

SHIFT WITH ORBIT BASIS AND REALIZATION OF ONE DIMENSIONAL MAPS

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0. Introduction

A special structure of the orbit space is known for typical examples of piecewise continuous maps of intervals to themselves, such as beta transformations among number theoretic transformations, and some unimodal linear maps among continuous maps of intervals. That structure is a kind of irreducible Markov property but not of finite type. The order of Markov property is finite locally but unbounded globally. In such typical examples, there appears a function of the form

$$(1) \quad D(t) = 1 - a_0 t - a_1 t^2 - \dots$$

with nonnegative integer coefficients a_n ([4,12]), which is the reciprocal of the Artin-Mazur zeta function, whose minimal nonnegative zero is given as e^{-h} in terms of the topological entropy h of the given system, and which can be interpreted as the Fredholm determinant of the Perron-Frobenius operator associated with the system ([12]).

We shall define a class of one sided shift transformations in Section 1, for the special cases of which the properties stated above are simultaneously verified. These special cases were introduced in a previous paper [11]. They are images of the towers (or sky-scrapers) over shifts as is stated in Theorem 1 in Section 2. The formula (1) is valid for these special cases called shifts with free orbit basis. There the coefficient a_n has the meaning that it is the number of words of length n in the orbit basis. And the power series $D(t)$ in the typical examples are never meromorphic unless they are rational functions as is shown in Remark 3 of Section 1.

The topological entropy of shifts with free orbit basis and the towers associated with them is computed in Theorem 2 of Section 3, after the definition of topological entropy for noncompact dynamical systems is given.

Next, it turns out that any piecewise continuous map f can be realized as a union of shifts with orbit basis (Theorem 3 stated in Section 4). Some concrete examples are given at the end of that section. Furthermore, we shall

define the "boundary" $\partial\bar{X}$ of the two sided sequence \bar{X} space realizing f and show that it is negligible if it is measured by topological entropy. The boundary comes from the gap between \bar{X} and the image of the associated tower in the two sided sequence space (Theorem 4 stated in Section 4). The boundary $\partial\bar{X}$ is something like the Martin boundary of Markov chain. The final section 5 is devoted to the proof of Theorems 3 and 4.

The results stated in Theorem 3 are obtained in [4, 9, 10] for beta transformations and in [7] for unimodal continuous maps by Shinada. Furthermore, the explicit form of the orbit basis is obtained from these works. For example, the realization of a beta transformation admits an orbit basis

$$(2) \quad B = \{(\zeta_0, \dots, \zeta_{n-1}, a); a = 0, 1, \dots, \zeta_n - 1, n \geq 1\}$$

where $\zeta = (\zeta_n)$ is the sequence called the expansion of one.

Another representation of the orbit basis (in our sense) is obtained in [5] for unimodal continuous maps of intervals. Finally, the author admits that the proof of a corollary to the main theorem in [10] was not complete because, as it was pointed out by Hofbauer [2], the boundary $\partial\bar{X}$ was not discussed there and its negligibility was considered to be obvious, which turned out not to be the case in closer scrutiny.

1. Definitions and notations

We are concerned with shifts which are not necessarily topological.

DEFINITION 1. Let A be a finite or countably infinite set and σ the *shift transformation* on $A^{\mathbb{N}} = \{x = (x_n)_{n \in \mathbb{N}}; x_n \in A\}$, i.e.

$$(1) \quad (\sigma x)_n = x_{n+1} \quad (n \in \mathbb{N}).$$

The shift transformation on $A^{\mathbb{Z}}$ will be denoted by $\bar{\sigma}$. A pair (X, σ) or $(\bar{X}, \bar{\sigma})$ consisting of a σ -invariant or $\bar{\sigma}$ -invariant set X or \bar{X} and the restriction of σ or $\bar{\sigma}$ to X or \bar{X} , denoted again by σ or $\bar{\sigma}$, is called (one-sided or two-sided) shift, respectively. If A is endowed with a topology and X (or \bar{X}) is compact, then it is called a *topological shift*.

The following notations are necessary. For a one or two sided shift (X, σ) ,

$$(2) \quad W_n(X) = \{(x_0, x_1, \dots, x_{n-1}); x = (x_m) \in X\} \quad (n\text{-word set}),$$

$$W(X) = \bigcup_{n=1}^{\infty} W_n(X) \quad (\text{word set of } X),$$

$$(3) \quad \text{Fix}(X, \sigma^n) = \{x \in X; \sigma^n x = x\},$$

$$\text{Per}_n(X, \sigma) = \text{Fix}(X, \sigma^n) \cap \left(\bigcup_{k < n} \text{Fix}(X, \sigma^k) \right)^c,$$

$$\text{Per}(X, \sigma) = \bigcup_{n=1}^{\infty} \text{Per}_n(X, \sigma).$$

For a word $u=(a_1, \dots, a_n) \in A^n$,

$$\begin{aligned}
 (4) \quad & \sigma_1(a_1, \dots, a_n) = (a_2, \dots, a_n), \\
 & |u| = n \quad (\text{length of } u), \\
 & [u] = \{x; (x_0, \dots, x_{n-1}) = u\} \quad (u\text{-cylinder set}).
 \end{aligned}$$

Here the notation $[u]$ will be used both in one sided and two sided cases. If $u=(a_1, \dots, a_n)$, $v=(b_1, \dots, b_m)$ and $x=(x_n)_{n \in \mathbb{N}}$,

$$\begin{aligned}
 (5) \quad & uv = (a_1, \dots, a_n, b_1, \dots, b_m), \\
 & ux = (a_1, \dots, a_n, x_0, x_1, \dots).
 \end{aligned}$$

For a subset W of the union $\bigcup_{n=1}^{\infty} A^n$,

$$\begin{aligned}
 (6) \quad & |W| = \text{the number of elements of } W \in \mathbb{N} \cup \{\infty\}, \\
 & [W] = \bigcup_{w \in W} [w],
 \end{aligned}$$

$$\begin{aligned}
 (7) \quad & M(W) = \{x = (x_n); (x_{i+n})_{n \geq 0} \in [W] \text{ for any } i\} \\
 & \hspace{15em} (\text{Markov hull of } W).
 \end{aligned}$$

DEFINITION 2. If a shift (X, σ) satisfies

$$(8) \quad X = M(W), \quad W = W_{p+1}(X),$$

then, (X, σ) is called a p -Markov shift. The minimal p that satisfies (8) will be called the *order* of (X, σ) and the set $W=W_{p+1}(X)$ is then called the *structure set* of (X, σ) .

If A is a discrete topological space, then Markov shifts satisfy the following properties:

- (a) If (X, σ) is Markov, then X is a closed shift invariant set.
- (b) For any topological shift (X, σ) ,

$$(9) \quad X = \bigcap_p M(W_p(X)), \quad X \subset M(W_{p+1}(X)) \subset M(W_p(X)) \quad (p \geq 0).$$

DEFINITION 3. For one or two sided shift (X, σ) , the following quantity is called the *word entropy* of (X, σ) :

$$(10) \quad \text{ent}(X, \sigma) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |W_n(X)| \in [0, \infty].$$

It is obvious that $\text{ent}(X, \sigma) = \lim \text{ent}(M(W_n(X)), \sigma)$. The ‘‘lim sup’’ in the right hand side of (10) can be replaced by ‘‘lim’’ and ‘‘inf’’ in virtue of the inclusion relation

$$(11) \quad W_{n+m}(X) \subset \{uv; u \in W_n(X), v \in W_m(X)\}.$$

DEFINITION 4. Let (X, σ) be a shift and $U: X \rightarrow (-\infty, +\infty]$. Assume

that

$$(12) \quad Q_n(X, U) = \sum_{x \in \text{Fix}(X, \sigma^n)} \exp\left[-\sum_{i=0}^{n-1} U(\sigma^i x)\right]$$

are well-defined. Then the following power series will be called the *D-function* of the pair (X, U) :

$$(13) \quad D_{X,U}(t) = \exp\left[-\sum_{n=1}^{\infty} \frac{t^n}{n} Q_n(X, U)\right].$$

The radius of convergence of $D_{X,U}(t)$ is given by $\exp[-P^0(X, U)]$, where

$$(14) \quad P^0(X, U) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log Q_n(X, U).$$

REMARK 1. The reciprocal $1/D_{X,0}(t)$ of the *D-function* for $(X, 0)$ (0 standing for the constant function with value 0) is called *Artin-Mazur zeta function* and the reciprocal $1/D_{X,U}(t)$ is called *Artin-Mazur-Ruelle zeta function*, (cf. [6]). But we shall call them as above for simplicity. They have their own meaning as the *Fredholm determinant* in certain cases ([12, 4]).

DEFINITION 5. For a given one sided shift (X, σ) , define

$$(15) \quad \bar{X} = \{x = (x_n)_{n \in \mathbb{Z}}; (x_{n+m})_{m \in \mathbb{N}} \in X \text{ for all } n \in \mathbb{Z}\}.$$

The two sided shift thus obtained is called the *natural extension* of (X, σ) to the two sided shift.

Let us denote the natural projection of $A^{\mathbb{Z}}$ to $A^{\mathbb{N}}$ by

$$(16) \quad \pi(x) = (x_n)_{n \in \mathbb{N}} \text{ if } x = (x_n)_{n \in \mathbb{Z}}.$$

Lemma 1. *Let $(\bar{X}, \bar{\sigma})$ be the natural extension of a one sided shift (X, σ) . Then the following statements are valid:*

- (i) $\pi(\bar{X}) = X$
- (ii) *If (X, σ) is a topological shift, then, so is $(\bar{X}, \bar{\sigma})$.*
- (iii) *The projection π maps $\text{Per}(\bar{X}, \bar{\sigma})$ bijectively to $\text{Per}(X, \sigma)$.*
- (iv) $\text{ent}(\bar{X}, \bar{\sigma}) = \text{ent}(X, \sigma)$.

Proof. Since $\bar{\sigma}\bar{X} = \bar{X}$, for each $x \in X$, one can find a sequence $x^k = (x_n^k)_{n \in \mathbb{N}}$, $k \in \mathbb{N}$, such that $\sigma x^k = x^{k-1}$ for $k \geq 1$ with $x^0 = x$. Define $\bar{x} = (\bar{x}_n)_{n \in \mathbb{Z}}$ by $\bar{x}_n = x_0^{-n}$ ($n < 0$) and $\bar{x}_n = x_n$ ($n \geq 0$). Then $\bar{x} \in \bar{X}$ and $\pi(\bar{x}) = x$. To prove (ii), let us introduce a map $j: A^{\mathbb{Z}} \ni \bar{x} \rightarrow (\pi(\bar{\sigma}^{-n}\bar{x}))_{n \in \mathbb{N}} \in (A^{\mathbb{N}})^{\mathbb{N}}$. Then j is an into homeomorphism and so the set \bar{X} is homeomorphic to the closed subset $j(\bar{X}) = \{(x^n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}; \sigma x^n = x^{n-1} (n \geq 1)\}$ of the compact set $X^{\mathbb{N}}$. Hence, \bar{X} is compact. Consequently, (ii) is proved. The other assertions are obvious.

The term "natural extension" will sometimes be used for extensions of

other quantities defined for a one sided shift to corresponding quantities for the natural extension.

