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ON THE NORMAL BUNDLES OF S^2 MINIMALLY IMMERSED INTO THE UNIT SPHERES

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Introduction

Let F: $S^2 \to S^m(1)$ be a full minimal isometric immersion of the 2-dimensional sphere S^2 into the m-dimensional unit sphere $S^m(1)$. Let $N(S^2)$ be the normal bundle of S^2 and $\Gamma(N(S^2))$ the space of all C^∞ cross-sections of $N(S^2)$. We denote by $\tilde{J}$ the Jacobi operator acting on $\Gamma(N(S^2))$. The operator $\tilde{J}$ is diagonalisable (Simons [6]).

The 2-dimensional sphere S^2 may be considered as the homogeneous space $SU(2)/S(U(1)\times U(1))$. Then the isometric immersion F is $SU(2)$-equivariant (Calabi [1], Do Carmo & Wallach [2]). Let V_λ be the complexification of the λ-eigenspace of $\tilde{J}$. Then V_λ is a $SU(2)$-module and the multiplicities of any complex irreducible $SU(2)$-modules contained in V_λ are all equal to 2 (Nagura [4]).

In this paper we show that the normal bundle $N(S^2)$ has a holomorphic vector bundle structure (Proposition 2). Therefore $\Gamma(N(S^2))$ is a complex vector space. Secondly we show that the Jacobi operator $\tilde{J}$ is complex linear (Proposition 3). Hence every eigenspace of $\tilde{J}$ is a complex linear subspace of $\Gamma(N(S^2))$. Thirdly we show that if we decompose an eigenspace of $\tilde{J}$ into a direct sum of complex irreducible $SU(2)$-modules, then any pairs of the components are not $SU(2)$-isomorphic (Proposition 4). This result explains the above fact on the multiplicities.

1. Preliminaries

We denote by G (resp. by K) the special unitary group $SU(2)$ of degree 2 (resp. the subgroup $S(U(1)\times U(1))$ of $SU(2)$), i.e.

$$K=\left\{\begin{pmatrix} b & 0 \\ 0 & \bar{b}\end{pmatrix};\ b\in \boldsymbol{C},\ \ |b|=1\right\},$$

where $\bar{b}$ is the complex conjugate of b. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the Lie subalgebra of $\mathfrak{g}$ corresponding to the subgroup K of G, i.e.

$$\mathfrak{g}=\left\{\begin{pmatrix}\sqrt{-1}a & b \\ -\bar{b} & -\sqrt{-1}a\end{pmatrix};\ a\in\boldsymbol{R},\ b\in\boldsymbol{C}\right\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix}\sqrt{-1}a & 0 \\ 0 & -\sqrt{-1}a\end{pmatrix};\ a\in\boldsymbol{R}\right\}.$$

Then $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$. We define an $Ad(G)$-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$ by

$$(X,\ Y)=-\frac{1}{2}Tr(XY)\qquad \text{for } X,\ Y\in\mathfrak{g},$$

where $Tr(XY)$ denotes the trace of the matrix XY. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$. Then

$$\mathfrak{p}=\left\{\begin{pmatrix}0 & b \\ -\bar{b} & 0\end{pmatrix};\ b\in\boldsymbol{C}\right\}.$$

We may consider $\mathfrak{p}$ as the tangent space $T_0(S^2)$ of S^2 at $o=\pi(e)$, where π is the natural projection of G onto $S^2=G/K$. The inner product $(\ ,\ )$ defines a G-invariant Riemannian metric on S^2 which coincides with the inner product $(\ ,\ )$ on $\mathfrak{p}=T_0(S^2)$. We also denote by $(\ ,\ )$ this G-invariant Riemannian metric. Then the Riemannian manifold $(S^2,\ (\ ,\ ))$ is of constant sectional curvature 4.

