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ON KRULL-SCHMIDT'S THEOREM AND THE INDECOMPOSABILITY OF AMALGAMATED SUM

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Let R be a ring and M an R-module. If M has direct decompositions $M=\bigoplus_{i\in I}L_i=\bigoplus_{j\in J}M_j$ into completely indecomposable R-modules L_i and M_j. Then, by Krull-Remak-Schmidt-Azumaya's theorem ([1]), it holds that for any finite subset J' of J, there exists a finite subset I' of I such that (1) $M=\bigoplus_{i\in I'}L_i\oplus(\bigoplus_{j\in J-J'}M_j)$. Then however it is not necessarily satisfied (2) $M=\bigoplus_{j\in J'}M_j\oplus(\bigoplus_{i\in I-I'}L_i)$. In § 1, we shall show that (1) and (2) hold simultaneously for suitable subset I' of I. The assertion is first showed in case M is semi-simple (in any completely reducible Grothendieck category). Then it is valid in general case, using the method of Harada and Sai [3, Corollary 1, p. 334]. But when the index set I is finite, we give an elementary proof for it.

Next, we consider finitely generated indecomposable modules over right artinian rings. In [4] and [5], Tachikawa investigated algebras of right local type (i.e. every indecomposable right module has the simple top) and of local or colocal type. To prove his assertions, he constructed indecomposable modules which were obtained by amalgamated sums (they were called interlacings there). In § 2, we shall slightly generalize his method, and give sufficient conditions for amalgamated sums to be indecomposable.

The authors wish to express their appreciation to Professor M. Harada and Mr. T. Inoue. The former suggested them that Theorem 1.3 (Theorem 1.7) is obtained from Lemma 1.1 (Lemma 1.6) using the method in [3], and the latter simplified their proof of Lemma 1.1 by his own method. As its proof, we take his own.

Throughout this note, R denotes a ring with unity and R-modules are (unital) right R-modules unless otherwise stated. For R-modules $L_i (i\in I)$, we use a notation $\bigoplus_I L_i$ instead of $\bigoplus_{i\in I}L_i$. If f: $L\to M$ is a homomorphism and L' is a submodule of L, we also denote the restriction map to L' of f: $L\to M$ by f: $L'\to M$. Let I be a set and I_j a subset of I for each $j=1, \cdots, n$.

If $I=I_1\cup\cdots\cup I_n$ and $I_j\cap I_k=\phi$ for all j and $k(j\neq k)$, we say that the union $I_1\cup\cdots\cup I_n$ is a *partition* (of I), and denote it by $I=I_1\coprod\cdots\coprod I_n$.

1. Krull-Remak-Schmidt-Azumaya's theorem

In this section, we study a generalization of Krull-Remak-Schmidt-Azumaya's theorem. The following lemma is basic for our results.

Lemma 1.1. *Let $M=L_1\oplus\cdots\oplus L_n$ be a simple decomposition of a semi-simple R-module M. Then for any direct decomposition $M=M_1\oplus M_2$, there exists a partition $\{1,\cdots,n\}=I_1\coprod I_2$ such that*

$$M=(\oplus_{I_1}L_i)\oplus M_2=M_1\oplus(\oplus_{I_2}L_i)\,.$$

Proof. We prove the assertion by induction on n. Let L_1' and L_1'' denote the images of L_1 under the projections $M\to M_1$ and $M\to M_2$, respectively. Then it holds either $M=L_1'\oplus L_2\oplus\cdots\oplus L_n$ or $M=L_1''\oplus L_2\oplus\cdots\oplus L_n$. We may assume $M=L_1'\oplus L_2\oplus\cdots\oplus L_n$. Under the canonical homomorphism $M\to M/L_1'$, we denote the image of N by $\bar{N}$ for every submodule N of M. Then $\bar{M}=\bar{L}_2\oplus\cdots\oplus\bar{L}_n=\bar{M}_1\oplus\bar{M}_2$. By inductional assumption, there exists a partition $\{2,\cdots,n\}=I_1'\coprod I_2'$ such that $\bar{M}=(\oplus_{I_1'}\bar{L}_i)\oplus\bar{M}_2=\bar{M}_1\oplus(_{I_2'}\bar{L}_i)$. Hence we have $M=L_1'+(\oplus_{I_1'}L_i)+M_2=M_1+(\oplus_{I_2'}L_i)$. Comparing the composition lengths of the three terms, we see the sums are direct. Then it follows $M=L_1\oplus(\oplus_{I_1'}L_i)\oplus M_2$ and hence our assertion is satisfied for $I_1=\{1\}\cup I_1'$ and $I_2=I_2'$.