For instance, if (X, σ) is given a Borel structure and μ is an invariant probability measure for (X, σ) , then there may be the natural extension $\bar{\mu}$ of μ as a shift invariant measure. In fact, the measure $\bar{\mu}$ is, if any, characterized by the conditions

$$(17) \quad \pi_*\bar{\mu} = \mu \text{ and } \bar{\sigma}_*\bar{\mu} = \bar{\mu}.$$

If (M, f) is an invertible dynamical system and h is a homomorphism of (M, f) to a one sided shift (X, σ) , then there is the natural extension $\bar{h}: (M, f) \rightarrow (\bar{X}, \bar{\sigma})$.

Now let us introduce the following notion.

DEFINITION 6. Let (X, σ) be a one sided shift over an alphabet set A and $r \in \mathbb{N}$. The shift (X, σ) is said to admit an orbit basis (B, V) of order r if the following five conditions are satisfied:

(a) B is a subset of the word set $W(X)$ of X and V is a subset of the product set B^{r+1} .

(b) Let

$$(18) \quad V_n = \{(b_1, \dots, b_n) \in B^n; b_1 \dots b_n \in W(X) \ (n \geq 1)\}.$$

Then, for $n > r$,

$$(19) \quad V_n = \{(b_1, \dots, b_n) \in B^n; (b_i, \dots, b_{i+r}) \in V \ (1 \leq i \leq n-r)\}.$$

(c) Each sequence x in X can be represented as

$$(20) \quad x = (\sigma^i b_0) b_1 b_2 \dots$$

for some $(b_n)_{n \geq 0} \in M(V)$ and some $0 \leq i \leq |b_0|$.

(d) Conversely, the sequence defined by (20) belongs to X whenever $(b_n)_{n \geq 0} \in M(V)$ and $0 \leq i < |b_0|$.

(e) If $(b_n)_{n \geq 0}, (b'_n)_{n \geq 0} \in M(V)$, $0 \leq i < |b_0|$ and

$$(21) \quad (\sigma^i b_0) b_1 b_2 \dots = b'_0 b'_1 b'_2 \dots,$$

then, $b_0 = b'_0$ when $i = 0$ and $|\sigma^i b_0| \geq |b'_0|$ when $i \neq 0$.

If $r = 0$, i.e. if $V = B$, then we shall call that a shift (X, σ) admits a free orbit basis B . If $r = 1$, then the orbit basis (B, V) is said to be simple Markov. The notion of free orbit basis is the only one that is essential as it will be shown in Lemma 2 below.

REMARK 2. The condition (e) is a sort of uniqueness condition for the

expression (20). The precise formulation will be given as Theorem 1 in terms of two sided shifts.

Let us give simple examples of shifts with free orbit basis. The proofs of these examples are trivial and so are omitted but they show the essence of the proof of Lemma 2 below.

Let us denote, for a given two sided shift $(\bar{X}, \bar{\sigma})$ and a word $u \in W(\bar{X})$,

$$(22) \quad \bar{X}(u) = \{x \in \bar{X}; (x_n, \dots, x_{n+|u|-1}) = u \text{ both for infinitely many } n \geq 0 \text{ and } n \leq 0\} .$$

EXAMPLE 1. Let $(\bar{X}, \bar{\sigma}) = (\bar{M}(W), \bar{\sigma})$ be a two sided 1-Markov shift with structure set W . Fix an alphabet a and denote $X(a) = \pi(\bar{X}(a))$. Then the shift $(X(a), \sigma)$ admits a free orbit basis

$$B = \{(a_0, \dots, a_n); n \geq 0, a_0 = a, (a_n, a) \in W, a_m \neq a \text{ and } (a_{m-1}, a_m) \in W (m = 1, \dots, n)\} .$$

EXAMPLE 2. Let $(\bar{X}, \bar{\sigma}) = (\bar{M}(W), \bar{\sigma})$ be an r -Markov shift with $r \geq 2$. $p \geq r$ and $u \in W_p(X)$. Put $X(u) = \pi(\bar{X}(u))$. Then the shift $(X(u), \sigma)$ admits a free orbit basis B , where B is the collection of words $(a_0, \dots, a_n) \in W(X)$ which satisfy the following three conditions:

- (a) $n \geq p$ and $(a_0, \dots, a_{p-1}) = u$.
- (b) $(a_m, \dots, a_{m+p-1}) \neq u \quad (m = 1, \dots, n)$.
- (c) $(a_m, \dots, a_{m+r}) \in W \quad (m = 0, 1, \dots, n)$.

Here we put $a_{n+i} = a_{i-1}$ for $i \geq 1$.

For a given two sided shift $(\bar{X}, \bar{\sigma})$, denote

$$(23) \quad \bar{X}_{\text{rec}} = \{x \in \bar{X}; x \in \bar{X}(u) \text{ for each } u \in W(\{\sigma^n x; n \in \mathbf{Z}\})\} .$$

Lemma 2. *Let (X, σ) be a shift with orbit basis (B, V) of order $r \geq 1$ and $(\bar{X}, \bar{\sigma})$ its natural extension. Put $X_{\text{rec}} = \pi(\bar{X}_{\text{rec}})$. Then (X_{rec}, σ) is the union of its subshifts (X_n, σ) which admit free orbit bases.*

Proof. It is obvious that

$$\bar{X}_{\text{rec}} = \bigcup_{u \in V_r} (\bar{X}_{\text{rec}})(u) .$$

Fix $(b_1^0, \dots, b_r^0) \in V_r$ and put $X(b_1^0, \dots, b_r^0) = \pi[(\bar{X}_{\text{rec}})(b_1^0, \dots, b_r^0)]$. Now let C be the collection of words (b_0, \dots, b_n) over B which satisfy the following three conditions with $b_{n+i} = b_i^0$ ($i > 0$):

- (a) $n \geq r$ and $(b_0, \dots, b_{r-1}) = (b_1^0, \dots, b_r^0)$.
- (b) $(b_m, \dots, b_{m+r-1}) \neq (b_1^0, \dots, b_r^0) \quad (m = 1, \dots, n)$.
- (c) $(b_m, \dots, b_{m+r}) \in V \quad (m = 0, 1, \dots, n)$.

Finally, put

$$B' = \{b_0 \cdots b_n; (b_0, \dots, b_n) \in C\} .$$

Then it is obvious that B' is a free orbit basis for the shift $(X(b_1^0, \dots, b_r^0), \sigma)$.

2. Tower over a shift

In this section, we shall characterize shifts with orbit basis in terms of tower construction. We shall consider shifts over alphabet sets A and B , and the shifts will be denoted by σ and τ , respectively.

DEFINITION 7. Let (Y, τ) be a shift over alphabet set B , and $\theta: Y \rightarrow N$ be a function. Define

$$(1) \quad Y^\theta = \{(y, n); y \in Y, n \in N \quad 0 \leq n < \theta(y)\}$$

and $\tau^\theta: Y^\theta \rightarrow Y^\theta$ by

$$(2) \quad \tau^\theta(y, n) = \begin{cases} (y, n+1) & \text{if } (y, n+1) \in Y^\theta \\ (\tau y, 0) & \text{otherwise .} \end{cases}$$

The pair (Y^θ, τ^θ) is called a *tower over (Y, τ) with the ceiling function θ* .

Let (X, σ) be a shift over alphabet set A which admits an orbit basis (B, V) . Put

$$(3) \quad Y = M(V) = \{y = (y_n)_{n \in Z} \in B^Z; (y_n, \dots, y_{n+r}) \in V \quad (n \in Z)\}$$

$$(4) \quad \theta(y) = |y_0| \quad \text{if } y = (y_n)$$

and define a map $i: Y^\theta \rightarrow A^Z$ as follows:

$$(5) \quad i(y, n) = \sigma^n i_0(y)$$

where

$$(6) \quad i_0(y)(n) = \begin{cases} y_k(n - \sum_{0 \leq j < k} |y_j|) & \text{if } 0 \leq n - \sum_{0 \leq j < k} |y_j| < |y_k| \\ & \text{and } n \geq 0 \\ y_i(n + \sum_{0 > j \geq k} |y_j|) & \text{if } 0 > n + \sum_{0 > j \geq k} |y_j| > -|y_k| \\ & \text{and } n < 0 \end{cases}$$

From now on, we shall assume that the alphabet set A is a Hausdorff space and introduce a topology on the set $\cup A^n$ which is given by

$$\bigcup_{n=1}^N U_n \cup \bigcup_{n > N} A^n \quad \text{with } U_n \subset A^n \text{ open and } N \geq 1$$

as the basis of open sets. Its subset B is always given the relative topology. The shifts over A or B are given the relative topology from the product topol-

ogy. Finally, the tower Y^θ is given the relative topology from the product topology on $Y \times N$, where the set N of nonnegative integers is given the usual topology (so that $N \cup \{\infty\}$ is compact).

Theorem 1. *Let (X, σ) be a one sided shift over alphabet set A which admits an orbit basis (B, V) , $(\bar{X}, \bar{\sigma})$ its natural extension to the two sided shift, Y and Y^θ the spaces defined by (3) and (1) with (4), respectively, and i the map given by (6) and (5). Then the map i is a homeomorphism of Y^θ into \bar{X} such that*

$$(7) \quad (\pi \circ i)(Y^\theta) = X \text{ and } i \circ \tau^\theta = \bar{\sigma} \circ i.$$

REMARK 1. The map i is not surjective in general. For instance, in Example 1 in the previous section, it is immediate to see that

$$(8) \quad i(Y^\theta) = \{x \in \bar{X}(\bar{a}); x_n = a \text{ for infinitely many } n \leq 0\}$$

while the natural extension $\bar{X}(\bar{a})$ contains sequences x such that $x_n \neq a$ for all $n \leq 0$ but a finite number of n 's. The set

$$\partial \bar{X} = \bar{X} \cap i(Y^\theta)^c$$

will be referred to as the boundary of \bar{X} and discussed in Theorem 4 in Section 4 in a concrete situation.

Proof of Theorem 1.

It is obvious that $\pi(i(y, n)) \in X$ for any $(y, n) \in Y^\theta$ and that $\bar{\sigma}(i(y, n)) = i(\tau^\theta(y, n))$. Consequently, $i(y, n) \in \bar{X}$ for each $(y, n) \in Y^\theta$.

First let us prove that i is an injection. Let $(y, n), (y', n') \in Y^\theta$ and assume that $i(y, n) = i(y', n')$. It suffices to prove $y = y'$ and $n = n'$ under the additional assumption that $n \geq n' = 0$. Then it follows from (5) that

$$\pi(i(y', 0)) = y'_0 y'_1 \cdots = \pi(i(y, n)) = (\sigma^n y_0) y_1 y_2 \cdots.$$

If $n \neq 0$, then one would get

$$y_0 y_1 \cdots = (\sigma^m y'_{-k}) y'_{-k+1} \cdots y'_0 y'_1 \cdots.$$

for some $k \geq 1$ where m is defined so that

$$|\sigma^m y'_{-k}| = n - |y'_{-k+1}| - \cdots - |y'_{-1}|.$$

In particular, $|\sigma^m y'_{-k}| \leq n < |y_0|$. But the condition (e) implies $|\sigma^m y'_{-k}| \geq |y_0|$. Hence, a contradiction. Consequently, $n = 0$ and it follows again from (e) that $y'_0 = y_0$. A similar argument shows that $y'_n = y_n$ for each $n \in \mathbb{Z}$.

