamma|})]\,,\\
&'d_{|\gamma|+n}(1\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\gamma|},\ \lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\beta|}))])\\
&\quad=(1\times\iota^{|\gamma|+n})\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\iota^{|\beta|})] \text{ and}\\
&'d_{|\beta|+|\gamma|+2n}((\iota_{|\alpha|+n}\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\beta|},\ 1\times\iota^{|\gamma|})])\\
&\quad=\pm((\iota_{|\alpha|+n}\times 1)\cup(1\times\iota^{|\beta|+n})\cup(1\times\iota^{|\gamma|+n}))\otimes 1\\
&\quad=0\,,
\end{aligned}$$

since $\iota^{|\beta|+n}\cup\iota^{|\gamma|+n}=0$ by the definition of $M(Z_p;\ |\beta|+n,\ |\gamma|+n)$. Hence, by (6.7), $(\iota_{|\alpha|+n}\times 1)\otimes\sigma^n[\lambda_{n-1}(1\times\iota^{|\beta|},\ 1\times\iota^{|\gamma|})]$ is killed by some element of $'E_2^{0,|\alpha|+|\beta|+|\gamma|+3n-2}$. It must be

$$1\otimes\sigma^n[\lambda_{n-1}(\iota_{|\alpha|}\times 1,\ 1\times\lambda^{n;\,|\beta|,|\gamma|})]\,,$$

since the behavior of other elements in $'E_r$ has been determined by the formula (4) and the naturality argument. So (9) (a)$'$ follows.

REMARK. In the above proof we have supposed that $\alpha<\beta<\gamma$ and have taken the set of basic λ_{n-1}-products as a part of a Z_p-basis for $\tilde{H}^*(G_nX)$. But if we take a different order among α, β, γ (e.g., $\beta<\alpha<\gamma$) and work in the same way, we find that (9) (a) does not hold as a formula. This trouble is overcomed by the following idea: we do not specify a Z_p-basis for $\tilde{H}^*(G_nX)$ and assert that (9) (a) holds in any case.

For the proof of (9) (b) we need some notations.

Let $M'(Z_p;\ i,\ j,\ k)$ denote the mapping fibre of

$$(\iota_i\times\iota_j\times 1,\ 1\times\iota_j\times\iota_k): K(Z_p,\ i)\times K(Z_p,\ j)\times K(Z_p,\ k)\to K(Z_p,\ i+j)\times K(Z_p,\ j+k)\,.$$

Then there is a fibration

$$K(Z_p,\ i+j-1)\times K(Z_p,\ j+k-1)\xrightarrow{\ \varepsilon_{M'}\ } M'(Z_p;\ i,\ j,\ k)\xrightarrow{\ \zeta_{M'}\ }$$

$$K(Z_p, i)\times K(Z_p, j)\times K(Z_p, k).$$

Application of Ω^n yields a fibration

$$K(Z_p, i+j-n-1)\times K(Z_p, j+k-n-1)\xrightarrow{\varepsilon_{M'}}\Omega^n M'(Z_p; i, j, k)$$
$$\xrightarrow{\zeta_{M'}} K(Z_p, i-n)\times K(Z_p, j-n)\times K(Z_p, k-n).$$

which is induced by $((\sigma^*)^n(\iota_i\times\iota_j\times 1), (\sigma^*)^n(1\times\iota_j\times\iota_k))$ for $n\geq 0$. Put $\iota^{i-n}=\zeta^*_{M'}(\iota_{i-n}\times 1\times 1)$, $\iota^{j-n}=\zeta^*_{M'}(1\times\iota_{j-n}\times 1)$ and $\iota^{k-n}=\zeta^*_{M'}(1\times 1\times\iota_{k-n})$.

Suppose $n\geq 1$. Then $(\sigma^*)^n(\iota_i\times\iota_j\times 1)=(\sigma^*)^n(1\times\iota_j\times\iota_k)=0$ and therefore

$$\Omega^n M'(Z_p; i, j, k)\simeq K(Z_p, i-n)\times K(Z_p, j-n)\times K(Z_p, k-n)$$
$$\times K(Z_p, i+j-n-1)\times K(Z_p, j+k-n-1).$$

Let $\lambda^{n;\, i-n, j-n}\in H^{i+j-n-1}(\Omega^n M'(Z_p; i, j, k))$ (resp. $\lambda^{n;\, j-n,k-n}\in H^{j+k-n-1}(\Omega^n M'(Z_p; i, j, k)))$ be the element such that

$$\varepsilon^*_{M'}(\lambda^{n;\, i-n, j-n}) = \iota_{i+j-n-1}\times 1$$
$$(\text{resp. } \varepsilon^*_{M'}(\lambda^{n;\, j-n, k-n}) = 1\times\iota_{j+k-n-1}).$$

We have fibrations

(6.9) $\quad K(Z_p, i+j-1)\xrightarrow{{}^L\varepsilon_{M'}} M'(Z_p; i, j, k)\xrightarrow{{}^L\zeta_{M'}} K(Z_p, i)\times M(Z_p; j, k)$,

(6.10) $\quad K(Z_p, j+k-1)\xrightarrow{{}^R\varepsilon_{M'}} M'(Z_p; i, j, k)\xrightarrow{{}^R\zeta_{M'}} M(Z_p; i, j)\times K(Z_p, k)$

such that $(1\times\zeta_M){}^L\zeta_{M'}\simeq\zeta_{M'}$ and $(\zeta_M\times 1){}^R\zeta_{M'}\simeq\zeta_{M'}$. Then ${}^L\zeta_{M'}{}^*(1\times\lambda^{n;\, j-n,k-n})=\lambda^{n;\, j-n,k-n}$ and ${}^R\zeta^*_{M'}(\lambda^{n;\, i-n,j-n}\times 1)=\lambda^{n;\, i-n,j-n}$.