From Lemma 1.1, we can obtain Theorem 1.3 using the argument in the proof of Harada and Sai [3, Corollary 1, p. 334]. We shall however give an elementary proof of it. (See [3, Lemma 2] for the following Lemma 1.2 and Remark.)

Lemma 1.2. *Let e and f be idempotents of a ring R. If $\bar{R}=\bar{e}\bar{R}\oplus\bar{f}\bar{R}$, then we have $R=eR\oplus fR$, where $\bar{R}=R/J$, J the Jacobson radical of R and $\bar{e}=e+J$, $\bar{f}=f+J\in R/J$.*

Proof. Assume $\bar{R}=\bar{e}\bar{R}\oplus\bar{f}\bar{R}$. Then $R=eR+fR$ since J_R is small in R_R. Hence $(1-f)R=(1-f)eR$, which implies that a left multiplication map $eR\to(1-f)R$ $(ea\mapsto(1-f)ea)$ by $(1-f)$ is an epimorphism. Since $(1-f)R$ is projective, we have a split exact sequence

$$0\to eR\cap fR\to eR\to(1-f)R\to 0\,.$$

By the assumption, however, $eR\cap fR\subset eJ$ and hence $eR\cap fR$ is small in eR. Therefore $eR\cap fR=0$ and so $R=eR\oplus fR$.

Remark. Lemma 1.2 holds more generally. Let $e_1,\cdots,e_n$ be idempotents of R. Using the notation of Lemma 1.2, if $\bar{R}=\bar{e}_1\bar{R}\oplus\cdots\oplus\bar{e}_n\bar{R}$, then $R=e_1R\oplus\cdots\oplus e_nR$. In fact, put $P_i=e_iR$ and consider an external direct sum $\bigoplus_{i=1}^{n}P_i$. Then we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^{n} P_i & \xrightarrow{\tau} & \bigoplus_{i=1}^{n} \bar{e}_i\bar{R} \\ \big\downarrow \rho & & \Big\| \\ R & \xrightarrow{\sigma} & \bar{R} \end{array}$$

with canonical maps. Since σ and τ are the projective covers of $\bar{R}=\bigoplus_{i=1}^{n}\bar{e}_i\bar{R}$, ρ is an isomorphism. This shows $R=e_1R\oplus\cdots\oplus e_nR$.

Recall an R-module M is *completely indecomposable* provided its endmorphism ring $\mathrm{End}_R(M)$ is a local ring. A direct decomposition $M=\oplus_I L_i$ is called a *completely indecomposable decomposition* if L_i is completely indecomposable for each $i\in I$.

Theorem 1.3. *Let $M=L_1\oplus\cdots\oplus L_n$ be a completely indecomposable decomposition of an R-module M. Then for any direct decomposition $M=M_1\oplus M_2$ there exists a partition $\{1, \cdots, n\}=I_1\coprod I_2$ such that*

$$M=(\oplus_{I_1} L_i)\oplus M_2=M_1\oplus(\oplus_{I_2} L_i)\,.$$

Proof. Let e_i denote the composition map of a projection $M\to L_i$ and an injection $L_i\to M$, and g_i also the composition map $M\to M_j\to M$ of canonical maps. Put $S=\mathrm{End}_R(M)$ and $\bar{S}=S/J(S)$, and denote $x+J(S)\in S/J(S)$ by $\bar{x}$ for $x\in S$, where $J(S)$ is the Jacobson radical of S. Then, since S is a semiperfect ring and $e_1, \cdots, e_n$ are mutually orthogonal primitive idempotents, and g_1, g_2 are orthogonal idempotents, we have $\bar{S}=\bar{e}_1\bar{S}\oplus\cdots\oplus\bar{e}_n\bar{S}=\bar{g}_1\bar{S}\oplus\bar{g}_2\bar{S}$ and each $\bar{e}_i\bar{S}$ is a simple $\bar{S}$-module. Therefore by Lemma 1.1 there exists a partition $\{1, \cdots, n\}=I_1\coprod I_2$ such that $\bar{S}=\bar{f}_1\bar{S}\oplus\bar{g}_2\bar{S}=\bar{g}_1\bar{S}\oplus\bar{f}_2\bar{S}$, where $f_1=\Sigma_{I_1}e_i$. $f_2=\Sigma_{I_2}e_i$. Hence by Lemma 1.2, $S=f_1S\oplus g_2S=g_1S\oplus f_2S$. Then it is easy to see that $M=(\oplus_{I_1} L_i)\oplus M_2=M_1\oplus(\oplus_{I_2} L_i)$.