Next let us show that i is an open map. Since Y^θ is a subset of $Y \times N$, the open sets of Y^θ are generated by the sets $U \times \{n\}$ where $n \in N$ and U are open subsets of Y . Furthermore, U can be taken among the sets

$$U_k(V_0, \dots, V_m) = \tau^k \{y \in Y; y_j \in V_j (j = 0, \dots, m)\}$$

where $k \in Z$ and V_j 's are open subsets of B . Here, V_j can be chosen among the sets $B \cap (W_1 \times \dots \times W_p)$ where W_j 's are open subsets of A . Hence it suffices to prove that the sets

$$i(U_k(W_{01} \times \dots \times W_{0p_0}, \dots, W_{m1} \times \dots \times W_{mp_m}) \times \{n\})$$

are open subsets of \bar{X} . But it is trivial since they are of the form

$$\{x \in \bar{X}; x_{p+j} \in W_j (j = 0, \dots, q)\}$$

for some p, q and some open subsets W_j of A .

Finally, the relation $\pi(i(Y^\theta)) = X$ is trivial from Definition 6 and the continuity of i follows immediately from the fact that each coordinate of $i(y, n)$ depends only on one coordinate of the sequence y . Hence the proof is completed.

The following properties are valid as formal power series.

Corollary 1. *Let $U: X \rightarrow (-\infty, +\infty]$ and (X, σ) be a shift with orbit basis.*

(i) *The D-function satisfies the following relation:*

$$(9) \quad D_{X,U}(t) = D_{Y,U^\theta - (\log t)\theta}(1),$$

where

$$(10) \quad U^\theta(y) = \sum_{0 \leq n < \theta(y)} U(i(y, n)).$$

(ii) *If (X, σ) admits a free orbit basis B and if the function $U(x)$ depends only on the zero-th coordinate x_0 , then,*

$$(11) \quad D_{X,U}(t) = 1 - \sum_{b \in B} \exp \left[- \sum_{n=0}^{|b|-1} U(\sigma^n b) \right] \cdot t^{|b|}.$$

(iii) *In particular, if (X, σ) admits a free orbit basis B , then,*

$$(12) \quad D_{X,0}(t) = 1 - \sum_{b \in B} t^{|b|}.$$

Proof. Let $x \in \text{Per}_p(X, \sigma)$. Then, for some $b_n \in B$ and $0 \leq k < |b_0|$,

$$\begin{aligned} x &= (\sigma^k b_0) b_1 b_2 \dots &= \sigma^p x \\ &= \begin{cases} (\sigma^{k+p} b_0) b_1 b_2 \dots & \text{if } k+p < |b_0| \\ (\sigma^j b_m) b_{m+1} b_{m+2} \dots & \text{if } j = k+p - |b_0| - \dots - |b_{m-1}| \end{cases} \\ &\text{satisfies } 0 \leq j < |b_m| (m \geq 1). \end{aligned}$$

It follows from (e) of Definition 6 that the first case never takes place and that $m \geq 1, \sigma^k b_0 = \sigma^j b_m$ and $b_{m+n} = b_n$ for $n \geq 1$. Consequently, each $x \in \text{Per}(X, \sigma)$ is expressed in a unique manner as

(13) $x = i(y, n)$ with $(y, n) \in Y^\theta$ such that $y \in \text{Per}(Y, \tau)$.

Conversely, it is evident that (13) defines an $x \in \text{Per}(X, \sigma)$.

Now one gets

$$\begin{aligned} & \sum_{n=1}^{\infty} t^{n-1} \sum_{x \in \text{Per}_n(X, \sigma)} \exp[-\sum_{m=0}^{n-1} U(\sigma^m x)] \\ &= \sum_{i=1}^{\infty} \sum_{b_0, \dots, b_{k-1}} \frac{|b_0|}{k} t^{\sum_{j=0}^{k-1} |b_j| - 1} \exp[-\sum_{j=0}^{k-1} \sum_{m=0}^{b_j-1} U(i(\tau^j y, m))] \\ &= \frac{d}{dt} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{y \in \text{Per}_k(Y, \tau)} t^{\sum_{m=0}^{k-1} \theta(\tau^m y)} \exp[-\sum_{n=0}^{k-1} U^\theta(\tau^n y)]. \end{aligned}$$

Consequently,

$$\begin{aligned} D_{X,U}(t) &= \exp[-\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{x \in \text{Per}_n(X, \sigma)} \exp\{-\sum_{m=0}^{n-1} U(\sigma^m x)\}] \\ &= \exp[-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \text{Per}_n(Y, \tau)} \exp\{-\sum_{m=0}^{n-1} (U^\theta - \log t \theta)(\tau^m y)\}] \\ &= D_{Y,U^\theta - (\log t)\theta}(1). \end{aligned}$$

Thus, (i) is proved.

Next let us note: if $V(y)$ is a function of $y(0)$ and $Y = B^Z$, then,

(14) $D_{Y,V}(t) = 1 - \sum_{b \in B} \exp[-V(b)] \cdot t.$

In fact, (14) follows from the Taylor expansion of $\log(1-t)$. Hence, we obtain (ii) and (iii).

REMARK 2. (i) The D -function for general shifts with orbit basis is given under the same assumption on U in (ii) of Corollary 1 by the formula

(15) $D_{X,U}(t) = \prod_n (1 - \sum_{b \in B_n} e^{-U^\theta(b)} t^{|b|})$

where B_n are the free orbit bases of the subshifts (X_n, σ) which give a decomposition of the recurrent part of (X, σ) stated in Lemma 2 of Section 1.

In fact, (15) follows from the property of D -functions that, if X_n 's are mutually disjoint shift invariant subsets of X and if

$$\text{Per}(X, \sigma) = \bigcup_n \text{Per}(X_n, \sigma)$$

then, for any function U ,

(16) $D_{X,U}(t) = \prod_n D_{X_n,U}(t).$

(ii) As a special case of (15), the following theorem on matrices is obtained. Let $M = (M_{ij})_{i,j=1, \dots, N}$ be a matrix ($N \leq \infty$). Then,

$$(17) \quad \det(I-tM) = \prod_{i=1}^N (1 - \sum_{n=1}^{\infty} T_{in} t^n)$$

where $T_{ii}=M_{ii}$ and

$$(18) \quad T_{in} = \sum_{i_1, \dots, i_{n-1} > i} M_{ii_1} M_{i_1 i_2} \cdots M_{i_{n-1} i} \quad (n \geq 2).$$

In fact, let W be the set of pairs (i, j) such that $M_{ij} \neq 0$ and consider the Markov shifts $X_i = M(W) \cap \{i, i+1, \dots, N\}^N$. Let us employ the method and notation given in Examples 1-2 of Section 1. Then the set of recurrent points of $X = X_1$ is the union of the sets $X_i(i), i=1, 2, \dots, N$, and each sub-shift $(X_i(i), \sigma)$ admits the free orbit basis $B_i = \{(i, i_1, \dots, i_{n-1}); i+1 \leq i_k \leq N \text{ for all } k\}$. Take $U(x) = -\log M_{x_0 x_1} (x = (x_n))$ as the potential. Then the formula (17) is nothing but (15) since

$$\det(I-tM) = \exp \left[- \sum_{n=1}^{\infty} \frac{t^n}{n} \text{tr } M^n \right].$$

REMARK 3. The power series $D(t)$ satisfies the following properties in many typical examples:

- (a) The coefficients, say c_n , belong to a finite set, or
- (b) $c_n = a_0 \cdots a_{n-1}$ and $a_n, n \geq 0$, belong to a finite set. In such cases, one of the following (i) or (ii) is valid:
 - (i) For some n_0 , the sequence c_{n+n_0} (or a_{n+n_0}) is periodic and $D(t)$ is rational.
 - (ii) c_{n+n_0} (or a_{n+n_0}) never form a periodic sequence for any n_0 and the power series $D(t)$ has no meromorphic extension beyond its domain of convergence.

Let us give a proof by computing the meromorphic radius of Hadamard [1]. Put

$$(19) \quad A_{n,p} = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+p} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+p+1} \\ \vdots & \vdots & & \vdots \\ c_{n+p} & c_{n+p+1} & \cdots & c_{n+2p} \end{vmatrix}.$$

Then the following condition is sufficient for the non-extendability as meromorphic function:

$$(20) \quad \lim_{p \rightarrow \infty} \limsup_{n \rightarrow \infty} |A_{n,p}|^{1/n(p+1)} = \limsup_{n \rightarrow \infty} |c_n|^{1/n}.$$

Since c_n or a_n takes a value in a finite set, (20) fails if and only if $A_{n,p} = 0$ for all sufficient large n for some p . Here we used the relation: $A_{n,p+1} = 0$ for $n \geq n_0 + 1$ if $A_{n,p} = 0$ for $n \geq n_0$. This relation follows from the following identity obtained from the Laplace expansion of the determinant $A_{n,p}$ with respect to the minor matrix of the four corner elements:

$$(21) \quad A_{n+1,p-1}A_{n-1,p-1}-A_{n,p-1}^2 = A_{n,p}A_{n+1,p-2}.$$

Thus, it suffices to show the claim: if there is a p such that $A_{n,p}=0$ for any sufficiently large n , say, $n \geq n_0$, then, the sequence c_n (or a_n), $n \geq n_1$, is periodic for some n_1 .

If $p=1$, then, $c_{n-1}c_{n+1}=c_n^2$. Hence, if $c_{n-1}=0$, then, $c_n=0$, and so, $c_m=0$ for any $m \geq n$. If $c_{n-1}c_n \neq 0$, then, $c_{n-1} \neq 0$ and $c_{n+1}/c_n=c_n/c_{n-1}$. In either case, c_n can be written as c^n for some c . Consequently, $c_n=0, 1$, or -1 in case (a) and $a_n=c$ in case (b).

Now let $p \geq 2$ be the minimal number for which $A_{n,p}=0$ for $n \geq n_0$. Then, $A_{n,p-1} \neq 0$ for $n \geq n_0$. In virtue of the identity (21) the sequence $A_{n,p-1}$ satisfies the relation $A_{n,p-1}=c^n$ for $n \geq n_0$ and for some c . It follows from the choice of p that $c \neq 0$.

Consequently, c_{n+2p} can be solved from $A_{n,p}=0$ by Cramer's formula. In case (a), it then follows that c_{n+2p} is a fixed function of c_n, \dots, c_{n+2p-1} on a finite set. Consequently, c_n forms a periodic sequence from some n_1 on. In case (b), a_{n+2p} is then solved as a function of a_n, \dots, a_{n+2p-1} if $a_n \neq 0$ for any n . Consequently, a_n also form a periodic sequence from some n_1 on. (If $a_m=0$ for some m , the choice of $a_n, n \geq m$, loses its meaning.)

As to the invariant measures we obtain the following.

Corollary 2. *Let μ be an ergodic invariant measure of (X, σ) . Assume that (X, σ) admits an orbit basis and the assumptions of Theorem 1 are satisfied. Then one of the following two statements is valid for the natural extension $\bar{\mu}$ of μ :*

(a) $\bar{\mu}$ is supported by $\bar{X} \cap i(Y^\theta)^c$, i.e. $\bar{\mu}(\bar{X} \cap i(Y^\theta)^c)=1$.

