MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS
AND THE FUNDAMENTAL SOLUTION FOR A
HYPERBOLIC SYSTEM WITH INVOLUTIVE
CHARACTERISTICS

Dedicated to the memory of Professor Hitoshi Kumano-go

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Introduction. Let $\mathcal{L}$ be a hyperbolic system with the diagonal principal part

$$\mathcal{L} = D_t - \lambda(t, X, D_x) \begin{bmatrix} \lambda(t, X, D_x) & 0 \\ 0 & \lambda(t, X, D_x) \end{bmatrix} + (b_{\alpha\beta}(t, X, D_x)).$$

In order to consider the propagation of singularities of solutions of an equation $\mathcal{L}U(t) = 0$, we frequently employ a method of constructing the fundamental solution $E(t, s)$ and investigating its properties. In Kumano-go-Taniguchi-Tozaki [11] and Kumano-go-Taniguchi [10] the fundamental solution $E(t, s)$ of the hyperbolic system $\mathcal{L}$ has been constructed in the form

$$E(t, s) = I_\phi(t, s) + \int_s^t I_\phi(t, \theta) \{ W_\phi(\theta, s)$$

$$+ \sum_{v=1}^\infty \int_{t_v}^\theta \int_{t_{v-1}}^{t_v} \cdots \int_{t_1}^{t_2} W_\phi(\theta, t_v) W_\phi(t_v, t_{v-1}) \cdots$$

$$\times W_\phi(t_{v-1}, s) dt_{v-1} \cdots dt_1 \} d\theta \quad (t_0 = \theta),$$

where $I_\phi(t, s)$ and $W_\phi(t, s)$ are $l \times l$ matrices of Fourier integral operators $P_\phi(t, s)$ defined by $P_\phi(t, s)u = \int e^{i\phi(t, s; x, \xi)} p(t, s; x, \xi) \hat{u}(\xi) d\xi$. The expression (2) is obtained by constructing, first, an approximate fundamental solution $I_\phi(t, s)$ and next applying the method of the successive approximation. When we want to derive some properties of $E(t, s)$ from (2), it is necessary to estimate the multi-product

$$Q_{v+1} = P_{1, \phi_1} P_{2, \phi_2} \cdots P_{v+1, \phi_{v+1}}$$
of Fourier integral operators $P_{j,t}$. In the present paper, we will show an estimate of $Q_{v+1}$ and apply it to reduce $E(t, s)$ of (2) to a finite sum expression

$$E(t, s) = W_0^0(t, s) + \sum_{j=2}^{v+1} \int_{t_v}^{t_{v-1}} \cdots \int_{t_2}^{t_1} W_0^0(t, t_1, \ldots, t_{v-1}, s) dt_{v-1} \cdots dt_1$$

when the operator (1) is involutive. The expression (4) gives us information on the propagation of singularities.

Let $S_\rho^m (\rho > 1/2)$ denote a class of symbols $p(x, \xi)$ of pseudo-differential operators in $\mathbb{R}^n$ which is defined in Definition 1.1 of Chap. 2 in [8], and set $S_\rho^m = S_{\rho-1}^m$ for $1/2 < \rho \leq 1$, $S_{\rho,\delta} = S_{\rho-\delta}^m \cup S_{\rho,\delta}^m$ and $S_{\rho,\delta}^m = \bigcap_m S_{\rho,\delta}^m$. The class $S_{\rho,\delta}^m$ is a Fréchet space with semi-norms

$$|p|_{1,1} = \max_{|\alpha|=1, |\beta|\leq 1} \sup \{ |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \langle \xi \rangle^{-|\alpha|-|\beta|} \} ,$$

where $p(x, \xi) = -i \xi D_x p(x, \xi)$, $D_x = (-i)^{\beta \delta}$ and $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ (c.f. § 1 of Chap. 2 and (1.13) of Chap. 7 in [8]). Let $S_\rho^m (\tau, l) (0 < \tau < 1, 1/2 < \rho \leq 1, l = 0, 1, 2, \ldots)$ be the class of phase functions $\phi(x, \xi)$ such that $J(x, \xi) = \phi(x, \xi) - x \cdot \xi (x \cdot \xi = x_1 \xi_1 + \cdots + x_n \xi_n)$ satisfy $J_{\rho,\delta}^m \in S_{\rho-\delta}^{1-|\alpha|}$ for $|\alpha| + |\beta| \leq 2$ and

$$|J|_{1,l} = \sum_{|\alpha| = 0}^{l} \sup \{ |J_{\rho,\delta}^m(x, \xi)| \langle \xi \rangle^{-(1-|\alpha|-(1-\rho)(|\alpha|+\beta)^2)} \} \leq \tau ,$$

where $a_\rho = \max(a, 0)$ for a real $a$. We set $S_\rho^m (\tau, l) = S_\rho^m (\tau, 0)$. For $\phi(x, \xi) \in S_\rho^m (\tau, l)$ and $p(x, \xi) \in S_\rho^m (\tau, l)$, we define a Fourier integral operator $P_{\phi} = p(\phi, X, D_x)$ with phase function $\phi(x, \xi)$ and symbol $\sigma(P_{\phi}) = p(x, \xi)$ by

$$P_{\phi}u = \int e^{i\phi(x, \xi)} p(x, \xi) \hat{u}(\xi) d\xi \quad \text{for} \ u \in S .$$

Here, $d\xi = (2\pi)^{-n} d\xi$, $S$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and $\hat{u}(\xi) = \int e^{-ix\xi} u(x) dx$ is the Fourier transform of $u(x)$. In (7) $P_\phi$ is a pseudo-differential operator when $\phi(x, \xi) = x \cdot \xi$. In this case we write $P_\phi = p(\phi, X, D_x)$ simply by $P = p(X, D_x)$ and we often say that $P$ is a pseudo-differential operator in $S_\rho^m$. For $p_j(x, \xi) \in S_\rho^m (\tau, l)$ ($j = 1, 2, \ldots$) we say that $\{p_j\}$ is bounded in $S_\rho^m (\tau, l)$ if the set $\{|p_j|_{1,l}^{(m)}\}$ of semi-norms $|p_j|_{1,l}^{(m)}$ is bounded for any $l_1$ and $l_2$.

Concerning the multi-products of Fourier integral operators the following is shown in Kumano-go-Taniguchi [10] for the case $\rho > 1/2$.

Let $\phi_j(x, \xi)$ belong to $S_\rho (\tau_j)$ and let $p_j(x, \xi)$ belong to $S_{\rho_j}^m (j = 1, 2, \ldots)$. Suppose that

1) Their proof is also valid for $\rho = 1/2$. 


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(*) \[ \sum_{j=1}^{\infty} \tau_j \leq \tau^0 \] for some positive constant \( \tau^0 \) and for \( f_j(x, \xi) = \phi_j(x, \xi) - x \cdot \xi \) the set \( \{ J_{j(\beta)} / \tau_j \} \) is bounded in \( S^1_{\beta} \) when \( |\alpha + \beta| \leq 2 \).

Then, for any \( v \) the multi-product \( \bar{Q}_{v+1} \) of (3) is a Fourier integral operator \( Q_{v+1, \Phi_{v+1}} \) with a phase function \( \Phi_{v+1}(x, \xi) \) in \( \mathcal{D}_\rho(\tau_{v+1}) \) \((\tau_{v+1} = \tau_1 + \tau_2 + \cdots + \tau_{v+1}) \) for some constant \( c_0 \) and with a symbol \( q_{v+1}(x, \xi) \) in \( S_{\rho_{v+1}}^m \) for \( \bar{m}_{v+1} = m_1 + m_2 + \cdots + m_{v+1} \). (c.f. Theorem 2.3 of [10]).

The result we want to show on \( \bar{Q}_{v+1} \) of (3) is the following:

**Theorem 1.** Suppose that \( \phi_j(x, \xi) \) belongs to \( \mathcal{D}_\rho(\tau_j, \tau_\delta) \), \( j=1, 2, \ldots \), and (*) holds, where \( \tau_\delta \) is an integer determined only by \( \rho \) and \( n \). Then, for each bounded set \( \{ p_j \} \) in \( \{ S_{\rho}^m \} \) there exists a constant \( C_0 \) such that the set \( \{ p_{v+1}q_{v+1} \} \) is bounded in \( \{ S_{\rho_{v+1}}^m \} \) if we assume \( \sum_{j=1}^{\infty} |m_j| < \infty \).

Concerning estimates of multi-products (3) Kumano-go–Taniguchi [10] gave only operator norms in Sobolev spaces, but they did not show estimates of symbols. To obtain their estimates they used essentially asymptotic expansions of products of Fourier integral operators, and it seems to us that it is almost impossible to obtain the estimates including the case \( \rho = 1/2 \).

In order to prove Theorem 1 we must employ a method completely different from [10]. First we show the fact that there exist pseudo-differential operators \( R \) and \( R' \) in \( S_{\rho}^0 \) such that

\[
\begin{align*}
I_\phi R I_{\phi^*} &= I, \\
I_\phi R' I_{\phi^*} &= I
\end{align*}
\]

hold for a phase function \( \phi(x, \xi) \) in \( \mathcal{D}_\rho(\tau, \tau_\delta) \) if \( \tau \) is small enough, where \( I_\phi \) [resp. \( I_{\phi^*} \)] is the Fourier [resp. conjugate Fourier] integral operator with phase function \( \phi(x, \xi) \) and symbol 1. Then, the multi-product (3) can be written in the form

\[
\begin{align*}
\{ &i \} \quad \bar{Q}_{v+1} = P_1' P_2' \cdots P_{v+1} I_{\Phi_{v+1}}, \\
\{ &ii \} \quad \bar{Q}_{v+1} = I_{\Phi_{v+1}} P_1'' P_2'' \cdots P_{v+1}'
\end{align*}
\]

with pseudo-differential operators \( P_j' \) and \( P_j'' \) in \( S_{\rho_j}^m \) \((j=1, 2, \ldots) \). Thus, the problem to estimate the symbol \( q_{v+1}(x, \xi) \) of a multi-product \( \bar{Q}_{v+1} = Q_{v+1, \Phi_{v+1}} \) of Fourier integral operators is reduced to the problem of obtaining an estimate of a multi-product of pseudo-differential operators. Therefore, it is the key point in the proof of Theorem 1 to show the existence of pseudo-differential operators \( R \) and \( R' \) verifying (8). To show their existence, it is necessary to obtain a sharp estimate of symbols of multi-products of pseudo-differential
operators. Our theorem concerning multi-products of pseudo-differential operators is the following.