By the definition of $M'(Z_p; i, j, k)$, $\iota^i\cup\iota^j=\iota^j\cup\iota^k=0$. So the Massey product $\langle\iota^i, \iota^j, \iota^k\rangle(=(-1)^{ij+ik+jk+1}\langle\iota^k, \iota^j, \iota^i\rangle)$ is defined. Consider the mod p cohomology spectral sequence $\{{}^LE_r, {}^Ld_r\}$ (resp. $\{{}^RE_r, {}^Rd_r\}$) of the fibration (6.9) (resp. (6.10)). Since ${}^L\tau(\iota_{i+j-1})=\iota_i\times\iota^j$ (resp. ${}^R\tau(\iota_{j+k-1})=\iota^j\times\iota_k$, it follows that

$${}^Ld_{i+j}((1\times\iota^k)\otimes\iota_{i+j-1}) = \pm(\iota_i\times(\iota^j\cup\iota^k))\otimes 1 = 0$$
$$(\text{resp. } {}^Rd_{j+k}((\iota^i\times 1)\otimes\iota_{j+k-1}) = \pm((\iota^i\cup\iota^j)\times\iota_k)\otimes 1 = 0).$$

Thus we find that $(1\times\iota^k)\otimes\iota_{i+j-1}$(resp. $(\iota^i\times 1)\otimes\iota_{j+k-1}$) survives to ${}^LE_\infty$ (resp. ${}^RE_\infty$). Let ${}^L\lambda^{i,j,k}$ (resp. ${}^R\lambda^{i,j,k}$)$\in H^{i+j+k-1}(M'(Z_p; i, j, k))$ be its representative.

Lemma 10. $\langle\iota^i, \iota^j, \iota^k\rangle=\pm{}^R\lambda^{i,j,k}$ (resp. $\langle\iota^i, \iota^j, \iota^k\rangle=\pm{}^L\lambda^{i,j,k}$).

Proof. Consider the map

$$1\times\varepsilon_M\colon K(Z_p, i)\times K(Z_p, j+k-1)\to K(Z_p, i)\times M(Z_p; j, k)$$
$$(\text{resp. } \varepsilon_M\times 1\colon K(Z_p, i+j-1)\times K(Z_p, k)\to M(Z_p; i, j)\times K(Z_p, k)).$$

Then we have a map

$$\begin{aligned}&{}^Lf\colon K(Z_p, i)\times K(Z_p, j+k-1)\to M'(Z_p\,; i, j, k)\\&(\text{resp. } {}^Rf\colon K(Z_p\,, i+j-1)\times K(Z_p, k)\to M'(Z_p\,; i, j, k))\end{aligned}$$

such that ${}^L\zeta_{M'}{}^Lf\simeq 1\times\varepsilon_M$ (resp. ${}^R\zeta_{M'}{}^Rf\simeq\varepsilon_M\times 1$). It is clear that ${}^Lf^*(\iota^i)=\iota_i\times 1$ (resp. ${}^Rf^*(\iota^i)=0$), ${}^Lf^*(\iota^j)=0$ (resp. ${}^Rf^*(\iota^j)=0$), ${}^Lf^*(\iota^k)=0$ (resp. ${}^Rf^*(\iota^k)=1\times\iota_k$) and for $v\in H^{i+j+k-1}(M'(Z_p\,; i, j, k))$,

$${}^Lf^*(v)=\begin{cases}\iota_i\times\iota_{j+k-1} & \text{if } v={}^R\lambda^{i,j,k}\\ 0 & \text{otherwise}\end{cases}$$

$$(\text{resp. } {}^Rf^*(v)=\begin{cases}\iota_{i+j-1}\times\iota_k & \text{if } v={}^L\lambda^{i,j,k}\\ 0 & \text{otherwise}\end{cases}).$$

By the same argument as in the proof of Lemma 7 of [15], we have

$$\begin{aligned}&{}^Lf^*\langle\iota^i, \iota^j, \iota^k\rangle=\pm\iota_i\times\iota_{j+k-1}\\&(\text{resp. } {}^Rf^*\langle\iota^i, \iota^j, \iota^k\rangle=\pm\iota_{i+j-1}\times\iota_k)\,.\end{aligned}$$

So the result follows.

Proof of (9) (b).

By hypothesis there is a lifting of $({}^n\tilde\alpha, {}^n\tilde\beta, {}^n\tilde\gamma)$, i.e., a map

$$({}^n\tilde\alpha, {}^n\tilde\beta, {}^n\tilde\gamma)^\wedge\colon Y\to M'(Z_p\,; |\alpha|+n, |\beta|+n, |\gamma|+n)$$

such that $\zeta_{M'}({}^n\tilde\alpha, {}^n\tilde\beta, {}^n\tilde\gamma)^\wedge\simeq({}^n\tilde\alpha, {}^n\tilde\beta, {}^n\tilde\gamma)$. Consider the homomorphism (6.8) for the case $g=({}^n\tilde\alpha, {}^n\tilde\beta, {}^n\tilde\gamma)^\wedge$. Then by (6.6) we see that to show (9) (b) it suffices to prove

$$\text{(9) (b)}' \qquad \begin{aligned}&\tau(\sigma^n[\lambda_{n-1}(\iota^{|\alpha|}, \lambda^{n;\,|\beta|,|\gamma|})]+c'\cdot\sigma^n[\lambda_{n-1}(\iota^{|\gamma|}, \lambda^{n;\,|\alpha|,|\beta|})])\\&=\langle\iota^{|\alpha|+n}, \iota^{|\beta|+n}, \iota^{|\gamma|+n}\rangle\end{aligned}$$

in the case $Y=M'(Z_p\,; |\alpha|+n, |\beta|+n, |\gamma|+n)$.

