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DUALITY IN GENERALIZED HOMOGENEOUS PROGRAMMING

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

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1. Introduction with problem setting

Homogeneous programming problems were first studied by Eisenberg [1] in finite dimensional spaces and next by Schechter [7]. In this paper we shall be concerned with more generalized homogeneous programming problems and their duality relations.

More precisely, let X and Y be real linear spaces which are in duality with respect to a bilinear functional $\langle\cdot, \cdot\rangle_1$ and let Z and W be real linear spaces which are in duality with respect to a bilinear functional $\langle\cdot, \cdot\rangle_2$. Hereafter we denote $\langle\cdot, \cdot\rangle_1$ and $\langle\cdot, \cdot\rangle_2$ by $\langle\cdot, \cdot\rangle$ for simplicity. In this paper, we assume that each one of the paired spaces is assigned the weak topology unless otherwise stated. We denote by $\tau(X, Y)$ the Mackey topology on X. We also assume that the cones considered have their vertices at the origin of the space.

Let P and Q be closed convex cones in X and Z respectively and denote by P° and Q° the dual cones of P and Q. Let f be an extended real valued function on X which is lower semicontinuous and sublinear, i.e., the epigraph $\{(x, r)\in X\times R; f(x)\leq r\}$ of f is a closed convex cone or the empty set, and let g be an extended real valued function on W which is upper semicontinuous and superlinear, i.e., $-g$ is sublinear. Note that if f is finite at some point, then f does not take the value $-\infty$. Let Ψ be an extended real valued function on $X\times W$ such that $\Psi_x=\Psi(x, \cdot)$ is lower semicontinuous and sublinear on W for every fixed $x\in X$ and $\Psi_w=\Psi(\cdot, w)$ is upper semicontinuous and superlinear on X for every fixed $w\in W$. We assume that $\Psi(0, 0)=f(0)=g(0)=0$.

For the quintuple (Ψ, P, Q°, f, g), we consider the following generalized homogeneous programming problems (=HP) and its dual problem (=DHP):

(HP) Find $M=\inf\{f(x);\ x\in S\}$,

where $S=\{x\in P; g(w)\leq\Psi(x, w)$ for all $w\in Q^\circ\}$.

(DHP) Find $M^*=\sup\{g(w);\ w\in S^*\}$,

where $S^*=\{w\in Q^\circ; f(x)\geq\Psi(x, w)$ for all $x\in P\}$.

Here we use the convention that the infimum of a real function on the empty set $\emptyset$ is equal to $+\infty$.

Our aim is to find some conditions which assure that the above two problems have the same value and have optimal solutions. In the case where Ψ is continuous and bilinear, Schechter [7] investigated duality relations for these problems. In the next section, we introduce programming problems with constraints of convex processes studied in [5], and state some relations between those problems. In §3 and §4, we give main results. In §4, we deal with the case where Ψ is bilinear and improve a result in [7].

2. Reduction of HP and DHP

In order to obtain a convex process and its adjoint process from Ψ, we consider the following two sets:

$$\mathrm{dom}_X \Psi = \{x \in X;\ \Psi(x, w) \text{ is finite for some } w \in W\}\ ,$$
$$\mathrm{dom}_W \Psi = \{w \in W;\ \Psi(x, w) \text{ is finite for some } x \in X\}\ .$$

If $x \in \mathrm{dom}_X \Psi$, then $\Psi(x, 0)=0$ and $\Psi(x, w) \neq -\infty$ for all $w \in W$. Thus $\mathrm{dom}_X \Psi = \{x \in X;\ \Psi(x, 0)=0\}$ and this set is closed, since $\Psi(\cdot, 0)$ is upper semicontinuous on X. If $w \in \mathrm{dom}_X \Psi$, then $\Psi(0, w)=0$ and $\Psi(x, w) \neq +\infty$ for all $x \in X$. Thus $\mathrm{dom}_W \Psi = \{w \in W;\ \Psi(0, w)=0\}$ and this set is closed. Note that $\Psi(x, w)$ is finite if and only if $x \in \mathrm{dom}_X \Psi$ and $w \in \mathrm{dom}_W \Psi$.

We recall the subdifferential $\partial f(0)$ of f and the superdifferential $\partial g(0)$ of g at the origins:

$$\partial f(0) = \{y \in Y;\ \langle x, y\rangle \leq f(x) \quad \text{for all } x \in X\}\ ,$$
$$\partial g(0) = \{z \in Z;\ \langle z, w\rangle \geq g(w) \quad \text{for all } w \in W\}\ .$$

It is well-known that $\partial f(0)$ and $\partial g(0)$ are nonempty closed convex sets, and that $f(x) = \sup_{y \in \partial f(0)} \langle x, y\rangle$ for all $x \in X$ and $g(w) = \inf_{z \in \partial g(0)} \langle z, w\rangle$ for all $w \in W$. If f is $\tau(X, Y)$-continuous, then $\partial f(0)$ is weakly compact (cf. [5; Lemma 1]).