**Theorem 2.** Let \(0 \leq \delta \leq \rho \leq 1\), \(\delta < 1\) and let \(p_j(x, \xi) \in S^{\rho, \delta}_{m, 0}\), \(j = 1, 2, \ldots\). Consider the multi-product

\[
Q_{v+1} = P_1 P_2 \cdots P_{v+1}
\]

of \(P_j = p_j(X, D_x)\). Denote by \(q_{v+1}(x, \xi)\) the symbol of \(Q_{v+1}\). Then, there exists a constant \(A\) determined only by \(\delta\) and \(n\) such that \(M = \sum |m_j| < \infty\) and the boundedness of \(\{p_j\}\) in \(\{S^{\rho, \delta}_{m, 0}\}\) imply the boundedness of \(\{C_0^{-\nu} q_{v+1}\}\) in \(\{S^{\rho, \delta}_{m, 0}\}\) with

\[
C_0 = A \max_j |p_j|_{m+1, \nu}
\]

for

\[
l_0 = \lceil n/(1-\delta) + 1 \rceil.
\]

Since the \((v+1)\)-st power \(P^{v+1}\) of a pseudo-differential operator \(P\) with symbol \(p(x, \xi)\) in \(S^{\rho, \delta}_{m, 0}\) satisfies

\[
|\sigma(P^{v+1})|_{l_1, l_2}^{(0)} \leq C_{l_1, l_2}(A |p|_{m+1, \nu})^\nu,
\]

we get immediately

**Theorem 3.** Assume that \(p(x, \xi)\) in \(S^{0, \delta}_{\rho, 0}\) satisfies

\[
|p|_{m+1, \nu}^{(0)} < 1/A
\]

for the constant \(A\) in Theorem 2. Then, the inverse \(Q\) of the operator \(I - P\) exists and is a pseudo-differential operator in \(S^{\rho, \delta}_{m, 0}\) represented by the Neumann series

\[
\sum_{v=0}^\infty P^v.
\]

The existence of \(R\) and \(R'\) in (8) is derived by applying Theorem 3 to \(I \phi I \phi - I\) and \(I \phi I \phi - I\). Concerning the estimate of multi-products of pseudo-differential operators Kumano-go obtained in [6] a semi-norm estimate

\[
|q_{v+1}|_{l_1, l_2}^{(m+1)} \leq C_{l_1, l_2}^\nu.
\]

The estimate (15) is effectively used for the construction of the fundamental solution of a parabolic equation (see, for example, § 4 of Chap. 7 in [8]) and also used for the \(L^2\)-boundedness of a pseudo-differential operator (see [6]). But the estimate (15) is not sufficient for the proof of the convergence of the Neumann series \(\sum P^v\). Hence, we need the estimate (13) sharper than (15). Using this estimate (13) we prove the convergence of the Neumann series.
We like to emphasize that by virtue of Theorem 2 the inverse of a pseudo-differential operator may be obtained only by the symbol calculus when (14) is satisfied. In [1] Beals has proved that the inverse of a pseudo-differential operator is also a pseudo-differential operator, but he showed it by a discussion in Sobolev spaces, not by the symbol calculus. In Appendix of [8] Kumano-go has given another proof of the convergence of Neumann series by using the commutator theory and the symbol calculus.

Now, we return to the problem of the reduction of the fundamental solution $E(t, s)$ of (2) for $\mathcal{L}$ to the expression (4). Let $M^0([0, T]; S^m_p((k)))$ [resp. $M([0, T]; S^m_p((k)))$] be the set of symbols $p(t, x, \xi)$ such that $p^{[\gamma]}(t, x, \xi)$ [resp. $\partial^\gamma p^{[\gamma]}(t, x, \xi)$ for any $\gamma$] are bounded in $S^m_{-|\alpha|}$ for any $t \in [0, T]$ when $|\alpha| + |\beta| \leq k$; and we also set $M^0([0, T]; S^m_p) = M^0([0, T]; S^m_p((0)))$ and $M([0, T]; S^m_p) = M([0, T]; S^m_p((0)))$ (for details, see Definition 2.4 and Definition 3.1). In the present paper, we shall consider a system (1) under the following condition (I) or (II).

(I) The characteristic roots $\lambda_m(t, x, \xi)$ belong to $M^0([0, T]; S^m_p((2))) \cap C^1([0, T] \times \mathbb{R}^n_\xi)$ and the symbols $b_{mk}(t, x, \xi)$ in (1) belong to $M^0([0, T]; S^m_p)$. For any $m$ and $k$ there exists a continuous function $a_{m,k}(t)$ such that the Poisson bracket $\{\tau - \lambda_m, \tau - \lambda_k\}$ of $\tau - \lambda_m$ and $\tau - \lambda_k$ satisfies

$$\{\tau - \lambda_m, \tau - \lambda_k\} = a_{m,k}(t)(\lambda_m - \lambda_k).$$

(II) The characteristic roots $\lambda_m(t, x, \xi)$ belong to $M([0, T]; S^1_p((3)))$ and the symbols $b_{mk}(t, x, \xi)$ in (1) belong to $M([0, T]; S^m_p)$. For any $m$ and $k$ there exist real symbols $a_{m,k}(t, x, \xi)$ and $a'_{m,k}(t, x, \xi)$ with

$$\begin{cases} a_{m,k}(t, x, \xi) \in M([0, T]; S^m_p((1))), \\ a'_{m,k}(t, x, \xi) \in M([0, T]; S^m_p) \end{cases}$$

such that

$$\begin{cases} \tau - \lambda_m, \tau - \lambda_k \} = a_{m,k}(t, x, \xi)(\lambda_m - \lambda_k) + a'_{m,k}(t, x, \xi) \end{cases}$$

holds.

By using Theorem 1 and the commutative law for $\#$-products of phase functions (Theorem 3.9) we obtain

**Theorem 4.** Under the condition (I) or (II) the fundamental solution $E(t, s)$ of (2) can be reduced to the expression (4).

The expression (4) of $E(t, s)$ gives us much information on the propagation of singularities of the solution of $\mathcal{L}U(t) = 0$. For example, the estimate of singularities obtained in [13] follows immediately from (4) (see Corollary 4.5). Concerning the expression (4) of $E(t, s)$ Ludwig-Granoff [12], Hata [2] and
Nosmas [14] obtained it only for \( \rho = 1 \). They constructed it by a method of solving transport equations. On the other hand, Kumano-go, Taniguchi and Tozaki have proved in [10]–[11] Theorem 4 without solving transport equations under a stronger assumption than (I), that is, \( \rho = 1 \) and \( a_{m,k}(t) \) in (16) are identically zero, and Morimoto [13] has also obtained it under the assumption (I) with \( \rho = 1 \) and \( C^\infty \)-functions \( a_{m,k}(t) \) in (16). We note that Ichinose [5] also showed Theorem 4 in the case of \( l = 2, \rho > 1/2 \) and (I).

Theorem 4 with \( \rho < 1 \) makes it possible to treat hyperbolic equations with characteristic roots which are not necessary \( C^\infty \) differentiable. Namely, with the aid of the approximation theory by [9] the hyperbolic equations can be reduced to the hyperbolic systems (1) with symbols in the class \( S_\rho = S_{\rho,1-\rho} \) (see [5], for details). The less differentiable the characteristic roots are, the smaller \( \rho \) (\( \geq 1/2 \)) we need. For example, we consider a hyperbolic operator \( L^i \) in \( \mathbb{R}^2 \):

\[
(19) \quad L^1 = D^2_x - a_3(x)(D^2_{x_1} + D^2_{x_2}),
\]

where \( a_3(x) \) (\( k \geq 2 \)) is a \( C^\infty \)-function satisfying

\[
\left\{ \begin{array}{ll}
a_3(x) &= x_1^{2k} + x_2^{2k} \quad (|x| \leq 1), \\
0 < a_3 &\leq 2 \quad (|x| \geq 2), \\
0 < a_3 &\leq |a_3(x)| \leq 2 \quad (|x| \geq 1) \quad \text{for some} \ a_3.
\end{array} \right.
\]

The operator \( L^1 \) has characteristic roots \( \chi^1(x, \xi) = \pm \sqrt{a_3(x)}|\xi| \) which are \( C^{k-1} \)-class with Lipschitz derivatives of \( (k-1) \)-st order for \( |\xi| \geq 1 \). The operator (19) with \( k \geq 5 \) was considered in [9] and (19) with \( k = 4 \) in [5]. Including the cases \( k = 2 \) and 3 we shall show that (19) can be reduced to a system (1) with symbols in \( S_\rho \) for \( \rho = 1 - 1/k \) and investigate the propagation of singularities. For the case \( k = 2 \) we need \( \rho = 1/2 \). Other examples which can be reduced to the system with \( \rho = 1/2 \) are

\[
(20) \quad L^2 = D^2_x - a(x_1)^2(D^2_{x_1} + a(x_1)^2D^2_{x_2}),
\]

\[
(21) \quad L^3 = D^2_x - 2a(x_1)^2D^2_{x_1}D_x - a(x_1)^6D^2_{x_2},
\]

where \( a(x_1) \) is a \( C^\infty \)-function in \( \mathbb{R}^1 \) satisfying

\[
\left\{ \begin{array}{ll}
a(x_1) &= x_1 \quad (|x_1| \leq 1), \\
0 < a_3 &\leq |a(x_1)| \leq 2 \quad (|x_1| \geq 1) \quad \text{for some} \ a_3.
\end{array} \right.
\]

The reduction of \( L^j \) (\( j = 1, 2, 3 \)) to the system (1) and the information on the propagation of singularities are given at the end of Section 4.

The outline of the present paper is the following: In Section 1 we shall study multi-products of pseudo-differential operators. Section 2 is devoted to the proof of Theorem 1. In Section 3 we shall prove the commutative law for \#-products of phase functions and in Section 4 we shall construct the
fundamental solution of (1) and prove Theorem 4.

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1. Multi-products of pseudo-differential operators and Neumann series. Let \((x^0, \xi^0) = (x^0, x^1, \cdots, x^\nu)\) be a \((\nu+1)\)-tuple of points \(x^0, x^1, \cdots, x^\nu\) in \(R^n\) and \(\xi^{\nu+1} = (\xi^1, \cdots, \xi^{\nu+1})\) be a \((\nu+1)\)-tuple of points \(\xi^1, \cdots, \xi^{\nu+1}\) in \(R^\nu\).

**Definition 1.1.** Let \(0 \leq \delta \leq \rho \leq 1\), \(\delta < 1\) and let \(\bar{m}_{\nu+1} = (m_1, \cdots, m_{\nu+1})\) be a real vector. We say that a \(C^\infty\)-function \(\rho(x^0, \xi^0, \xi^{\nu+1}, x^{\nu+1}) = \rho(x^1, \xi^1, x^1, \xi^2, \cdots, x^\nu, \xi^{\nu+1}, x^{\nu+1})\) in \(R^{(\nu+3)n}\) belongs to a multiple symbol class \(S^\bar{m}_{\nu+1}\) when

\[
|\partial^{\alpha_1}_{\xi^{\nu+1}} \cdots \partial^{\alpha_{\nu+1}}_{\xi^{\nu+1}} \partial^{\beta_0}_{x^0} \partial^{\beta_1}_{x^1} \cdots \partial^{\beta_{\nu+1}}_{x^{\nu+1}} \rho(x^0, \xi^0, \xi^{\nu+1}, x^{\nu+1})| 
\leq C^\bar{m}_{\nu+1, \rho, \bar{\beta}^{\nu+1}} \prod_{j=1}^{\nu+1} |\xi^j|^{-\rho |\alpha_j| - \beta_0} \prod_{j=1}^{\nu+1} |\xi^j|^{-\beta_j} \prod_{j=1}^{\nu+1} |\xi^{\nu+1}|^{-\beta_{\nu+1}}
\]

holds for any \((\nu+1)\)-tuple \(\alpha^{\nu+1} = (\alpha_1, \cdots, \alpha^{\nu+1})\) and \((\nu+2)\)-tuple \((\beta^0, \bar{\beta}^{\nu+1}) = (\beta_0, \beta_1, \cdots, \beta^{\nu+1})\) of multi-indices \(\alpha_1, \cdots, \alpha^{\nu+1}\) and \(\beta_0, \beta_1, \cdots, \beta^{\nu+1}\) of \(R^n\), where \(\langle \xi; \xi^\rho \rangle = \langle \xi^\rho \rangle + \langle \xi^\rho \rangle^\rho\).