We use the homomorphism (6.8) for the case $g={}^L\zeta_{M'}\colon M'(Z_p\,; |\alpha|+n, |\beta|+n, |\gamma|+n)\to K(Z_p, |\alpha|+n)\times M(Z_p\,; |\beta|+n, |\gamma|+n)$. Then, in ${}'E_2^{0,|\alpha|+|\beta|+|\gamma|+3n-2}$ there are elements

$$\begin{aligned}&1\otimes\sigma^n[\lambda_{n-1}(\iota^{|\alpha|}, \lambda^{n;\,|\beta|,|\gamma|})] \quad\text{and}\\&1\otimes\sigma^n[\lambda_{n-1}(\iota^{|\gamma|}, \lambda^{n;\,|\alpha|,|\beta|})]\,.\end{aligned}$$

On the other hand, in ${}'E_2^{|\alpha|+|\beta|+|\gamma|+3n-1,0}$ there is an element

$$\langle\iota^{|\alpha|+n}, \iota^{|\beta|+n}, \iota^{|\gamma|+n}\rangle\otimes 1$$

(which is non-zero by Lemma 10). By [6], it must be in the image of ${}'d_r$ (for some r). It follows from the naturality argument (cf. the proof of (9) (a)) that

$$'d_{|\alpha|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota^{|\alpha|},\lambda^{n;|\beta|,|\gamma|})]) \\ =\iota^{|\alpha|+n}\otimes\sigma^n[\lambda_{n-1}(\iota^{|\beta|},\iota^{|\gamma|})].$$

So we may conclude that

$$(6.11)\qquad '\tau(\sigma^n[\lambda_{n-1}(\iota^{|\gamma|},\lambda^{n;|\alpha|,|\beta|})]+\textit{other terms}) \\ =\langle\iota^{|\alpha|+n},\iota^{|\beta|+n},\iota^{|\gamma|+n}\rangle.$$

Similarly from the naturality argument with respect to the map ${}^R\zeta_{M'}\colon M'(Z_p;|\alpha|+n,|\beta|+n,|\gamma|+n)\to M(Z_p;|\alpha|+n,|\beta|+n)\times K(Z_p,|\gamma|+n)$ it follows that

$$'d_{|\gamma|+n}(1\otimes\sigma^n[\lambda_{n-1}(\iota^{|\gamma|},\lambda^{n;|\alpha|,|\beta|})]) \\ =\iota^{|\gamma|+n}\otimes\sigma^n[\lambda_{n-1}(\iota^{|\alpha|},\iota^{|\beta|})]$$

and

$$(6.12)\qquad '\tau(\sigma^n[\lambda_{n-1}(\iota^{|\alpha|},\lambda^{n;|\beta|,|\gamma|})]+\textit{other terms}) \\ =\langle\iota^{|\alpha|+n},\iota^{|\beta|+n},\iota^{|\gamma|+n}\rangle.$$

Thus equations (6.11) and (6.12) imply (9) (b)$'$.

Proof of (9) (c).

Let $M''(Z_p;i,j,k)$ denote the mapping fibre of

$$(\iota_i\times\iota_j\times 1,\ \iota_i\times 1\times\iota_k,\ 1\times\iota_j\times\iota_k)\colon K(Z_p,i)\times K(Z_p,j)\times K(Z_p,k) \\ \to K(Z_p,i+j)\times K(Z_p,i+k)\times K(Z_p,j+k).$$

Then ι^i, ι^j and ι^k are defined similarly. We have fibrations

$$K(Z_p,i+j-1)\to M''(Z_p;i,j,k)\to M'(Z_p;j,k,i),$$
$$K(Z_p,i+k-1)\to M''(Z_p;i,j,k)\to M'(Z_p;i,j,k)\quad\text{and}$$
$$K(Z_p,j+k-1)\to M''(Z_p;i,j,k)\to M'(Z_p;k,i,j)$$

which are induced by $\iota^j\cup\iota^i$, $\iota^i\cup\iota^k$ and $\iota^k\cup\iota^j$ respectively. By definition, all Massey products $\langle\iota^i,\iota^j,\iota^k\rangle$, $\langle\iota^j,\iota^k,\iota^i\rangle$ and $\langle\iota^k,\iota^i,\iota^j\rangle$ are defined and non-zero; this follows from the same argument as in Lemma 10. Furthermore, by [15] there is a relation

$$(-1)^{ik}\langle\iota^i,\iota^j,\iota^k\rangle+(-1)^{ij}\langle\iota^j,\iota^k,\iota^i\rangle+(-1)^{jk}\langle\iota^k,\iota^i,\iota^j\rangle=0.$$

Taking this into consideration, we see that (the universal example for (9) (c) is $M''(Z_p;i,j,k)$ and) (9) (c) follows from the naturality arguments with respect to the maps $M''(Z_p;i,j,k)\to M'(Z_p;j,k,i)$, $M''(Z_p;i,j,k)\to M'(Z_p;i,j,k)$ and so on.

REMARK. We can go without (9) (c), because it is essentially a copy of (9) (b).

Proof of (10).

Consider the homomorphism (6.8) for the case $g={}^n\tilde{\alpha}\colon Y\to K(Z_3,\ |\alpha|+n)$. Then we see that to show (10) it suffices to prove

$$(10)' \qquad \tau(\sigma^n[Q^s(\iota_{|\alpha|})]) = \Delta^*\mathfrak{P}^s(\iota_{|\alpha|+n})$$

in the case $Y=K(Z_3,\ |\alpha|+n)$.