(b) *There exists a unique probability invariant measure $\nu=\alpha(\mu)$ of (Y, τ) , with respect to which the ceiling function θ is integrable, such that $i_{\bar{x}}^{-1}(\nu^\theta)=\bar{\mu}$. Here ν^θ denotes the invariant measure of (Y^θ, τ^θ) defined by the formula:*

$$(22) \quad \nu^\theta(E \times \{n\}) = \nu \{y \in E; (y, n) \in Y\} / \int_Y \theta d\nu$$

for Borel subsets E of Y .

Proof. Since the set $i(Y^\theta)$ is an invariant subset, the natural extension $\bar{\mu}$ is concentrated on $i(Y^\theta)$ unless (a) holds. In virtue of the injectivity of the map i , there is a unique invariant measure ν^θ on Y^θ . Let ν be the probability measure obtained by renormalizing the restriction of ν^θ to the set $Y \times \{0\}$ identified with the set Y . Then it is an invariant measure of (Y, τ) and the relation (22) is immediate.

3. Topological entropy

Let us begin with a few comments on the relation between the topological

and the word entropy.

If M is a compact Hausdorff space and $f: M \rightarrow M$ a continuous map, let us denote for a given finite open cover \mathcal{U} of M

$$(1) \quad X(\mathcal{U}) = \{(U_n)_{n \geq 0}; U_n \in \mathcal{U}, \bigcap_{n=0}^{\infty} f^{-n} U_n \neq \emptyset\}.$$

Then, the topological entropy is given by

$$(2) \quad h(M, f) = \sup \text{ent}(X(\mathcal{U}), \sigma).$$

In general the definition of topological entropy is given based upon the uniform structure on the space M . For instance, if M is a normal topological space, then a uniform topology compatible with the given topology on M is defined by the collection of finite open covers of M as system of uniform covers and a continuous map f is a morphism of uniform topological space M to itself.

DEFINITION 8. Let M be a normal topological space and $f: M \rightarrow M$ be a continuous map. Denote the minimal number of elements of subcovers of a finite open cover \mathcal{U} by $N(\mathcal{U})$ and put

$$(3) \quad h(M, f, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N(\mathcal{U} \vee f^{-1}\mathcal{U} \vee \dots \vee f^{-n+1}\mathcal{U}).$$

Here $\mathcal{U} \vee \mathcal{C}$ is the refinement of covers \mathcal{U} and \mathcal{C} , and the limit exists in virtue of the inequality: $N(\mathcal{U} \vee \mathcal{C}) \leq N(\mathcal{U})N(\mathcal{C})$. Then the following quantity is called *topological entropy* of the dynamical system (M, f) :

$$(4) \quad h(M, f) = \sup \{h(M, f, \mathcal{U}); \mathcal{U} \text{ is a finite open cover}\}.$$

REMARK 1. (a) It is easy to see that, if (X, σ) is a shift over a finite alphabet set A (regarded as a discrete topological space), then the topological and the word entropies coincide with each other:

$$(5) \quad h(X, \sigma) = \text{ent}(X, \sigma).$$

(b) It is also immediate to see that, if (X, σ) is a shift over a compact Hausdorff alphabet set A , then,

$$(6) \quad h(X, \sigma) = \sup h(X(\mathcal{U}), \sigma),$$

where the supremum is taken over finite open covers \mathcal{U} of X which can be expressed in the form

$$(7) \quad \mathcal{U} = \{\bigcup_{a \in V} [a]; V \in \mathcal{C}\} \quad \text{for some finite open cover } \mathcal{C} \text{ of } A.$$

(c) Although we shall employ Definition 8, there are many possibilities for the choice of uniform topology on M which may depend on f , and we do not know the relationship among the resultant topological entropies.

Theorem 2. *Let (X, σ) be a shift over a countable compact Hausdorff alphabet set which admits a free orbit basis B and (Y^θ, τ^θ) the tower associated with it. Then,*

$$(8) \quad h(Y^\theta, \tau^\theta) = h(X, \sigma) = P^0(X, 0) = -\log t(B),$$

where

$$(9) \quad t(B) = \sup \{t \geq 0; D(t) \text{ converges and is nonnegative}\}$$

$$(10) \quad D(t) = D_B(t) = 1 - \sum_{b \in B} t^{|b|}.$$

The proof will be given after a series of lemmas.

Lemma 1. *Let $a_{kn}, k \geq 1, n \geq 1$, be nonnegative real numbers such that $a_{kn} \leq a_{k+1,n}$ and $\lim_{k \rightarrow \infty} a_{kn} = a_{\infty n}$. Define*

$$(11) \quad D_k(t) = 1 - \sum_{n=1}^{\infty} a_{kn} t^n \quad (1 \leq k \leq \infty) \quad \text{and}$$

$$(12) \quad t_k = \sup \{t \geq 0; D_k(t) \text{ converges and is nonnegative}\}$$

Then (i) $t_k \geq t_{k+1} \geq t_\infty$. (ii) $\lim_{k \rightarrow \infty} t_k = t_\infty$.

Proof. It is obvious that

$$D_k(t) \geq D_j(t) \quad 1 \leq k \geq j \geq \infty \quad \text{for } t \geq 0$$

so far as they are convergent series. Hence, (i) follows.

Now put $s = \inf \{t_k; 1 \leq k < \infty\}$. Then it follows from $s \leq t_k$

$$D_k(s) \geq D_k(t_k) = 0 \quad \text{i.e.} \quad \sum_{n=1}^{\infty} a_{kn} s^n \leq 1.$$

Hence,

$$\sum_{n=1}^{\infty} a_{\infty n} s^n \leq 1 \quad \text{i.e.} \quad D_\infty(s) \geq 0.$$

Consequently, $s \leq t_\infty$ and so $s = t_\infty$.

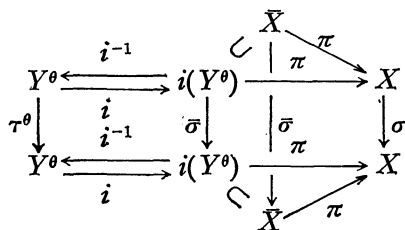
Lemma 2. *Let f_i be a continuous map of a normal topological space M_i to itself ($i=1, 2$) and assume the existence of a continuous map g of M_1 onto M_2 such that $g \circ f_1 = f_2 \circ g$. Then,*

$$(12) \quad h(M_1, f_1) \geq h(M_2, f_2).$$

Proof. Obvious from the definition of topological entropy.

Lemma 3. $h(Y^\theta, \tau^\theta) = h(X, \sigma)$.

Proof. Let us recall that the following diagram is commutative and all the maps are surjective:



Consequently, it follows from Lemma 2 and Theorem 1 that

$$\begin{aligned}
 h(Y^\theta, \tau^\theta) &= h(i(Y^\theta), \bar{\sigma}) \\
 &\geq h(X, \sigma) = h(\bar{X}, \bar{\sigma}) \geq h(i(Y^\theta), \bar{\sigma})
 \end{aligned}$$

since the topological entropy does not change its value under the natural extension.

Lemma 4. Assume that \$A\$ is a finite set and that there are an alphabet \$a^* \in A\$ and a number \$N\$ such that

$$(13) \quad \{b \in B; |b| = n\} = \{(a^*, \dots, a^*)\} \quad \text{for } n > N.$$

Then,

$$(14) \quad h(X, \sigma) = \text{ent}(X, \sigma) = -\log t(B).$$

Proof. Let

$$\begin{aligned}
 R^* &= \{(a_0, \dots, a_n) \in W(X); a_0 = \dots = a_n = a^*, n \geq 0\}, \\
 R_0 &= \{(a_i, \dots, a_j); (a_0, \dots, a_{m-1}) \in B, 0 \leq i \leq j < m \leq N\}, \\
 R_1 &= \{(a_i, \dots, a_{m-1}); (a_0, \dots, a_{m-1}) \in B, 0 \leq i < m \leq N\} \cup \{e\}, \\
 R_2 &= \{(a_0, \dots, a_j); (a_0, \dots, a_{m-1}) \in B, 0 \leq j < m \leq N\} \cup \{e\},
 \end{aligned}$$

where the letter \$e\$ stands for the empty word, i.e., \$e\$ is a formal word of length \$|e|=0\$ such that \$eu=ue=u\$ for any word \$u\$. Then any word \$u \in W_n(X)\$ satisfies one of the following two conditions:

- (a) \$u \in R_0 \cup R^*\$
- (b) \$u = vb_1 \dots b_p w\$, \$v \in R_1 \cup R^*\$, \$w \in R_2 \cup R^*\$, \$p \ge 0\$ and \$b_k \in B\$.

Now let us use the notation

$$F(t) \ll G(t)$$

when \$G(t) - F(t)\$ is a formal power series with nonnegative coefficients. Then,

$$\begin{aligned}
 &\sum t^n W_n(X) \\
 (15) \quad &\ll \sum_{u \in R_0 \cup R^*} t^{|u|} + \sum_{v \in R_1 \cup R^*} \sum_{w \in R_2 \cup R^*} \sum_{p=0}^{\infty} \sum_{b_1 \in B} \dots \sum_{b_p \in B} t^{|v| + |w| + \sum |b_k|}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{u \in R_0} t^{|u|} + t^{N+1}(1-t)^{-1} \\
 &\quad + \left(\sum_{v \in R_1} t^{|v|} + t^{N+1}/(1-t) \right) \left(\sum_{w \in R_2} t^{|w|} + t^{N+1}/(1-t) \right) / \left(1 - \sum_{b \in B} t^{|b|} \right).
 \end{aligned}$$

On the other hand, since $b_1 \cdots b_p \in W(X)$ for any $p \geq 1$ and $b_k \in B$,

$$\begin{aligned}
 (16) \quad &\sum t^n W_n(X) \gg \sum_{p=1}^{\infty} \sum_{b_1 \in B} \cdots \sum_{b_p \in B} t^{|b_1| + \cdots + |b_p|} \\
 &= \left(\sum_{b \in B} t^{|b|} \right) / \left(1 - \sum_{b \in B} t^{|b|} \right).
 \end{aligned}$$

Consequently, it follows from (15), (16) and the finiteness of the sets R_i , $i=1, 2, 3$, that the radius of convergence of the left hand side of (15) and (16) coincides with $t(B)$.

Lemma 5. *Assume that A is finite. Then (14) holds.*

Proof. Let us add an extra alphabet a^* to A and put $A^* = A \cup \{a^*\}$. Define a map $f = f_N: B \rightarrow \cup (A^*)^n$ for each $N \geq 1$ by

$$f(b) = \begin{cases} b & \text{if } |b| \leq N \\ (a^*, \dots, a^*) & \text{otherwise} \end{cases}$$

so that $|f(b)| = |b|$. Then f is a continuous map in virtue of our definition of the topology on B (given before Theorem 1 in Section 2).

Next, let $F(X)$ be the shift with orbit basis $f(B)$.