Since Ψ_x is lower semicontinuous and sublinear on W, we can define the subdifferential $\partial \Psi_x(0)$ of Ψ_x at the origin for $x \in \mathrm{dom}_X \Psi$:

$$\partial \Psi_x(0) = \{z \in Z;\ \langle z, w\rangle \leq \Psi(x, w) \quad \text{for all } w \in W\}\ .$$

Now we define a set-valued mapping A from X to Z by

(2.1) $\quad Ax = \partial \Psi_x(0)$ if $x \in \mathrm{dom}_X \Psi$, and $Ax = \emptyset$ if $x \notin \mathrm{dom}_X \Psi$.

As an infinite version of [6; Theorem 39.4], we have

Proposition 1. *The mapping A is a closed convex process from X to Z, i.e.,* graph $A = \{(x, z);\ x \in \mathrm{dom}_X \Psi,\ z \in Ax\}$ *is a closed convex cone in $X \times Z$.*

Proof. It is easy to check that $tz\in A(tx)$ if $z\in Ax$ and $t>0$. Let $x_1, x_2\in \mathrm{dom}_X\Psi$, $z_1\in Ax_1$ and $z_2\in Ax_2$. Since $\Psi(x_1+x_2, 0)\geq\Psi(x_1, 0)+\Psi(x_2, 0)=0$ and $\Psi(\cdot, 0)$ does not take the value $+\infty$, $x_1+x_2\in\mathrm{dom}_X\Psi$. For all $w\in W$, $\Psi(x_1+x_2, w)\geq\Psi(x_1, w)+\Psi(x_2, w)\geq\langle z_1, w\rangle+\langle z_2, w\rangle=\langle z_1+z_2, w\rangle$. Thus $z_1+z_2\in A(x_1+x_2)$ and graph A is a convex cone.

Let $\{(x_\alpha, z_\alpha)\}$ be a net in graph A which converges to (x_0, z_0). Since $\mathrm{dom}_X\Psi$ is closed, $x_0\in\mathrm{dom}_X\Psi$. For all $w\in W$, $\Psi(x_0, w)\geq\limsup\Psi(x_\alpha, w)\geq\limsup\langle z_\alpha, w\rangle=\langle z_0, w\rangle$. Thus $z_0\in Ax_0$ and graph A is closed.

We regard A as a supremum oriented convex process (see [5] or [6]). Then the adjoint A^* of A is defined by $A^*w=\{y\in Y; \langle x, y\rangle\geq\langle z, w\rangle$ for all $(x, z)\in$ graph $A\}$.

Proposition 2. *$A^*w=\partial\Psi_w(0)=\{y\in Y; \langle x, y\rangle\geq\Psi(x, w)$ for all $x\in X\}$ if $w\in\mathrm{dom}_W\Psi$, and $A^*w=\emptyset$ if $w\notin\mathrm{dom}_W\Psi$.*

Proof. Note that $\Psi(x, w)=\sup_{z\in Ax}\langle z, w\rangle=\inf_{y\in\partial\Psi_w(0)}\langle x, y\rangle$ for all $x\in\mathrm{dom}_X\Psi$ and $w\in\mathrm{dom}_W\Psi$ (cf. [5; Lemma 1]). Let $w_0\in\mathrm{dom}_W\Psi$. If $y_0\in\partial\Psi_{w_0}(0)$, then $\langle x, y_0\rangle\geq\Psi(x, w_0)\geq\langle z, w_0\rangle$ for all $(x, z)\in$graph A. Thus $\partial\Psi_{w_0}(0)\subset A^*w_0$. Conversely if $y_0\in A^*w_0$, then $\langle x, y_0\rangle\geq\langle z, w_0\rangle$ for all $x\in\mathrm{dom}_X\Psi$ and $z\in Ax$. Thus $\langle x, y_0\rangle\geq\sup_{z\in Ax}\langle z, w_0\rangle=\Psi(x, w_0)$ for all $x\in\mathrm{dom}_X\Psi$. Since $\Psi(x, w_0)=-\infty$ if $x\notin\mathrm{dom}_X\Psi$, $\langle x, y_0\rangle\geq\Psi(x, w_0)$ for all $x\in X$. Therefore $y_0\in\partial\Psi_{w_0}(0)$ and $A^*w_0=\partial\Psi_{w_0}(0)$.

Let $w_0\notin\mathrm{dom}_W\Psi$. If $y_0\in A^*w_0$, then similarly we see that $\langle x, y_0\rangle\geq\sup_{z\in Ax}\langle z, w_0\rangle=\Psi(x, w_0)=+\infty$ for all $x\in\mathrm{dom}_X\Psi$. This is a contradiction, since $\mathrm{dom}_X\Psi$ is nonempty. Thus $A^*w_0=\emptyset$. This completes the proof.

Corollary. *If $x\in\mathrm{dom}_X\Psi$ or $w\in\mathrm{dom}_W\Psi$, then $\Psi(x, w)=\sup_{z\in Ax}\langle z, w\rangle=\inf_{y\in A^*w}\langle x, y\rangle$.*

In connection with HP and DHP, we consider the following extremum problems defined by the quintuple (A, P, Q, f, g):

(2.2) Find $\hat{M}=\inf\{f(x); x\in\hat{S}\}$,

where $\hat{S}=\{x\in P; (Ax-\partial g(0))\cap Q\neq\emptyset\}$.

(2.3) Find $\hat{M}^*=\sup\{g(w); w\in\hat{S}^*\}$,

where $\hat{S}^*=\{w\in Q^\circ; (\partial f(0)-A^*w)\cap P^\circ\neq\emptyset\}$.