We choose an orthonormal basis $\{h,\ x,\ y\}$ of $\mathfrak{g}$ as follows:

$$h=\begin{pmatrix}\sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{pmatrix},\quad x=\begin{pmatrix}0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{pmatrix},\quad y=\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}.$$

An irreducible orthogonal representation $\rho\colon G\to GL(V)$ is said to be a *real spherical representation* of the pair $(G,\ K)$, if there exists a unit vector $v\in V$ such that $\rho(k)v=v$ for any $k\in K$. We have

Lemma 1 (cf. Serre [5]). *Let $\rho\colon G\to GL(V)$ be a real spherical representation of $(G,\ K)$. Then*

(1) *The complexification $\rho\colon G\to GL(V^C)$ of ρ is a complex irreducible representation with highest weight $2nh$, where V^C is the complexification of the vector space V and n is a non-negative integer.*

(2) *We can choose an orthogonal basis $\{u,\ v_i,\ w_i;\ i=1,\ 2,\cdots,n\}$ of V with the following properties:*

$$d\rho(h)u=0\,,\quad d\rho(h)v_i=2iw_i\,,\quad d\rho(h)w_i=-2iv_i\,,\qquad i=1,\ 2,\cdots,n\,.$$

$$d\rho(x)u=2nv_1\,,\quad d\rho(y)u=-2nw_1\,.$$

If i is even,

$$\begin{aligned} d\rho(x)v_i &= (n+i)v_{i-1}+(n-i)v_{i+1}\,, \\ d\rho(x)w_i &= (n+i)w_{i-1}+(n-i)w_{i+1}\,, \\ d\rho(y)v_i &= (n+i)w_{i-1}-(n-i)w_{i+1}\,, \\ d\rho(y)w_i &= -(n+i)v_{i-1}+(n-i)v_{i+1}\,. \end{aligned}$$

If i is odd,

$$d\rho(x)v_i = \begin{cases} -(n+1)u-(n-1)v_2 & i=1\,, \\ -(n+i)v_{i-1}-(n-i)v_{i+1} & i>1\,, \end{cases}$$

$$d\rho(x)w_i = \begin{cases} -(n-1)w_2 & i=1\,, \\ -(n+i)w_{i-1}-(n-i)w_{i+1} & i>1\,, \end{cases}$$

$$d\rho(y)v_i = \begin{cases} (n-1)w_2 & i=1\,, \\ -(n+i)w_{i-1}+(n-i)w_{i+1} & i>1\,, \end{cases}$$

$$d\rho(y)w_i = \begin{cases} (n+1)u-(n-1)v_2 & i=1\,, \\ (n+i)v_{i-1}-(n-i)v_{i+1} & i<1\,. \end{cases}$$

Here $d\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)$ is the differential of the representation ρ.

For

$$k = \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} \in K,\ \theta\in\boldsymbol{R}\,,$$

we have by the above lemma

$$\begin{cases} \rho(k)v_i = \cos(2i\theta)v_i+\sin(2i\theta)w_i\,, \\ \rho(k)w_i = -\sin(2i\theta)v_i+\cos(2i\theta)w_i\,. \end{cases} \tag{1.1}$$

Let (M, g) (resp. $(\bar{M}, \bar{g})$) be a Riemannian manifold of dimension k (resp. of dimension m). Let $F\colon M\to\bar{M}$ be an isometric immersion of M into $\bar{M}$. We identify the tangent space $T_p(M)$ of M at $p\in M$ with a linear subspace of the tangent space $T_{F(p)}(\bar{M})$ of $\bar{M}$ at $F(p)\in\bar{M}$. We denote by $N_p(M)$ the orthogonal complement of $T_p(M)$ in $T_{F(p)}(\bar{M})$. Let $T(M)$ (resp. $N(M)$) be the tangent bundle (resp. the normal bundle) of M. We denote by $\mathfrak{X}(M)$ (resp. by $\Gamma(N(M))$) the space of all C^∞ cross-sections of $T(M)$ (resp. of $N(M)$). Let ∇ (resp. $\bar{\nabla}$) be the Riemannian connection of M (resp. of $\bar{M}$). Let D be the normal connection of F. Let $B\colon T_p(M)\times T_p(M)\to N_p(M)$ be the second fundamental form of F, and $A\colon N_p(M)\times T_p(M)\to T_p(M)$ the Weingarten form of F. For any vector fields $X,\ Y\in\mathfrak{X}(M)$ and for any normal vector field $\xi\in\Gamma(N(M))$, we have the followings (cf. Kobayashi & Nomizu [3] Vol. II Chap. 7 section 3):

$$\bar{\nabla}_X Y = \nabla_X Y+B(X,\ Y)\,,$$

$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi ,$$
$$\bar{g}(\xi , B(X, Y)) = g(A_\xi X, Y) .$$

We denote by H the mean curvature of F. Let $\{e_1, e_2, \cdots, e_k\}$ be an orthonormal basis of $T_p(M)$. Then we have

$$H_p = \sum_{i=1}^{k} B(e_i, e_i) .$$

The isometric immersion F: $M \to \bar{M}$ is said to be *minimal*, if the mean curvature H of F vanishes identically.

