CHARACTER CORRESPONDENCES IN $p$-SOLVABLE GROUPS

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Introduction

Let $G$ and $A$ be finite groups and suppose that $A$ acts on $G$ by automorphisms. We write $\text{Irr}(G)$ to denote the set of all irreducible characters of $G$ over the complex number field. Then $A$ induces permutation action on $\text{Irr}(G)$. For $\chi \in \text{Irr}(G)$ and $a \in A$, the character $\chi^a$ is defined by $\chi^a(g) = \chi(g^a)$ for $g \in G$. The set of all $A$-invariant characters in $\text{Irr}(G)$ is denoted by $\text{Irr}_A(G)$.

Assume further that $(|G|, |A|) = 1$. G. Glauberman [2] first showed that if $A$ is solvable then there is a bijection $\pi(G, A) : \text{Irr}_A(G) \rightarrow \text{Irr}(C_G(A))$ which is uniquely defined by the action of $A$ on $G$.

When $A$ is not solvable, the Odd-Order Theorem of Feit and Thompson implies that $|A|$ is even and hence $|G|$ is odd. E.C. Dade and I.M. Isaacs [3] developed the correspondence when $|G|$ is odd, and T.R. Wolf [7] showed the correspondences of Glauberman and Isaacs are equal when both are defined.

For a fixed prime $p$, $\text{IBr}(G)$ denotes the set of all irreducible $p$-modular characters of $G$, chosen with respect to some fixed pullback of the $p$-modular roots of unity to the complex numbers. Then $A$ also induces permutation action on $\text{IBr}(G)$ by the same manner as on $\text{Irr}(G)$. Now the question arises whether there is a bijection from $\text{IBr}_A(G)$ onto $\text{IBr}(C_G(A))$ or not. The purpose of this paper is to show that it exists when $G$ is $p$-solvable, namely, we shall prove the following.

**Theorem.** Let $A$ act on $G$ such that $(|G|, |A|) = 1$. Suppose that $G$ is $p$-solvable. Then there exists a bijection

$$\tilde{\pi}(G, A) : \text{IBr}_A(G) \rightarrow \text{IBr}(C_G(A)).$$

And the following hold.

(i) If $B \triangleleft A$, then $\tilde{\pi}(G, A) = \tilde{\pi}(G, B)\tilde{\pi}(C_G(B), A/B)$.

(ii) If $A$ is a $q$-group for a prime $q$, then, for $\phi \in \text{IBr}_A(G)$, $(\phi)\tilde{\pi}(G, A)$ is the unique irreducible constituent of $\phi_{C_G(A)}$ with multiplicity prime to $q$. 

The proof of the above Theorem is divided into two parts. It is proved when $A$ is solvable in Section 3 (Theorem 3.10). If $A$ is nonsolvable, then $2|A|$ by the Odd-Order Theorem. Thus $|G|$ is odd and we may assume $p \neq 2$. In this case it is done in Section 4 (Theorem 4.3).

We follow the notation of [5].

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1. Preliminaries

In this section, we mention some properties of co-prime actions.

The first lemma, which can be proved via the Schur-Zassenhaus Theorem, is quite useful when looking at co-prime actions. It is due to Glauberman [1]. Also a proof can be found in Lemma 13.8 and Corollary 13.9 of [5].

**Lemma 1.1.** Suppose that a group $A$ acts on a group $G$ with $(|G|, |A|) = 1$. Let $A$ and $G$ both act on a set $\Omega$ and assume

(i) $(x \cdot g) \cdot a = (x \cdot a) \cdot g^a$ for all $x \in \Omega$, $g \in G$ and $a \in A$.

(ii) $G$ is transitive on $\Omega$.

Then $A$ fixes a point of $\Omega$ and $C_G(A)$ acts transitively on the set of fixed points of $A$.

The following lemma is easily seen by using the above.

**Lemma 1.2.** Assume $A$ acts on $G$, $N \triangleleft G$, $N$ is $A$-invariant, $(|G|/|N|, |A|) = 1$, and $\chi \in \text{Irr}_A(G)$. Then

(i) $\chi_N$ has an $A$-invariant irreducible constituent $\theta$.

(ii) If $C_{G/N}(A) = 1$, then the above $\theta$ is unique.

(iii) If $C_{G/N}(A) = G/N$, then every irreducible constituent of $\chi_N$ is $A$-invariant.

The next result is in some sense dual to the above lemma.

**Lemma 1.3.** Assume $A$ acts on $G$, $N \triangleleft G$, $N$ is $A$-invariant, $(|G|/|N|, |A|) = 1$, and $\theta \in \text{Irr}_N(A)$. Then

(i) $\theta^G$ has an $A$-invariant irreducible constituent $\chi$.

(ii) If $C_{G/N}(A) = 1$, then the above $\chi$ is unique.

(iii) If $C_{G/N}(A) = G/N$ then every irreducible constituent of $\theta^G$ is $A$-invariant.

Proof. This Lemma follows from Theorem 13.31 and Problems 13.10 and 13.13 of [5].
2. Preliminaries for character correspondence

In this section, we recall some properties of the character correspondence of Glauberman and Dade-Isaacs. Since we will be frequently looking at co-prime actions, we make the following hypothesis.

**Hypothesis 2.1.** Let \( A \) act on \( G \) such that \(|G|, |A|=1\). Let \( C=C_{o}(A) \) and let \( \Gamma=GA \) be the semi-direct product of \( G \) and \( A \).

The results of Glauberman, Isaacs and Wolf may be summarized as follows.

**Theorem 2.2.** Assume Hypothesis 2.1. Then there is a uniquely defined map

\[
\pi(G, A): \text{Irr}_A(G) \rightarrow \text{Irr}(C)
\]

and the following hold.