We have

Proposition 3. (1) *$\hat{S}\subset S$, $\hat{S}^*\subset S^*$ and $\hat{M}^*\leq M^*\leq M\leq\hat{M}$.*

(2) *If $Q+\partial g(0)-Ax$ is closed for every $x\in P$, then $\hat{S}=S$.*

(3) *If $P^\circ-\partial f(0)+A^*w$ is closed for every $w\in Q^\circ$, then $\hat{S}^*=S^*$.*

Proof. (1) Let $x\in\hat{S}$. Then there exist $z_1\in Ax$, $z_2\in\partial g(0)$ and $q\in Q$ such that $q=z_1-z_2$. For all $w\in Q^\circ$, $g(w)\leq\langle z_2, w\rangle=\langle z_1-q, w\rangle\leq\langle z_1, w\rangle\leq\Psi(x, w)$. Thus $x\in S$. Similarly we see that $\hat{S}^*\subset S^*$. It is easy to check that $M^*\leq M$. Therefore $\hat{M}^*\leq M^*\leq M\leq\hat{M}$.

(2) We assume that $x\in S$ and $x\notin\hat{S}$. Then $(Ax-\partial g(0))\cap Q=\emptyset$. If Ax is emtpy, then $\Psi(x, 0)=-\infty$. This is impossible since $\Psi(x, 0)\geq g(0)=0$. Thus Ax is nonempty. Since $0\notin Q+\partial g(0)-Ax$, by the separation theorem there exist $w_0\in W$ and $\mu>0$ such that $\langle q+z_1-z_2, w_0\rangle\geq\mu$ for all $q\in Q$, $z_1\in\partial g(0)$ and $z_2\in Ax$. Then $w_0\in Q^\circ$, $\langle z_1, w_0\rangle\geq\mu+\langle z_2, w_0\rangle$ and thus $g(w_0)=\inf_{z_1\in\partial g(0)}\langle z_1, w_0\rangle>\sup_{z_2\in Ax}\langle z_2, w_0\rangle=\Psi(x, w_0)$. This is a contradiction. Thus $\hat{S}\supset S$. By (1), we see that $\hat{S}=S$.

(3) By Proposition 2, we can similarly see that $\hat{S}^*=S^*$.

By the aid of Proposition 3, the following duality theorem for (2.2) and (2.3) is also applicable to HP and DHP in the case where f is $\tau(X, Y)$-continuous on X. See [5; Theorem 1].

Theorem A. *Assume that f is $\tau(X, Y)$-continuous on X and the following two conditions are satisfied:*

(2.4) *The set $G=\{(x, -z, f(x)+r);\ x\in\mathrm{dom}_X\Psi,\ z\in Ax,\ r\geq 0\}+(-P)\times(Q+\partial g(0))\times\{0\}$ is a closed subset of $X\times Z\times R$.*

(2.5) $\hat{S}\neq\emptyset$ or $\hat{S}^*\neq\emptyset$.

Then $\hat{M}=\hat{M}^*$. Furthermore if $\hat{M}$ is finite, then there exists $x_0\in\hat{S}$ such that $f(x_0)=\hat{M}$, i.e., problem (2.2) has an optimal solution.

3. First duality theorem

In this section, we establish a duality theorem by using the method of Rockafellar as in [7].

Theorem 1. *Assume that the following two conditions hold:*

(3.1) $\mathrm{dom}_X\Psi\supset P$ *or* $\mathrm{dom}_W\Psi\supset Q^\circ$.

(3.2) *There exists $w_0\in Q^\circ$ such that $g(w_0)\neq-\infty$ and the $\tau(Y,X)$-interior of $(\partial f(0)-P^\circ-A^*w_0)$ contains the origin. Then $\hat{M}^*=M^*=M$. Furthermore if $S\neq\emptyset$, then HP has an optimal solution.*

Proof. Condition (3.2) implies $\hat{M}^*\neq-\infty$. Since $\hat{M}^*\leq M^*\leq M$, we may assume that $\hat{M}^*$ is finite. We define a convex function Φ on $W\times Y$ by

$$\Phi(w, y)=-g(w)+\delta(w\,|\,Q^\circ)+\delta(y\,|\,\partial f(0)-P^\circ-A^*w)\,,$$

where $\delta(w\,|\,Q^\circ)=0$ for $w\in Q^\circ$ and $\delta(w\,|\,Q^\circ)=+\infty$ for $w\notin Q^\circ$. Then $-\hat{M}^*=$

$\inf\{\Phi(w, 0);\, w\in W\}$. Let Φ^* be the conjugate function of Φ:

$$\Phi^*(z, x) = \sup\{\langle z, w\rangle + \langle x, y\rangle - \Phi(w, y);\, w\in W, y\in Y\}\ ,$$

for $z\in Z$ and $x\in X$. Then

$$\begin{aligned}\Phi^*(0, x) &= \sup\{\langle x, y\rangle + g(w);\, w\in Q^\circ \cap \mathrm{dom}_W\Psi,\, y\in\partial f(0) - P^\circ - A^*w\}\\ &= \sup\{\langle x, y_1\rangle - \langle x, y_2\rangle + (g(w) - \langle x, y_3\rangle);\\ &\qquad w\in Q^\circ \cap \mathrm{dom}_W\Psi,\, y_1\in\partial f(0),\, y_2\in P^\circ,\, y_3\in A^*w\}\ .\end{aligned}$$