**Remark.** The multiple symbol class was introduced in Kumano-go [6]. But the semi-norms (1.2) are slightly different from semi-norms (2.4) of [6]. Corresponding the multiple symbol class, the class \(S^m_{\nu, \delta}\) in Introduction is often called a single symbol class.

For \(\rho(x^0, \xi^0, \xi^{\nu+1}, x^{\nu+1}) = \rho(x^0, \xi^0, x^1, \xi^2, \cdots, x^\nu, \xi^{\nu+1}, x^{\nu+1})\) in \(S^\bar{m}_{\nu+1}\) \(\rho(X, D_x^1, X^1, D_x^1, \cdots, X^\nu, D_x^\nu, X^{\nu+1})\) denotes a pseudo-differential operator \(P\) defined by

\[
(Pu)(x^0) = O_{\nu+1} \int \exp \left\{ \sum_{j=1}^{\nu+1} (x^{j-1} - x^j) \cdot \xi^j \right\} \times \rho(x^0, \xi^0, x^1, \xi^2, \cdots, x^\nu, \xi^{\nu+1}, x^{\nu+1}) \times u(x^{\nu+1}) \, dx^1 \cdots dx^{\nu+1} \, d\xi^1 \cdots d\xi^{\nu+1} \quad \text{for} \quad u \in \mathcal{C},
\]

and \(\sigma(P) = \rho(x^0, \xi^0, \xi^{\nu+1}, x^{\nu+1})\) is called a symbol of \(P\). Here, the right hand side of (1.3) is the oscillatory integral defined in Section 6 of Chap. 1 in [8].
Throughout this paper, we shall often use the result there. Following Kumano-go [8], we write \( p(X, D_x, X'), p(X, D_x, X', D_x) \) and \( p(X, X', D_x, X') \) by \( p(X, D_x, X') \), \( p(X, D_x, X', D_x) \) and \( p(X, X', D_x, X') \), respectively. For \( p(x^0, \xi^1, \ldots, \xi^v, x^1, \xi^2, \ldots, x^{v+1}) \) in \( S_{m, b+1} \) we write
\[
(1.4) \quad p_L(x, \xi, x') = \frac{\partial^v}{\partial y \partial \eta} \int e^{-iy \cdot p(x, \xi + \eta, x + y, \xi + \eta)} d\eta d\eta^v,
\]
called a simplified symbol of \( p(x^0, \xi^1, x^1, \xi^2, x^2, \ldots, \xi^v, x^{v+1}, x^{v+1}) \), where
\[
(1.5) \quad \psi = \frac{\partial^v}{\partial y \partial \eta} (\eta^v - \eta^{v+1}) = \sum_{j=1}^v (y^j - y^{j-1}) \eta^j \quad (y^0 = \eta^{v+1} = 0),
\]
\( \eta^v = (y^1, \ldots, y^v) \in R^v, \eta^v = (\eta^1, \ldots, \eta^v) \in R^v \) and \( d\eta^v d\eta = dy^1 dy^v d\eta^1 \cdots d\eta^v d\eta^1 \cdots d\eta^v \). It is well-known that \( p_L(x, \xi, x') \) belongs to \( S_{m, v+1} \) \((m = m_1 + m_2 + \cdots + m_{v+1})\) and
\[
(1.6) \quad p(X, D_x, X', X^2, X^{v+1}) = p_L(X, D_x, X')
\]
\((c.f. \S 2 \text{ of Chap. 7 of } [8])\).

Now, we begin to prove Theorem 2. It is well-known from (2.6) and (2.8) of Chap. 7 in [8] that the symbol \( q_{v+1}(x, \xi) \) of the multi-product \( Q_{v+1} = P_1 P_2 \cdots P_{v+1} \) has the form
\[
(1.7) \quad q_{v+1}(x, \xi) = \frac{\partial^v}{\partial y \partial \eta} \int e^{-iy \cdot p(x, \xi + \eta, x + y, \xi + \eta)} d\eta d\eta^v \quad (y^0 = \eta^{v+1} = 0)
\]
with \( \psi \) in (1.5). Differentiating \( q_{v+1}(x, \xi) \) with respect to \( x \) and \( \xi \) we have
\[
(1.8) \quad q_{v+1, \langle \bar{\alpha}^{v+1}, \bar{\beta}^{v+1} \rangle}(x, \xi) = \sum_{\alpha, \beta, \gamma} \frac{\alpha! \beta!}{\gamma^{v+1}!} q_{v+1, \langle \alpha^{v+1}, \beta^{v+1} \rangle}(x, \xi)
\]
for
\[
(1.9) \quad q_{v+1, \langle \alpha^{v+1}, \beta^{v+1} \rangle}(x, \xi) = \frac{\partial^v}{\partial y \partial \eta} \int e^{-iy \cdot p(\alpha, \beta)} d\eta d\eta^v.
\]
Here, \( \bar{\alpha}^{v+1} = \alpha^1 \cdots \alpha^{v+1}, \bar{\beta}^{v+1} = \beta^1 \cdots \beta^{v+1} \) for \( \alpha^{v+1} = (\alpha^1, \ldots, \alpha^{v+1}), \beta^{v+1} = (\beta^1, \ldots, \beta^{v+1}) \) and \( \sum_{\alpha, \beta, v+1} \) means that the summation is taken over all \( (\alpha^{v+1}, \beta^{v+1}) \) satisfying \( \alpha^1 + \cdots + \alpha^{v+1} = \alpha \) and \( \beta^1 + \cdots + \beta^{v+1} = \beta \). Note that (1.9) means
\[
(1.10) \quad Q_{v+1, \langle \alpha^{v+1}, \beta^{v+1} \rangle} = P_{\alpha^1} P_{\beta^1} \cdots P_{\alpha^{v+1}}
\]
Set
\[
(1.11) \quad \varepsilon_\tau = (1 - \delta)(l_\tau - n/(1 - \delta)) \quad (>0)
\]
for the integer $l_0$ in (12). In this section $l_0$ and $\varepsilon_0$ always mean the numbers defined by (12) and (1.11).

**Proposition 1.2.** Let $\{m_j\}$ and $\{m'_j\}$ be sequences satisfying

\begin{align}
(1.12) \quad & \varepsilon_1 \equiv \sum_{j=1}^{\infty} |m_j| < \varepsilon_0, \\
(1.13) \quad & M' \equiv \sum_{j=1}^{\infty} |m'_j| < \infty, \\
(1.14) \quad & N^0 \equiv \text{the number of } \{j; m'_j > 0\} < \infty.
\end{align}

Let $Q_{v+1}^0 = P_1^0 P_2^0 \cdots P_{v+1}^0$ for $P_j^0 = P_j^0(X, D_x)$, $P_j^0(x, \xi) \in S_{m_j + m'_j}^{m_j + m'_j}$. Then, there exist a constant $A_0$ independent of $M'$, $N^0$ and $v$ and a constant $C$ depending only on $M'$, $N^0$, $n$, and $\delta$ (but independent of $v$) such that the symbol $q_{v+1}^0(x, \xi)$ of $Q_{v+1}^0$ satisfies

\begin{align}
(1.15) \quad |q_{v+1}^0(x, \xi)| \leq C A_0 \max_{k_j=0,1} \left\{ |p_j^0|^{(m_j + m'_j)(v+1)} \prod_{j=2}^{v+1} |p_j^0|^{(m_j + m'_j)(v+1)} \right\}^\frac{1}{v+1} \leq C A_0 \max_{k_j=0,1} \left\{ |p_j^0|^{(m_j + m'_j)(v+1)} \prod_{j=2}^{v+1} |p_j^0|^{(m_j + m'_j)(v+1)} \right\}^\frac{1}{v+1}
\end{align}

with $\mu^0 = [M'/((1-\delta)\xi)]$, $m_{v+1} = m_1 + \cdots + m_{v+1}$ and $m'_{v+1} = m'_1 + \cdots + m'_{v+1}$. Here, for a real $a$ we denote by $[a]^*$ the smallest integer not less than $a$.

Admitting this proposition, we apply it to each multi-product (1.10) by setting $P_j^0 = P_j^0(\alpha_j)$. Take an integer $N_0$ satisfying

\begin{align}
(1.16) \quad \sum_{j=N_0+1}^{\infty} |m_j| < \varepsilon_0
\end{align}

and set for fixed $\alpha^{v+1} = (\alpha_1, \ldots, \alpha^{v+1})$ and $\beta^{v+1} = (\beta_1, \ldots, \beta^{v+1})$

\begin{align}
(1.17) \quad \mu_j = \begin{cases} 
0 & j \leq N_0, \\
m_j & j > N_0.
\end{cases}
\end{align}

Then, $\mu_j + \mu'_j = m_j - \rho |\alpha^j| + \delta |\beta^j|$ and for $p_j^0 = P_j^0(\alpha_j)$ the set $\{p_j^0\}$ satisfies the assumption of Proposition 1.2 with $\{m_j\}$ and $\{m'_j\}$ replaced by $\{\mu_j\}$ and $\{\mu'_j\}$, respectively. The number $N^0$ in the proposition does not exceed $N_0 + \delta^* |\beta|$ if we set $\delta^* = [\delta]^*$, that is, $\delta^* = 0$ when $\delta = 0$ and $\delta^* = 1$ when $0 < \delta < 1$. Hence, we obtain from (1.15)

\begin{align}
(1.18) \quad |q_{v+1}(x, (\alpha^{v+1}, \beta^{v+1}))(x, \xi)| \leq C_{\alpha, \beta} A_0 \max_{j=1}^{v+1} \left\{ |p_j^0(\alpha_j)|^{(m_j - \rho |\alpha^j| + \delta |\beta^j|)} \right\}.
\end{align}
\[
\kappa = (k_1, \ldots, k_{v+1}) \in K_{v+1}(N_0 + \delta^*|\beta| + 1) \)

\[
\times \langle \xi \rangle^{\infty_{v+1} - p(\alpha + \delta|\beta|)} 
\leq C_{\alpha, \beta} A_0^{\nu} \max \prod_{j=0}^{v+1} |p_j| \left( m_j \right) ; 
\kappa \in K_{v+1}(N_0 + |\alpha| + |\beta| + \delta^*|\beta| + 1) \)

\[
\times \langle \xi \rangle^{\infty_{v+1} - p(\alpha + \delta|\beta|)} 
\leq C_{\alpha, \beta} A_0^{\nu} \max \prod_{j=0}^{v+1} |p_j| \left( m_j \right) ; 
\kappa \in K_{v+1}(N_0 + |\alpha| + |\beta| + \delta^*|\beta| + 1) \)

with a constant \( C_{\alpha, \beta} \) depending only on \( \alpha \) and \( \beta \), where \( K_{v+1}(l) = \{ \kappa = (k_1, \ldots, k_{v+1}); \ k_j = 0, 1, \sum_{j=1}^{v+1} k_j \leq l \} \). If we use \( \sum_{\alpha, \beta, v} \alpha \beta \left( \alpha^{v+1} \beta^{v+1} \right) = (v+1)^{\alpha+\beta} \), we obtain from (1.8) and (1.18)

\[
(1.19) \ |q_{v+1}(x, \xi)| \leq C_{\alpha, \beta} A_0^{\nu} (v+1)^{\alpha+\beta} 
\times \max \prod_{j=0}^{v+1} |p_j| \left( m_j \right) ; 
\kappa \in K_{v+1}(N_0 + |\alpha| + |\beta| + \delta^*|\beta| + 1) \)

\[
\times \langle \xi \rangle^{\infty_{v+1} - p(\alpha + \delta|\beta|)} ,
\]

For any fixed \( \sigma > 1 \) we take a constant \( C_\sigma \) independent of \( v \) such that for all \( v \)

\[
(v+1)^{\alpha+\beta} \leq C_\sigma^\nu .
\]

Combining this with (1.19) we get

\[
(1.20) \ |q_{v+1}|^{(\text{max})} \]

\[
\leq C_{t_1, t_2}^{t_1, t_2} A^{t_1, t_2} \max \prod_{j=0}^{v+1} |p_j| \left( m_j \right) ;
\kappa \in K_{v+1}(N_0 + \delta^*|\beta| + 1) \)

\[
(\tilde{I} = I_1 + I_2 + \delta^*l_2, \ \tilde{l}'' = [(M + \rho l_1 + \delta l_2)/(1-\delta)]^*),
\]

if we set \( A = A_0^{\tilde{I}} \) and \( C_{t_1, t_2} = \max_{|\alpha| \leq l_1, |\beta| \leq l_2} (C_{\alpha, \beta} C_{|\alpha+\beta|}) \). Consequently, for the constant \( C_0 \) in (11) the set \( \{ C_0^{-\nu} q_{v+1} \} \) is bounded in \( \{ S^{\infty_{v+1}} \} \). This concludes the proof of Theorem 2.