(i) \( \pi(G, A) \) is bijective.

(ii) If \( B \leq A \), then \( \pi(G, A)=\pi(G, B)\pi(C_{o}(B), A/B) \).

(iii) If \( A \) is a \( q \)-group for a prime \( q \) and \( \chi \in \text{Irr}_A(G) \), then \( (\chi)\pi(G, A) \) is the unique \( \xi \in \text{Irr}(C) \) such that \( qG_{\chi}[X_{G}, \xi] \).

(iv) If \( |G| \) is odd and \( \chi \in \text{Irr}_A(G) \), then there exists the unique \( \xi \in \text{Irr}_A([G, A]C) \) such that \( 2G_{\chi}[X_{G}, A, \xi] \). Also \( (\chi)\pi(G, A)=(\xi)\pi([G, A]C, A) \).

Moreover suppose \( \alpha \) is an automorphism of \( \Gamma \) which leaves \( G \) and \( A \) invariant. Then \( C \) is \( \alpha \)-invariant and we have

\[
(\chi^{\alpha})\pi(G, A) = \{(\chi)\pi(G, A)\}^{\alpha} \quad \text{for all } \chi \in \text{Irr}_A(G).
\]

**Proof.** See Corollary 5.2 of [7] for (i)\( \sim \)(iv). The last statement holds since \( \pi(G, A) \) is ultimately determined uniquely by multiplicities. A similar argument can be found, for example, in the discussion preceding Corollary 13.19 of [5].

By saying that \( \pi(G, A) \) is uniquely defined, we mean that \( \pi(G, A) \) is determined by the action of \( A \) on \( G \). If \( A \) is solvable, then (ii) and (iii) give an algorithm for computing \( \pi(G, A) \). Suppose that \(|G| \) is odd. If \([G, A]=1\), then \( C=G \) and \( \pi(G, A) \) is the identity map on \( \text{Irr}(G) \). Assume that \([G, A] \neq 1\). The Odd-Order Theorem implies \([G, A]<[G, A] \) and hence \([G, A]<C<G \). Thus (iv) provides an algorithm for computing \( \pi(G, A) \) when \(|G| \) is odd.

In the next lemma, we mention some useful properties of \( \pi(G, A) \), which relate \( \pi(G, A) \) and \( \pi(N, A) \) for an \( A \)-invariant normal subgroup \( N \) of \( G \).

**Lemma 2.3.** Assume Hypothesis 2.1 and that \( N \) is an \( A \)-invariant normal subgroup of \( G \). Let \( \chi \in \text{Irr}_A(G), \theta \in \text{Irr}_A(N), T=I_{\alpha}(\theta), \xi=(\chi)\pi(G, A), \) and \( \phi = \text{null map} \). Then

\[
(\chi_{\phi})\pi(G, A) = \{(\chi)\pi(G, A)\}^{\phi} \quad \text{for all } \chi \in \text{Irr}_A(G).
\]
(θ)π(\mathcal{N}, A), where I_G(θ) denotes the inertia group of θ in G. Then

(i) \[ [\chi_N, \theta] = 0 \text{ if and only if } [\xi_{N \cap C}, \phi] = 0. \]

(ii) \( T \cap C = I_C(\phi) \) and \((\psi^0)\pi(G, A) = ((\psi^0)\pi(T, A))^c \) for \( \psi \in \text{Irr}_A(T | \theta). \)

Proof. See Lemma 2.5 of [8].

Assume Hypothesis 2.1. For \( \chi \in \text{Irr}_A(G) \) there exists the unique extension \( \chi^* \in \text{Irr}(\Gamma) \) of \( \chi \) such that \( A \subseteq \ker(\det\chi^*) \). (See Lemma 13.3 of [5].) \( \chi^* \) is called the canonical extension of \( \chi \).

**Lemma 2.4.** Assume Hypothesis 2.1 and that \( A \) is cyclic. Let \( \chi \in \text{Irr}_A(G) \) and \( \xi = (\chi)\pi(G, A) \). Let \( \chi^* \) be the canonical extension of \( \chi \) to \( \Gamma \). Then there exists \( \varepsilon = \pm 1 \) such that

\[ \chi^*(ca) = \varepsilon \xi(c) \text{ for all } c \in C \text{ and all generators } a \text{ of } A. \]

Proof. See Theorem 13.6 of [5].

3. **Correspondence of Brauer characters**

Let \( p \) be a fixed prime. In this section, we construct, under Hypothesis 2.1, a bijection from \( \text{IBr}_A(G) \) onto \( \text{IBr}(C) \) when \( G \) is \( p \)-solvable. We begin with two useful results of Isaacs [4], [6]. For a character \( \chi \) of \( G \) let \( \chi \) denote the restriction of \( \chi \) to the \( p \)-regular elements of \( G \).

**Lemma 3.1.** Let \( N \subseteq G \) with \( p \mid |G : N| \). Let \( \theta \in \text{Irr}(N) \) and assume

(i) \( \hat{\theta} \in \text{IBr}(N) \) and

(ii) \( \hat{\theta} = \theta \) for those \( g \in G \) with \( \hat{\theta} = \hat{\theta} \).

Then \( \hat{\theta} \) defines a bijection from \( \text{Irr}(G | \theta) \) onto \( \text{IBr}(G | \hat{\theta}) \).

Proof. See Lemma 2.6 of [6].

**Lemma 3.2.** Let \( N \subseteq G \) with \( G/N \) a \( p \)-group. Let \( \phi \in \text{IBr}(N) \). Then \( \text{IBr}(G | \phi) \) consists of a single element \( \psi \). Moreover if \( I_0(\phi) = \{ g \in G | \phi^g = \phi \} = G \), then \( \psi_N = \phi \).