In case $x\in S$, $-\langle x, y_2\rangle \leq 0$ and $g(w) - \langle x, y_3\rangle \leq 0$ so that $\Phi^*(0, x) = \sup_{y\in\partial f(0)} \langle x, y\rangle = f(x)$. In case $x\notin P$, $\sup_{y\in P^\circ} - \langle x, y\rangle = +\infty$ so that $\Phi^*(0, x) = +\infty$. We consider the case where $x\in P$ and $\Psi(x, \bar{w}) < g(\bar{w})$ for some $\bar{w}\in Q^\circ$. If $\bar{w}\neq \mathrm{dom}_W\Psi$, then $x\in\mathrm{dom}_W\Psi$ by (3.1) so that $\Psi(x, \bar{w}) = +\infty$. This is a contradiction. Therefore $\bar{w}\in\mathrm{dom}_W\Psi$ and there exists $\bar{y}\in A^*\bar{w}$ such that $\langle x, \bar{y}\rangle < g(\bar{w})$ by Corollary of Proposition 2. Since $t\bar{y}\in A^*(t\bar{w})$ for all $t>0$, we have $\Phi^*(0, x) = +\infty$. Thus $-M = \sup_{x\in X} - \Phi^*(0, x)$.

Condition (3.2) implies that $\Phi(w_0, y)$ is bounded above by $-g(w_0)$ in a $\tau(Y, X)$-neighborhood of 0. By [2; Proposition 2.5 in Chapter I], we see that $\Phi(w_0, y)$ is continuous in a $\tau(Y, X)$-neighborhood of 0. Thus by [2; Proposition 2.3 in Chapter III], we have $\hat{M}^* = M$ and HP has an optimal solution. Since $\hat{M}^* \leq M^* \leq M$, this completes the proof.

Now we examine condition (3.2). First we define a closed convex process $\tilde{A}$ from X to Z which is obtained by a modification of Ψ. We set $\tilde{\Psi}(x, w) = \Psi(x, w)$ if $x\in P$ and $w\in Q^\circ$, $\tilde{\Psi}(x, w) = +\infty$ if $x\in P$ and $w\notin Q^\circ$, and $\tilde{\Psi}(x, w) = -\infty$ if $x\notin P$. We define $\tilde{A}$ by replacing Ψ by $\tilde{\Psi}$ in (2.1).

Proposition 4. *Assume that Ψ is finite on $P\times Q^\circ$. If the $\tau(Y, X)$-interior* $\mathrm{int}(\partial f(0) - P^\circ)$ *of* $\partial f(0) - P^\circ$ *is nonempty, then the following three conditions are equivalent:*

(3.3) *There exists $w_0\in Q^\circ$ such that $A^*w_0 \cap \mathrm{int}(\partial f(0) - P^\circ) \neq \emptyset$.*

(3.4) *There exists $w_0\in Q^\circ$ such that $\Psi(x, w_0) < f(x)$ for all $x\in P$ with $x\neq 0$.*

(3.5) *$x\in P$, $\tilde{A}x \cap Q \neq \emptyset$ and $f(x)\leq 0$ imply $x=0$.*

Proof. First we assume that (3.3) holds. Let $y_0\in A^*w_0 \cap \mathrm{int}(\partial f(0) - P^\circ)$ and $x\in P$ with $x\neq 0$. Then there exist $y\in Y$ and $t>0$ such that $\langle x, y\rangle > 0$ and $y_0 + ty\in\partial f(0) - P^\circ$. Then $y_0 + ty = y' - y''$ for some $y'\in\partial f(0)$ and $y''\in P^\circ$. We have $\Psi(x, w_0) \leq \langle x, y_0\rangle = \langle x, y' - y'' - ty\rangle \leq \langle x, y'\rangle - t\langle x, y\rangle < f(x)$. Thus (3.4) holds.

Next we assume that (3.4) holds. Let x be an element in P such that $\tilde{A}x \cap Q \neq \emptyset$ and $f(x)\leq 0$. Then for $z\in\tilde{A}x\cap Q$, $\Psi(x, w_0) \geq \langle z, w_0\rangle \geq 0 \geq f(x)$. Thus from (3.4) it follows that $x=0$.

Finally we assume that (3.5) holds. If (3.3) does not hold, then $A^*(Q^\circ)\cap \mathrm{int}(\partial f(0)-P^\circ)=\emptyset$. Then by the separation theorem, there exists $x_0\in X$ with $x_0\neq 0$ such that $\langle x_0, y'-y''\rangle\leq 0$ for all $y'\in\partial f(0)$ and $y''\in P^\circ$ and $\langle x_0, y\rangle\geq 0$ for all $w\in Q^\circ\cap \mathrm{dom}_W\Psi$ and $y\in A^*w$. From the first inequality, it follows that $x_0\in P^{\circ\circ}=P$ and $f(x_0)\leq 0$. By the second inequality, we have $\Psi(x_0, w)\geq 0$ for all $w\in Q^\circ$. Thus $0\in\tilde{A}x_0\cap Q$ and this is a contradiction. Hence (3.5) implies (3.3). This completes the proof.