**Remark.** In Proposition 1.2 the constant \( A_0 \) depends also on \( \varepsilon_1 \). But, when \( N_0 \) in (1.16) satisfies \( \sum_{j=N_0+1}^{\infty} |m_j| \leq \varepsilon_0/2 \), we can take the constant \( A \) in Theorem 2 depending only on \( n \) and \( \delta \).

As a special case of Theorem 2 we have

**Corollary 1.3.** Let \( P \) be a pseudo-differential operator with symbol \( p(x, \xi) \) in \( S^p_{\rho, \delta} \). Then, the symbol \( q_{v+1}(x, \xi) \) of the \((v+1)\)-th power \( Q_{v+1} = P^{v+1} \) of \( P \) satisfies for any \( l_1 \) and \( l_2 \)
MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS

\[ (1.21) \quad |q_{v+1}| \leq C_{t, t_2} C_0 \left( |p|_{m+1, t_2} \right)^{t_1+t_2+\delta_* t+1} \]

\[ (I' = l_0 + [(p_1 + l_2)/(1-\delta)]^*, \ \delta_* = [\delta]^*) \]

if \( v \geq l_1 + l_2 + \delta_* l_1 + 1 \) holds. The constant \( C_0 \) is determined by

\[ (1.22) \quad C_0 = A |p|_{m+1, t} \]

for the constant \( A \) in Theorem 2.

Using this corollary we prove Theorem 3. Set

\[ q_v(x, \xi) = \sigma(P^v). \]

Suppose (14). Then, taking account of (1.21) the series \( \sum_{v=0}^\infty q_v(x, \xi) \) converges to a symbol \( q(x, \xi) \) in \( S_{p,0}^0 \) because of \( C_0 < 1 \). Since

\[ (1.23) \begin{cases} (\sum_{j=1}^\infty Q_j)(I - P) = I - Q_{v+1} \\ (I - P)(\sum_{j=1}^\infty Q_j) = I - Q_{v+1} \end{cases} \]

hold, by tending \( v \) in (1.23) to the infinity we see that the pseudo-differential operator \( Q = q(X, D_x) \) is the inverse of \( I - P \).

We note that the symbol \( q(x, \xi) \) has a semi-norm estimate

\[ (1.24) \quad |q|_{m, l_1, l_2} \leq C_{t, t_2} \left( \max \left( |p|_{m+1, t_2}, 1 \right) \right)^{t_1+t_2+\delta_* t_1+1} \]

with a constant \( C_{t, t_2} \) depending only on \( A |p|_{m+1, t_2} \), \( l_1 \) and \( l_2 \). In fact, writing

\[ q(x, \xi) = \sum_{v=0}^N q_v(x, \xi) + \sum_{v=N+1}^\infty q_v(x, \xi) \quad (N = l_1 + l_2 + \delta_* l_2) \]

we obtain (1.24) by applying (1.21) to the second term.

We turn to the proof of Proposition 1.2.

DEFINITION 1.4. For an integer \( N \) we say that a symbol \( p(x, \xi, x') \in S_{p,0}^m \) belongs to a class \( SX_{p,0}^m; N \) when \( p(x, \xi, x') \) satisfies

\[ (1.25) \quad |\delta_\xi^\alpha D_\xi^\beta D_\xi^\gamma p(x, \xi, x')| \leq C_{\alpha, \beta, \gamma} \langle \xi \rangle^{m-n|\alpha|+\delta|\beta|+\gamma\delta^*}(1 + \langle \xi \rangle^\delta |x-x'|)^{-N}. \]

For \( p(x, \xi, x') \in SX_{p,0}^m; N \) we define semi-norms \( ||p||_{m, l_1, l_2, l_2^*; N} \) by

\[ (1.26) \quad ||p||_{m, l_1, l_2; N} = \max \inf \left\{ C_{\alpha, \beta, \gamma} \text{ of (1.25)} \right\}, \]

where the maximum is taken over all \( (\alpha, \beta, \beta') \) satisfying \( |\alpha| \leq l_1, |\beta| \leq l_2 \) and \( |\beta'| \leq l_2^* \). Then, \( SX_{p,0}^m; N \) is a Fréchet space.

Lemma 1.5. Setting
we define for an integer \( N \) a mapping \( F_N \) from a single symbol class \( S^m_{p, \delta} \) to a class \( SX^m_{p, \delta; N} \) by

\[
F_N(p)(x, \xi, x') = (L_d)^N p(x, \xi) \quad \text{for } p(x, \xi) \in S^m_{p, \delta},
\]

where \( \nabla_\xi = (\partial/\partial \xi_1, \ldots, \partial/\partial \xi_m) \) and \( L_d \) denotes a transposed operator of \( L_\delta \). Then, we have

\[
\begin{cases}
F_N(p)(x, \xi, x') \in SX^m_{p, \delta; N}, \\
\|F_N(p)\|_{S^m_{p, \delta; N}}^{(m)} \leq C_{l_1, l_2, l_3, N} |p|_{S^m_{p, \delta; N}}^{(m)}
\end{cases}
\]

with a constant \( C_{l_1, l_2, l_3, N} \) independent of \( m \) and

\[
(1.29) \quad F_N(p)(X, D_x, X') = p(X, D_x).
\]

Proof. From (1.27) we get (1.28) easily. So, we have only to prove (1.29). For simplicity we denote \( p^{(N)}(x, \xi, x') = F_N(p)(x, \xi, x') \). Set \( \tilde{L}_\delta = (1 + \langle \xi + \eta \rangle^2 y^2)^{-1}(1 + i \langle \xi + \eta \rangle y \cdot \nabla_y) \). Then, we have \( p^{(N)}(x, \xi + \eta, x + y) = (L_d)^N p(x, \xi + \eta) \). Hence, using \( \tilde{L}_\delta e^{-iy \eta} = e^{-iy \eta} \) we obtain

\[
(p^{(N)})_{L_o}(x, \xi) = O_s - \int e^{-iy \eta, p^{(N)}(x, \xi + \eta, x + y)dy \overline{d} \eta}
\]

\[
= O_s - \int e^{-iy \eta} (L_d)^N p(x, \xi + \eta) dy d \eta
\]

\[
= O_s - \int e^{-iy \eta} p(x, \xi + \eta) dy d \eta
\]

\[
= p(x, \xi).
\]

This proves (1.29). Q.E.D.

**Lemma 1.6.** Let \( \delta \) satisfy \( 0 \leq \delta < 1 \). Then, the following hold:

i) Set

\[
\mathcal{J}(\xi, \xi') = (1 + \langle \xi ' \rangle^{-\delta} |\xi - \xi'|)^{-1/(1-\delta)} \langle \xi ' \rangle^{-\delta} \langle \xi; \xi ' \rangle_0^{1/(1-\delta)}
\]

Then, we have for \( \mu \) with \( |\mu| \leq 1 \)

\[
(1.30) \quad \mathcal{J}(\xi, \xi') |\xi |^{\mu} \leq C_1 |\xi ' |^{\mu}.
\]

ii) Set

\[
\mathcal{J}(\xi, \xi) = (1 + \langle \xi ' \rangle^{-\delta} |\xi - \xi'|)^{-\delta} \langle \xi ' \rangle^{-\delta} \langle \xi; \xi ' \rangle_0.
\]

Then, we have for \( \theta \) with \( 0 \leq \theta \leq 1 \)

\[
(1.31) \quad \mathcal{J}(\xi, \xi') \leq C_2 (\langle \xi ' \rangle^\delta \langle \xi ' \rangle^{-\delta})^{(1-\delta)\theta}.
\]
Proof. i) First, we assume $|\xi - \xi'| \leq <\xi >/2$. Then, we get $(1/2)<\xi > \leq <\xi >\leq 2<\xi >$ and get (1.30) immediately. So, we may assume $|\xi - \xi'| \geq <\xi >/2$. Then, from $<\xi ; \xi > \leq 5 |\xi - \xi'|^8$ we get

$$J(\xi, \xi')<\xi >^\mu \leq C_1 (|<\xi >| - |\xi - \xi'| - 1/2)|<\xi >|^{\mu - 1} \leq C_2<\xi >^\mu.$$  

Hence, we get (1.30).

ii) By the same way as in i), we get (1.31) when $|\xi - \xi'| \leq <\xi >/2$. So, we may assume $|\xi - \xi'| \geq <\xi >/2$. Then, we have

$$J(\xi, \xi') \leq C_2 (<\xi > - |\xi - \xi'|)^{\mu - 1} (|<\xi >| - |\xi - \xi'|)^{\mu - 1} \leq C_2<\xi >^\mu.$$  

Hence, we get (1.31). Q.E.D.

The following proposition is the first step to the study of multi-products of pseudo-differential operators.

**Proposition 1.7.** Let $\{m_j\}_{j=1}^\infty$ be a sequence of real numbers satisfying (1.12). Suppose that a multiple symbol $p_{v+1}(x^0, x^v, \xi^{v+1})$ in $S^{m_{v+1}}$, $m_{v+1}=(m_1, \cdots, m_{v+1}),$ satisfies

(1.32) \[ |D^\mu_0 \cdots D^\mu_v p_{v+1}(x^0, x^v, \xi^{v+1})| \leq B \prod_{j=1}^{v+1} <\xi >^m \prod_{j=1}^{v+1} <\xi >^j \prod_{j=1}^{v+1} \beta (1 + <\xi >^j |x^j - x|)^{-n} \]

for $|\beta| \leq l$, $j=1, \cdots, v$.

Then, for a simplified symbol $(p_{v+1})_{L}(x, \xi)$ of $p_{v+1}(x^0, x^v, \xi^{v+1})$ an estimate

(1.33) \[ |(p_{v+1})_{L}(x, \xi)| \leq A_0 <\xi >^{m_{v+1}} \]  

holds for a constant $A_0$ determined only by $n$, $\delta$ and $\xi$.