Proof. See Lemma 4.4 of [6].

To construct the bijection, we need a definition. If \( \Omega \subseteq \text{Irr}(G) \) and \( H \leq G \), we write \( \Omega(H) \) to denote \( \{ \theta \in \text{Irr}(H) | [\chi_H, \theta] = 0 \text{ for some } \chi \in \Omega \} \). Note that if \( K \leq H \leq G \) and \( \psi \in \Omega(H) \), then every irreducible constituent of \( \psi_K \) lies in \( \Omega(K) \).

**Definition 3.3.** Assume \( A \) acts on \( G \). Let \( \Omega \) be a subset of \( \text{Irr}(G) \) and let \( G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{ 1 \} \) be a normal series of \( G \). We say that \( \Omega \) has the lifting property with respect to \( A \) and \( \{ G_i \}_{i=0}^n \) if the following is satisfied.
(i) $\Omega$ is $A$-invariant.

(ii) $\wedge$ defines the bijection from $\Omega(G_i)$ onto $IBr(G_i)$ for each $i$, $0 \leq i \leq n$. Furthermore, we simply say that $\Omega$ has the lifting property with respect to $A$ if $\Omega$ has the lifting property with respect to $A$ and every normal series of $G$.

It is easily seen that an $A$-invariant subset $\Omega$ of $Irr(G)$ has the lifting property with respect to $A$ if and only if $\wedge$ defines the bijection from $\Omega(N)$ onto $IBr(N)$ for any subnormal subgroup $N$ of $G$.

**Remark.** For each $p$-solvable group $G$, Isaacs [4], [6] constructed a characteristic subset $\mathcal{J}(G)$ of $Irr(G)$ such that

(i) $\wedge$ defines a bijection from $\mathcal{J}(G)$ onto $IBr(G)$, and

(ii) if $N \lhd G$ and $\chi \in \mathcal{J}(G)$, then every irreducible constituent of $\chi_N$ lies in $\mathcal{J}(N)$.

Moreover the above properties (i) and (ii) of $\mathcal{J}(G)$ imply that for each subnormal subgroup $N$ of $G$ and $\theta \in \mathcal{J}(N)$, $\theta^G$ has an irreducible constituent which lies in $\mathcal{J}(G)$. This can be shown using the same argument as in Lemma 3.4. So $\mathcal{J}(G)$ has the lifting property with respect to any $A$.

**Lemma 3.4.** Assume $A$ acts on $G$. Let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$ be a normal series of $G$. Let $\Omega \subset Irr(G)$ have the lifting property with respect to $A$ and $\{G_i\}_{i=0}^n$.

(i) If $p \neq |G_k: G_l|$ for $0 \leq k \leq l \leq n$, then $Irr(G_k \mid \theta) \subset \Omega(G_k)$ for all $\theta \in \Omega(G_k)$.

(ii) If $|G_k: G_l|$ is a power of $p$ for $0 \leq k \leq l \leq n$, then $Irr(G_k \mid \theta) \cap \Omega(G_k)$ consists of a single element for every $\theta \in \Omega(G_k)$.

To prove Lemma 3.4 we need another lemma.

**Lemma 3.5.** Under the hypothesis of Lemma 3.4 let $\theta \in \Omega(G_i)$ and $\psi \in \Omega(G_k)$. Suppose $\psi \in IBr(G_k \mid \hat{\theta})$. Then $\psi \in Irr(G_k \mid \theta)$.

**Proof.** We have $\psi = \sum_{i=1}^{\Omega_i} \eta_i$, where the $\eta_i$ are irreducible. We also have $\psi = \sum_{i=1}^{\Omega_i} \eta_i^G$. Since $\eta_i \in \Omega(G_i)$, $\eta_i \in IBr(G_i)$ by the lifting property of $\Omega$. Since $\psi \in IBr(G_k \mid \hat{\theta})$, it follows that $\hat{\theta} = \eta_i$ for some $i$. Since $\wedge$ is the bijection from $\Omega(G_i)$ onto $IBr(G_i)$, we have $\theta = \eta_i$ and the result follows.

**Proof of Lemma 3.4.** It suffices to prove only when $l = k + 1$. Thus we may assume that $G_l$ is normal in $G_k$.

(i) Let $\theta \in \Omega(G_i)$ and $\chi \in Irr(G_k \mid \theta)$. For $g \in G_k$, $\theta^G \in \Omega(G_i)$ by the definition of $\Omega(G_i)$ and it follows by the lifting property of $\Omega$ that $\theta^G = \theta$ for those $g \in G_k$ with $\hat{\theta} = \hat{\theta}$. From Lemma 3.1, $\hat{\chi}$ belongs to $IBr(G_k \mid \hat{\theta})$, and then we can find $\psi \in \Omega(G_k)$ such that $\psi = \hat{\chi}$. Since $\psi \in IBr(G_k \mid \hat{\theta})$, Lemma 3.4 yields
that $\varphi \in \text{Irr}(G_1|\theta)$. Thus by Lemma 3.1 again, we have $\chi = \varphi \in \Omega(G_1)$.

(ii) Let $\theta \in \Omega(G_1)$. Let $\phi \in \text{IBr}(G_1|\theta)$. Then by the lifting property of $\Omega$, there exists $\psi \in \Omega(G_1)$ such that $\psi = \phi$. Since $\psi \in \text{IBr}(G_1|\theta)$, we have $\psi \in \text{Irr}(G_1|\theta)$ by Lemma 3.5. Thus $\psi \in \text{Irr}(G_1|\theta) \cap \Omega(G_1)$. It follows from Lemma 3.2 and the lifting property of $\Omega$ that such a $\psi$ is unique. Now the proof is complete.