Let $M=M_1\oplus M_2$ be a decomposition and let g_1: $M\to M_1$ denote the projection. For a submodule L of M the restriction map g_1: $L\to M_1$ is isomorphic if and only if $M=L\oplus M_2$. Therefore putting Theorem 1.3 in this way, we have

Lemma 1.4 *Let $L=\bigoplus_{i=1}^{n}L_i$ and $M=M_1\oplus M_2$ be decompositions of R-modules such that each L_i is completely indecomposable, and let g_j: $M\to M_j$ denote the projection. If f: $L\to M$ is an isomorphism, then there exists a partition $\{1, \cdots, n\}=I_1\coprod I_2$ such that restriction map g_jf: $\oplus_{I_j}L_i\to M_j$ is isomorphic for each $j=1, 2$.*

The following corollary is a generalization of Krull-Remak-Schmidt's theorem.

Corollary 1.5. *Let* $M=\overset{n}{\underset{i=1}{\oplus}}L_i=\overset{r}{\underset{j=1}{\oplus}}M_j$ *be direct decompositions of an* R-*module* M *with completely indecomposable modules* L_i. *Then there exists a partition* $\{1, \cdots, n\}=I_1 \amalg \cdots \amalg I_r$ *such that the induced map* $g_j\colon N_j \rightarrow M_j$ *is isomorphic and* $M=M_1\oplus\cdots\oplus M_j\oplus N_{j+1}\oplus\cdots\oplus N_r$ *for each* $j=1, \cdots, r$, *where* $N_j=\oplus_{I_j}L_i$ *and* $g_j\colon M\rightarrow M_j$ *is a projection.*

Proof. Regard $\overset{r}{\underset{j=k}{\oplus}}M_j$ as $M_k\oplus(\overset{r}{\underset{j=k+1}{\oplus}}M_j)$, $k=1, \cdots, r-1$. Then we get the assertion applying Lemma 1.4 inductively from case $k=1$ and f is the identity map.

Lemma 1.6. *Let* $M=\oplus_I L_i$ *be a simple decomposition of a semi-simple* R-*module* M. *Then for any direct decomposition* $M=M_1\oplus M_2$ *where* M_1 *has a finite composition length, there exists a partition* $I=I_1\amalg I_2$ *such that* $M=(\oplus_{I_1}L_i)\oplus M_2=M_1\oplus(\oplus_{I_2}L_i)$.

Proof. Since $M_1\subset\oplus_I L_i$, we have $M_1\subset\oplus_{I'}L_i$ for some finite subset I' of I. Put $M'=\oplus_{I'}L_i$ and $I''=I-I'$. Since $M_1\subset M'$, $M'=M_1\oplus M_2'$ ($M_2'=M'\cap M_2$) and $M_2=M_2'\oplus M_2''$ for some submodules M_2' and M_2'' of M_2. It is clear $M=M'\oplus M_2''=M'\oplus(\oplus_{I''}L_i)$. Applying Lemma 1.1 to $M'=\oplus_{I'}L_i=M_1\oplus M_2'$, there exists a partition $I'=I_1\amalg I_2'$ such that $M'=\oplus_{I_1}L_i\oplus M_2'=M_1\oplus(\oplus_{I_2'}L_i)$. Thus, for $I_2=I_2'\cup I''$ it holds $I=I_1\amalg I_2$ and $M=(\oplus_{I_1}L_i)\oplus M_2=M_1\oplus(\oplus_{I_2}L_i)$.