**Proof.** Integrating the oscillatory integral (1.4) for $p=p_{v+1}$ by parts with respect to $y^v$ we have

(1.34) \[ (p_{v+1})_{L}(x, \xi) = (r_{v+1})_{L}(x, \xi) \]

= $O_s - \int e^{-i\cdot r_{v+1}(x, \xi, y, \eta, x+y, \xi+y, x+y, \xi, y) dy^v d\eta^v}$

for a multiple symbol

(1.35) \[ r_{v+1}(x^0, x^v, \xi^{v+1}) = \prod_{j=1}^{v+1} (1 + <\xi > |x^j - x^j|)^{-2\delta} \cdot \nabla_{x^j} \cdot p_{v+1}(x^0, x^v, \xi^{v+1}). \]
Expanding \((1 - i\xi^{j+1})^{-2\delta}(\xi^j - \xi^{j+1})\cdot \nabla_x\) by the polynomial theorem and applying (1.32) to the derivatives of \(p^{v+1}\), we have

\[
(1.36) \quad |r^{v+1}(x^0, x^v, \xi^{v+1})| \leq BA_{1}G_v(x^0, x^v, \xi^{v+1})
\]

for a constant \(A_1\) depending only on \(n\) and \(\delta\), where

\[
(1.37) \quad G_v(x^0, x^v, \xi^{v+1}) = \prod_{j=1}^{v}(1 + \langle \xi^j \rangle^\delta |x^j - x^j|)^{-(n+1)} \times \prod_{j=1}^{v} \{1 + \langle \xi^{j+1}\rangle^{-\delta} |\xi^j - \xi^{j+1}|\}^{-\delta}(\langle \xi^j \rangle^{-\delta} \langle \xi^{j+1}\rangle_\delta)^{\delta}
\]

Set

\[
(1.38) \quad \begin{cases} 
\bar{\varepsilon}_0 = \varepsilon_n - \varepsilon_1 \quad (< 0), \\
\ell'_o = (n+\bar{\varepsilon}_n)/(1-\delta)
\end{cases}
\]

and set

\[
H_v(\xi^{v+1}) = \left\{ \prod_{j=1}^{v} J(\xi^j, \xi^{j+1}) \prod_{j=1}^{v} \langle \xi^j \rangle^{\gamma_j} \right\}^\gamma,
\]

\[
H_v(\xi^{v+1}) = \left\{ \prod_{j=1}^{v} J(\xi^j, \xi^{j+1}) \right\}^\gamma
\]

with \(J(\xi, \xi')\) and \(J(\xi, \xi')\) in the preceding lemma. Then, since \(\ell_o = \ell'_o + \varepsilon_1/(1-\delta) = (n+\bar{\varepsilon}_n) + \delta\ell'_o + \varepsilon_1/(1-\delta)\) we have

\[
(1.40) \quad G_v(x^0, x^v, \xi^{v+1}) = \prod_{j=1}^{v}(1 + \langle \xi^j \rangle^\delta |x^j - x^j|)^{-(n+1)} \times \prod_{j=1}^{v} \{1 + \langle \xi^{j+1}\rangle^{-\delta} |\xi^j - \xi^{j+1}|\}^{-\delta}(\langle \xi^j \rangle^{-\delta} \langle \xi^{j+1}\rangle_\delta)^{\delta} H_v(\xi^{v+1}) H_v(\xi^{v+1}) .
\]

Using Lemma 1.6–i) repeatedly, we have

\[
\begin{align*}
J(\xi^1, \xi^2)\langle \xi^1 \rangle^{m_1/\gamma_1} &\leq C_{1} \langle \xi^v \rangle^{m_1/\gamma_1} \quad (|m_1/\varepsilon_1| \leq 1), \\
J(\xi^2, \xi^3)\langle \xi^2 \rangle^{m_2/\gamma_1} &\leq C_{1} \langle \xi^v \rangle^{m_2/\gamma_1} \quad (|m_2/\varepsilon_1| \leq 1), \\
\ldots &\\
J(\xi^v, \xi^{v+1})\langle \xi^v \rangle^{m_v/\gamma_1} &\leq C_{1} \langle \xi^{v+1} \rangle^{m_v/\gamma_1} \quad (|m_v/\varepsilon_1| \leq 1).
\end{align*}
\]

Then, we have

\[
(1.41) \quad H_v(\xi^{v+1}) \leq C_1 \langle \xi^{v+1} \rangle^{m_v+1} .
\]

Applying Lemma 1.6–ii) with \(\theta = n/(n+\bar{\varepsilon}_n)\) (< 1), we have

\[
(1.42) \quad H_v(\xi^{v+1}) \leq C_2 \langle \xi^{v+1} \rangle \left\{ \prod_{j=1}^{v} \langle \xi^j \rangle^{\delta} \langle \xi^{j+1}\rangle^{-\delta} \right\}^{\delta}
\]

\[
= C_2 \langle \xi^{v+1} \rangle \left\{ \prod_{j=1}^{v} \langle \xi^j \rangle^{\delta} \langle \xi^{j+1}\rangle^{-\delta} \right\}^{\delta}
\]
by virtue of (1.38). Consequently, we have

\[
G_{\nu}(x^0, x^\nu, \xi^{\nu+1}) \leq A_2^\nu \prod_{j=1}^y (1 + |\xi_j^{\nu+1}| - |x_j^{\nu+1}|)^{-(\epsilon_j^{\nu+1})} \\
\times \prod_{j=1}^y (1 + |\xi_j^{\nu+1} - \xi_j| - |x_j^{\nu+1} - x_j|)^{-(\epsilon_j^{\nu+1})} \\
\times \langle \xi^{\nu+1} \rangle^{-n_0} \langle \xi^{\nu+1} \rangle^{n_0+1}
\]

with a constant \(A_2\) determined only by \(n, \delta\) and \(\epsilon_1\). Set

\[
W_\nu = W_\nu(\xi) = \int \prod_{j=1}^y \left(1 + |\xi_j + \eta_j| - |y_j| - |y_j'|\right)^{-(\epsilon_j + \eta_j)} \langle \xi + \eta \rangle^{n_0}
\]

Then, from (1.34), (1.36) and (1.43) we have

\[
|p_{\nu+1}(x, \xi)| \leq B(A_1A_2)^\nu W_\nu \langle \xi \rangle^{n_0+1}.
\]

Since \(W_\nu\) has an estimate

\[
W_\nu \leq A_3^\nu \prod_{j=1}^y \left(1 + |\xi_j + \eta_j^{\nu+1}| - |y_j^{\nu+1}| - |y_j'|\right)^{-(\epsilon_j^{\nu+1} + \eta_j^{\nu+1})} \langle \xi + \eta^{\nu+1} \rangle^{n_0}\]

with constants \(A_3\), \(A_3'\) and \(A_3 (=A_1A_2')\) independent of \(\nu\), we get (1.33) from (1.45) if we set \(A_0 = A_1A_2A_3\).

Q.E.D.

We fix a \(C^\infty\)-function \(\chi_\nu(\xi)\) satisfying

\[
0 \leq \chi_\nu \leq 1, \quad \chi_\nu(\xi) = 1 \left( |\xi| \leq 1/4 \right), \quad \chi_\nu(\xi) = 0 \left( |\xi| \geq 1/2 \right).
\]

Set for \(p_j(x, \xi, x') \in S^{m_j}_{\rho, \delta}, j = 1, 2, \) and \(\epsilon \in (0, 1]\)

\[
q_0^\nu(x, \xi, x', \xi', x'') = p_1(x, \xi, x') \chi_0^\nu(\xi, \xi') p_2(x', \xi', x''),
\]

\[
q_1^\nu(x, \xi, x', \xi', x'') = p_1(x, \xi, x') \chi_1^\nu(\xi, \xi') p_2(x', \xi', x'')
\]

and

\[
q_1^\nu(\mu)(x, \xi, x', \xi', x'') = \{ -i |\xi - \xi'| - 2(\xi - \xi') \cdot \nabla_x \}^\mu q_1^\nu(x, \xi, x', \xi', x'')
\]

\(\mu = 0, 1, \cdots\),

where \(\chi_0^\nu(\xi, \xi') = \chi_0((\xi - \xi')/(\xi < \xi'))\) and \(\chi_1^\nu(\xi, \xi') = 1 - \chi_0^\nu(\xi, \xi')\). We define for pseudo-differential operators \(P_j = p_j(X, D_x, x'), j = 1, 2, \) the products \(P_1 \square P_2\)

\((k = 0, 1)\) and \(P_1 \square P_2, (\mu = 0, 1, \cdots)\) by
We also denote for $P_j=p_j(X, D_x, X')$, $j=1, 2$, the pseudo-differential operator with symbol $p_j(x, \xi, x')p_2(x, \xi, x')$ by $P_1 \otimes P_2$. Then, we have obtained the following results in Section 2 of [16].

**Proposition 1.8.** i) *It holds that*

\begin{equation}
\left\{
\begin{array}{l}
P_1 \Box P_2 = q_1(X, D_x, X', D_x', X''), \quad k = 0, 1, \\
P_1 \Box^\mu P_2 = (q_1(\mu))(X, D_x, X, D_x', X'',) \quad \mu = 0, 1, \ldots.
\end{array}
\right.
\end{equation}

We denote for $P_j=p_j(X, D_x, X')$, $j=1, 2$, the pseudo-differential operator with symbol $p_j(x, \xi, x')p_2(x, \xi, x')$ by $P_1 \otimes P_2$. Then, we have obtained the following results in Section 2 of [16].

**Proposition 1.8.** ii) *Let $p_j(x, \xi, x')$ belong to $S\chi_{m_j}^{\alpha_j}$ $(j=1, 2)$. Set for real numbers $s, s_1$ and $s_2$

\begin{equation}
\left\{
\begin{array}{l}
Q_0^*(\mu) = (P_1 \otimes \Lambda^*) \Box^\mu (\Lambda^* \otimes P_2), \\
Q_1^*(\mu) = (P_1 \otimes \Lambda^*) \Box^\mu (\Lambda^* \otimes P_2), \\
Q_2^*(\mu) = (P_1 \otimes \Lambda^*) \Box^\mu (\Lambda^* \otimes P_2),
\end{array}
\right.
\end{equation}

where $\Lambda^i = \langle D_x \rangle^i$. Then, $q_2^0(0, \xi, x', \xi', x'') = \sigma(Q_0^*(\mu)), k = 0, 1$ and $(q_2^0(\mu))(x, \xi, x', \xi', x'') = \sigma(Q_2^*(\mu))$ satisfy for any $\beta, \beta', \beta''$ with $|\beta'| \leq l, |\beta| \leq l$ and $|\beta''| \leq l$.

\begin{equation}
\left| D_x^\beta \partial_x^\beta \partial_x^{\beta''} q_0^0(x, \xi, x', \xi', x'') \right|
\end{equation}

\begin{equation}
\begin{aligned}
&\leq \{(1+\varepsilon)^{m_0}\|p_1\|_{0, l+1; N_1}\|p_2\|_{0, l+1; N_2} \\
&\times \langle \xi \rangle^{m_0+1}\langle \xi \rangle^k \langle \xi \rangle^l \langle x-x' \rangle^k \langle x-x'' \rangle^l \langle x-x'' \rangle \langle x-x'' \rangle^l \langle x-x'' \rangle^l \langle x-x'' \rangle^l,
\end{aligned}
\end{equation}

\begin{equation}
\left| D_x^\beta \partial_x^\beta \partial_x^{\beta''} q_1^1(\mu)(x, \xi, x', \xi', x'') \right|
\end{equation}

\begin{equation}
\begin{aligned}
&\leq \{(1+\varepsilon)^{m_0}\|p_1\|_{0, l+1; N_1}\|p_2\|_{0, l+1; N_2} \\
&\times \langle \xi \rangle^{m_0+1}\langle \xi \rangle^k \langle \xi \rangle^l \langle x-x' \rangle^k \langle x-x'' \rangle^l \langle x-x'' \rangle^l \langle x-x'' \rangle^l \langle x-x'' \rangle^l,
\end{aligned}
\end{equation}

\begin{equation}
\left| D_x^\beta \partial_x^\beta \partial_x^{\beta''} q_2^2(\mu)(x, \xi, x', \xi', x'') \right|
\end{equation}

\begin{equation}
\begin{aligned}
&\leq \{(1+\varepsilon)^{m_0}\|p_1\|_{0, l+1; N_1}\|p_2\|_{0, l+1; N_2} \\
&\times \langle \xi \rangle^{m_0+1}\langle \xi \rangle^k \langle \xi \rangle^l \langle x-x' \rangle^k \langle x-x'' \rangle^l \langle x-x'' \rangle^l \langle x-x'' \rangle^l,
\end{aligned}
\end{equation}

when $\mu \geq (s_1+\ldots+s_2)/(1-\delta)$.