Consider the spectral sequence (3.8) for the case that $Y=K(Z_3,\ 2s+1)$ and $n=1$. Since

$$\begin{aligned}\xi_1^*(\Delta^*\mathfrak{P}^s(\iota_{2s+1})) &= \Delta^*\mathfrak{P}^s(\xi_1^*(\iota_{2s+1})) = \Delta^*\mathfrak{P}^s(\sigma(\iota_{2s}))\\ &= \sigma(\Delta^*\mathfrak{P}^s(\iota_{2s})) = \sigma(\Delta^*(\iota_{2s}\cup\iota_{2s}\cup\iota_{2s})) = \sigma(0) = 0\,,\end{aligned}$$

$\Delta^*\mathfrak{P}^s(\iota_{2s+1})\otimes 1\in E_2^{6s+2,0}$ must be in the image of d_r (for some r). (Describe $E_r^{*,*}$, especially, $E_2^{0,*}=H^*(G_1K(Z_3,\ 2s))$.) In view of the formulas (1), (6) and (7), we find that the only element which may kill it is $1\otimes\sigma[Q^s(\iota_s)]\in E_2^{0,6s+1}$; that is,

$$\tau(\sigma[Q^s(\iota_{2s})]) = \Delta^*\mathfrak{P}^s(\iota_{2s+1})\,.$$

Consider the diagram (4.4) for the case that $Y=K(Z_3,\ 2s+1)$, $n=-|\alpha|+2s+1$ and $k=n-1=-|\alpha|+2s$; then we have the commutative diagram (5.4) (where $X=K(Z_3,\ |\alpha|)$ and $\Omega Y=K(Z_3,\ 2s)$), and by (2) of Lemma 9,

$$\begin{aligned}\tau(\sigma^{-|\alpha|+2s+1}[Q^s(\iota_{|\alpha|})]) &= \tau(\tilde{\xi}'_{-|\alpha|+2s})^*(\sigma[Q^s(\iota_{2s})])\\ &= \tau(\sigma[Q^s(\iota_{2s})])\\ &= \Delta^*\mathfrak{P}^s(\iota_{2s+1})\,.\end{aligned}$$

Consider the diagram (4.2) for the case that $Y=K(Z_3,\ |\alpha|+n)$ and $k=|\alpha|+n-2s-1$; then we have the commutative diagram (5.5) (where $X=K(Z_3,\ |\alpha|)$ and $\Omega^k Y=K(Z_3,\ 2s+1)$), and by (2) of Lemma 8,

$$\begin{aligned}(\sigma^*)^{|\alpha|+n-2s-1}\tau(\sigma^n[Q^s(\iota_{|\alpha|})]) &= \tau(\sigma^*)^{|\alpha|+n-2s-1}(\sigma^n[Q^s(\iota_{|\alpha|})])\\ &= \tau(\tilde{\eta}'_{|\alpha|+n-2s-1})^*(\sigma^*)^{|\alpha|+n-2s-1}(\sigma^n[Q^s(\iota_{|\alpha|})])\\ &= \tau(\sigma^{-|\alpha|+2s+1}[Q^s(\iota_{|\alpha|})])\\ &= \Delta^*\mathfrak{P}^s(\iota_{2s+1})\\ &= (\sigma^*)^{|\alpha|+n-2s-1}(\Delta^*\mathfrak{P}^s(\iota_{|\alpha|+n}))\,.\end{aligned}$$

Since $(\sigma^*)^{|\alpha|+n-2s-1}\colon H^{|\alpha|+n+4s+1}(K(Z_3,\ |\alpha|+n))\to H^{6s+2}(K(Z_3,\ 2s+1))$ is monomorphic (see [4]), (10)′ follows.

Proof of (11).

Consider the homomorphism (6.8) for the case $g={}^n\tilde{\alpha}\colon Y\to K(Z_3,\ |\alpha|+n)$. Then we see that to show (11) it suffices to prove

$$(11)' \qquad \tau(\sigma^n[\Delta Q^s(\iota_{|\alpha|})])=\mathfrak{P}^s(\iota_{|\alpha|+n})$$

in the case $Y=K(Z_3,\ |\alpha|+n)$.

Consider the spectral sequence (3.8) for the case that $Y=K(Z_3,\ 2s+1)$ and $n=2$. Since

$$\begin{aligned}\xi_2^*(\mathfrak{P}^s(\iota_{2s+1}))&=\mathfrak{P}^s(\xi_2^*(\iota_{2s+1}))=\mathfrak{P}^s(\sigma^2(\iota_{2s-1}))\\&=\sigma^2(\mathfrak{P}^s(\iota_{2s-1}))=\sigma^2(0)=0\,,\end{aligned}$$

$\mathfrak{P}^s(\iota_{2s+1})\otimes 1\in E_2^{6s+1,0}$ must be in the image of d_r (for some r). (Describe $E_r^{*,*}$, especially, $E_2^{0,*}=H^*(G_2K(Z_3,\ 2s-1))$.) In view of the formulas (1) and (6), we find that the only element which may kill it is $1\otimes\sigma^2[\Delta Q^s(\iota_{2s-1})]\in E_2^{0,6s}$; that is,

$$\tau(\sigma^2[\Delta Q^s(\iota_{2s-1})])=\mathfrak{P}^s(\iota_{2s+1})\,.$$

Consider the diagram (4.4) for the case that $Y=K(Z_3,\ 2s+1)$, $n=-|\alpha|+2s+1$ and $k=n-2=-|\alpha|+2s-1$; then we have the commutative diagram analogous to (5.4), and by (2) of Lemma 9,

$$\begin{aligned}\tau(\sigma^{-|\alpha|+2s+1}[\Delta Q^s(\iota_{|\alpha|})])&=\tau(\tilde{\xi}'_{-|\alpha+2s-1})^*(\sigma^2[\Delta Q^s(\iota_{2s-1})])\\&=\tau(\sigma^2[\Delta Q^s(\iota_{2s-1})])\\&=\mathfrak{P}^s(\iota_{2s+1})\,.\end{aligned}$$