In the next proposition, we see that $\pi(G, A)$ preserves the lifting property with respect to $A$ and any given $A$-composition series of $G$.

**Proposition 3.6.** Assume Hypothesis 2.1 and that $G$ is $p$-solvable. Let $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1\}$ be an $A$-composition series of $G$ and let $\Omega \subset \text{Irr}(G)$ have the lifting property with respect to $A$ and $\{G_i\}_{i=0}^n$. Then the image of $\Omega_A = \Omega \cap \text{Irr}_A(G)$ by $\pi(G, A)$ has the lifting property with respect to $\{1\}$ (the trivial automorphism of $C$) and $\{G_i \cap C\}_{i=0}^n$.

Proof. Use induction on $|G|$. Let $\Lambda$ be the image of $\Omega_A$ by $\pi(G, A)$. Let $\eta \in \Omega(G_i)_A = \Omega(G_i) \cap \text{Irr}_A(G_i)$ for $i \geq 1$. If $G_{i-1}/G_i$ is a $p'$-group, then it is clear from Lemma 1.3 and Lemma 3.4 (i) that $\Omega(G_{i-1}) \cap \text{Irr}(G_{i-1}|\eta)$ is nonempty. If $G_{i-1}/G_i$ is a $p$-group, then by Lemma 3.4 (ii) $\Omega(G_{i-1}) \cap \text{Irr}(G_{i-1}|\eta)$ has exactly one element and since both $\Omega(G_{i-1})$ and $\text{Irr}(G_{i-1}|\eta)$ are $A$-invariant, it must be $A$-invariant. So $\Omega(G_{i-1}) \cap \text{Irr}(G_{i-1}|\eta)$ is nonempty in any case.

Suppose $\psi \in \Omega(G_{i-1}) \cap \text{Irr}(G_{i-1}|\eta)$. Applying Lemma 2.3 (i) repeatedly we can find $\chi \in \Omega_A \cap \text{Irr}(G_1|\eta)$ such that $(\chi)\pi(G, A) \in \text{Irr}(C|\eta)\pi(G, A))$. Since $(\chi)\pi(G, A) \in \Lambda$, it follows that $(\eta)\pi(G_i, A) \in \Lambda(G_i \cap C)$. Conversely suppose $\xi \in \Lambda(G_i \cap C)$ and set $\eta = (\xi)\pi^{-1}(G_i, A)$. By the definition of $\Lambda$ and $\Lambda(G_i \cap C)$ there exists $\chi \in \Omega_A$ such that $(\chi)\pi(G, A) \in \text{Irr}(C|\eta)$. From Lemma 2.3 (i), we have $\chi \in \text{Irr}(G_1|\eta)$, so $\eta \in \Omega(G_i)_A$. Thus we can conclude that the image of $\Omega(G_i)_A$ by $\pi(G_i, A)$ is precisely $\Lambda(G_i \cap C)$ for each $i$, $0 \leq i \leq n$. Since $\Omega(G_i)_A$ has the lifting property with respect to $A$ and $\{G_i \cap C\}_{i=1}^n$ and $\{G_i \cap C\}_{i=1}^n$ is an $A$-composition series of $G_i$, it follows from the inductive hypothesis that for each $i$, $1 \leq i \leq n$,

$\Lambda: \Lambda(G_i \cap C) \rightarrow \text{IBr}(G_i \cap C)$

is a bijection. Therefore the proof will be complete if we show that $\Lambda$ gives a bijection from $\Lambda$ onto $\text{IBr}(C)$.

If $C \leq G_i$, then $G_i \cap C = C = G_0 \cap C$ and $\Lambda(G_i \cap C) = \Lambda$. Thus we have nothing to prove.

Now assume $G_i \nmid C$. Let $\theta_1, \cdots, \theta_k$ be representatives of $C$-orbits of $\Omega(G_i)_A$. For each $i$, $1 \leq i \leq k$, set $\phi_i = (\theta_i)\pi(G_i, A)$. If $g \in C$, then $\theta_i^g$ is also $A$-invariant and from Theorem 2.2 we have
Thus \( \phi_1, \ldots, \phi_k \) are representatives of \( C \)-orbits of \( \Lambda(G_1 \cap C) = (\Omega(G_1) \Delta \pi(G_1, A) \phi_i \). Furthermore \( \theta_i \in \Omega(G_i) \) for \( g \in G \) and \( i, 1 \leq i \leq k \), by the definition of \( \Omega(G_i) \). Since \( \phi \) gives the bijection from \( \Omega(G_i) \) onto \( IBr(G_i) \), it follows that \( \theta_i = \phi_i \) for those \( g \in G \) with \( \theta_i = \hat{\theta}_i \). Thus we have \( I_c(\theta_i) = I_c(\hat{\theta}_i) \) for \( i, 1 \leq i \leq k \). Also we obtain \( I_c(\phi_i) = I_c(\phi_i) \) for each \( i, 1 \leq i \leq k \).

We distinguish two cases.

Case 1. \( G/G_1 \) is a \( p \)-group.

Since \( G/G_1 \) is abelian, we have \( G_1 C = G \). Then \( Irr(G \mid \theta_i) \cap Irr(G \mid \theta_j) \) is empty for \( i \neq j \). And by Lemma 3.4 (ii), \( Irr(G \mid \theta_i) \cap \Omega \) has exactly one element which is of course \( A \)-invariant. So we have \( \Omega_A = \bigcup_{i=1}^k Irr(G \mid \theta_i) \cap \Omega \) and especially \( |\Omega_A| = k \). Thus \( |\Lambda| = k \).