Theorem 1.7. *Let* $M=\oplus_I L_i=\oplus_J M_j$ *be completely indecomposable decompositions of an* R-*module* M. *Then for any finite subset* $J'=\{j_1, \cdots, j_n\}$ *of* J, *there exists a subset* $I'=\{i_1, \cdots, i_n\}$ *of* I *such that* $L_{i_k}\cong M_{j_k}$ *for each* $k=1, \cdots, n$ *and*

$$M=(\oplus_{I'}L_i)\oplus(\oplus_{J-J'}M_j)=(\oplus_{J'}M_j)\oplus(\oplus_{I-I'}L_i).$$

Proof. We see easily that the proof of Lemma 1.6 is valid in any completely reducible and Grothendieck category. Hence by the method of Harada and Sai [3, Corollary 1, p. 334], the assertion holds.

EXAMPLE. Let $M=\overset{n}{\underset{i=1}{\oplus}}L_i=\oplus_J M_j$ be completely indecomposable decompositions of M and $J=J_1\amalg J_2$ a partition of J. Then by Krull-Remak-Schmidt's theorem, for some subset I_1 of $I=\{1, \cdots, n\}$, $M=(\oplus_{I_1}L_i)\oplus(\oplus_{J_2}M_j)$. But it is not necessarily satisfied that for $I_2=I-I_1$, $M=(\oplus_{J_1}M_j)\oplus(\oplus_{I_2}L_i)$.

Let R be a field and M a vector space with dimension 3 over R. That is M $(=R^3)=\{(a_1, a_2, a_3)^t\,|\,a_i\in R\}$, where $(a_1, a_2, a_3)^t$ expresses the transposed matrix of (a_1, a_2, a_3). Put

$$v_1=(1,1,1)^t,\quad v_2=(1,1,0)^t,\quad v_3=(1,0,1)^t,$$
$$u_1=(1,0,0)^t,\quad u_2=(0,1,0)^t,\quad u_3=(0,0,1)^t;$$

and $L_i=v_iR$, $M_1=u_iR$, $i=1, 2, 3$. Then $M=L_1\oplus L_2\oplus L_3=(M_1\oplus M_2)\oplus M_3$ and moreover

$$\begin{aligned} M &= L_1\oplus M_2\oplus M_3 = M_1\oplus L_2\oplus L_3 \\ &= M_1\oplus L_2\oplus M_3=(M_1\oplus M_2)\oplus L_3\,. \end{aligned}$$

On the other hand, $(L_1\oplus L_2)+M_3=(u_1+u_2)R\oplus u_3R(\neq M)$.

2. Indecomposability of amalgamated sums

In this section, we assume a ring R is always right artinian and R-modules mean finitely generated right R-modules except for Proposition 2.2.

Let (E) $0\to K\xrightarrow{\alpha} L\xrightarrow{\beta} M\to 0$ be an exact sequence of R-modules. We consider the following condition:

(*) If X is a (non-zero) R-module and $\varphi\colon L\to X$ is a retraction (i.e. split epimorphism) then there is no homomorphism $\psi\colon M\to X$ such that $\varphi=\psi\beta$.

If an exact sequence (E) satisfies the condition (*), we say that (E) is a (*)-*sequence*. Consider the following commutative diagrams of R-modules such that σ is a retraction and τ' is a section (i.e. split monomorphism):

$$\text{(D)}\quad \begin{tikzcd} X \arrow[rr, "\rho"] \arrow[dr, "\sigma"'] & & Y \arrow[dl, "\tau"] \\ & Z & \end{tikzcd} \qquad \text{(D}'\text{)}\quad \begin{tikzcd} X \arrow[rr, "\rho"] & & Y \\ & Z \arrow[ul, "\sigma'"] \arrow[ur, "\tau'"'] & \end{tikzcd}.$$

In (D), let σ' be a right inverse of σ and put $\tau'=\rho\sigma'$. Then τ' is a right inverse of τ, and so we get the diagram (D$'$). Conversely, we can get (D) from (D$'$) and hence the condition (*) is equivalent to the following condition.

(*)$'$ If X is a (non-zero) R-module and $\psi'\colon X\to M$ is a section, then there is no homomorphism $\varphi'\colon X\to L$ such that $\psi'=\beta\varphi'$.

REMARK 1. It is easy to see that (E) is a (*)-sequence if M has no direct summand which is isomorphic to a direct summand of L. In particular, if (E) does not split and M is indecomposable, then (E) is a (*)-sequence.

2. For an exact sequence (E), we can consider the following dual condition (**) of (*), and show the duals of all the results in Section 2 except for Propositions 2.2 and 2.3.