Consider the diagram (4.2) for the case that $Y=K(Z_3,\ |\alpha|+n)$ and $k=|\alpha|+n-2s-1$; then we have the commutative diagram (5.5) (where $X=K(Z_3,\ |\alpha|)$ and $\Omega^kY=K(Z_3,\ 2s+1)$), and by (2) of Lemma 8,

$$\begin{aligned}(\sigma^*)^{|\alpha|+n-2s-1}\tau(\sigma^n[\Delta Q^s(\iota_{|\alpha|})])&=\tau(\sigma^*)^{|\alpha|+n-2s-1}(\sigma^n[\Delta Q^s(\iota_{|\alpha|})])\\&=\tau(\tilde{\eta}'_{|\alpha|+n-2s-1})^*(\sigma^*)^{|\alpha|+n-2s-1}(\sigma^n[\Delta Q^s(\iota_{|\alpha|})])\\&=\tau(\sigma^{-|\alpha|+2s+1}[\Delta Q^s(\iota_{|\alpha|})])\\&=\mathfrak{P}^s(\iota_{2s+1})\\&=(\sigma^*)^{|\alpha|+n-2s-1}(\mathfrak{P}^s(\iota_{|\alpha|+n}))\,.\end{aligned}$$

Since $(\sigma^*)^{|\alpha|+n-2s-1}\colon H^{|\alpha|+n+4s}(K(Z_3,\ |\alpha|+n))\to H^{6s+1}(K(Z_3,\ 2s+1))$ is monomorphic (see [4]), (11)$'$ follows.

Furthering the assertion of the Remark below Theorem 7, we find that, for example, in view of (1.10) and the diagram (5.4) together with Lemma 9 (3), the formula (6) of Theorem 7 should be rewritten as follows:

$$\tau(\sigma^n[\lambda_{n-1}(\alpha,\ \theta)])=(-1)^{|\alpha|+n}\,{}^n\tilde{\alpha}\cup{}^n\tilde{\theta}\,.$$

But here we shall not pursue this discussion.

7. Several remarks

In this section we collect miscellaneous remarks on the results of the previous sections.

First we have

Proposition 11. *Let $n\geq 1$ and $i, j>n$. Then*

(i) *In $H_*(L(Z_2, i; i-n))$, $Q^{i-1}(\iota_*^{i-n})=\kappa_*^{i;\,i-n}$.*

(ii) *In $H_*(\Omega^n M(Z_p; i, j))$, $\lambda_{n-1}(\iota_*^{i-n}, \iota_*^{j-n})=\lambda_*^{n;\,i-n,j-n}$.*

Proof. We use induction on n. To prove (i) for $n=1$, we first consider the mod 2 cohomology spectral sequence $\{E_r, d_r\}$ of the path fibration

$$L(Z_2, i; i-1) \rightarrow PL(Z_2, i; i) \rightarrow L(Z_2, i; i)\,.$$

Then by the well-known argument [10, Lemma 3.1.1], $\tau(\iota^{i-1})=\iota^i$ and $d_i(1\otimes\kappa^{i;\,i-1})=\iota^i\otimes\iota^{i-1}$. We next consider the mod 2 homology spectral sequence $\{E^r, d^r\}$ of the same fibration. It follows from the duality between E_r and E^r that $\tau_*(\iota_*^i)=\iota_*^{i-1}$ and $d^i(\iota_*^i\otimes\iota_*^{i-1})=1\otimes\kappa_*^{i;\,i-1}$. According to [3, Theorem II. 5.A], these equations imply that $\iota_*^{i-1}*\iota_*^{i-1}=\kappa_*^{i;\,i-1}$ in $H_*(L(Z_2, i; i-1))$. By (1.3), this proves (i) for $n=1$.

Assume that $Q^{i-1}(\iota_*^{i-n+1})=\kappa_*^{i;\,i-n+1}$ in $H_*(L(Z_2, i; i-n+1))$. Consider the mod 2 homology spectral sequence of the path fibration

$$L(Z_2, i; i-n) \rightarrow PL(Z_2, i; i-n+1) \rightarrow L(Z_2, i; i-n+1)\,.$$

In view of (6.1), we find that ι_*^{i-n+1} and $\kappa_*^{i;\,i-n+1}$ transgress to ι_*^{i-n} and $\kappa_*^{i;\,i-n}$ respectively. So

$$\begin{aligned}\kappa_*^{i;\,i-n} &= \tau_*(\kappa_*^{i;\,i-n+1})\\ &= \tau_*(Q^{i-1}(\iota_*^{i-n+1}))\\ &= Q^{i-1}(\tau_*(\iota_*^{i-n+1})) \qquad \text{(by (1.16))}\\ &= Q^{i-1}(\iota_*^{i-n})\,.\end{aligned}$$

To prove (ii) for $n=1$, we first consider the mod p cohomology spectral sequence $\{E_r, d_r\}$ of the path fibration

$$\Omega M(Z_p; i, j) \rightarrow PM(Z_p; i, j) \rightarrow M(Z_p; i, j)\,.$$

Then $\tau(\iota^{i-1})=\iota^i$ and $\tau(\iota^{j-1})=\iota^j$. Therefore $d_i(1\otimes(\iota^{i-1}\cup\iota^{j-1}))=\iota^i\otimes\iota^{j-1}$ and $d_i(\iota^j\otimes\iota^{i-1})=(\iota^i\cup\iota^j)\otimes 1=0$ by the definition of $M(Z_p; i, j)$. So $\iota^j\otimes\iota^{i-1}$ must be in the image of d_j. In view of (6.3), we find that

$$d_j(1\otimes\lambda^{1;\,i-1,j-1}+\text{ other terms}) = \iota^j\otimes\iota^{i-1}\,.$$

We next consider the mod p homology spectral sequence $\{E^r, d^r\}$ of the same fibration. It follows from the duality and [3, Theorem II. 5. A] that $d^i(\iota_*^i\otimes\iota_*^{j-1})=1\otimes(\iota_*^{i-1}*\iota_*^{j-1})$ and $d^j(\iota_*^j\otimes\iota_*^{i-1})=1\otimes(\iota_*^{j-1}*\iota_*^{i-1})$. This implies that

$$H_{i+j-2}(\Omega M(Z_p\,;\,i,j)) = Z_p\{\iota_*^{i-1}*\iota_*^{j-1},\ \iota_*^{j-1}*\iota_*^{i-1},\ \cdots\}\,.$$