Recall that \( \phi_1, \ldots, \phi_k \) are representatives of \( C \)-orbits of \( \Lambda(G_1 \cap C) \). By the lifting property of \( \Lambda(G_1 \cap C) \), \( \phi_1, \ldots, \phi_k \) are representatives of \( C \)-orbits of \( IBr(G_1 \cap C) \) and thus by Lemma 3.2 we have \( |IBr(C)| = k \). Therefore it suffices to prove that, for \( \chi \in \Omega_A \cap Irr(G \mid \theta_i) \), \( (\chi) \pi(G, A) \) is modularly irreducible. Let \( \chi \in \Omega_A \cap Irr(G \mid \theta_i) \). Then there exists \( \xi \in Irr(\chi(\theta_1), \theta_1) \) such that \( \xi = \chi \).

(See Theorem 6.11 of [5].) Since \( \xi = \chi \), \( \xi \) must be irreducible. Also \( [\theta_i, \xi] \neq 0 \) yields that \( \theta_i \) is an irreducible constituent of \( \xi \). By Lemma 2.3, \( (\xi) \pi(I_c(\theta_i), A) \) \( \in \) \( Irr(I_c(\phi_i) \mid \phi_i) \) and \( (\chi) \pi(G, A) = ((\xi) \pi(I_c(\theta_i), A)) \). And by Lemma 3.2, \( \xi \) is the extension of \( \hat{\theta}_i \). Since \( I_c(\theta_i) \mid \theta_1 \) is abelian, \( |Irr(I_c(\theta_i) \mid \theta_1)| = |I_c(\theta_i) : G_1| \). (See Corollary 6.17 of [5].) By Lemma 1.3 (iii), we have \( Irr(I_c(\theta_i)) \cap \Omega \subset IBr(G_1 \cap C) \) and thus by Lemma 2.3 we have \( |Irr(I_c(\theta_i) \mid \theta_1)| = |Irr(I_c(\phi_i) \mid \phi_i)| \). Since \( I_c(\theta_i) : G_1 = |I_c(\phi_i) : G_1 \cap C| \), we get \( |Irr(I_c(\phi_i)) \mid \phi_i)| \) and hence each element in \( Irr(I_c(\phi_i)) \mid \phi_i) \) is an extension of \( \phi_i \) to \( I_c(\phi_i) \) and so is modularly irreducible. This applies, in particular, to \( (\xi) \pi(I_c(\theta_i), A) \) \( \in \) \( Irr(I_c(\phi_i) \mid \phi_i) \). The equality \( I_c(\phi_i) = I_c(\phi_i) \) implies that

\[
(\chi) \pi(G, A) = (\xi) \pi(I_c(\theta_i), A) = (\xi) \pi(I_c(\theta_i), A)
\]

belongs to \( IBr(C \mid \phi_i) \). (See also Lemma 3.3 of [6].)

Thus the proof is complete.

Case 2. \( G/G_1 \) is a \( p' \)-group.

If \( \chi \in \Omega_A \), then by Lemma 1.2 (i), \( \chi \in Irr(G \mid \theta_i) \) for some \( i, 1 \leq i \leq k \). Thus by Lemma 2.3 (i) it follows that \( (\chi) \pi(G, A) \) \( \in \) \( Irr(C \mid \phi_i) \), so we have \( \Lambda \subset \bigcup_{i=1}^k Irr(C \mid \phi_i) \). Conversely if \( \mu \in Irr(C \mid \phi_i) \), then set \( \chi = (\mu) \pi^{-1}(G, A) \) \( \in \) \( IBr(G_1) \) and by Lemma 2.3 (ii) again, we have \( \chi \in Irr(G \mid \theta_i) \). Since \( \theta_i \in \Omega(G_1) \), it follows from Lemma 3.4 (i) that \( \chi \in \Omega_A \). Thus \( \mu = (\chi) \pi(G, A) \) \( \in \Lambda \) and we
conclude $\Lambda = \bigcup_{i=1}^{k} \text{Irr}(C|\phi_i)$. Since $I_C(\phi_i) = I_C(\phi)$, by Lemma 3.1 $\wedge$ defines a bijection

$$\wedge : \text{Irr}(C|\phi_i) \rightarrow \text{IBr}(C|\phi_i)$$

for each $i$, $1 \leq i \leq k$.

Now if $i \neq j$, then $\text{Irr}(C|\phi_i) \cap \text{Irr}(C|\phi_j)$ and $\text{IBr}(C|\phi_i) \cap \text{IBr}(C|\phi_j)$ are both empty. Recall that $\phi_1, \ldots, \phi_k$ are representatives of $C$-orbits of $\text{IBr}(G_1 \cap C)$.

Since $\text{IBr}(C) = \bigcup_{i=1}^{k} \text{IBr}(C|\phi_i)$, we can conclude that $\wedge$ gives the bijection from $\Lambda$ onto $\text{IBr}(C)$. This completes the proof.

This proposition implies immediately the following.

**Corollary 3.7.** Assume Hypothesis 2.1 and that $G$ is $p$-solvable. Let $\Omega \subset \text{Irr}(G)$. If $\Omega$ has the lifting property with respect to $A$, then $\wedge$ gives a bijection from $(\Omega_A)\pi(G, A)$ onto $\text{IBr}(C)$.

**Remark.** Under the hypotheses in Proposition 3.6 $\Lambda = (\Omega_A)\pi(G, A)$ has the lifting property with respect to a composition series of $C$ obtained as a refinement of $\{G_t \cap C\}_t$. This can be shown using Lemma 3.1 and Lemma 3.4 (i). Also if $B \leq A$, it can be proved that $\Omega_B)\pi(G, B)$ has the lifting property with respect to $A/B$ and $\{G_t \cap C(B)\}_t$. But in general $\pi(G, A)$ does not preserve the lifting property with respect to $A$. (See Appendix.)