Let $\bar{R}$ be the curvature tensor of $\bar{M}$. We define linear mappings $\tilde{A}$, $\tilde{R}$, of $N_p(M)$ as follows:

$$\tilde{A}(v) = \sum_{i,j=1}^{k} \bar{g}(v, B(e_i, e_j))B(e_i, e_j) , \tag{1.2}$$

$$\tilde{R}(v) = \sum_{i=1}^{k} (\bar{R}(e_i, v)e_i)^N \qquad \text{for } v \in N_p(M) , \tag{1.3}$$

where $\{e_1, e_2, \cdots, e_k\}$ is an orthonormal basis of $T_p(M)$ and $(\bar{R}(e_i, v)e_i)^N$ denotes the normal component of $\bar{R}(e_i, v)e_i$. The linear mappings $\tilde{A}$ and $\tilde{R}$ are independent of the choice of an orthonormal basis. We denote by Δ the Laplace operator on $N(M)$. Let $\{E_1, E_2, \cdots, E_k\}$ be an orthonormal local basis of $T(M)$ on a neighborhood of $p \in M$. Then we have

$$\Delta f(p) = \sum_{i=1}^{k} (D_{E_i} D_{E_i} f)(p) - \sum_{i=1}^{k} (D_{\nabla_{E_i} E_i} f)(p) \qquad \text{for } f \in \Gamma(N(M)) .$$

The Jacobi operator $\tilde{J}$ is the operator on $N(M)$ defined by

$$\tilde{J} = -\Delta - \tilde{A} + \tilde{R} . \tag{1.4}$$

2. A complex structure on the normal bundle $N(S^2)$

In the rest of this paper we assume that F: $S^2 \to S^m(1)$ is a full minimal isometric immersion of $(S^2, c(\ , \ ))$, $c > 0$, into the m-dimensional unit sphere $S^m(1)$. We may consider $S^m(1)$ as the unit sphere of an $(m+1)$-dimensional Euclidean vector space V with the center 0. Then the following results are known (Calabi [1] p. 123, Do Carmo & Wallach [2] p. 103): The minimal immersion F is rigid, and there exist a real spherical representation ρ: $G \to GL(V)$ of (G, K) and a unit vector $u_0 \in V$ such that

$$F(gK) = \rho(g) u_0 \qquad \text{for any } g \in G .$$

Let $\{u, v_i, w_i;\ i=1, 2, \cdots, n\}$ $(m=2n)$ be the orthogonal basis of V in Lemma 1. We identify the tangent space of V with V itself in a canonical way. Then we have

$$T_0(S^2) = \{v_1, w_1\}_{\boldsymbol{R}},$$
$$N_0(S^2) = \{v_i, w_i; i = 2, 3, \cdots, n\}_{\boldsymbol{R}},$$

where $N_0(S^2)$ is the normal space of S^2 at o in the unit sphere $S^m(1)$. Put

$$V^0 = \boldsymbol{R}u_0 = \boldsymbol{R}u, \quad V^T = T_0(S^2), \quad V^N = N_0(S^2).$$

Then we have the following orthogonal decomposition:

$$T_{u_0}(V) = V^0 + V^T + V^N. \tag{2.1}$$

REMARK. The number c can be explicitely computed (cf. Nagura [4] I p. 128). We have

$$c = 2n(n+1).$$

Let ϕ: $K \to GL(V^N)$ be the representation of K defined by

$$\phi(k) = \rho(k)_{|V^N} \qquad \text{for } k \in K.$$

Let $G \times_K V^N$ be the vector bundle associated to G by ϕ. The vector bundle homomorphism ι: $G \times_K V^N \to N(S^2)$ defined by

$$\iota(g \circ v) = \rho(g)v \qquad \text{for } g \in G \text{ and } v \in V$$

is isomorphic (Nagura [4] I p. 123). Here $x \circ v$ is the image of $(x, v) \in G \times V^N$ by the natural projection of $G \times V^N$ onto $G \times_K V^N$. Put

$$\boldsymbol{C}^\infty(G; V^N)_K = \left\{f: G \to V^N \quad C^\infty \text{ mapping}; f(gk) = \phi(k)^{-1} f(g) \atop \text{for } g \in G \text{ and } k \in K \right\}.$$