Here $\iota_*^{j-1}*\iota_*^{i-1}$ can be replaced by $\iota_*^{i-1}*\iota_*^{j-1}-(-1)^{(i-1)(j-1)}\iota_*^{j-1}*\iota_*^{i-1}=\lambda_0(\iota_*^{i-1},\ \iota_*^{j-1})$ (see (1.9)). Since $\lambda_0(\iota_*^{i-1},\ \iota_*^{j-1})$ is primitive, we may conclude that

(7.1) $\quad\lambda_0(\iota_*^{i-1},\ \iota_*^{j-1})$ *(resp.* $\iota_*^{i-1}*\iota_*^{j-1}$*) is dual to* $\lambda^{1;\,i-1,j-1}$ *(resp.* $\iota^{i-1}\cup\iota^{j-1}$*)*.

This proves (ii) for $n=1$.

Assume that $\lambda_{n-2}(\iota_*^{i-n+1},\ \iota_*^{j-n+1})=\lambda_*^{n-1;\,i-n+1,j-n+1}$ in $H_*(\Omega^{n-1}M(Z_p;\,i,j))$. Consider the mod p homology spectral sequence of the path fibration

$$\Omega^n M(Z_p;\,i,j) \to P\Omega^{n-1}M(Z_p\,;\,i,j) \to \Omega^{n-1}M(Z_p;\,i,j)\,.$$

In view of (6.3), we find that ι_*^{i-n+1}, ι_*^{j-n+1} and $\lambda_*^{n-1;\,i-n+1,j-n+1}$ transgress to ι_*^{i-n}, ι_*^{j-n} and $\lambda_*^{n;\,i-n,j-n}$ respectively. So

$$\begin{aligned}\lambda_*^{n;\,i-n,j-n} &= \tau_*(\lambda_*^{n-1;\,i-n+1,j-n+1})\\ &= \tau_*(\lambda_{n-2}(\iota_*^{i-n+1},\ \iota_*^{j-n+1}))\\ &= \lambda_{n-1}(\tau_*(\iota_*^{i-n+1}),\ \tau_*(\iota_*^{j-n+1})) \qquad \text{(by (1.16))}\\ &= \lambda_{n-1}(\iota_*^{i-n},\ \iota_*^{j-n})\,.\end{aligned}$$

REMARK. This Proposition assures us that

$$\{Q^s(\alpha)\}\,(p=2),\ \{\lambda_{n-1}(\alpha,\,\beta)\}\in H^*(X)$$

are dual to

$$Q^s(\alpha_*)(p=2),\ \lambda_{n-1}(\alpha_*,\,\beta_*)\in H_*(X)$$

respectively.

Suppose $X=\Omega^n Y$ for $n\geq 1$. Let $\mu\colon X\times X\to X$ be the loop multiplication. Then

$$H^*(X)\xrightarrow{\mu^*} H^*(X\times X)\xleftarrow[\cong]{} H^*(X)\otimes H^*(X)$$

gives a coproduct in $H^*(X)$.

Corollary 12. *In the notations of Corollary* 4,

(1) $\mu^*(\theta) = \theta\otimes 1+1\otimes\theta$;

(2) $\mu^*(\alpha\cup\beta) = (\alpha\cup\beta)\otimes 1+\alpha\otimes\beta+(-1)^{|\alpha||\beta|}\beta\otimes\alpha+1\otimes(\alpha\cup\beta)$;

(3) $\mu^*(\{Q^s(\alpha)\}) = \begin{cases}\{Q^{|\alpha|}(\alpha)\}\otimes 1+\alpha\otimes\alpha+1\otimes\{Q^{|\alpha|}(\alpha)\} & \text{if } s=|\alpha|\\ \{Q^s(\alpha)\}\otimes 1+1\otimes\{Q^s(\alpha)\} & \text{if } s>|\alpha|\,;\end{cases}$

$$(4)\quad \mu^*(\{\lambda_{n-1}(\alpha,\beta)\}) = \begin{cases} \{\lambda_0(\alpha,\beta)\}\otimes 1-(-1)^{|\alpha||\beta|}\beta\otimes\alpha \\ \qquad\qquad +1\otimes\{\lambda_0(\alpha,\beta)\} & \text{if } n=1 \\ \{\lambda_{n-1}(\alpha,\beta)\}\otimes 1+1\otimes\{\lambda_{n-1}(\alpha,\beta)\} & \text{if } n>1\,. \end{cases}$$

Proof. (1) is a consequence of

(7.2) *Every element of* Im σ^* *is primitive.*

(See (3.3*) of [16, VIII].)

For (2), since α and β are primitive, the result follows.

Proposition 11 (i) and (1.5) imply that for $i>j$,

$$(7.3)\qquad \mu^*(\kappa^{i;j}) = \begin{cases} \kappa^{j+1;j}\otimes 1+\iota^j\otimes\iota^j+1\otimes\kappa^{j+1;j} & \text{if } i=j+1 \\ \kappa^{i;j}\otimes 1+1\otimes\kappa^{i;j} & \text{if } i>j+1\,. \end{cases}$$

So (3) follows from (6.5).

From (7.1) we deduce that

$$\langle\mu^*(\lambda^{1;i,j}),\ \iota_*^i\otimes\iota_*^j\rangle = \langle\lambda^{1;i,j},\ \iota_*^i*\iota_*^j\rangle = 0 \quad\text{and}$$
$$\langle\mu^*(\lambda^{1;i,j}),\ \iota_*^j\otimes\iota_*^i\rangle = \langle\lambda^{1;i,j},\ \iota_*^j*\iota_*^i\rangle = -(-1)^{ij}\,.$$

This, together with Proposition 11 (ii) and (1.12), implies that for $n\geq 1$,

$$\mu^*(\lambda^{n;i,j}) = \begin{cases} \lambda^{1;i,j}\otimes 1-(-1)^{ij}\iota^j\otimes\iota^i+1\otimes\lambda^{1;i,j} & \text{if } n=1 \\ \lambda^{n;i,j}\otimes 1+1\otimes\lambda^{n;i,j} & \text{if } n>1\,. \end{cases}$$

So (4) follows from (6.6).