Then F is also continuous and it follows from Lemmas 3 and 4 that

$$(17) \quad h(f(Y)^\theta, \tau^\theta) = h(F(\bar{X}), \sigma) = -\log t(f(B)).$$

Since the topology on Y is generated by the induced topologies from sets $f_N(B)$ by f_N , $N=1, 2, \dots$,

$$(18) \quad h(Y^\theta, \tau^\theta) = \sup_X h(f_N(Y)^\theta, \tau^\theta).$$

On the other hand, it follows from Lemma 1 that

$$(19) \quad \sup_X \{-\log t(f_N(B))\} = -\log t(B).$$

Consequently, one obtains

$$h(X, \sigma) = h(Y^\theta, \tau^\theta) = -\log t(B).$$

Proof of Theorem 2.

We have proved it in Lemma 5 when A is finite. Since A is a countable compact Hausdorff space, any open cover admits its refinement of the form $\{g^{-1}(c); c \in A(g)\}$ where g is a continuous map of A to a finite set $A(g)$. Consequently,

$$(20) \quad h(X, \sigma) = h(\bar{X}, \bar{\sigma}) = \sup_g h(G(\bar{X}), \bar{\sigma})$$

where G is the map of \bar{X} to $A(g)^Z$ defined by $G((x_n)) = (g(x_n))$.

The shifts $(G(\bar{X}), \bar{\sigma})$ are, of course, not necessarily shifts with orbit basis. However, the map $\pi \circ i$ of $(G(B)^Z)^\theta$ to $\pi(G(\bar{X}))$ is still surjective. Thus, Lemmas 4 and 5 work as the upper estimate of $h(G(\bar{X}), \sigma)$ and one obtains

$$(21) \quad h(G(\bar{X}), \bar{\sigma}) \leq h((G(B)^Z)^\theta, \tau^\theta) = -\log t(G(B)) \leq h(Y^\theta, \tau^\theta).$$

Then Lemma 1 is applicable and it follows

$$(22) \quad -\log t(B) = \sup \{-\log t(G(B))\}.$$

Consequently, Theorem 2 is obtained from (20), (21) and (22).

4. Structure of the realization of a map of interval

In this section we shall show that the shifts which realize piecewise continuous maps of bounded closed intervals are unions of shifts with orbit basis. This result is obtained in [7] for unimodal continuous maps and in [4, 9] for beta transformations.

Let f be a piecewise continuous map of a bounded closed interval J and assume, for the sake of simplicity, that it is surjective on J and that there are finite number of lap intervals. Denote the lap intervals by $I_a, a \in A$. Then, by definition, the map f is nonotone on each I_a , the intersection of I_a and $I_{a'}$, for $a \neq a'$ is, if any, a one point set and the union of I_a 's is the interval J . If f is nondecreasing on I_a , then we shall denote $\varepsilon(a) = +1$ and, otherwise, $\varepsilon(a) = -1$.

DEFINITION 9. Let us define a linear order $< = <_f$ on the product space A^N as follows: $x = (x_n) > x' = (x'_n)$ if $x = x'$ or if there is a number $k \in N$ such that

$$(1) \quad \begin{aligned} x(n) &= x'(n) && \text{for } 0 \leq n < k \text{ and} \\ x(k) &\geq x'(k) && \text{according as } \varepsilon(x_0 \cdots x_{k-1}) \geq 0, \end{aligned}$$

where $\varepsilon(x_0 \cdots x_{k-1}) = \varepsilon(x_0) \cdots \varepsilon(x_{k-1})$, and the order $a > a'$ for $a, a' \in A$ is defined by $\max I_{a'} \leq \min I_a$.

In a previous paper [11] the following theorem is proved:

Theorem 0. *Let f be a piecewise continuous map of interval J . Then there exists a realization (X, σ, ρ) in the following sense:*

(a) *The set X is a σ -invariant closed subset of A^N which is characterized by its elements $\zeta_a^+, \zeta_a^-, a \in A$, as*

$$(2) \quad X = \{x \in A^N; \zeta_{x(n)}^- < \sigma^n x < \zeta_{x(n)}^+ \quad (n \in N)\}.$$

(b) *The map ρ is defined on X and takes subintervals of J as its values. It is monotone in the sense that*

$$(3) \quad \max \rho(x) \leq \min \rho(x') \quad \text{if } x < x' \text{ and } x \neq x'$$

and is surjective in the sense that

$$(4) \quad \bigcup_{x \in X} \rho(x) = J.$$

Furthermore, it is a conjugacy map in the sense that

$$(5) \quad \rho(\sigma x) \supset f(\rho(x)), \quad x \in X.$$

(c) *The set X is the disjoint union of two subsets X^0 and X^1 , where X^1 is at most countable. For each $x \in X^0$, the subinterval $\rho(x)$ consists of a single point, which is denoted, again, by $\rho(x)$. The map $\rho: X^0 \rightarrow J$ is continuous and is one-to-one except for at most countably many exceptional points where it is two-to-one.*

(d) *The topological entropy of (J, f) is the same as that of (X, σ) and the periodic orbits of (X, σ) are in one-to-one correspondence with the connected components of the set of periodic points of (J, f) .*

The proof of Theorem 0 is given in [11] but we shall sketch the proof.

1° Let $t \in J$ be such that $f^n t$ never falls into the boundaries of monotone intervals I_a , $a \in A$, and define a sequence $x(t) = (x_n(t))$ by

$$x_0(t) = a \text{ if } t \in I_a, \text{ and } x_n(t) = x_0(f^n t) \text{ (} n \geq 1 \text{)}.$$

Put $X =$ the closure of $x(t)$'s in $A^{\mathbb{N}}$
 $=$ the closure of $x(t)$'s w.r.t. $<$,

$$(6) \quad \zeta_a^+ = \sup \{x(t); x_0(t) = a\}, \\ \zeta_a^- = \inf \{x(t); x_0(t) = a\}.$$

Then it turns out that (a) is valid.

2° Let $x \in X$ and put

$$(7) \quad \rho(x) = \bigcap f^{-n} I_{x(n)}.$$

Then (b) holds.

3° (c) follows from the Baire's category theorem.

4° (d) follows from the relation: $\rho(x(t)) \ni t$.

REMARK 1. If the boundaries of I_a , $a \in A$, fall into a single orbit under the iteration of f , then, only one of the sequences ζ_a^+ , ζ_a^- is sufficient to define X . It is the case for beta transformations ([4, 8, 9]) and unimodal continuous maps.

REMARK 2. Theorem 0 is valid even if there are infinitely many monotone intervals I_a . In this general case, the set A is taken as the closure of the set $\{I_a\}$ of the monotone intervals with respect to the order structure among the subintervals of J . Hence, the set A is always countable and compact Hausdorff. Under this interpretation, Theorems 3 and 4 below are valid for general cases except possibly the property (9) in Theorem 3 (ii).

Now let us state the main theorem.

Theorem 3. *Let f be a piecewise continuous map of a bounded closed interval J onto itself and (X, σ, ρ) its realization by a shift over alphabet set A stated in Theorem 2. Then there exist two subshifts X^* and X^{**} of X which satisfy the following three properties:*

(i) *The shift (X^*, σ) admits a simple Markov orbit basis. Moreover, for each $a \in A$, the subshift $(X^*(a), \sigma)$, where*

$$(8) \quad X^*(a) = \{x \in X^*; x(n) = a \text{ for infinitely many } n\}$$

admits a free orbit basis.

(ii) *The shift (X^{**}, σ) also admits an orbit basis. The subshift defined similarly by the set $X^{**}(a)$ admits a simple Markov basis (B, V) with the following property but it does not admit a free orbit basis in general: there is a finite subset B_0 such that*

$$(9) \quad B = \{b; (b_0, b) \in V \text{ for some } b_0 \in B_0\}.$$

(iii) *The set X^* is the intersection of the inverse images of the complement $(X^{**})^c$ under the iterates of σ :*

$$(10) \quad X^* = \bigcap \sigma^{-n}(X^{**})^c.$$

In particular, the complement $X \cap (X^ \cup X^{**})^c$ contains no recurrent points of (X, σ) .*

The proof will be given in the next section.

REMARK 3. The second subshift (X^{**}, σ) in Theorem 3 appears when there are mutually disjoint closed subintervals J_i of J such that

$$(11) \quad f(\cup J_i) \subset \cup J_i.$$

Furthermore, the union of J_i 's is then an attractor of the map f . It corresponds to the case called window phenomenon by R. May or to the case called island phenomenon. (cf. [3], [12])

The conclusion above follows from the construction of the sets X^* , X^{**} given in Section 5, the definitions of which are found in Lemmas 7 and 8. A

typical concrete example will be found in Example 1 and a more complicated one in Example 2 at the end of this section.

Let $(\bar{X}, \bar{\sigma})$ be the natural extension of (X, σ) . As it was already pointed out in Remark after Theorem 1, there may exist the boundary $\partial\bar{X}$. But the example there was trivial since the sequence was not recurrent. A recurrent point x does belong to the boundary when $x(-\infty, 0)$ does not admit a unique representation by orbit basis. In fact, if there is a unique representation, then $x \in \tau(\bar{Y}^\theta)$. Therefore it is necessary to consider the following sets, which are the essential parts of the boundary. $\partial\bar{X} = \partial\bar{X}^* \cup \partial\bar{X}^{**}$.

For $\zeta = \zeta_a^d, a \in A, d \in \{+1, -1\}$, let

$$(12) \quad T_0(\zeta) = \{x \in \bar{X} : x[-n, 0) = \zeta[0, n) \text{ for infinitely many } n\},$$

$$(13) \quad T(\zeta) = \bigcap_{n \in \mathbb{N}} \sigma^n T_0(\zeta).$$

Theorem 4. *Assume the same hypotheses as in Theorem 3. Then the following statements are valid:*

- (i) *Let $x \in \partial\bar{X}$ and assume that x is a recurrent point. Then, the point x belongs to the set $T(\zeta_a^d)$ for some $a \in A$ and some $d \in \{+1, -1\}$.*
- (ii) *For each $\zeta = \zeta_a^d, \text{ent}(T(\zeta), \sigma) = 0$.*

The proof will be given in the next section.

REMARK 4. The second statement of Theorem 4 shows that the set $\partial\bar{X}$, which is the complement of the image of the tower associated with (X, σ) , is small enough to be negligible in entropy analysis of map f . In [10] the samllness was thought to be self-evident and left unproved as it was pointed out by Hopfbauer [2].