The isomorphism ι: $G \times_K V^N \to N(S^2)$ induces an isomorphism of $C^\infty(G; V^N)_K$ onto $\Gamma(N(S^2))$. We also denote by ι this isomorphism

We denote by $\tilde{G}$ the complex special linear group $SL(2, \mathbf{C})$ of degree 2. Let $\tilde{U}$ be the subgroup of $\tilde{G}$ defined by

$$\tilde{U} = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}; a, b \in \boldsymbol{C}, \ a \neq 0 \right\}.$$

The 2-dimensional sphere S^2 may be considered as the 1-dimensional complex projective space. In fact, the mapping i: $S^2 = G/K \to P^1(\boldsymbol{C}) = \tilde{G}/\tilde{U}$, $i(gK) = g\tilde{U}$ for $g \in G$, gives this identification. We define a complex structure I on V^N by

$$Iv_i = w_i, \quad Iw_i = -v_i \qquad i = 2, 3, \cdots, n.$$

We denote by $\tilde{V}^N$ this complex vector space (V^N, I). We have by

$$(1.1) \qquad \phi(k)\circ I = I\circ\phi(k) \qquad \text{for } k\in K\,.$$

Therefore the bundle $G\times_K V^N$ has a complex vector bundle structure. In addition the following proposition asserts that $G\times_K V^N$ is a holomorphic vector bundle.

Proposition 2. *Let F: $(S^2, c(\ ,\ ))\to S^m(1)$, $c>0$, be a full minimal isometric immersion. Then the normal bundle $N(S^2)$ has a holomorphic vector bundle structure.*

Proof. We shall show that $G\times_K V^N$ has a holomorphic vector bundle structure. We define a mapping ψ: $\tilde{U}\to GL(\tilde{V}^N)$ by

$$\psi(\tilde{u})v_i = (Re\, a^{2i})v_i+(Im\, a^{2i})w_i\,,$$
$$\psi(\tilde{u})w_i = -(Im\, a^{2i})v_i+(Re\, a^{2i})w_i$$
$$\text{for } \tilde{u} = \begin{pmatrix} a & b \\ 0 & \dfrac{1}{a}\end{pmatrix}\in\tilde{U}\,,$$

where $Re\, a^{2i}$ (resp. $\mathrm{Im}\, a^{2i}$) is the real part (resp. the imagenary part) of $a^{2i}\in\boldsymbol{C}$. Since $\psi(\tilde{u}_1\tilde{u}_2)=\psi(\tilde{u}_1)\psi(\tilde{u}_2)$ for $\tilde{u}_1$, $\tilde{u}_2\in\tilde{U}$, ψ is a holomorphic representation of $\tilde{U}$. Let $\tilde{G}\times_{\tilde{U}}\tilde{V}^N$ be the vector bundle associated to $\tilde{G}$ by ψ. This vector bundle $\tilde{G}\times_{\tilde{U}}\tilde{V}^N$ is a holomorphic vector bundle. Since the restriction $\psi_{|K}$ of ψ to K coincides with ϕ, the bundle homomorphism i: $G\times_K V^N\to\tilde{G}\times_{\tilde{U}}\tilde{V}^N$, $i(x\circ v)=x\circ v$, is an isomorphism as C^∞ vector bundle. Hence $G\times_K V^N$ has a holomorphic vector bundle structure. Q.E.D.

3. On the Jacobi operator $\tilde{J}$

We also denote by I the complex structure on $C^\infty(G;\ V^N)_K$ induced from the complex structure I on V^N. Let $\hat{I}$ be the complex structure on $\Gamma(N(S^2))$ corresponding to this complex structure I on $C^\infty(G;\ V^N)_K$ under the isomorphism ι: $C^\infty(G;\ V^N)_K\to\Gamma(N(S^2))$. We define an action L (resp. an action σ) of G on $C^\infty(G;\ V^N)_K$ (resp. on $\Gamma(N(S^2))$) as follows:

$$(L_g f)(h) = f(g^{-1}h) \qquad \text{for } g, h\in G \text{ and } f\in C^\infty(G;\ V^N)_K\,,$$
$$(\sigma_g\hat{f})(hK) = d(\rho(g))\hat{f}(hK) \qquad \text{for } g, h\in G \text{ and } \hat{f}\in\Gamma(N(S^2))\,,$$

where $d(\rho(g))$ is the differential of the isometry $\rho(g)$ of $S^m(1)$. Then we have (Nagura [4] I p. 124)

$$\iota\circ L_g = \sigma_g\circ\iota \qquad \text{for } g\in G\,.$$

We have easily

$$(3.1) \qquad I\circ L_g = L_g\circ I\,, \quad \hat{I}\circ\sigma_g = \sigma_g\circ\hat{I} \qquad \text{for } g\in G\,.$$

The above result shows that the action L (resp. σ) is a complex representation of G with representation space $(C^\infty(G; V^N)_K, I)$ (resp. $(\Gamma(N(S^2)), \tilde{I})$).