Let $X=\Omega^n Y$. In certain situations the secondary operation problem in $H^*(Y)$ is equivalent to the primary operation problem in $H^*(X)$. We describe such situations by the following examples whose origin is [1, Addendum].

EXAMPLE 1. Throughout this example, coefficients will be Z_2. Let Φ be the secondary cohomology operation associated with the relation

$$Sq^1Sq^{2s+1}=0\,.$$

The universal example for Φ consists of pairs $(E_j,\ \phi_j)$, $j\geq 1$, where E_j is the total space of the fibration

$$K(Z_2, j+2s)\xrightarrow{\varepsilon_j} E_j\xrightarrow{\zeta_j} K(Z_2, j)$$

which is induced by $Sq^{2s+1}(\iota_j)$: $K(Z_2, j)\to K(Z_2, j+2s+1)$, i.e., $E_j=L(Z_2, 2s+1; j)$, and ϕ_j is an element of $H^{j+2s+1}(E_j)$ such that

(1) $(\sigma^*)^n(\phi_{j+n})=\phi_j$ for all n, in particular, ϕ_j is primitive (by (7.2));

(2) $\varepsilon_j^*(\phi_j)=Sq^1(\iota_{j+2s})$.

If $j<2s+1$, these conditions determine ϕ_j uniquely. In fact, from (7.3) and the definition of $\kappa^{2s+1;j}$ it follows that

$$(7.4)\qquad \phi_j = \begin{cases} Sq^1(\kappa^{2s+1;2s})+\iota^{2s}\cup Sq^1(\iota^{2s}) & \text{if } j=2s \\ Sq^1(\kappa^{2s+1;j}) & \text{if } j<2s\,. \end{cases}$$

Suppose that an element $\alpha\in H^{2s}(X)$ such that $Sq^{2s+1}({}^n\tilde{\alpha})=0$ is given. Then we can consider the element $Sq^1\{Q^{2s}(\alpha)\}\in H^{4s+1}(X)$. By using (1.7) we see that $\sigma^n(Sq^1\{Q^{2s}(\alpha)\})\in\operatorname{Ker}\nu_n^*$ if and only if $Sq^1(\alpha)=0$. Assume that $Sq^1(\alpha)=0$. Then

$$\begin{aligned} Sq^1\{Q^{2s}(\alpha)\} &= Sq^1(\Omega^{nn}\tilde{\alpha}^\wedge)^*(\kappa^{2s+1;2s}) \quad \text{(by (6.5))} \\ &= (\Omega^{nn}\tilde{\alpha}^\wedge)^*Sq^1(\kappa^{2s+1;2s}) \\ &= (\Omega^{nn}\tilde{\alpha}^\wedge)^*(\phi_{2s}+\iota^{2s}\cup Sq^1(\iota^{2s})) \quad \text{(by (7.4))} \\ &= (\Omega^{nn}\tilde{\alpha}^\wedge)^*(\phi_{2s})+\alpha\cup Sq^1(\alpha) \\ &= (\Omega^{nn}\tilde{\alpha}^\wedge)^*(\phi_{2s}) \\ &= (\Omega^{nn}\tilde{\alpha}^\wedge)^*(\sigma^*)^n(\phi_{n+2s}) \quad \text{(by (1))} \\ &= (\sigma^*)^n({}^n\tilde{\alpha}^\wedge)^*(\phi_{n+2s}) \\ &= (\sigma^*)^n\Phi({}^n\tilde{\alpha})\,. \end{aligned}$$

Thus $\Phi({}^n\tilde{\alpha})={}^n\tilde{\theta}$ if and only if $Sq^1\{Q^{2s}(\alpha)\}=\theta$.

EXAMPLE 2. Throughout this example, coefficients will be Z_3. Let Φ be the secondary cohomology operation associated with the relation

$$-\mathfrak{P}^2\Delta^*+\mathfrak{P}^1(\Delta^*\mathfrak{P}^1)-\Delta^*\mathfrak{P}^2=0\,.$$

The universal example for Φ consists of pairs $(E_j,\ \phi_j)$, $j\geq 1$, where E_j is the total space of the fibration

$$K(Z_3,j)\times K(Z_3,j+4)\times K(Z_3,j+7)\xrightarrow{\ \varepsilon_j\ } E_j\xrightarrow{\ \zeta_j\ } K(Z_3,j)$$

which is induced by $(\Delta^*(\iota_j),\ \Delta^*\mathfrak{P}^1(\iota_j),\ \mathfrak{P}^2(\iota_j))$: $K(Z_3,j)\to K(Z_3,j+1)\times K(Z_3,j+5)\times K(Z_3,j+8)$ (so $\Omega^nE_{j+n}\simeq E_j$), and ϕ_j is an element of $H^{j+8}(E_j)$ such that

(1) $(\sigma^*)^n(\phi_{j+n})=\phi_j$ for all n, in particular, ϕ_j is primitive;

(2) $\varepsilon_j^*(\phi_j)=-\mathfrak{P}^2(\iota_j)\times 1\times 1+1\times\mathfrak{P}^1(\iota_{j+4})\times 1-1\times 1\times\Delta^*(\iota_{j+7})$.

Put $\alpha_j=\zeta_j^*(\iota_j)$. Then $(\sigma^*)^n(\alpha_{j+n})=\alpha_j$ for all n.