**Remark.** A product $P_1 \otimes P_2$ is denoted as $P_1 \otimes P_2$ in [16] and a slightly different estimates are derived there, but (1.51)–(1.53) follow by their proof.
Now, we are prepared to prove Proposition 1.2. We divide the proof into four steps.

I) Let \( F_{n+1} \) be a mapping from \( S_{p,\delta}^m \) to \( SX_{p,\delta}^m \) defined in Lemma 1.5 with \( N=n+1 \). Denote for simplicity
\[
(1.54) \quad p_j(x, \xi, x') = F_{n+1}(p^*_j)(x, \xi, x') \quad \text{for} \quad j \leq 2.
\]
From Lemma 1.5 we note that \( p_j(x, \xi, x') \) belongs to \( SX_{p,\delta}^{m_j+m'_j} \) and satisfies
\[
(1.55) \quad \|p^*_j\|_{S_{p,\delta}^m \to SX_{p,\delta}^{m_j+m'_j}} \leq C_{p,\delta} \|p^*_j\|_{S_{p,\delta}^{m_j+m'_j}}
\]
for a constant \( C_{p,\delta} \) independent of \( j \). From (1.54) and (1.29) we can write
\[
(1.56) \quad Q_{v+1}^c = P_1^cP_2^c \cdots P_{v-1}^cP_v^c.
\]
Set
\[
K_v = \{k = (k_1, k_2, \ldots, k_v); k_j = 0, 1\}.
\]
Then, from (1.49) we have
\[
(1.57) \quad Q_{v+1}^c = \sum_{k \in K_v} Q_{v+1}(\omega)\Lambda^{m_{v+1}}
\]
for
\[
(1.58) \quad Q_{v+1}(\omega) = P_1^c \Box_k^c P_2^c \Box_k^c \cdots \Box_{k_{v-1}} P_{v-1}^c P_v^c \Lambda^{m_{v+1}}.
\]
II) Set \( \omega' \equiv \omega = (0, 0, \ldots, 0) \in K_v \) and consider \( Q_{v+1}(\omega') \) in (1.58). We set
\[
(1.59) \quad \left\{ \begin{array}{l}
q_{v+1}(x, \xi) = q_v(x, \xi, x') \langle \xi \rangle^{-m_j} \in SX_{p,\delta}^{m_j+m'_j} \\
q_{v+1}(x, \xi) = q_v(x, \xi) \langle \xi \rangle^{-m_j} \in SX_{p,\delta}^{m_j+m'_j}
\end{array} \right.
\]
Then, we can write \( Q_{v+1}(\omega') \) in the form
\[
Q_{v+1}(\omega') = (P_1' \circ \Lambda^{m_1}) \Box_0 \circ (\Lambda^{-m_1} \circ P_1' \circ \Lambda^{m_2}) \Box_0 \cdots \\
\cdot \Box_0 \circ (\Lambda^{-m_{v-1}} \circ P_v' \circ \Lambda^{m_v}) \Box_0 \circ (\Lambda^{-m_{v+1}} \circ P_{v+1}')
\]
Let \( q_{v+1}(\omega')(x, \xi, \xi^{v+1}) \) be the multiple symbol corresponding to \( Q_{v+1}(\omega') \) and set
\[
(1.60) \quad B_{v+1}(\omega') = (1 + \lambda)|^{\omega'} \|p_j'\|_{S_{p,\delta}^m \to SX_{p,\delta}^{m_j+m'_j}} \prod_{j=2}^v \|p_j'\|_{S_{p,\delta}^m \to SX_{p,\delta}^{m_j+m'_j}} \|p_{v+1}'\|_{S_{p,\delta}^{m_{v+1}}}.
\]
Then, applying (1.51) we obtain for \( |\beta'| \leq l' \) (\( j = 1, \ldots, v \)),
\[
(1.61) \quad |D^{\beta_1}_x \cdots D^{\beta_v}_{x^v} q_{v+1}(\omega')(x^0, x^v, \xi^{v+1})| \leq B_{v+1}(\omega') \prod_{j=1}^v \langle \xi \rangle^{-m_j} \prod_{j=1}^v \langle \xi \rangle^{-m_j} \prod_{j=1}^v (1 + \langle \xi \rangle^j x^j - x^j)^{-(n+1)}.
\]
that is, \( q_{v+1,0}((x, \xi)) \) satisfies (1.32) with \( B = B_{v+1,0} \). Hence, we can apply Proposition 1.7 to obtain

\[
|q_{v+1,0}((x, \xi))| \leq A_0 B_{v+1,0} |\xi|^\alpha.
\]

III) Next, we consider

\[
Q_{v+1,0} = P_1^1 P_2^0 P_3^1 \cdots P_{v+1}^1 P_{v+1}^0 \Lambda^{-\alpha_{v+1}}
\]

for \( \kappa = (k_1, \ldots, k_v) + \kappa_0 \).

Set for \( \iota < \iota' \)

\[
Q_{v+1,0} = P_1^1 P_2^0 P_3^1 \cdots P_{v+1}^1 P_{v+1}^0 \Lambda^{-\alpha_{v+1}}
\]

For \( j \) with \( j < v \) and \( k_j = 1 \) we set \( \theta = \max \{ j' ; 0 \leq j' < j, k_j = 1 \} \) \((k_0 = 1)\) and consider the part \( Q_{v+1,0} ; \theta+1, \iota \). Using (1.59) we write it in the form

\[
Q_{v+1,0} ; \theta+1, \iota = (P_{\theta+1,0}^0 \Lambda^{\alpha_{\theta+1}}) \boxdot \xi_0 (\Lambda^{-\alpha_{\theta+1}} \circ P_{\theta+1,0}^0 \Lambda^{\alpha_{\theta+1}}) \boxdot \xi_0 \cdots
\]

where \( \bar{m}_{\theta+1} = m_{\theta+1}^j + \cdots + m_{\theta+1}^j \) \((\theta + 1 \leq j \leq j') \). In the case of \( \bar{m}_{\theta+1}^j \leq 0 \), applying (1.51) and (1.52), we have

\[
|q_{v+1,0} ; \theta+1, \iota| \leq (1 + \epsilon) \mu_0^{j-1} \prod_{i=0}^{j-1} \| p_i^0 \|_{l_0, l_0, \alpha; i+1} \| p_{j+1}^0 \|_{l_0, l_0, \alpha; \alpha+1} (\bar{m}_{\theta+1, j} \leq 0).
\]

In the case of \( \bar{m}_{\theta+1}^j > 0 \) we write by using (1.50)

\[
Q_{v+1,0} ; \theta+1, \iota = (P_{\theta+1,0}^0 \Lambda^{\alpha_{\theta+1}}) \boxdot \xi_0 (\Lambda^{-\alpha_{\theta+1}} \circ P_{\theta+1,0}^0 \Lambda^{\alpha_{\theta+1}}) \boxdot \xi_0 \cdots
\]

with \( \mu_0 = [M'/(1-\delta)]^* \). Then, from (1.51) and (1.53) we have

\[
|q_{v+1,0} ; \theta+1, \iota| \leq C_{l_0, \mu_0, \epsilon} (1 + \epsilon) \mu_0^{j-1} \prod_{i=0}^{j-1} \| p_i^0 \|_{l_0, l_0, \alpha; i+1} \times \| p_j^0 \|_{l_0, l_0, \mu_0; \alpha+1} \| p_{j+1}^0 \|_{l_0, l_0, \mu_0, i; \alpha+1} (\bar{m}_{\theta+1, j} > 0).
\]

For \( j^0 = \max \{ j ; k_j = 1 \} \) we write \( Q_{v+1,0} ; \theta, v+1 \) in the form

\[
Q_{v+1,0} ; \theta, v+1 = (P_{\theta+1,0}^0 \Lambda^{\alpha_{\theta+1}}) \boxdot \xi_0 \mu_0 \Lambda (\Lambda^{-\alpha_{\theta+1}} \circ P_{\theta+1,0}^0 \Lambda^{\alpha_{\theta+1}}) \boxdot \xi_0 \cdots
\]

with \( \mu_0 = [M'/(1-\delta)]^* \). Then, from (1.51) and (1.53) we have

\[
|q_{v+1,0} ; \theta, v+1| \leq C_{l_0, \mu_0, \epsilon} (1 + \epsilon) \mu_0^{j-1} \prod_{i=0}^{j-1} \| p_i^0 \|_{l_0, l_0, \alpha; i+1} \times \| p_j^0 \|_{l_0, l_0, \mu_0; \alpha+1} \| p_{j+1}^0 \|_{l_0, l_0, \mu_0, i; \alpha+1} (\bar{m}_{\theta, v+1} > 0).
\]
Then, the multiple symbol \( q_{v+1}(x) : P_{v+1}^0(x^v, \xi^v) \) of \( Q_{v+1}(x, \xi) \) satisfies

\[
\begin{align*}
|q_{v+1}(x) : P_{v+1}^0(x^v, \xi^v)| &= C_{L_{v+1}, 0, v} \left| P_{v+1}^0(x^v, \xi^v) \right|^{(m_{v+1})} \\
&\leq C_{L_{v+1}, 0, v} (1 + \varepsilon)^{M^v} \left| P_{v+1}^0(x^v, \xi^v) \right|^{(m_{v+1})} \\
&\times \prod_{i=1}^{\infty} \left| p_{i+1}^{\nu} \right|^{(m_{v+1})} \\
&\leq C_{L_{v+1}, 0, v} \left| p_{v+1}^{\nu} \right|^{(m_{v+1})} \\
&\leq C_{L_{v+1}, 0, v} \left| p_{v+1}^{\nu} \right|^{(m_{v+1})}
\end{align*}
\]

(1.66)

Now, we set

\[
\Gamma(\kappa) = \{ j; 1 \leq j < \nu, k_j = 1, \sum_{i=1}^{j} m_j > 0 \} \cup \{ j^0 \}
\]