If A is continuous and linear, it is easy to check that $\tilde{A}$ can be replaced by A in (3.5). Thus Proposition 4 is an improvement of [7; Lemma 3.2]. From Theorem 1 and Proposition 4, we have

Corollary. *Assume that g is finite on Q°, Ψ is finite on $P\times Q^\circ$ and $\tau(Y, X)$-interior of $\partial f(0)-P^\circ$ is nonempty. If (3.5) holds and $\hat{M}^*$ is finite, then $\hat{M}^*=M^*=M$ and HP has an optimal solution.*

We shall show that $\tilde{A}$ cannot be replaced by A in (3.5) in general.

Example. We take $X=Y=R^2$, $Z=W=R^3$, $P=R^2_+=\{(x_1, x_2);\ x_1\geq 0,\ x_2\geq 0\}$ and $Q=\{(z_1, z_2, z_3);\ z_1\leq 0,\ z_2\leq 0,\ -\infty<z_3<+\infty\}$. We set $f(x)=-x_1x_2/(x_1+x_2)$ if $x=(x_1, x_2)\in P$ with $x\neq 0$, $f(x)=0$ if $x=0$, $f(x)=+\infty$ if $x\notin P$ and $g(w)=w_1+w_2$ for all $w=(w_1, w_2, w_3)\in W$. Then $P^\circ=\{(y_1, y_2);\ y_1\geq 0, y_2\geq 0\}$, $Q^\circ=\{(w_1, w_2, w_3);\ w_1\leq 0,\ w_2\leq 0,\ w_3=0\}$ and $\partial g(0)=\{(1, 1, 0)\}$. By the definition of $\partial f(0)$, $(y_1, y_2)\in\partial f(0)$ if and only if $-x_1x_2\geq(x_1y_1+x_2y_2)(x_1+x_2)$ for all positive numbers x_1 and x_2. By setting $t=x_1/x_2$, $(y_1, y_2)\in\partial f(0)$ if and only if $t^2y_1+t(y_1+y_2+1)+y_2\leq 0$ for all $t\geq 0$. From this we easily see that $\partial f(0)=\{(y_1, y_2)\in -R^2_+;\ y_1+y_2+1\leq 0$ or $4y_1y_2\geq(y_1+y_2+1)^2\}$.

Next we set $\Psi(x, w)=-2[(x_1^2w_2+x_2^2w_3)w_1]^{1/2}$ if $x=(x_1, x_2)\in R^2_+$ and $w=(w_1, w_2, w_3)\in -R^3_+$, $\Psi(x, w)=+\infty$ if $x\in R^2_+$ and $w\notin -R^3_+$, and $\Psi(x, w)=-\infty$ if $x\notin R^2_+$.

We show that $Ax=\{(z_1, z_2, z_3);\ z_1z_2\geq x_1^2,\ z_1z_3\geq x_2^2,\ z_1\geq 0\}$ if $x=(x_1, x_2)\in R^2_+$ and $Ax=\emptyset$ if $x\notin R^2_+$. Let $x_1\geq 0$ and $x_2\geq 0$. If $(z_1, z_2, z_3)\in Ax$, then $-2[(x_1^2w_2+x_2^2w_3)w_1]^{1/2}\geq w_1z_1+w_2z_2+w_3z_3$ for all negative numbers w_1, w_2 and w_3. We easily see that $z_1\geq 0$, $z_2\geq 0$ and $z_3\geq 0$. Furthermore we have $\psi(\alpha, \beta)=[(w_1z_1+w_2z_2+w_3z_3)^2-4(x_1^2w_2+x_2^2w_3)w_1]/w_1^2=\alpha^2z_2^2+2\alpha(z_1z_2-2x_1^2+z_1z_3\beta)+\beta^2z_3^2+2\beta(z_1z_3-2x_2^2)+z_1^2\geq 0$ where $\alpha=w_2/w_1$ and $\beta=w_3/w_1$. Since $\psi(\alpha, 0)\geq 0$ for all $\alpha\geq 0$, we have $z_1z_2\geq x_1^2$. Similarly $z_1z_3\geq x_2^2$. Conversely if z_1, z_2 and z_3 are nonnegative, $z_1z_2\geq x_1^2$ and $z_1z_3\geq x_2^2$, then

$$\begin{aligned}(w_1z_1+w_2z_2+w_3z_3)^2&\geq 4w_1w_2z_1z_2+4w_1w_3z_1z_3\\ &\geq 4(x_1^2w_2+x_2^2w_3)w_1\end{aligned}$$

for all negative numbers w_1, w_2 and w_3, and thus $(z_1, z_2, z_3)\in Ax$.

Similarly we have $\tilde{A}x=\{(z_1, z_2, z_3);\ z_1z_2\geq x_1^2,\ z_1\geq 0\}$. Thus we see that $x\in P$ and $Ax\cap Q\neq\emptyset$ imply $x=0$, but condition (3.5) is not satisfied.

We can easily see that $A^*w=\hat{A}^*w=\{(y_1,y_2);\ y_1\geq -2(w_1w_2)^{1/2}, y_2\geq 0\}$. Thus $M=M^*=\hat{M}^*=-1$. Since $x=(x_1, x_2)\in S$ if and only if $0\leq x_1\leq 1$ and $x_2\geq 0$, we see that HP has no optimal solution. Finally we note that all the conditions except (3.5) in Corollary hold.