Let J be the operator on $C^\infty(G; V^N)_K$ corresponding to the Jacobi operator $\tilde{J}$. We have (Nagura [4] I p. 131)

$$Jf = -\frac{1}{c}\Big[\sum_{i=1}^{3} E_i E_i f - 2c_\rho f + 2\sum_{i=1}^{3} \{d\rho(E_i)(E_i f)\}^N + 2\sum_{i=1}^{3} \{d\rho(E_i)(d\rho(E_i)f)^N\}^N\Big] \quad \text{for } f \in C^\infty(G;\ V^N)_K, \tag{3.2}$$

where $E_1=h$, $E_2=x$, $E_3=y$, $c_\rho=-2c$ and $(v)^N$ denotes the V^N-component of $v\in V$ with respect to the decomposition (2.1). In (3.2) we consider $\mathfrak{g}$ as the Lie algebra of left invariant vector fields on G.

Proposition 3. *The Jacobi operator $\tilde{J}$ is complex linear on* $(\Gamma(N(S^2)), \tilde{I})$.

Proof. We shall show that $J\circ I = I\circ J$. Since $Z\circ I = I\circ Z$ for $Z\in\mathfrak{g}$, it is sufficient to show that

$$\{d\rho(Z)(I(v))\}^N = I(d\rho(Z)v)^N \qquad \text{for } Z\in\mathfrak{g} \text{ and } v\in V^N. \tag{3.3}$$

Applying Lemma 1, we have

$$\{d\rho(Z)(I(v))\}^N = I(d\rho(Z)v)^N$$
$$\text{for } Z = h,\ x,\ y \text{ and } v = v_i,\ w_i,\ i = 2, 3, \cdots, n\,.$$

This proves (3.3). Q.E.D.

Let U_λ be the λ-eigenspace of $\tilde{J}$ in $\Gamma(N(S^2))$. Since the space U_λ is G-invariant (Nagura [4] I p. 119), U_λ is a complex G-invariant subspace of $(\Gamma(N(S^2)), I)$ by Proposition 3. Therefore we have the following proposition by Nagura [4] (III (2) of Theorem 12.3.3).

Proposition 4. *If we decompose an eigenspace of $\tilde{J}$ into a direct sum of complex irreducible G-modules, then any pairs of the irreducible components are not G-isomorphic.*

Let $\tilde{L}$ be the space of Killing vector fields on the unit sphere $S^m(1)$. Put

$$\tilde{W} = \{(\tilde{k}_{|S^2})^N;\ \tilde{k}\in\tilde{L}\}\,,$$

where $(\tilde{k}_{|S^2})^N$ is an element of $\Gamma(N(S^2))$ obtained by the normal projection of $\tilde{k}\in\tilde{L}$. This space $\tilde{W}$ is a G-module. A cross-section $\tilde{f}\in\Gamma(N(S^2))$ is called a *Jacobi field,* if it satisfied the equation $\tilde{J}\tilde{f}=0$. An element of $\tilde{W}$ is a Jacobi field (Simons [6] p. 74). Let $\tilde{W}^C$ be the complexification of the space $\tilde{W}$. Then the multiplicities of any complex irreducible G-modules contained in $\tilde{W}^C$ are equal to 1 (Nagura [4] III Lemma 12.4.2). Hence we have

$$\tilde{I}\tilde{W}\cap\tilde{W}=\{0\}\,. \tag{3.4}$$

Let U_0 be the space of all Jacobi fields. Then we have by (3.4) and Nagura [4] (III Theorem 12.4.1 and Lemma 12.4.2)

$$U_0=\tilde{W}+\tilde{I}\tilde{W}\,(\text{direct sum})\,.$$

Thus we could obtain the space U_0. However the author does not know the geometric meaning of this decomposition.

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