EXAMPLE 1 (beta transformations). Let $\beta > 1, J = [0, 1]$ and

$$(14) \quad \begin{aligned} f(t) &= \beta t - [\beta t] && \text{if } t \in [0, 1), \\ f^n(1) &= \lim_{t \rightarrow 1} f^n(t) \end{aligned}$$

where $[s]$ denotes the integer part of a real number s . The monotone intervals are $[a/\beta, (a+1)/\beta], a \in A = \{0, 1, \dots, r\}$ and r is the integer such that $\beta - 1 \leq r < \beta$. It is easy to see that the sequences ζ_a^+, ζ_a^- are given as

$$(15) \quad \begin{aligned} \zeta_a^- &= a\zeta_0^-, \zeta_0^- = (0, 0, \dots), \\ \zeta_a^+ &= a\zeta_r^+ \quad (a \neq r). \end{aligned}$$

Consequently, the realization (X, ρ, σ) of f is characterized by the sequence $\zeta = \zeta_r^+$ and is given by

$$(16) \quad X = \{x \in A^N; \sigma^n x < \zeta\}, \quad \text{and} \\ \rho(x) = \sum_{n=0}^{\infty} x(n)/\beta^{n+1} \text{ (\beta-expansion of } t = \rho(x)\text{)}.$$

As it was stated in Introduction, the shift (X, σ) admits a free orbit basis

$$(17) \quad B = \{(\zeta_0, \dots, \zeta_{n-1}, a); n \geq 0, a < \zeta_n\}$$

when the sequence ζ is not periodic. If ζ is periodic with period p , then,

$$(18) \quad B = \{(\zeta_0, \dots, \zeta_{n-1}, a); 0 \leq n \leq p-2, a \neq \zeta_{p-1}\} \\ \cup \{(\zeta_0, \dots, \zeta_{p-2}, a); a \leq \zeta_{p-1}\}$$

is a free orbit basis. The proof is contained implicitly in [4, 10] but the direct proof is immediate from the following two observations:

(a) $[b] \cap X, b \in B$, form a partition of X .

(b) Any $b \in B$ is a free word. In other words, f^{n+1} maps $f^{-n}\rho[\zeta_0] \cap \dots \cap f^{-1}\rho[\zeta_{n-1}] \cap \rho[a]$ bijectively onto J .

Consequently,

$$(19) \quad D_{X,0}(t) = 1 - \sum_{b \in B} t^{|b|} = 1 - \sum_{n=0}^{\infty} \zeta_n t^{n+1}$$

when ζ is not periodic, and

$$(20) \quad D_{X,0}(t) = 1 - \zeta_0 t - \dots - \zeta_{p-2} t^{p-1} - (\zeta_{p-1} + 1)t^p$$

when it is periodic. Hence we can obtain the results in [4, 9] and [10]. In particular, the β -expansion of one,

$$1 = \sum_{n=0}^{\infty} \zeta_n / \beta^{n+1},$$

shows that $h(J, f) = h(X, \sigma) = \log \beta$.

EXAMPLE 2 (unimodal continuous maps). Let f be a unimodal continuous map of the unit interval $J = [0, 1]$ onto itself i.e. the inverse image $f^{-1}(1)$ consists of a single point c and $f(1) = 0$. Then,

$$(21) \quad \zeta_0^+ = 01\zeta, \zeta_1^- = 11\zeta, \text{ and } \zeta_1^+ = 1\zeta \text{ with } \zeta = \zeta_0^-.$$

Consequently, the realization shift (X, σ) is characterized by

$$(22) \quad X = \{x \in A^N; \sigma^n x > \zeta (n \in N)\}.$$

We shall not discuss the form of orbit basis but the D -function has the following form ([12]):

$$(23) \quad D_{X,0}(t) = \begin{cases} 1 - \sum_{n=0}^{\infty} \varepsilon_n t^{n+1} & \text{if } \zeta \text{ is not periodic} \\ 1 - \sum_{n=0}^{p-2} \varepsilon_n t^{n+1} - (\varepsilon_{p-1} + \varepsilon_p)t^p & \text{if } \zeta \text{ has period } p \end{cases}$$

where

$$\varepsilon_n = \begin{cases} (-1)^{\xi_0 + \dots + \xi_{n-1}} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0. \end{cases}$$

Here the right hand side of (23) must be multiplied by some factor in pathological cases when f does not satisfy the Schwarzian condition.

Now let us state some results on unimodal linear maps:

$$(24) \quad f(x) = \begin{cases} (x+a+b-1)/b & \text{if } 0 \leq x \leq c = 1-a \\ (1-x)/a & \text{if } c \leq x \leq 1 \end{cases}$$

where a and b are parameters which satisfy $0 < a < 1, b > 0, a + b > 1$. It was observed in [3] that the unimodal linear map f shows window phenomenon or island phenomenon according as

$$(25) \quad ab^{p-1} > 1 \quad \text{or} \quad a(a+b)b^{p-2} \geq 1, \text{ respectively,}$$

when it satisfies the condition

$$(26) \quad a(1+b+\dots+b^{p-2}) < 1 < a(1+\dots+b^{p-1}).$$

In fact, it follows from the condition (26) that

$$(27) \quad f^2c = 0 < f^{p+2}c < f^3c < f^{p+3}c < \dots < f^pc < c < f^{p+1}c < f^{2p+1}c < fc = 1.$$

Consequently, the subintervals

$$(28) \quad J_1 = [f^{2p+1}c, fc] \quad \text{and} \quad J_i = [f^i c, f^{i+p}c] \quad (i = 2, 3, \dots, p)$$

are mapped cyclically to each other under f , and their union J^{**} is an attractor of f^p . Depending on which of the alternatives in (25) takes place, the map f^p on each J_i has an attracting periodic orbit or is chaotic. In this sense, the map f shows window or island phenomenon.

In our terminologies, the subintervals J_i correspond to the cylinder sets $[b]$, $b \in B_0$, the orbits in J^{**} to the sequences in X^{**} and the orbits contained in the complement $J^* = J \cap (J^{**})^c$ to the sequences in X^* . The statement (iii) of Theorem 3 is reflected in the fact that J^{**} is an attractor and the sequences in $X \cap (X^* \cup X^{**})^c$ correspond to the orbits of f which wander out from J^* to J^{**} . Window phenomenon can be verified by the fact that the orbit basis for X^{**} consists of words of the periodic sequence ζ .

EXAMPLE 3 (critical unimodal continuous maps). The (N_n) -critical case defined in [12] corresponds to the following situation. Let $N_n, n \geq 0$, be integers greater than 1, and f be an (N_n) -critical map. Then the map f has N_0 subintervals which are mapped cyclically onto each other and the map f^{N_0} has

N_1 subintervals in each of the N_0 subintervals which are mapped by f^{N_1} cyclically onto each other and so on. As a consequence of this “self-similarity”, the recurrent part of X is decomposed into countably many subshifts $(X^{(n)}, \sigma)$ which admit free orbit bases.

Let N be an integer ≥ 2 and f be a unimodal continuous map of the unit interval $[0, 1]$ which corresponds to N^∞ -critical case ($N_n = N$ for all n). Let $c = f^{-1}\{1\} \in (0, 1)$ and $f(1) = 0$. Then the realization (X, σ, ρ) is characterized by the sequences

$$(29) \quad \zeta_0^- = \zeta, \zeta_0^+ = 01\zeta, \zeta_1^- = 11\zeta, \text{ and } \zeta_1^+ = 1\zeta$$

and, therefore, by the single sequence ζ . In this case f shows many self-similarities. Let us state the conclusions without proofs.

The sequence ζ is generated by the following two rules:

Rule 0° $\zeta[0, N) = (0, \dots, 0, 1)$.

Rule 1° $\zeta[0, N^{m+1}) = \zeta[0, N^m)\zeta[0, N^m)^* \dots \zeta[0, N^m)^*$.

Here,

$$(a_0, \dots, a_n)^* = (a_0, \dots, a_{n-2}, a_{n-1}^*, a_n), \quad 0^* = 1, \quad 1^* = 0.$$

In this case, the structure of the set X^{**} is known: there are subshifts $(X^{(m)}, \sigma)$, $m \geq 0$, $X^{(0)} = X^*$, which admit finite free orbit bases

$$(30) \quad \begin{aligned} B_m &= \{\zeta[0, nN^m); n = 1, \dots, N-1\} \quad (m \geq 1) \\ B_0 &= \{\zeta[0, n); n = 2, \dots, N-1\} \cup \{1\}. \end{aligned}$$

The sets $X^{(m)}$ are mutually disjoint and the systems $(X^{(m)}, \sigma^{N^m})$ are mutually conjugate to each other. Note that, for $N=2$, each basis B_m consists of a single word. It means that $X^{(m)}$ is the orbit of some periodic point. The complement $X^{(\infty)}$ of the union of $X^{(m)}$, $m \geq 0$, also admit an orbit basis (B, V) given by

$$(31) \quad \begin{aligned} B &= \{\zeta[0, N^m); m \leq 0\} \quad \text{and} \\ V &= \{(\zeta[0, N^m), \zeta[0, N^n)); m < n\}. \end{aligned}$$

The natural extension $(\bar{X}^{(\infty)}, \sigma)$ of $(X^{(\infty)}, \sigma)$ satisfies a strange property: $\partial \bar{X}^{(\infty)} = \bar{X}^{(\infty)}$. In particular, the topological entropy of $(\bar{X}^{(\infty)}, \sigma)$ is zero. It seems that the orbit closure of the almost periodic sequence ζ coincides with $X^{(\infty)}$ but we have no proof.

5. Proof of Theorems 3 and 4

The key to the proof of Theorem 3 is the following notion.

DEFINITION 10. Let $w \in W(X)$. A word u in X is called w -free if $u \in$

$W(X)$ and

$$(1) \quad ux = (u_0, \dots, u_{|u|-1}, x_0, x_1, \dots) \in X \quad \text{whenever} \quad x \in X \cap [w].$$

The totality of w -free words will be denoted by $F(w)$.

Let us use the following abbreviations and notations:

For a sequence $x = (x_0, x_1, \dots)$ or a word $u = (u_0, \dots, u_{m-1})$ ($m = |u|$), let $x(n)$ and $u(n)$ denote the n -th coordinate x_n and u_n , respectively, and

$$(2) \quad \begin{aligned} x[n, m] &= (x_n, \dots, x_{m-1}), & u[n, m] &= (u_n, \dots, u_{m-1}), \\ \sigma u &= (u_1, \dots, u_{m-1}), & u' &= (u_0, \dots, u_{m-2}), \end{aligned}$$

$$(3) \quad Z = \{\zeta_a^d[0, n]; a \in A, d \in \{-1, +1\}, n \in N\},$$

$$(4) \quad W^\circ(w) = \{u \in W; uw \in W \cap Z^c\} \quad \text{where } W = W(X),$$

$$(5) \quad F^\circ(w) = \{u \in W; \sigma^n u \in W^\circ(w), n = 0, \dots, |u| - 2\},$$

$$(6) \quad Z(w) = \{u \in F(w); uw \in Z\},$$

$$(7) \quad F(a, w) = \{u \in F(w); u(0) = a\},$$

$$(8) \quad F_0(a, b) = \{u \in F(a, b); u[0, n] \in F(a, u(n))^c \text{ for } n \leq |u| - 1\}.$$

Furthermore,

$$(9) \quad x <^d x' \text{ if } d = +1 \text{ and } x < x' \text{ or if } d = -1 \text{ and } x' > x.$$

For $z \in Z$,

$$(10) \quad d(z) = d \text{ and } \zeta_z = \zeta_a^d \text{ if } z = \zeta_a^d[0, |z|].$$

Here the map $d(\cdot)$ and ζ_z are considered to be double-valued when they are not uniquely determined, i.e., when $z = \zeta_a^+[0, n] = \zeta_a^-[0, n]$ for some a and n .