(**) If X_1 is a (non-zero) R-module and $\varphi_1\colon X_1\to L$ is a section, then there is no homomorphism $\psi_1\colon X_1\to K$ with $\varphi_1=\alpha\psi_1$.

Let $L=\bigoplus_{i=1}^{n}L_i$ be a direct decomposition of L into indecomposable modules

L_i, $1 \leq i \leq n$. Put $S = \mathrm{End}_R(L)$ and denote by $J(S)$ the Jacobson radical of S. Then every element φ in S is expressed by a matrix with coefficients $\varphi_{ij}: L_j \to L_i$; $\varphi = (\varphi_{ij})$. As is well known, $\varphi \in J(S)$ if and only if each φ_{ij} is not an isomorphism $1 \leq i, j \leq n$. Moreover, let $\alpha = (\alpha_1, \cdots, \alpha_n)^t: K \to \bigoplus_{i=1}^{n} L_i$ be a matrix expression of $\alpha: K \to L$. Then the following proposition is immediate.

Proposition 2.1. *Let* (E) $0 \to K \overset{\alpha}{\to} L \overset{\beta}{\to} M \to 0$ *be an exact sequence and* $L = \bigoplus_{i=1}^{n} L_i$ *a decomposition with indecomposable modules* L_i, $1 \leq i \leq n$. *Then the following conditions are equivalent.*

(*a*) (E) *is a* (*)*-sequence.*

(*b*) $\{\varphi \in S \mid \varphi\alpha = 0\} \subset J(S)$.

(*c*) *For* $\varphi = (\varphi_{ij})$ *in* S, *each* φ_{ij} *is not isomorphic, whenever* $\varphi_{i1}\alpha_1 + \cdots + \varphi_{in}\alpha_n = 0$ $(i = 1, \cdots, n)$.

REMARK 3. Monomorphisms α with the property (b) in Proposition 2.1 were investigated by Dickson and Kelly [2].

Let e and f be primitive idempotents of R and u_i an element of fJe such that $u_i \notin fJ^2e$ for $i = 1, \cdots, n$, where J is the Jacobson radical of R. If an element m of a module M and N is a submodule of M, a notation "$\bar{m} \in M/N$" means $\bar{m} = m + N$ $(\in M/N)$. Then for $\bar{u}_i \in Je/J^2e$, $R\bar{u}_i \simeq Rf/Jf$ (simple). Moreover, put $L_i = fR/fJ^2$, $i = 1, \cdots, n$ and $K = eR/eJ$. Consider (external) direct sum $L = \bigoplus_{i=1}^{n} L_i$ and define a map $\alpha: K \to L$ by $\alpha(\overline{ea}) = \sum_{i=1}^{n} \overline{u_i ea}$, where $\overline{ea} \in eR/eJ = K$ and $\overline{u_i ea} \in fR/fJ^2 = L_i$. Now put $M = \mathrm{Coker}\,\alpha$. Then the following proposition is immediate from the definition of (*)-sequences. (See [5, Propositions 3.3 and 3.5] for Propositions 2.2 and 2.3.)

Proposition 2.2. *Under the above notation, the following conditions are equivalent.*

(*a*) *For* $\bar{u}_i \in Je/J^2e$, $1 \leq i \leq n$, *we have* $R\bar{u}_1 \oplus \cdots \oplus R\bar{u}_n \subset Je/J^2e$.

(*b*) *For* $\bar{u}_i \in fJe/fJ^2e$, $1 \leq i \leq n$, $\bar{u}_1, \cdots, \bar{u}_n$ *are linearly independent over a division ring* fRf/fJf *(considering* fJe/fJ^2e *as a left mldule).*

(*c*) *The exact sequence* $0 \to K \to L \to M \to 0$ *is a* (*)*-sequence.*

Next, let $L_1, \cdots, L_r$ be indecomposable R-modules such that every homomorphism $\varphi: L_i \to L_j$ vanishes the socle of L_i for each pair i and j $(i \neq j)$, and let K be a simple R-module. Put $N_i = L_i^{(k_i)}$ (k_i-times direct sum of copies of L_i) and $L = \bigoplus_{i=1}^{r} N_i$. Let $\alpha_i': K \to N_i (1 \leq i \leq r)$, and $\alpha: K \to L$ be maps with $\alpha = (\alpha_1', \cdots, \alpha_r')^t$, and put $M_i = \mathrm{Coker}\,\alpha_i'$ and $M = \mathrm{Coker}\,\alpha$. Then the following proposition is immediate.