Then, from the definition of \( N^0 \) and the relation

\[
\Gamma(\kappa) \subseteq \{ j; k_j = 1, m_j > 0 \} \cup \{ j; k_j = 1, \ldots, k_{j-1} = 0, m_j > 0 \text{ for some } \theta < j \} \cup \{ j^0 \}
\]

the number \( l \) of the elements in \( \Gamma(\kappa) \) does not exceed \( N^0 + 1 \). Set

\[
\Gamma(\kappa) = \{ j_1, j_2, \ldots, j_l \} \quad (j_1 < j_2 < \cdots < j_l = j^0)
\]

and write the multi-product \( Q_{v+1}(x, \xi) \) of (1.63) in the form

\[
Q_{v+1}(x, \xi) = P_{v+1}^0 \prod_{i=1}^{\nu} P_{i}^{s_j} \prod_{i=1}^{\nu} P_{i}^{s_j} \cdots P_{v+1}^0 \prod_{i=1}^{\nu} P_{i}^{s_j} \\
\times \prod_{i=1}^{\nu} \left| p_{i+1}^{\nu} \right|^{(m_{v+1})} \left| p_{i+1}^{\nu} \right|^{(m_{v+1})} \\
\]

Then, using the discussions in the preceding paragraph and Proposition 1.8–ii), we get the following: There exists a constant \( C_1 \) depending on \( M^v, N^0 \) and \( \varepsilon \) (but independent of \( \nu \)) such that the multiple symbol \( q_{v+1}(x, \xi) \) of \( Q_{v+1}(x, \xi) \) satisfies (1.32) with \( B \) replaced by

\[
B_{v+1}(x, \xi) = C_1 (1 + \varepsilon)^{M^v} \left| p_{v+1}^{\nu} \right|^{(m_{v+1})} \\
\times \prod_{i=1}^{\nu} \left| p_{i+1}^{\nu} \right|^{(m_{v+1})}
\]

(1.67)

Here, \( s_j = 0 \) for \( j \notin \Gamma(\kappa) \) and \( s_j = 1 \) for \( j \in \Gamma(\kappa) \). Hence, by Proposition 1.7 we get

\[
|q_{v+1}(x, \xi)| \leq A_0 B_{v+1}(x, \xi) \left| \xi \right|^{M^v_{v+1}}
\]

(1.68)

IV) From (1.57) we have

\[
q_{v+1}^0(x, \xi) = \sum_{k \in K} (q_{v+1}(x, \xi)) \left[ \xi \right] \left| \xi \right|^{M^v_{v+1}}.
\]
Hence, we obtain from (1.62) and (1.68)

\[(1.69) \quad |q_{v+1}(x, \xi)| \leq (2A_0)^v \max_{\kappa \in \mathbb{K}_\nu} B_{v+1}^{-1} < \delta > \cdot \max_{\nu+1} \nu+1.\]

On the other hand, (1.59) and (1.55) imply

\[(1.70) \quad \left\{ \begin{array}{l}
\|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \leq C \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \\
\|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \leq C A_{\nu} \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \\
\|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \leq C A_{\nu} \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \\
\end{array} \right. (2 \leq j \leq \nu),
\]

for a constant $A_{\nu}$ independent of $M'$ and $\nu$ and a constant $C$ independent of $\nu$. Hence, from (1.60) and (1.67) we have for any $\kappa \in \mathbb{K}_\nu$

\[B_{v+1}\leq C_\nu C^{(N-1)} \cdot A'\|1+\varepsilon\|^{\nu} \cdot \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \times \prod_{j=1}^{\nu} \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} (\Gamma(\kappa_0) = \phi).\]

For any fixed $\sigma > 1$ we take $\varepsilon = \varepsilon_{M'} \in (0, 1]$ satisfying

\[(1+\varepsilon)^{M'} \leq \sigma.\]

Then, there exists a constant $C_\nu$ independent of $\nu$ such that

\[(1.71) \quad B_{v+1} \leq C_\nu (A_{\nu} \sigma)^{\nu} \max_{\kappa \in \mathbb{K}_\nu} \{ \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \|p_j^\circ\|_{\nu, \nu, \mu, \mu^0; \nu+1} \}.\]

Consequently, setting $A_{\nu} = 2\sigma A_{\nu} A_\nu$, we get (1.15) from (1.69) and (1.71). This concludes the proof of Proposition 1.2.

2. **Multi-products of Fourier integral operators.** Throughout this section we denote by $I_{\phi}$ the Fourier integral operator with phase function $\phi(x, \xi) \in \mathcal{D}_\rho(\tau)$ and symbol 1. Following [7] we define for $\phi(x, \xi) \in \mathcal{D}_\rho(\tau)$ the conjugate Fourier integral operator $I_{\phi^*}$ (with symbol 1) by

\[(2.1) \quad (I_{\phi^*} u)(x) = \Omega - \int e^{i(x\cdot \xi - \phi(x', \xi'))} u(x') dx' d\xi \quad \text{for} \quad u \in S.\]

Set for $\phi(x, \xi) \in \mathcal{D}_\rho(\tau)$

\[(2.2) \quad \left\{ \begin{array}{l}
\nabla_x \phi(x, \xi, x') = \int_0^1 \nabla_x \phi(x' + \theta(x-x'), \xi) d\theta, \\
\nabla_\xi \phi(\xi, x', \xi') = \int_0^1 \nabla_\xi \phi(x', \xi' + \theta(\xi-\xi')) d\theta.
\end{array} \right.\]
We employ the following lemma, which is a slightly different version of Proposition 1.5 of Chap. 10 in [8], but it can be proved by a similar way.

**Lemma 2.1.** Let \( \phi(x, \xi) \) belong to \( \mathcal{P}_\rho(\tau) \), \( 0 \leq \tau < 1 \), \( 1/2 \leq \rho \leq 1 \). Then, we have the following:

i) The equation \( \eta = \nabla \phi(x, \xi, x') \) has the unique solution \( \xi = \nabla \phi^{-1}(x, \eta, x') \) and it satisfies

\[
\begin{align*}
&\text{a) } |\xi - \eta| \leq \tau |\eta| \quad \text{with} \quad \xi = \nabla \phi^{-1}(x, \eta, x'), \\
&\text{b) } C^{-1} |\eta| \leq \langle \nabla \phi^{-1}(x, \eta, x') \rangle \leq C |\eta|, \\
&\text{c) } |\partial_x^\alpha D^\beta_x \nabla \phi^{-1}(x, \eta, x')| \leq C_{\alpha, \beta} |\eta|^{1 - |\alpha + \beta| + (1 - \rho)(|\alpha + \beta'| - 1)} (|\alpha + \beta + \beta'| \geq 1).
\end{align*}
\]

(2.3)

ii) The equation \( y' = \nabla \epsilon \phi(\xi, x', \xi') \) has the unique solution \( x' = \nabla \epsilon \phi^{-1}(\xi, y', \xi') \) and it satisfies

\[
\begin{align*}
&\text{a) } |\nabla \epsilon \phi^{-1}(\xi, y', \xi') - y'| \leq C, \\
&\text{b) } |\partial_x^\alpha D^\beta_x \nabla \epsilon \phi^{-1}(\xi, y', \xi')| \\
&\quad \leq C_{\alpha, \beta} |\xi; \xi'|^{(1 - \rho)(|\alpha + \alpha' + \beta' - 1)} (|\alpha + \alpha' + \beta'| + 1), \\
&\text{c) } |\partial_x^\alpha D^\beta_x \chi((\xi - \xi')/\langle \xi' \rangle) \nabla \epsilon \phi^{-1}(\xi, y', \xi')| \\
&\quad \leq C_{\alpha, \beta} |\xi; \xi'|^{1 - |\alpha + \alpha'| + (1 - \rho)(|\alpha + \alpha' + \beta' - 1)} (|\alpha + \alpha' + \beta'| \geq 1),
\end{align*}
\]

where \( \langle \xi; \xi' \rangle = \langle \xi \rangle + \langle \xi' \rangle \) and \( \chi \) is a \( C^\omega \)-function satisfying

\[
|\chi| \leq 1, \quad \chi = 1 \quad (|\xi| \leq 2/5), \quad \chi = 0 \quad (|\xi| \geq 1/2).
\]

Moreover, if \( \{\phi_\gamma\}_{\gamma \in \Gamma} \) is bounded in \( \mathcal{P}_\rho(\tau) \), we can take the constants \( C, C_{\alpha, \beta, \beta'} \) and \( C_{\alpha, \alpha', \beta'} \) in (2.3) and (2.4) independent of \( \gamma \in \Gamma \).

**Remark 1.** In the lemma and in what follows, we say that for \( \phi_\gamma \in \mathcal{P}_\rho(\tau) \) [resp. \( \phi \in \mathcal{P}_\rho(\tau, \ell) \)] the set \( \{\phi_\gamma\}_{\gamma \in \Gamma} \) is bounded in \( \mathcal{P}_\rho(\tau) \) [resp. \( \mathcal{P}_\rho(\tau, \ell) \)] if the corresponding set \( \{||J_\gamma||_\ell\}_{\gamma \in \Gamma} \) of semi-norms \( ||J_\gamma||_\ell \) of (6) in Introduction is bounded for any \( \ell' = 0, 1, 2, \ldots \).

**Remark 2.** Throughout this section we denote by \( \chi(x) \) a \( C^\omega \)-function satisfying (2.5).

Now, we show the existence of pseudo-differential operators \( R \) and \( R' \) satisfying (8).

**Proposition 2.2.** There exist a constant \( r \) \((< 1)\) and an integer \( I_\rho \) such that for a phase function \( \phi(x, \xi) \) in \( \mathcal{P}_\rho(\tau, I_\rho) \) we can find pseudo-differential operators \( R \) and \( R' \) in \( S^\omega_\rho \) satisfying

\[
I_\rho I_\rho' R = RI_\rho' I_\rho = I,
\]

(2.6)
\[
\begin{align*}
(2.7) \quad \mathbf{I}_\phi \mathbf{I}_\phi \mathbf{R}' &= \mathbf{R}' \mathbf{I}_\phi \mathbf{I}_\phi = \mathbf{I} \\
\text{and} \\
(2.8) \quad \begin{cases}
\text{i) } \mathbf{I}_\phi \mathbf{R} \mathbf{I}_\phi &= \mathbf{I} , \\
\text{ii) } \mathbf{I}_\phi \mathbf{R}' \mathbf{I}_\phi &= \mathbf{I} .
\end{cases}
\end{align*}
\]

If the set \( \{ \phi_\gamma \}_{\gamma \in \Gamma} \) is bounded in \( \mathcal{D}'(\tau, I_\phi) \), the corresponding sets \( \{ \sigma(\mathbf{R}_\gamma) \}_{\gamma \in \Gamma} \) and \( \{ \sigma(\mathbf{R}'_\gamma) \}_{\gamma \in \Gamma} \) are bounded in \( \mathcal{S}'_\phi \).

Proof. The property (2.8) follows immediately from (2.6) and (2.7). The existence of \( \mathbf{R}' \) satisfying (2.7) is proved in Theorem 6.1 of Chap. 10 in [8]. So, it remains to prove the existence of \( \mathbf{R} \) satisfying (2.6).