REMARK. Fujimoto's result [3; Theorem 2.1] follows from Proposition 4.

4. Second duality theorem

In this section, we give another duality theorem under the assumption that Ψ is bilinear, $\Psi(x, \cdot)$ is continuous on W for every $x\in X$ and $\Psi(\cdot, w)$ is continuous on X for every $w\in W$. This assumption is equivalent to that the mapping A defined by (2.1) is continuous and linear.

For a closed convex subset C of X, we recall the asymptotic cone ac C of C:

$$\mathrm{ac}\, C = \bigcap_{t>0} t(C-x), \text{ where } x\in C .$$

In connection with the asymptotic cone, we have two lemmas.

Lemma 1. *Let C and D be closed convex subsets of X. If C is locally compact and* $\mathrm{ac}\, C\cap(-\mathrm{ac}\, D)$ *is a linear subspace, then $C+D$ is closed.*

This lemma was proved by Zălinescu [8; Proposition 7] in the case where the projection of C to X/X' ($X'=\mathrm{ac}\, C\cap(-\mathrm{ac}\, D)$) is locally compact. It suffices to note that the projection of C is locally compact in this case.

Lemma 2. *Assume that $\{w\in Q^\circ;\ g(w)>-\infty\}$ is dense in Q°. Then* $\mathrm{ac}\, \partial g(0)$ *is contained in Q. Furthermore if $Q+\partial g(0)$ is closed, then* $\mathrm{ac}(Q+\partial g(0))=Q$.

Proof. If $Q+\partial g(0)$ is closed, then $\mathrm{ac}(Q+\partial g(0))$ is well-defined. Let $z\in \mathrm{ac}(Q+\partial g(0))$ and $z_0\in\partial g(0)$. Then $tz+z_0\in Q+\partial g(0)$ for all $t>0$. There exist $z_t\in\partial g(0)$ and $q_t\in Q$ such that $tz+z_0=z_t+q_t$. For all $w\in Q^\circ$ and $t>0$, $\langle tz+z_0, w\rangle=\langle z_t+q_t, w\rangle\geq\langle z_t, w\rangle\geq g(w)$. It follows that $\langle z, w\rangle\geq 0$ for all $w\in Q^\circ$ such that $g(w)>-\infty$ and hence for all $w\in Q^\circ$. Thus $z\in Q^{\circ\circ}=Q$. Since $\mathrm{ac}(Q+\partial g(0))\supset Q$, $\mathrm{ac}(Q+\partial g(0))=Q$. Similarly we can check that $\mathrm{ac}\, \partial g(0)\subset Q$.

As the first step toward the second duality theorem, we prove

Lemma 3. *The equality $M=M^*$ holds if the following four conditions are fulfilled:*

(4.1) *P is locally compact and $Q+\partial g(0)$ is closed.*

(4.2) *f is $\tau(X, Y)$-continuous on X and g is finite on Q°.*

(4.3) *$x\in P$, $Ax\in Q$ and $f(x)\leq 0$ imply $x=0$.*

(4.4) *$S \neq \emptyset$ or $S^* \neq \emptyset$.*

Proof. We apply Theorem A to (A, P, Q, f, g). Since f is $\tau(X, Y)$-continuous, $\partial f(0)$ is weakly compact and thus $P^\circ - \partial f(0)$ is closed. By Proposition 3, we see that $S = \hat{S}$ and $S^* = \hat{S}^*$. From (4.4) it follows that condition (2.5) in Theorem A is satisfied.

We set $G_0 = \{(x, -Ax, f(x)+r); x \in X\ r \geq 0\}$. We show that the set $G = G_0 + (-P) \times (Q + \partial g(0)) \times \{0\}$ is closed. By the continuity of A and the lower semicontinuity of f, we easily check that $G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}$ is closed. By (4.2) and Lemma 2, we see that $\mathrm{ac}[G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}] = G_0 + \{0\} \times \mathrm{ac}(Q + \partial g(0)) \times \{0\} = G_0 + \{0\} \times Q \times \{0\}$, and by (4.3), we see that $\mathrm{ac}[G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}] \cap P \times \{0\} \times \{0\} = \{(0, 0, 0)\}$. From (4.1) and Lemma 1 it follows that $G = [G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}] + (-P) \times \{0\} \times \{0\}$ is closed and thus condition (2.4) also holds. Thus by Theorem A and Proposition 3, we see that $M = M^*$.