Now assume Hypothesis 2.1 and that $G$ is $p$-solvable. Suppose $\Omega \subset \text{Irr}(G)$ has the lifting property with respect to $A$ and some $A$-composition series of $G$. We denote $\wedge^{-1}$ the inverse of the bijection

$$\wedge : \Omega \rightarrow \text{IBr}(G).$$

Since $\wedge$ obviously preserves the actions of $A$ on $\Omega$ and $\text{IBr}(G)$, Proposition 3.6 gives us the following diagram of bijections

$$\text{IBr}_A(G) \xrightarrow{\wedge^{-1}} \Omega_A \xrightarrow{\pi(G, A)} (\Omega_A)\pi(G, A) \xrightarrow{\wedge} \text{IBr}(C).$$

The composition $\pi(G, A) = \wedge^{-1} \pi(G, A) \wedge$ is a bijection from $\text{IBr}_A(G)$ onto $\text{IBr}(C)$.

From its construction, it appears that $\pi(G, A)$ depends on the choice of $\Omega$. In the rest of this section we shall show that it is independent of the choice of $\Omega$ with the lifting property with respect to $A$, if $A$ is solvable. If $A$ is non-solvable, by the Odd-Order Theorem we may assume $p = 2$. When $p$ is odd, we shall prove a stronger result in Section 4, namely, that $\pi(G, A)$ gives a bijection from $\Omega(G) \cap \text{Irr}_A(G)$ onto $\Omega(C)$ (see Theorem 4.1). So we shall have a uniquely defined bijection $\pi(G, A)$. 
Proposition 3.8. Assume Hypothesis 2.1 and that $G$ is $p$-solvable and $A$ is solvable. Let $B\trianglelefteq A$, $D=C_B(B)$, and assume that $\Omega\subseteq \text{Irr}(G)$ and $\Lambda\subseteq \text{Irr}(D)$ both have the lifting property with respect to $A$. Let $\chi\epsilon \Omega_A$ and let $\phi$ be the unique element of $\Lambda_{A/B}$ such that $\phi=(\overline{\chi})\pi(G, B)$. (Note that by Corollary 3.7 $(\chi)\pi(G, B)$ is modularly irreducible.) Then $(\overline{\chi})\pi(G, A)=(\phi)\pi(D, A/B)$.

We need a lemma.

Lemma 3.9. Assume Hypothesis 2.1 and that $A$ is cyclic. Let $\chi, \psi \in \text{Irr}_A(G)$. If $\chi$ and $\psi$ are both modularly irreducible and $\chi=\psi$, then $(\chi)\pi(G, A)=(\psi)\pi(G, A)$.

Proof. Let $\Gamma=GA$. We may assume $p\nmid |G|$, thus $p\nmid |A|$. Since $\chi$ and $\psi$ are $A$-invariant, it follows from Lemma 3.1 that

$\wedge: \text{Irr}(\Gamma|\chi) \rightarrow \text{IBr}(\Gamma|\chi)$ and

$\wedge: \text{Irr}(\Gamma|\psi) \rightarrow \text{IBr}(\Gamma|\psi)$

are bijections. Let $\chi^*$ (resp. $\psi^*$) be the canonical extension of $\chi$ (resp. $\psi$) to $\Gamma$. Then we have $\text{Irr}(\Gamma|\chi)=\{\chi^*\mu|\mu\in \text{Irr}(A)\}$. Since $\chi=\psi$, there exists $\mu\in \text{Irr}(A)$ such that $\chi^*\mu=\psi^*$. Let $a$ be a generator of $A$. By Lemma 2.4, there exist $\varepsilon=\pm 1$ and $\varepsilon'=\pm 1$ such that

$(\chi)\pi(G, A)(g)=\varepsilon\chi^*(ga)$ and

$(\psi)\pi(G, A)(g)=\varepsilon'\psi^*(ga)$

for all $g\in C$. Note that $a$ is $p$-regular. Thus we obtain

$\varepsilon\varepsilon'(\chi)\pi(G, A)(g)\mu(a)=(\psi)\pi(G, A)(g)$

for every $p$-regular element $g\in C$. Now evaluation at $g=1$ yields

$\varepsilon\varepsilon'(\chi)\pi(G, A)(1)\mu(a)=(\psi)\pi(G, A)(1)$.

Since $\mu(a)$ is a root of unity, $\varepsilon\varepsilon'\mu(a)=1$. Thus we obtain

$(\chi)\pi(G, A)=(\psi)\pi(G, A)$.

as desired.

Proof of Proposition 3.8. Use induction on $|A|$.

If $A=B$, there is nothing to prove. We may assume $A\neq B$. Let $H$ be a maximal normal subgroup of $A$ containing $B$. By the inductive hypothesis, we have
Also by Corollary 3.7, \((\chi)\pi(G, H)\) and \((\phi)\pi(D, H|B)\) are in \(\text{Irr}_A(C_0(H))\) and both of them are modularly irreducible. Since \(A/H\) is cyclic, it follows from Lemma 3.9 that \((\chi)\pi(G, H)\pi(C_0(H), A/H)\) and \((\phi)\pi(D, H|B)\pi(C_0(H), A/H)\) are equal on the set of \(p\)-regular elements of \(C\). Now the proof is completed by Theorem 2.2 (ii).

**Theorem 3.10.** Assume Hypothesis 2.1 and that \(G\) is \(p\)-solvable and \(A\) is solvable. Then there exists a bijection

\[ \pi(G, A): \text{IBr}_A(G) \to \text{IBr}(C) \]

which is independent of the choice of \(\Omega\) with the lifting property with respect to \(A\). And the following hold.