Proposition 2.3. *Under the above notation, the following conditions are equivalent.*

(*a*) *The exact sequence* $0 \to K \to N_i \to M_i \to 0$ *is a* (*)-*sequence for each* $i = 1, \cdots, r$.

(*b*) *The exact sequence* $0 \to K \to L \to M \to 0$ *is a* (*)-*sequence.*

Let M be an R-module. Recall that M is *local* (resp. *colocal*) if M has a unique maximal (resp. minimal) submodule. We denote the composition length of M by $|M|$.

Lemma 2.4. *Let* $L = \bigoplus_{i=1}^{n} L_i$ *and* $M = M_1 \oplus M_2$ *be decompositions of* R-*modules* L *and* M, *where* L_i'*s are local* (*resp. colocal*), *and* $\pi_j\colon M \to M_j$ *a projection for* $j=1, 2$. *If* $\beta\colon L \to M$ *is an epimorphism* (*resp. monomorphism*), *then there exists a partition* $\{1, \cdots, n\} = I_1 \coprod I_2$ *such that each* $\pi_j \beta\colon L \to M_j$ *induces an epimorphism* (*resp. monomorphism*) $\pi_j \beta\colon \oplus_{I_j} L_i \to M_j$ *for* $j=1, 2$.

Proof. We shall only show the assertion in case L_i's are local, because we can similarly do it in the other case. Let $\bar{M}$ denote the top M/MJ of M and put $\bar{N} = \sigma(N)$ for every submodule N of M, where σ is the canonical homomorphism $M \to \bar{M}$. Then for some subset I' of $\{1, \cdots, n\}$, we have $\bar{M} = \oplus_{I'} \overline{\beta(L_i)} = \bar{M}_1 \oplus \bar{M}_2$. Using Lemma 1.1, as easily seen, there exists a partition $I' = I_1' \coprod I_2'$ such that $M = \Sigma_{I_1'} \beta(L_i) + M_2 = \Sigma_{I_2'} \beta(L_i) + M_1$. Then the assertion is immediate from $\pi_1(M_2) = \pi_2(M_1) = 0$.

Theorem 2.5. *Let* (*E*) $0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$ *be an exact sequence of* R-*modules such that* $L = \bigoplus_{i=1}^{n} L_i$, L_i *is local but is not simple and* K *is simple. Then the following conditions are equivalent.*

(*a*) M *is indecomposable.*

(*b*) M *has no direct summand which is isomorphic to* L_i *for some* $i=1, \cdots, n$.

(*c*) *The exact sequence* (*E*) *is a* (*)-*sequence.*

Proof. We shall only prove that (*c*) implies (*a*), for the others are clear (see Remark 1). Assume M is decomposable, say $M = M_1 \oplus M_2$. By Lemma 2.4, there exists a partition $\{1, \cdots, n\} = I_1 \coprod I_2$ such that the restriction map $\psi_j\colon \oplus_{I_j} L_i \to M_j$ of $\pi_j \beta\colon \bigoplus_{i=1}^{n} L_i \to M_j$ is an epimorphism, $j=1, 2$. But $|L| = |M| + |K| = |M| + 1$. Hence ψ_j is an isomorphism for some j ($j=1$ or 2), say $j=1$. Put $\varphi = \psi_1^{-1} \pi_1 \beta$ and let $\kappa\colon \oplus_{I_1} L_i \to \bigoplus_{i=1}^{n} L_i$ denote the canonical monomorphism. Then $\varphi\kappa$ is clearly the identity map of $\oplus_{I_1} L_i$ and so (E) is not a (*)-sequence.

REMARK 4. Theorem 2.5 is essentially due to Tachikawa [5, Lemma 1.1] under Lemma 1.1. By Theorem 2.5 and Propositions 2.2 and 2.3, we can give simple proofs of Propositions 2.4, 3.1, 3.3, 3.5 in Tachikawa [4].