Lemma 1. *Let $w \in Z$.*

(i) *If $z \in Z$, $wz \in W$ and $d(z) = d(w)\varepsilon(w)$, then,*

$$(11) \quad wz = \zeta_w[0, |w| + |z|].$$

(ii) *If $w\zeta_a^d \in X$ for some $a \in A$ with $d = d(w)\varepsilon(w)$, then, $\sigma^{|w|}\zeta_w = \zeta_a^d$.*

Proof. Let $n = |w|$, $m = |z|$, $e = d(w)$ and $e' = d(w)\varepsilon(w)$. It follows from

$$wz <^e \zeta_w[0, n+m] = w(\sigma^n \zeta_w)[0, m],$$

that $z <^{e'} \sigma^n \zeta_w[0, m]$. Since $d(z) = e'$ and z is the largest word with respect to $<^{e'}$, it follows that $z = \sigma^n \zeta_w[0, m]$ and, hence, $wz = \zeta_w[0, m+n]$. The second assertion follows from (i) by considering the limit as $m \rightarrow \infty$.

Lemma 2. *Let $w \in W$.*

- (i) *If $u \in F(w)$ and $|u| \geq 2$, then, $\sigma u \in F(w)$.*
- (ii) *If $u \in F(v)$ and $v \in F(w)$, then, $uv \in F(w)$.*
- (iii) *$F^\circ(w)$ is a subset of $F(w)$ and*

$$F(w) \cap F^\circ(w)^c = \{u \in W; u = u_1 u_2, u_1 \in F^\circ(u_2 w), u_2 \in Z(w), |u_2| \geq 1\}.$$

- (iv) *$Z(a) = \{z \in Z; \sigma^{|z|} \zeta_z = \zeta_a^d \text{ if } \varepsilon(z) = \varepsilon(a)d\}$.*

Proof. The assertions (i) and (ii) are trivial. The condition $u \in F(w)$ means that

$$(12) \quad \zeta_{u(n)}^- < \sigma^n u w x < \zeta_{u(n)}^+ \quad \text{for } n = 0, \dots, |u| - 1$$

whenever $w x \in X$. An obvious sufficient condition for (12) is that

$$\sigma^n u w \in W \cap Z^c \quad \text{for } n = 0, \dots, |u| - 1$$

i.e., that $u \in F^\circ(w)$. Now suppose that $\sigma^m u w \in W \cap Z^c$ for every $m < n$ and that $\sigma^n u w \in Z$. Then, $u[0, n] \in F^\circ(\sigma^n u w)$ and $\sigma^n u \in Z(w)$. Hence, one obtains (iii).

Finally, assume that $z \in Z(w)$ and $w \in Z$. Then, $z \zeta_w \in X$ and $z \in Z$. Hence it follows from Lemma 1 that $\zeta_a^d = \sigma^{|z|} \zeta_z$ for $d = d(z)\varepsilon(z)$. Thus, (iv) is proved.

Lemma 3. *Each word $u \in F(a, b)$ is decomposed as*

$$(13) \quad u = u_1 \cdots u_m$$

for some $m \geq 1$, $u_i \in F_0(u_i(0), u_{i+1}(0))$, $0 < i < m$, and $u_m \in F_0(u_m(0), b)$.

Proof. Let $u = (a_0, \dots, a_n) \in F(a, b)$. If $n = 0$, then (13) is trivial. Put

$$k = \min \{i; (a_0, \dots, a_i) \in F(a, a_{i+1})\} \quad (a_{n+1} = b).$$

Then, if $k = n$, $u \in F_0(a, b)$. If $k < n$, then, $u_1 = (a_0, \dots, a_k)$ belongs to $F_0(a, a_{k+1})$ and $\sigma^{k+1} u \in F(a_{k+1}, b)$ in virtue of Lemma 2 (i). Consequently, (13) follows by induction on the length $|u|$ of u .

Let $F = F(A)$ be the union of $F(a)$, $a \in A$, and

$$(14) \quad [F] = \bigcup_{a \in A} \bigcup_{u \in F(a)} [u].$$

Lemma 4. *Let $x \in X \cap [F]^c$. Then, for each $n \geq 1$, there is an $m \geq 0$ such that $m < n$ and*

$$(15) \quad x[m, n] \in Z, \quad x[m, n] \in F(x(n))^c \quad \text{and} \quad x[0, m] \in F^\circ(x[m, n]).$$

(The last relation is not necessary when $m = 0$.)

Proof. It follows from the assumption that $x[0, n] \in F(x(n))^c$ for each $n \geq 1$. Fix n and put

$$(16) \quad m = \max \{i; 0 \leq i \leq n, x[j, n] \in W^\circ(x(n)) \text{ for every } 0 \leq j < i\}.$$

Then it is clear that $x[m, n] \in Z$ by the definition of $W^\circ(\cdot)$. It follows from Lemma 2 (i) that $x[0, m] \in F^\circ(x[m, n])$ when $m \geq 1$. Finally, if $x[m, n] \in F(x(n))$, then one would obtain $x[0, n] \in F(x(n))$ by Lemma 2 (ii). Consequently, $x[m, n] \in F(x(n))^c$.

Lemma 5. Let $x \in X \cap [F]^c$. Then, for each $n \geq 0$, there are an integer $k \geq 0$ and words $z_0, \dots, z_k \in Z$ such that

- (i) $x[0, n] = z'_0 \cdots z'_{k-1} z_k = z_0 \sigma z_1 \cdots \sigma z_k$.
- (ii) $z'_i \in F^\circ(z_{i+1}) \cap F(z_{i+1}(0))^c$, $i=0, \dots, k-1$, and $z'_k \in F(x(n))^c$.
- (iii) $z_0[0, j] \in F(z_0(j))^c$, $j=1, \dots, |z_0| - 1$.

Proof. Let us prove (i)–(iii) by induction on n . For $n=0$, they are trivial. Assume that they are true for $n < n_0$. Take the number m in (16) for $n=n_0$. It then follows from the induction assumption that the expression (i) is true with $n=m$ for some z_0, \dots, z_k satisfying (ii) and (iii). Now put $z_{k+1} = x[m, n]$. Then (i)–(iii) for z_j , $j \leq k+1$, follow from Lemma 4.

Lemma 6. Let $x \in X \cap [F]^c$. Then either (A) or (B) is valid:

- (A) $\sigma^n x = \zeta_{x(n)}^d$ for some n and $d \in \{+1, -1\}$.
- (B) The following three properties are satisfied:
 - (a) $x = z'_0 z'_1 \cdots = z_0 \sigma z_1 \sigma z_2 \cdots$.
 - (b) $z'_{i-1} \in Z \cap F^\circ(z_i) \cap F(z_i(0))^c$, $i \geq 1$.
 - (c) $z_0[0, j] \in F(z_0(j))^c$, $1 \leq j < |z_0| - 1$.

Proof. In virtue of Lemma 5, one obtains a number $k=k(n)$ and words $z_i = z_i^n$ for each n which satisfy (i)–(iii) of Lemma 5. For $i=0$, there are two cases. If the length of z_0^n is not bounded in n , then one can find a sequence, say $n_0(k)$, $k \geq 0$, $n_0(k) \rightarrow \infty$ along which $1^\circ |z_0^n| \rightarrow \infty$ and $2^\circ \varepsilon(z_0^n) = \text{const.}$, say d . Hence, $x = \zeta_{x(0)}^d$. On the other hand, if z_0^n is bounded in n , then one can find a sequence $n_0(k) \rightarrow \infty$ along which the word z_0^n itself is constant, say z_0 .

Now it follows from a similar argument that either of the following two takes place for each $i < 1$:

- (A_i) $z_j^n = z_j$ for any $j < i$ and $z_i^n \rightarrow \zeta_a^d$ for some a and d along some subsequence $n_i(k) \rightarrow \infty$ of $n_{i-1}(k)$.
- (B_i) $z_j^n = z_j$ for any $j \leq i$ (including i) along some subsequence $n_i(k) \rightarrow \infty$ of $n_{i-1}(k)$.

Thus Lemma 6 is proved.

REMARK. It is immediate to obtain the converse of Lemma 6: If a se-

quence x is defined by (a) with z_i 's satisfying (b) and (c), then, x belongs to the set $X \cap [F]^c$.

Lemma 7. *Let*

$$(17) \quad X^* = \bigcap_{n=0}^{\infty} \sigma^{-n}[F] \cap X$$

$$(18) \quad B^* = \bigcup_{a,b \in A} F_0(a, b)$$

$$(19) \quad V^* = \{(u, v) \in B^* \times B^*; u \in F(v(0))\} .$$

Then the shift (X^, σ) admits (B^*, V^*) as a simple Markov orbit basis.*

Proof. Let $x \in X^*$. Then it is clear that the sequence x can be expressed as

$$(20) \quad x = u_0 u_1 u_2 \dots, \quad u_{n-1} \in F(u_n(0)) \quad (n \geq 1)$$

by some $u_n \in F$ since $\sigma^n x \in [F]$ for each n . It follows from Lemma 3 that x can be expressed as (20) by some $u_n \in B^*$, which is unique. On the other hand, it is obvious that the sequence x defined by (20) with some $(u_n)_{n \geq 0} \in M(V^*)$ belongs to the set X^* .

Suppose that x given by (20) has another expression

$$x = (\sigma^i v_0) v_1 v_2 \dots, \quad (v_n)_{n \geq 0} \in M(V^*) .$$

Then, $\sigma^i v_0 \in F$ and so it is decomposed into several words in the sets $F_0(a, b)$'s in virtue of Lemma 3. Since the expression (20) with $u_n \in B^*$ is unique, $\sigma^i v_0 = u_0$ when $i=0$ and $|\sigma^i v_0| \geq |u_0|$ when $i \neq 0$.

Lemma 8. *Let*

$$(21) \quad X^{**} = \bigcup_{n=0}^{\infty} \sigma^n (X \cap [F]^c) ,$$

$$(22) \quad B^{**} = \bigcup_{z \in Z} F^o(z) \cap F(z(0))^c \cap Z ,$$

$$(23) \quad V^{**} = \{(u, v, w) \in B^{**} \times B^{**} \times B^{**}; u \in F^o(vw(0)) \cap F(v(0))^c, \\ v \in F(w(0))^c\} .$$

*Then the shift (X^{**}, σ) admits (B^{**}, V^{**}) as an orbit basis.*

Proof. It follows from Lemma 6 that any $x \in X \cap [F]^c$ can be expressed as

$$(24) \quad x = z'_0 z'_1 \dots, \quad (z'_n, z'_{n+1}, z'_{n+2}) \in V^{**} .$$

Conversely, if x is a sequence defined by (24), then it is evident that $x \in X$. Let us prove that $x \in [F]^c$. In fact, if $x[0, n] \in F(x(n))$ for some n , then it follows from Lemma 2 (i) that there are some m and p such that $0 < p < |z'_m|$

and $z_m[0, p] \in F(z_m(p))$. It contradicts Lemma 6.

Now let the sequence x given by (24) have another expression

$$(25) \quad x = (\sigma^k w'_0) w'_1 w'_2 \cdots, \quad (w'_n, w'_{n+1}, w'_{n+2}) \in V^{**}$$

and suppose that $p = |\sigma^k w'_0|$ satisfy $1 \leq p < |z'_0|$. Then

$$z'_0[0, p] w'_1[0, |z'_0| - p] = z'_0,$$

which contradicts the fact that $z'_0 \in F^\circ(z_1)$. Since $x \in \bigcup_{n=0}^{\infty} \sigma^{-n} X \cap [F]^c$ satisfies $\sigma^n x \in X \cap [F]^c$ for some $n \geq 1$, thus the proof is completed.