(i) If \(B \triangleleft A\), then \(\pi(G, A) = \pi(G, B)\pi(C_0(B), A/B)\).

(ii) If \(A\) is a \(q\)-group for a prime \(q\) and \(\phi \in \text{IBr}_A(G)\), then \((\phi)\pi(G, A)\) is the unique irreducible constituent of \(\phi_C\) with multiplicity prime to \(q\).

Proof. By putting \(B=1\) in Proposition 3.8, it follows that \(\pi(G, A)\) is independent of the choice of such an \(\Omega\). If \(B \triangleleft A\), it is easily seen by Proposition 3.8 that \(\pi(G, A) = \pi(G, B)\pi(C_0(B), A/B)\). Now assume \(\phi \in \text{IBr}_A(G)\) and fix \(\Omega\) with the lifting property with respect to \(A\). Then there exists \(\chi \in \Omega_A\) such that \(\hat{\chi} = \phi\). If \(A\) is a \(q\)-group, by Theorem 2.2 (iii) we have

\[ \chi_C = m(\chi)\pi(G, A) + q\psi, \]

where \(m\) is a positive integer prime to \(q\) and \(\psi\) is zero or a character of \(C\). Therefore by the definition of \(\pi(G, A)\), it follows that

\[ \phi_C = m(\phi)\pi(G, A) + q\psi, \]

and the rest of Theorem is obvious.

4. **The case: \(p\) is odd**

In this section, we consider the correspondence of \(p\)-modular characters for an odd prime \(p\). First we show the following.

**Theorem 4.1.** Assume Hypothesis 2.1 and that \(G\) is \(p\)-solvable. If \(p\) is odd, then \(\pi(G, A)\) gives a bijection from \(\mathcal{Q}_A(G) = \mathcal{Q}(G) \cap \text{Irr}_A(G)\) onto \(\mathcal{Q}(C)\).

Before proving the above theorem, we should mention the definition of \(\mathcal{Q}(G)\) for an odd prime \(p\). When \(p\) is odd, \(\mathcal{Q}(G)\) coincides with the set of subnormally \(p\)-rational irreducible characters of \(G\). Here a character \(\chi\) is called subnormally \(p\)-rational if upon restriction to every subnormal subgroup,
every irreducible constituent of $\chi$ is $p$-rational i.e. has values in some field of the form $\mathbb{Q}[\xi]$ where $\xi^n = 1$, $p \not| n$.

To prove Theorem 4.1 we need one more lemma about $\pi(G, A)$.

For $\chi \in \text{Irr}(G)$, let $Q(\chi)$ be the extension of $Q$ obtained by adjoining the values $\chi(g)$, $g \in G$, to $Q$.

**Lemma 4.2.** Assume Hypothesis 2.1. Let $K$ be a Galois extension of $Q$ containing a primitive $|G|$-th root of unity. Let $\chi \in \text{Irr}_A(G)$ and $\xi = (\chi)\pi(G, A)$. Then $(\chi')\pi(G, A) = \xi'$ for $\sigma$ in the Galois group of $K$ over $Q$. Moreover $Q(\chi) = Q(\xi)$.

Proof. Since the actions of $A$ and $\sigma$ on $\text{Irr}(G)$ commute with each other, it follows that $\chi' \in \text{Irr}_A(G)$. Thus $(\chi')\pi(G, A)$ is meaningful. First we show $(\chi')\pi(G, A) = \xi'$. When $A$ is solvable, we may assume that $|A|$ is a prime by Theorem 2.2 (ii) and induction on $|A|$. Then it follows from Theorem 2.2 (iii) that $\xi'$ is the unique irreducible constituent of $\chi'_e$ with multiplicity prime to $|A|$. Thus the result follows.

When $|G|$ is odd, by Theorem 2.2 (iv) $\chi_{\sigma, A \in C}$ has the unique $A$-invariant irreducible constituent $\eta$ with odd multiplicity and $(\chi)\pi(G, A) = (\eta)\pi([G, A]'C, A)$. An argument similar to the above one and induction on $|G|$ yield the result.

The rest of Lemma follows from the first statement; the field automorphisms of $K$ that fix $\chi$ coincide with those that fix $\xi$.

**Proof of Theorem 4.1.** Use induction on $|G|$. Let $N$ be a maximal $A$-invariant normal subgroup of $G$.

First we claim $(\mathfrak{q}_A(G))\pi(G, A) \subseteq \mathfrak{q}(C)$. Assume $\chi \in \mathfrak{q}_A(G)$. By Lemma 1.2, there exists $\theta \in \text{Irr}_A(N)$ such that $[\chi_N, \theta] \neq 0$. Since $\theta \in \mathfrak{q}_A(N)$, it follows from the inductive hypothesis that $(\theta)\pi(N, A) \in \mathfrak{q}(N \cap C)$. Also by Lemma 2.3 (i), we have $(\chi)\pi(G, A) \in \text{Irr}(C \{(\theta)\pi(N, A))$. If $G/N$ is a $p'$-group, then by Lemma 3.4 (i), we have $(\chi)\pi(G, A) \in \text{Irr}(C \{(\theta)\pi(N, A)) \subseteq \mathfrak{q}(C)$ as desired. Now assume that $G/N$ is a $p$-group. Then $((\theta)\pi(N, A)^c$ has a unique $p$-rational irreducible constituent. (See Corollary 7.3 of [4].) By Lemma 3.4 (ii), it lies in $\mathfrak{q}(C)$. On the other hand, since $\chi$ is $p$-rational, so is $(\chi)\pi(G, A)$ by Lemma 4.2. Thus we conclude that $(\chi)\pi(G, A)$ is just the element in $\mathfrak{q}(C) \cap \text{Irr}(C \{(\theta)\pi(N, A))$.