As for the existence of an optimal solution for HP, we obtain

Lemma 4. *Assume that* (4.1) *and* (4.3) *are satisfied. If g is finite on Q° and $S \neq \emptyset$, then HP has an optimal solution.*

Proof. We may assume that $M \neq +\infty$. Let $\{x_\alpha\} \subset S$ be a net such that $\{f(x_\alpha)\}$ converges to M. Since P is locally compact, there exists a neighborhood U of the origin of X such that $P \cap U$ is compact. We set

$$K = \{x \in P \cap U; x \notin 2^{-1} U^i\},$$

where U^i is the interior of U. Then there exist $t_\alpha > 0$ and $\bar{x}_\alpha \in K$ such that $x_\alpha = t_\alpha \bar{x}_\alpha$. Since K is compact, there exists a subnet of $\{\bar{x}_\alpha\}$ which converges to an element $\bar{x} \in K$. We may assume that $\{\bar{x}_\alpha\}$ converges to $\bar{x}$. We show that there exists a subnet of $\{t_\alpha\}$ which converges to a real number $t_0 \geq 0$. Otherwise, $\lim t_\alpha = +\infty$. Let $z_0 \in \partial g(0)$ and $s > 0$. Then $sA(t_\alpha^{-1} x_\alpha) + (1 - s t_\alpha^{-1}) z_0 \in Q + \partial g(0)$ for all α such that $s t_\alpha^{-1} < 1$. Since $Q + \partial g(0)$ is closed, $\lim \{sA(t_\alpha^{-1} x_\alpha) + (1 - s t_\alpha^{-1}) z_0\} = sA\bar{x} + z_0 \in Q + \partial g(0)$. Thus we see $A\bar{x} \in \mathrm{ac}(Q + \partial g(0)) = Q$ by Lemma 1. Since $f(\bar{x}) \leq \liminf f(\bar{x}_\alpha) = \liminf t_\alpha^{-1} f(x_\alpha) \leq 0$, from (4.3) it follows that $\bar{x} = 0$. This is a contradiction, since $0 \notin K$. Thus $\{t_\alpha\}$ contains a convergent subnet. Denote the subnet by $\{t_\alpha\}$ again and let t_0 be its limit. Then $\lim x_\alpha = \lim t_\alpha \bar{x}_\alpha = t_0 \bar{x}$ and $M = \lim f(x_\alpha) \geq f(t_0 \bar{x})$. Since S is closed, $t_0 \bar{x} \in S$ and $f(t_0 \bar{x}) \geq M$. Thus $M = f(t_0 \bar{x})$ and HP has an optimal solution. This completes the proof.

Now we prove the second duality theorem.

Theorem 2. *Assume that* (4.3), (4.4) *in Lemma* 3 *and the following* (4.1′) *and* (4.2′) *are satisfied:*

(4.1′) *P is locally compact and $P^\circ - \partial f(0)$, $Q + \partial g(0)$ are closed.*

(4.2′) *g is finite on Q°.*

Then $M=M^*$. Furthermore if $S\neq\emptyset$, then HP has an optimal solution.

Proof. For arbitrary $y_1, \cdots, y_k \in \partial f(0)$, the function $h(x)=\max\{\langle x, y_j\rangle; j=1, \cdots, k\}$ is continuous and sublinear on X. By J we denote the set of all such functions. Then J is directed by a natural ordering and increases to f at each point in X. For each $h\in J$, we set

$$\begin{aligned} M_h &= \inf\{h(x);\, x\in S\}\ , \\ S_h^* &= \{w\in Q^\circ;\, \langle Ax, w\rangle \leq h(x) \quad \text{for all } x\in P\}\ , \\ M_h^* &= \sup\{g(w);\, w\in S_h^*\}\ . \end{aligned}$$

By Proposition 3, we see that $S_h^*=\{w\in Q^\circ; A^*w\in\partial h(0)-P^\circ\}$ and $S^*=\{w\in Q^\circ; A^*w\in\partial f(0)-P^\circ\}$. Since $\{\partial h(0); h\in J\}$ is an increasing net of sets and $\bigcup_{h\in J}\partial h(0)=\partial f(0)$, $\{S_h^*; h\in J\}$ increases to S^* and $\lim_{h\in J}M_h^*=M^*$.

In order to show that $\lim_{h\in J}M_h=M$ and $M_h=M_h^*$ for all sufficiently large $h\in J$, we examine condition (4.3). Condition (4.3) is equivalent to the condition that $f(x)>0$ for all $x\in P\cap A^{-1}(Q)$ such that $x\neq 0$. Let K be the set as in the proof of Lemma 4. Then $f(x)>0$ for all $x\in K\cap A^{-1}(Q)$. Since f is lower semicontinuous and $K\cap A^{-1}(Q)$ is compact, $\inf_{x\in K\cap A^{-1}(Q)}f(x)>\mu$ for $\mu>0$. For any $x_0\in K\cap A^{-1}(Q)$ there exists $y_0\in\partial f(0)$ such that $\langle x_0, y_0\rangle>\mu$, since $\sup_{x\in\partial f(0)}\langle x, y\rangle=f(x)$ for all $x\in X$. Since $\langle\cdot, y_0\rangle$ is continuous on X, there exists a neighborhood V_0 of x_0 such that $\langle x, y_0\rangle>\mu$ for all $x\in V_0$. Since $K\cap A^{-1}(Q)$ is compact, there exist $x_1, \cdots, x_n\in K, y_1^0, \cdots, y_n^0\in\partial f(0)$ and $V_1, \cdots, V_n$ such that V_j is a neighborhood of x_j, $\langle x, y_j^0\rangle>\mu$ for each j and $x\in V_j$, and $\bigcup_{j=1}^n V_j\supset K\cap A^{-1}(Q)$. Then $h_0(x)=\max\{\langle x, y_j^0\rangle; j=1, \cdots, n\}>\mu$ on $K\cap A^{-1}(Q)$. We set $J'=\{h\in J; h\geq h_0\}$. Then $x\in P$, $Ax\in Q$ and $h(x)\leq 0$ imply $x=0$ for all $h\in J'$.