Proof of Theorem 3.

The assertion (i) follows from Lemma 7. The first half of (ii) if obtained as Lemma 8 and the second half comes from Lemma 2 of Section 1. Finally, (iii) is obvious from our choice of the sets X^* and X^{**} .

Now we are going to prepare several lemmas used for the proof of Theorem 4.

Let us fix $\zeta = \zeta_a^d$ ($a \in A$, $d \in \{+1, -1\}$) and denote $W = W(X)$,

$$(26) \quad Z = Z(\zeta) = W(T(\zeta)) = \{z_n; n \geq 1\}, \quad z_n = \zeta[0, n],$$

$$(27) \quad P = \{n \geq 1; \varepsilon(z_n) = +1\},$$

$$(28) \quad Q = \{n \in P; \sigma^{n-m} z_n \neq z_m \text{ if } m \in P\}.$$

Lemma 9. (i) If $n \in P$, $m \geq 1$, and $z_n z_m \in W$, then, $z_n z_m = z_{n+m}$.
(ii) If $n \in P$, $m \in Q$ and $z_n z_m \in W$, then, $n \geq m$.

Proof. The assertion (i) follows from Lemma 1 (i) since

$$d(z_n) \varepsilon(z_n) = d = d(z_m).$$

Suppose the contrary to (ii). Then it follows from $n < m$ that

$$z_n z_m = z_{n+m} = z_m \sigma^m z_{n+m} = z_n \sigma^n z_m \sigma^m z_{n+m}.$$

Hence, $z_m = \sigma^n z_m \sigma^m z_{n+m}$ and so $z_{m-n} = \sigma^n z_m$. Consequently, one would obtain $z_m = z_n z_{m-n}$ and $\varepsilon(z_{m-n}) = \varepsilon(\sigma^n z_m) = \varepsilon(z_n) \varepsilon(z_m) = +1$, a contradiction to the fact that $m \in Q$.

Let $x \in T = T(\zeta)$ and, for $N \geq 0$, define a function N^* of N by

$$(29) \quad N^* = \inf \{n \in D; n > N, x[-n, 0] \in Z\}$$

where $D = P$ or P^c . It is obvious that (29) defines a finite valued function of N either with $D = P$ or with $D = P^c$ or both. Let us denote P or P^c by D according as N^* is well-defined with $D = P$ or is not, respectively. Then define

$$(30) \quad N_0 = N_0(x) = 0, \quad N_j = N_j(x) = (N_{j-1})^* \quad (j \geq 1) \quad \text{and}$$

$$(31) \quad n_j = n_j(x) = N_j - N_{j-1} \quad (j \geq 1).$$

Lemma 10. *Let $x \in T_0 = T_0(\zeta)$. Then, $n_j(x) \in Q$ for each $j \geq 1$.*

Proof. Since $\varepsilon(z_n)$ takes a constant value for $n \in D$, $n_j \in P$ is evident. If $n = n_j \in Q^c$, then it would contradict the choice of N_j in virtue of Lemma 9 (i). In fact, if there were $m \in P$ such that $m < n$ and $\sigma^{n-m}z_n = z_m$, then, by Lemma 1.

$$z_m z_{N_{j-1}} = z_{m+N_{j-1}} \quad \text{and} \quad \varepsilon(z_{m+N_{j-1}}) = \varepsilon(z_{N_{j-1}}).$$

Consequently, $n = n_j \in Q$.

Lemma 11. *If ζ is not a periodic sequence, then,*

$$(32) \quad \lim_{j \rightarrow \infty} n_j(x) = \infty \quad \text{for any } x \in T_0 = T_0(\zeta).$$

Proof. It follows from Lemma 9 (ii) and Lemma 10 that the sequence n_j is nondecreasing. Thus, if n_j is bounded in j , then, $n_i = n_{i+1} = \dots$ for some i . Hence, $z_{N_j} = z_{n_i} z_{N_{j-1}}$ for all $j > i$ and so $\zeta = z_{n_i} \zeta$.

Lemma 12. *Let $M \geq 1$ and put*

$$(33) \quad Q_M = \{q \in Q; q \geq M\}.$$

Then the disjoint sets $\sigma^k[z_k] \cap T(\zeta)$, $k \in Q_M$, and the remaining set in $T(\zeta)$ form a partition α_M of $T(\zeta)$ which is a generator for the shift σ , i.e., for any distinct points x and x' , one can find an integer n such that either of the following is valid:

- (a) $\sigma^n x \in \sigma^k[z_k]$, $\sigma^n x' \in \sigma^{k'}[z_{k'}]$, $k \in Q_M$, $k' \in Q_M$ and $k \neq k'$.
- (b) one of $\sigma^n x$ and $\sigma^n x'$ belongs to the union of $\sigma^k[z_k]$, $k \in Q_M$, and the other does not.

Proof. The sequence x in T is determined in virtue of the definition of the set $T(\zeta)$ by any sequence $M_j \rightarrow \infty$ such that $x[-M_j, 0] \in Z$. Hence, it is determined by those $n_j(x)$'s for which $n_j \geq M$, where M is an arbitrary given number. In other words, if such n_j 's coincide with each other for x and x' in T , then, $x = x'$. But the n_j 's are determined by whether $\sigma^n x \in \sigma^k[z_k]$, $k \in Q_M$, n being an integer, or, equivalently, by the images of the partition α_M under the action of σ . Consequently, α_M is a generator.

Next let us apply the method of tower construction stated in Section 2. Let

$$(35) \quad B_M = \{z_n; n \in Q_M\}, \quad Y_M = B_M^Z$$

and τ be the shift on Y_M . The ceiling function is defined as before by

$$(36) \quad \theta(y) = |y_0| \quad \text{if } y = (y_n) \in Y_M.$$

The map i defined by the relation

$$i(y, n) = (\sigma^n y_0) y_1 \cdots \quad (y, n) \in Y_M^0$$

has a natural extension to the map $\bar{i}: Y_M \rightarrow A^Z$ which is given as follows:

$$(37) \quad \begin{aligned} \bar{i}(y, n) &= \sigma^n \bar{i}(y, 0) \quad \text{for } n \neq 0 \quad \text{and} \\ \bar{i}(y, 0) &= \cdots y_{-1} \dot{y}_0 y_1 \cdots \end{aligned}$$

where the dot means that the 0-th coordinate of the sequence is $y_0(0)$.

Lemma 13. *Under the situation above,*

(i) *The map \bar{i} defines a homomorphism of (Y_M^0, τ^0) onto $(T(\zeta), \sigma)$.*

(ii) *$\text{ent}(Y_M^0, \tau^0) \leq -\log t_M$ where t_M is the smallest nonnegative solution of the algebraic equation*

$$(37) \quad 1 - (1-t)^{-1} t^M = 0.$$

Proof. It follows from the iterated application of Lemma 9 (i) that the image of the map \bar{i} is contained in $T = T(\zeta)$. On the other hand, Lemma 10 shows that T is contained in the image. Hence one obtains (i). Recall Corollary 1 to Theorem 1 in Section 1 and Remark 1 after the proof of Corollary 2 of the same theorem. It follows that $\exp[-\text{ent}(Y_M^0, \tau^0)]$ is the smallest non-negative solution of the equation

$$(38) \quad 1 - \sum_{k \in Q_M} t^k = 0.$$

Since $k \geq M$ for $k \in Q_M$, the second assertion (ii) is obtained from the inequality

$$(39) \quad \sum_{k \in Q_M} t^k \leq t^M / (1-t).$$

Proof of Theorem 4.

Now the second assertion (ii) follows from Lemma 13 since t_M converges to 1 as M tends to infinitely

Let us prove the first assertion (i). Let $x \in \partial X = \partial \bar{X}^* \cup \partial \bar{X}^{**}$ be a recurrent sequence. Then, for $B = B^*$ or B^{**} , there are sequences (p_n) and (q_n) of integers with the following properties:

$$(a) \lim p_n = -\infty \quad (b) \inf q_n > -\infty \quad (c) x[p_n, q_n] \in B.$$

If $B = B^{**}$, then $B \subset Z$. Since x is recurrent, one can find $a \in A$ and $d = -1, +1$ such that, for infinitely many n , $x[p_n, q_n] = \zeta_a^d [0, q_n - p_n]$. Consequently, $x \in T(\zeta_a^d)$.

Now let us assume that $B = B^*$ and, for simplicity, $q_n = 0$. Then, $x[p_n, 0] \in F_0(x(p_n), x(1))$. Hence, $x[p_n, i] \in F(x(i))^c$ for $0 \leq i \leq p_n$ by the definition (8). It follows from Lemma 5 (note that Lemma 5 (i)-(iii) hold for all words $x[0, n] \in W \cap F^c$) that there exists a unique expression

$$x[p_n, 0) = z'_0 \cdots z'_{k-1} z'_k$$

where $k=k_n \geq 0$ and $z'_i = z_{n, k_n - i} \in Z$ satisfy Lemma 5 (ii)–(iii) for each n . Since $x \in \overline{X}^{**c}$, it follows from an argument similar to the argument in the proof of Lemma 6 that the length of the word $z_{n,i}$ cannot be bounded in n for some i . Consequently, $x \in T(\zeta)$ for some ζ , and the proof is completed.

References

- [1] J. Hadamard: *Essai sur l'étude des fonctions données par leur développement de Taylor*, in Oeuvres de Jacques Hadamard, tom I, 7–92, Centre National de la Recherche Scientifique, Paris.
- [2] F. Hofbauer: β -shifts have unique maximal measures, *Monatsh. Math.* **85** (1978), 189–198.
- [3] S. Ito, H. Nakada and S. Tanaka: *On unimodal linear transformations I, II*, *Tokyo J. Math.* **2** (1979), 221–259.
- [4] S. Ito and Y. Takahashi: *Markov shifts and realization of β -transformations*, *J. Math. Soc. Japan* **26** (1974), 33–55.
- [5] M. Mori: *The extension of Sarkovskii's results and the topological entropy in unimodal transformations*, *Tokyo J. Math.* **4** (1981), 133–152.
- [6] D. Ruelle: *Thermodynamical formalism*, *Encyclopedia of Mathematics*, vol. 5, Addison-Wesley, 1979.
- [7] S. Shinada: *Decomposition theorem for words in dynamical systems of closed intervals and its application* (in Japanese), Tokyo Inst. Technology, 1980, unpublished.
- [8] I.P. Cornfeld, Ya.G. Sinai and V.S. Fomin: *Ergodic theory*, Nauka, 1980; English translation, Springer-Verlag, 1982.
- [9] Y. Takahashi: *β -transformations and symbolic dynamics*, *Proc. Second Japan-USSR Symp. on Prob. Theory*, Springer Lecture Notes in Math. 330 (1973), 455–464.
- [10] ———: *Isomorphisms of β -automorphisms to Markov automorphisms*, *Osaka J. Math.* **10** (1973), 175–184.
- [11] ———: *A formula for topological entropy of one-dimensional dynamics*, *Sci. Papers College Gen. Educ. Univ. Tokyo* **30** (1980), 11–22.
- [12] ———: *Fredholm determinant of unimodal linear maps*, loc. *ibid.* **31** (1981), 61–87.

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