Now we prove Theorem. Since $\mathfrak{q}(G)$ has the lifting property with respect to $A$, it follows from Corollary 3.7 that $\wedge$ gives the bijection from $(\mathfrak{q}_A(G))\pi(G, A)$ onto $\text{IBr}(C)$. Especially we have $|(\mathfrak{q}_A(G))\pi(G, A)| = |\text{IBr}(C)|$. Since $|\mathfrak{q}(C)| = |\text{IBr}(C)|$, $(\mathfrak{q}_A(G))\pi(G, A)$ coincides with $\mathfrak{q}(C)$. This completes the proof.

Note that Theorem 4.1 is false if $p=2$. (See Appendix.)
By Theorem 4.1, we can define \( \pi(G, A) \) via \( \mathcal{J}(G) \) in the case where \( p \) is odd.

**Theorem 4.3.** Assume Hypothesis 2.1 and that \( G \) is \( p \)-solvable for an odd prime \( p \). Then there exists a uniquely defined bijection
\[
\pi(G, A) : \text{IBr}_A(G) \to \text{IBr}(C).
\]
Moreover if \( B \leq A \), then \( \pi(G, A) = \pi(G, B) \pi(C_G(B), A/B) \).

**Proof.** We have the following diagram
\[
\text{IBr}_A(G) \xrightarrow{\pi(G, A)} \text{IBr}(C).
\]
Since \( \mathcal{J}(G) \) is a characteristic subset of \( \text{Irr}(G) \),
\[
\pi(G, A) = \pi^{-1}(\pi(G, A)) : \text{IBr}_A(G) \to \text{IBr}(C)
\]
is a uniquely defined bijection. The last part of Theorem is clear.

**Remark.** In the case where \( A \) is solvable and \( p \) is odd, Theorem 3.10 permits us to take \( \mathcal{J}(G) \) for \( \Omega \), and hence the bijection \( \pi(G, A) \) in Theorem 3.10 and Theorem 4.3 are the same.

Under Hypothesis 2.1, we note that application of Lemma 1.1 shows that there is a bijection from the set of \( A \)-fixed \( \pi \)-regular conjugacy classes \( \mathcal{K} \) of \( G \) onto the set of all \( \pi \)-regular conjugacy classes of \( C \) sending any such \( \mathcal{K} \) into \( \mathcal{K} \cap C \).

The final result is analogous to Theorem 13.24 of [5].

**Corollary 4.4.** Assume Hypothesis 2.1 and that \( G \) is \( p \)-solvable. Then the actions of \( A \) on \( \text{IBr}(G) \) and on the set of \( p \)-regular conjugacy classes of \( G \) are permutation isomorphic.

**Proof.** The same proof as in Lemma 13.23 and Theorem 13.24 of [5] works for our case.

**Appendix**

Here we give an example which shows us that Theorem 4.1 is false when \( p=2 \).

Let \( q \) be a prime such that \( q \equiv \pm 5 \pmod{12} \). Let \( E \) be a nonabelian group of order \( q^3 \) and exponent \( q \) with \( E = \langle x, y \rangle \), \( x^q = y^q = [x, y]^q = 1 \). Define \( \gamma \in \text{Aut}(E) \) by \( x^\gamma = y^{-1}x, y^\gamma = x \) so that \( o(\gamma) = 6 \). Let \( G = E \rtimes \langle \gamma^3 \rangle \), the semi-direct product, and let \( A = \langle \gamma^2 \rangle \). Then \( A \) acts on \( G \) coprimely, centralizing \( C = Z(E) \rtimes \langle \gamma^3 \rangle \). Let \( \theta \) be a faithful irreducible character of \( E \). Since \( \theta \) vanishes
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on $E \setminus Z(E)$, it is $A$-invariant. And since $(|E|, |\langle \gamma^3 \rangle|) = 1$, there exists the canonical extension of $\theta$ to $G$, say $\chi$. Assume $p=2$. Then $\chi$ lies in $\mathcal{X}(G)$. For the definition of $\mathcal{X}(G)$, see Definition 2.2 of [6]. Thus $\chi$ lies in $\mathcal{H}_d(G)$. (See Definition 5.1 of [6].) Since $[\chi, \chi] = 1$, easy calculation yields $|\chi(\gamma^3)| = 1$. Let $\lambda$ be the unique irreducible constituent of $\chi_{\gamma^3}(E)$. Then we have

$$\chi_{\gamma} = \begin{cases} ((q+1)/2)\lambda + ((q-1)/2)\lambda_\mu, & \text{if } q \equiv 5 \pmod{12} \\ ((q-1)/2)\lambda + ((q+1)/2)\lambda_\mu, & \text{if } q \equiv -5 \pmod{12}, \end{cases}$$

where $\mu$ is the unique nontrivial character of $\langle \gamma^3 \rangle$.

In both cases, we have $3\gamma_1 [\chi, \lambda_\mu]$. It follows from Theorem 2.2 (iii) that $(\chi_\gamma)(G, A) = \lambda_\mu$. But since $(\lambda_\mu, \gamma^3) = \mu \in \mathcal{H}(<\gamma^3>) = \{1, \varphi_3\}$ and $<\gamma^3>_C \not\subseteq \mathcal{H}(C)$, $\lambda_\mu$ does not belong to $\mathcal{H}(C)$.

Moreover note that the image of $\mathcal{H}_d(G)$ by $\pi(G, A)$ does not have the lifting property with respect to $\{1\}$, the trivial automorphism of $C$, although $\mathcal{H}(G)$ has it with respect to $A$.

References


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