First we assume that $S\neq\emptyset$. Then from Lemmas 3 and 4 it follows that there exists a net $\{x_h; h\in J'\}\subset S$ such that $h(x_h)=M_h=M_h^*$ for all $h\in J'$. If $\lim_{h\in J'}h(x_h)=+\infty$, then $M=M^*=+\infty$. So we may assume that $\lim_{h\in J'}h(x_h)$ is finite. Then as in the proof of Lemma 4, we see that there exists a subnet of $\{x_h\}$ which converges to an element $x_0\in S$. We may assume that $\{x_h; h\in J'\}$ converges to x_0. If we fix an arbitrary $h_1\in J'$, then $h_1(x_0)=\lim_{h\in J'}h_1(x_h)\leq\lim_{h\in J'}h(x_h)=\lim_{h\in J'}M_h^*=M^*$. Thus $M\leq f(x_0)=\sup_{h\in J'}h(x_0)\leq M^*$. Since $M\geq M^*$, $M=M^*=f(x_0)$.

Next we assume that $S=\emptyset$ and $S^*\neq\emptyset$. Then $M_h=M_h^*=+\infty$ for $h\in J'$ such that $S_h^*\neq\emptyset$. Since $M_h^*\leq M^*\leq M$, we have $M=M^*=+\infty$. This completes the proof.

In the finite dimensional case, we can omit condition (4.1′) in Theorem 2. To prove this we prepare

Lemma 5. *Assume that X, Y, Z and W are all finite dimensional spaces and set* $\operatorname{dom} g=\{w\in W;\, g(w)$ *is finite*$\}$. *Then* $(\operatorname{dom} g)^\circ+\partial g(0)$ *is closed. Similarly* $(\operatorname{dom} f)^\circ-\partial f(0)$ *is also closed.*

Proof. Let $z\in(\operatorname{dom} g)^\circ$ and $z_0\in\partial g(0)$. Since $\langle tz+z_0, w\rangle\geq\langle z_0, w\rangle\geq g(w)$ for all $w\in\operatorname{dom} g$ and $t>0$, $z\in\operatorname{ac}\partial g(0)$. From Lemma 2 it follows that $\operatorname{ac}\partial g(0)=(\operatorname{dom} g)^\circ$. Hence $(-\operatorname{dom} g)^\circ\cap\operatorname{ac}\partial g(0)=(-\operatorname{dom} g)^\circ\cap(\operatorname{dom} g)^\circ$ and this is a linear subspace. Since Z and W are finite dimensional, $\partial g(0)$ is locally compact, so that we see by Lemma 1 that $(\operatorname{dom} g)^\circ+\partial g(0)$ is closed. The last statement can be similarly proved.

Corollary. *Assume that X, Y, Z and W are finite dimensional spaces and that conditions* (4.2′), (4.3) *and* (4.4) *are satisfied. Then $M=M^*$. Furthermore if $S\neq\emptyset$, then HP has an optimal solution.*

Proof. Let $\tilde{P}$ be the closure of $\{x\in P;\, f(x)<+\infty\}$. We set $\tilde{f}(x)=f(x)$ for $x\in\tilde{P}$, $\tilde{f}(x)=+\infty$ for $x\notin\tilde{P}$, $\tilde{g}(w)=g(w)$ for $u\in Q^\circ$ and $\tilde{g}(w)=-\infty$ for $w\notin Q^\circ$. Then $\tilde{P}$ is the closure of $\operatorname{dom}\tilde{f}$ and Q° is $\operatorname{dom}\tilde{g}$. Thus by Lemma 5, we see that $\tilde{P}^\circ-\partial\tilde{f}(0)$ and $Q+\partial\tilde{g}(0)$ are closed. By applying Theorem 2 to $(A, \tilde{P}, Q, \tilde{f}, \tilde{g})$, we complete the proof.

This is a more precise version of Corollary of Theorem 1 and an improvement of [7; Theorem 3.1]. In this corollary the assumption that all spaces are finite dimensional cannot be omitted. See [4; Example 3.1]. By the following example, we observe that condition (4.2′) cannot be omitted either.

Example. We take $X=Y=Z=W=R^2$, $P=R^2_+$ and $Q=\{(0, 0)\}$. We set $\Psi(x, w)=x_1w_1+x_2w_2$ and $f(x)=x_1$ for $x=(x_1, x_2)\in X$ and $w=(w_1, w_2)\in W$, $g(w)=2(w_1w_2)^{1/2}$ for $(w_1, w_2)\in R^2_+$ and $g(w)=-\infty$ for $(w_1, w_2)\notin R^2_+$. Then A is the identity mapping from X to Z so that condition (4.3) is satisfied. Since $x=(x_1, x_2)\in S$ if and only if $x_1x_2\geq 1$ and $x_1>0$, $M=\inf\{x_1;\, x\in S\}=0$ and HP has no optimal solution.

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Added in proof. Recently the author noticed that Lemma I was also proved in [9] J. Gwinner: *Closed images of convex multivalued mapping in linear topological spaces with applications*, J. Math. Anal. Appl. **60** (1977), 75–86.

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