Consider the case $j=2$. Since $\Delta^*\mathfrak{P}^1(\iota_2)=0$ and $\mathfrak{P}^2(\iota_2)=0$ in $H^*(K(Z_3,2))$, it follows that

$$(7.5)\qquad E_2\simeq K(Z_9,2)\times K(Z_3,6)\times K(Z_3,9)\,.$$

Let $\beta_6\in H^6(E_2)$ (resp. $\gamma_9\in H^9(E_2)$) be the element such that $\varepsilon_2^*(\beta_6)=1\times\iota_6\times 1$ (resp. $\varepsilon_2^*(\gamma_9)=1\times 1\times\iota_9$). Apply (10) of Theorem 7 to the case that $Y=E_3$, $n=1$

(so $X=E_2$ and $m=2$), $\alpha=\alpha_2$ and $s=1$; then $\tau(\sigma[Q^1(\alpha_2)])=\Delta^*\mathfrak{P}^1(\alpha_3)$, which is equal to zero by the definition of E_3. Thus we get an element $\{Q^1(\alpha_2)\}$ of $H^6(E_2)$. In view of (7.5), we find that $\beta_6=\{Q^1(\alpha_2)\}$ (up to a sign).

Consider the mod 3 cohomology spectral sequence $\{E_r, d_r\}$ of the path fibration

$$E_2 \to PE_3 \to E_3 .$$

Then $\tau(\alpha_2)=\alpha_3$. So, by the Kudo transgression theorem [7], $d_5(\alpha_3\otimes(\alpha_2\cup\alpha_2)) = -\Delta^*\mathfrak{P}^1(\alpha_3)\otimes 1=0$. Since $\tilde{H}^*(PE_3)=0$, $\alpha_3\otimes(\alpha_2\cup\alpha_2)$ must be in the image of d_3. By (7.5), $H^6(E_2)=Z_3\{\mathfrak{P}^1(\alpha_2), \beta_6\}$ and $\mathfrak{P}^1(\alpha_2)$ is transgressive. Hence the only remaining possibility is $d_3(1\otimes\beta_6)=\alpha_3\otimes(\alpha_2\cup\alpha_2)$. This implies that $Q^1(\alpha_{2*})=\beta_{6*}$ or equivalently,

$$\mu^*(\beta_6) = \beta_6\otimes 1-(\alpha_2\cup\alpha_2)\otimes\alpha_2-\alpha_2\otimes(\alpha_2\cup\alpha_2)+1\otimes\beta_6 . \tag{7.6}$$

The conditions (1) and (2) determine ϕ_2 uniquely. In fact, by using (7.5) and (7.6), we see that

$$PH^{10}(E_2) = Z_3\{\alpha_2^{\cup 5}-\mathfrak{P}^1(\beta_6), \Delta^*(\gamma_9)\}$$

(where P denotes the primitive module functor), and so

$$\phi_2 = -\alpha_2^{\cup 5}+\mathfrak{P}^1(\beta_6)-\Delta^*(\gamma_9) . \tag{7.7}$$

Let G_2 be the compact exceptional Lie group of rank 2. As is well known,

(7.8) $\quad H^*(G_2) = \Lambda(y_3, y_{11})$ *where* $|y_i| = i$.

(7.9) *In dimensions* ≤ 10, $H^*(\Omega G_2)=Z_3[x_2]/(x_2^{\cup 3})\otimes Z_3[x_6, x_{10}]$ *where* $|x_i|=i$.

(7.10) $\quad \sigma^*(y_3) = x_2$ *and* $\sigma^*(y_{11}) = x_{10}$.

Applying Theorem 7 to the case that $Y=G_2$ and $n=1$, we find that $x_6=\{Q^1(x_2)\}$. By (7.8) (resp. (7.9)), the map y_3: $G_2\to K(Z_3, 3)$ (resp. x_2: $\Omega G_2\to K(Z_3, 2)$) can be lifted to a map $\hat{y_3}$: $G_2\to E_3$ (resp. $\hat{x_2}$: $\Omega G_2\to E_2$). Furthermore, by (7.10) we may suppose that $\sigma^*(\hat{y_3})=\hat{x_2}$. Then we have the commutative diagram (5.1) for the case that $g=\hat{y_3}$ and $n=1$, and it follows that

$$\hat{x_2}^*(\alpha_2) = x_2,\ \hat{x_2}^*(\beta_6) = x_6 \quad\text{and}\quad \hat{x_2}^*(\gamma_9) = 0 .$$

Hence

$$\begin{aligned}
\mathfrak{P}^1(x_6) &= \mathfrak{P}^1\hat{x_2}^*(\beta_6) \\
&= \hat{x_2}^*\mathfrak{P}^1(\beta_6) \\
&= \hat{x_2}^*(\phi_2+\alpha_2^{\cup 5}+\Delta^*(\gamma_9)) \quad \text{(by (7.7))} \\
&= \hat{x_2}^*(\phi_2)+x_2^{\cup 5} \\
&= \hat{x_2}^*(\phi_2)
\end{aligned}$$

$$
\begin{aligned}
&= x_2^{\wedge *}\sigma^*(\phi_3) \quad \text{(by (1))} \\
&= \sigma^* y_3^{\wedge *}(\phi_3) \\
&= \sigma^*\Phi(y_3)\,.
\end{aligned}
$$

Thus $\mathfrak{P}^1(x_6)=x_{10}$ is equivalent to $\Phi(y_3)=y_{11}$.

Theorem 7 is applicable to the special case that $Y=G_nX$ and $X=F_nX$. In this case $H^*(F_nX)$ is to be known; it suffices to use (1.17) and Lemma 2. So, since F_nX is $(2m-1)$-connected, by using Theorem 3 (resp. Theorem 7), at least the additive structure of $\tilde{H}^*(G_nX)$ in dimensions $<6m+n-1$ (resp. $8m+n-1$) ought to be known. We conjecture that, on the Z_p-basis obtained as above, there are formulas for the differentials of the spectral sequence (3.8); that is, Theorem 7 will be extended.

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