Proposition 2.6. *Let* $L=\bigoplus_{i=1}^{n} L_i$ *be an indecomposable decomposition and* (*E*) $0 \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0$ *be a* (*)-*sequence such that the n-th coordinate map* $\alpha_n\colon K \to L_n$ *of* α *is monomorphic. If* $L_1, \cdots, L_n$ *are colocal and Coker* α_n *is simple, then* M *is indecomposable.*

Proof. Put $L'=\bigoplus_{i=1}^{n-1} L_i$ and $K_n=\alpha_n(K)$. Consider the following diagram with injection κ' and projection π_n.

$$\begin{array}{ccccccccc} 0 & \to & K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \to & 0 \\ & & & & \| & & & & \\ 0 & \to & L' & \xrightarrow{\kappa'} & L & \xrightarrow{\pi_n} & L_n & \to & 0\,. \end{array}$$

Then, since α_n (i.e. $\pi_n\alpha$) is monomorphic, the restriction map $\beta'\colon L' \to M$ (i.e. $\beta\kappa'$) of $\beta\colon L \to M$ is also monomorphic, and we have an exact sequence $0 \to L' \to M \to N \to 0$, where $N=\text{Coker }\beta'$. It is easy to see that $\beta(L') \cap \beta(L_n)=\beta(K_n)$ and so $N=(\beta(L')+\beta(L_n))/\beta(L') \cong \beta(L_n)/\beta(K_n)$. Therefore N is simple (or zero) since L_n/K_n is simple by the assumption. Suppose M is decomposable, say $M=M_1 \oplus M_2$, and let $\pi_j\colon M \to M_j$ denote the projection, $j=1, 2$. Then by Lemma 2.4, there exists a partition $\{1, \cdots, n-1\}=I_1 \amalg I_2$ such that the restriction map $\psi_k\colon \bigoplus_{I_k} L_i \to M_k$ of $\pi_k\beta'\colon L' \to M$ is monomorphic, $k=1, 2$. Since $|M|=|L'|+|N| \leq |L'|+1$, we may assume the monomorphism $\psi_1\colon \bigoplus_{I_1} L_i \to M_1$ is isomorphic. Then by the same way as in the proof of Theorem 2.5, we see that (E) is not a (*)-sequence. This verifies the assertion.

Proposition 2.7. *Let* $L=\bigoplus_{i=1}^{n} L_i$ *and* (*E*) *be as in the above proposition. If* $L_1, \cdots, L_n$ *are local and colocal and* $|L_n|$ *divides* $|L_i|$ *for each* i, $(1 \leq i \leq n)$, *then* M *is indecomposable.*

Proof. Suppose M is decomposable, say $M=M_1 \oplus M_2$. Let $\pi_k\colon M \to M_k$ denote the projection, $k=1, 2$. As in the proof of Theorem 2.5, for some partition $\{1, \cdots, n\}=J_1 \amalg J_2$, the restriction map $\pi_k\beta\colon \bigoplus_{J_k} L_i \to M_k$ of $\pi_k\beta\colon L \to M_k$ is epimorphic for each $k=1, 2$. On the other hand, as in the proof of Proposition 2.6, for some partition $\{1, \cdots, n-1\}=I_1 \amalg I_2$, the restriction map $\pi_k\beta\colon \bigoplus_{I_k} L_i \to M_k$ of $\pi_k\beta\colon \bigoplus_{i=1}^{n-1} L_i \to M_k$ is monomorphic. Put $c_i=|L_i|$. Then the above fact implies $\sum_{I_k} c_i \leq |M_k| \leq \sum_{J_k} c_i$, and hence $(\sum_{I_k} c_i)/c_n \leq (\sum_{J_k} c_i)/c_n$ for $k=1, 2$. But c_n divides c_i for each $i=1, \cdots, n$ and $(\sum_{I_1} c_i)/c_n+(\sum_{I_2} c_i)/c_n+1=(\sum_{i=1}^{n} c_i)/c_n=(\sum_{J_1} c_i)/c_n+(\sum_{J_2} c_i)/c_n$. This shows $(\sum_{I_k} c_i)/c_n=(\sum_{J_k} c_i)/c_n$ for some $k=1$ or 2, say $k=1$. Hence $\sum_{I_1} c_i=|M_1|=\sum_{J_1} c_i$ and the monomorphism

$\pi_1\beta: \oplus_{I_1} L_i \to M_1$ is isomorphic. Thus (E) is not a (*)-sequence. This verifies the assertion.

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