Set \( P = \mathbf{I}_\phi \mathbf{I}_\phi \). Then, we have

\[
(2.9) \quad p(\mathbf{x}, \mathbf{\xi}') (= \sigma(P)) = O_s - \iiint e^{-i\mathbf{x}' \cdot \mathbf{\xi}'} d\mathbf{x}' d\mathbf{\xi}'
\]

with \( \psi = \mathbf{x} \cdot \mathbf{\xi} - \phi(\mathbf{x}', \mathbf{\xi}) + \phi(\mathbf{x}', \mathbf{\xi}') - \mathbf{x} \cdot \mathbf{\xi}' \). Set

\[
(2.10) \quad \overline{p}(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}') = \left\{ \left[ \left| \frac{\text{det} \partial \nabla e \phi(\mathbf{\xi}, \mathbf{w}, \mathbf{\xi}')}{\text{det} \mathbf{E}_f} \right| \right]^{-1} \right\}_{\mathbf{w} = \nabla \phi^{-1}(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}')},
\]

where for a vector \( \mathbf{f} = (f_1, \ldots, f_n) \) of functions \( f_j(\mathbf{x}, \mathbf{\xi}) \frac{\partial}{\partial \mathbf{x}} f \) is \( (\frac{\partial f_j}{\partial \mathbf{x}} \mathbf{j} \downarrow 1, \ldots, n) \). In what follows we also use \( \frac{\partial}{\partial \mathbf{\xi}} \mathbf{f} = \left( \frac{\partial f_j}{\partial \mathbf{\xi}} \mathbf{j} \downarrow 1, \ldots, n \right) \). Since \( \psi = (\mathbf{x} - \nabla e \phi(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}')) \cdot (\mathbf{\xi} - \mathbf{\xi}') \), by a change of the variables \( \mathbf{y} = \nabla e \phi(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}') - \mathbf{x}, \mathbf{\eta} = \mathbf{\xi} - \mathbf{\xi}' \), we have from (2.9)–(2.10)

\[
(2.11) \quad p(\mathbf{x}, \mathbf{\xi}') = O_s - \iiint e^{-i\mathbf{\eta} \cdot \mathbf{\xi}' + i \mathbf{\eta} \cdot \mathbf{x} + \mathbf{\xi}'} dy d\mathbf{\eta}.
\]

Here, the oscillatory integral in (2.11) is well-defined because of

\[
|\partial_\mathbf{\eta}^\alpha D_\mathbf{x}^\beta \overline{p}(\mathbf{\xi}' + \mathbf{\eta}, \mathbf{x} + \mathbf{\eta}, \mathbf{\xi}')| \leq C_{\alpha, \beta} <\mathbf{\eta}>^{\delta |\alpha| + |\beta|} \quad \text{for any fixed } \mathbf{x} \text{ and } \mathbf{\xi}'
\]

with \( \delta = 1 - \rho \ (\rho < 1) \). Set

\[
(2.12) \quad \varrho(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}') = \overline{p}(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}') - 1.
\]

Since \( \frac{\partial}{\partial \mathbf{x}} \nabla e \phi = \mathbf{E} + \frac{\partial}{\partial \mathbf{x}} \nabla e \mathbf{J} \) (\( \mathbf{E} \) is an identity matrix), \( \varrho(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}') \) has the form

\[
(2.13) \quad \varrho(\mathbf{\xi}, \mathbf{x}', \mathbf{\xi}') = \left\{ \left[ 1 - \det(\mathbf{E} + \frac{\partial}{\partial \mathbf{x}} \nabla e \mathbf{J}(\mathbf{\xi}, \mathbf{w}, \mathbf{\xi}')) \right] \right\} / \det(\mathbf{E} + \frac{\partial}{\partial \mathbf{x}} \nabla e \mathbf{J}(\mathbf{\xi}, \mathbf{w}, \mathbf{\xi}')) \mathbf{w} = \nabla \phi^{-1}(\mathbf{\xi}, \mathbf{x}, \mathbf{\xi}').
\]
Fix a constant $\tau'$ satisfying $0 \leq \tau' < 1$. Then, if $\phi(x, \xi)$ belongs to $\mathcal{P}(\tau', I)$, we can prove by applying Lemma 2.1-ii) to $\nabla_t \phi^{-1}(\xi, x', \xi')$ that the symbol $\overline{\phi}(\xi, x', \xi')$ satisfies

$$
\begin{align*}
(2.14) & \quad \left\{ \begin{array}{l}
\text{i) } |\partial_{\xi}^{\sigma} \partial_{\xi'}^{\sigma'} D_{x}^{\alpha} D_{x'}^{\beta} \overline{\phi}(\xi, x', \xi')| \\
\quad \quad \leq C_{\tau', \alpha, \alpha', \beta'} \|J\| \langle \xi; \xi' \rangle^{|1-\rho|} \|x + x' + \xi + \xi'\| (|\alpha + \alpha' + \beta'| \leq l), \\
\text{ii) } |\partial_{\xi}^{\sigma} \partial_{\xi'}^{\sigma'} D_{x}^{\alpha} D_{x'}^{\beta} \{X(\xi - \xi') \langle \xi' \rangle \} \overline{\phi}(\xi, x', \xi')| \\
\quad \quad \leq C_{\tau', \alpha, \alpha', \beta'} \|J\| \langle \xi' \rangle^{-|\alpha + \alpha'| + (1-\rho)|\alpha + \beta'|} (|\alpha + \alpha' + \beta'| \leq l)
\end{array} \right.
\end{align*}
$$

with a constant $C_{\tau', \alpha, \alpha', \beta'}$ depending on $\tau'$. Write the simplified symbol $q(x, \xi)$ of $\overline{\phi}(\xi, x', \xi')$ as

$$
(2.15) \quad q(x, \xi') = O_{s} - \int e^{-iy \cdot \eta} \chi(\eta \langle \xi' \rangle) \overline{\phi}(\xi' + \eta, x' + y, \xi')dyd\eta \\
\quad + O_{s} - \int e^{-iy \cdot \eta}(1 - \chi(\eta \langle \xi' \rangle)) \overline{\phi}(\xi' + \eta, x' + y, \xi')dyd\eta
$$

and use (2.14)--ii) to the first term of (2.15) and (2.14)--i) to the second term of (2.15). Then, we can find a constant $A_{s}$ (depending on $\tau'$) and an integer $I_{s}$ such that we have for $I_{s} = [n/\rho + 1]$

$$
(2.16) \quad |q|^{(2)}_{\tau', I_{s}} \leq A_{s} \|J\|_{I_{s}}
$$

if $\phi(x, \xi)$ belongs to $\mathcal{P}(\tau', I_{s})$. Take a constant $\tau(\leq \tau')$ satisfying

$$
(2.17) \quad \tau < 1/(AA_{s})
$$

with a constant $A$ in Theorem 2. Then, $q(x, \xi)$ satisfies (14) and by means of Theorem 3 the inverse $R$ of the operator $p(X, D_{x}) = I + q(x, \xi)$ is obtained with the form $R = r(X, D_{x})$ for a symbol $r(x, \xi)$ in $S^{n}_{\rho}$. This $R$ satisfies (2.6). Finally, from the above discussions we obtain the last statement of the proposition.

Q.E.D.

From now on, for $p(x, \xi) \in S^{m}_{\rho}$ we shall use the semi-norms

$$
(2.18) \quad |p|^{(m)}_{\tau, I_{s}} = \max_{|\sigma| + |\mu| \leq m} \sup_{x, \xi} \{|p^{(\sigma)}_{\mu}(x, \xi)| < \xi^{-\nu_{\gamma}(m - |\sigma + (1-\rho)|\sigma + \beta)}\}
$$

instead of using (5).

**Proposition 2.3.** Let $\phi(x, \xi)$ belong to $\mathcal{P}(\tau, I_{s})$ for the constant $\tau$ and the integer $I_{s}$ in Proposition 2.2. Let $p(x, \xi)$ belong to $S^{m}_{\rho}$ ($-\infty < m < \infty$, $1/2 \leq \rho \leq 1$). Then, we have the following:

i) There exist pseudo-differential operators $P_{j} = p_{j}(X, D_{x})$, $j = 1, 2$, in $S^{n}_{\rho}$ such that

$$
(2.19) \quad P_{\phi} = P_{1}I_{\phi} ,
$$

$$
(2.20) \quad P_{\phi} = I_{\phi}P_{2}
$$
and estimates

\[(2.21) \quad |p_j|^{(n)} \leq C_l |p|^{(\rho)} \quad (j = 1, 2)\]

hold for any \(l\), where \(C_l\) is a constant depending only on \(m\), \(\rho\), \(l\) and \(|F|^{(l)}\) (for some \(l'\)) and \(l'\) is an integer depending only on \(m\), \(\rho\) and \(l\).

\(\text{ii) There exist pseudo-differential operators } P_j = p_j(X, D_x), j = 1, 2, 3, 4, \text{ in } S^m_\rho \text{ such that we have }\)

\[(2.22) \quad PI_\phi = I_\phi P_1, \quad I_\phi P = P_2 I_\phi, \]
\[(2.23) \quad PI_\phi^* = I_\phi P_3, \quad I_\phi P = P_4 I_\phi. \]

and the symbols \(p_j(x, \xi), j = 1, 2, 3, 4, \) have the semi-norm estimates similar to (2.21).

Proof. \(i)\) Set

\[
\begin{cases}
P_1 = P_\phi I_\phi R', \\
P_2 = R I_\phi P_\phi
\end{cases}
\]

with pseudo-differential operators \(R\) and \(R'\) constructed in Proposition 2.2. Note that from Theorem 1.6 and Theorem 1.7 of Chap. 10 in [8] the operators \(P_1\) and \(P_2\) are pseudo-differential operators in \(S^m\). From (2.8) they satisfy (2.19) and (2.20). If we go over the proof carefully once again, we obtain (2.21).

\(\text{ii) Set }\)

\[
\begin{cases}
P_1 = R I_\phi P_\phi, \\
P_2 = I_\phi P_\phi R', \\
P_3 = R' I_\phi P_\phi, \\
P_4 = I_\phi P_\phi.
\end{cases}
\]

Then, as in \(i)\) we see that \(P_j, j = 1, 2, 3, 4, \) are pseudo-differential operators in \(S^m\) and they satisfy (2.22), (2.23), and the last statement of \(\text{ii) }\). Q.E.D.

In order to study products of Fourier integral operators, we shall review some results of multi-products of phase functions. Proofs are found in Section 1 of [11] or Section 5 of Chap. 10 in [8]. First, we introduce

DEFINITION 2.4. For \(-\infty < m < \infty, 1/2 \leq \rho \leq 1\) and an integer \(k\) we define a class \(S^m_\rho((k))\) by the set of symbols \(p(x, \xi) \in S^m_\rho\) satisfying

\[(2.24) \quad p^{(\alpha)}(x, \xi) \in S^{m-|\alpha|}_\rho \quad \text{for } |\alpha| + |\beta| \leq k.\]

The class \(S^m_\rho((k))\) is a Fréchet space with semi-norms

\[(2.25) \quad |p|^{(m)} = \max_{|\alpha| + |\beta| \leq k} \sup_{x, \xi} \{ |p^{(\alpha)}(x, \xi)| \langle \xi \rangle^{-(m-|\alpha|+|\beta|)(\alpha+\beta-k)} \}.\]

Now, we begin with
Proposition 2.5. i) Let $\tau'_0$ be a constant satisfying $0 \leq \tau'_0 < 1/3$. Let
\[ \phi_j(x, \xi) \text{ belong to } \mathcal{P}_\rho(\tau_j), \quad j = 1, 2, \cdots, v+1, \cdots, \]
and suppose that $\sum_{j=1}^{v} \tau_j \leq \tau'_0$. Then, the equation
\begin{equation}
\begin{cases}
x^j = \nabla \phi_j(x, \xi^j), \\
\xi^j = \nabla x \phi(x^j, \xi),
\end{cases}
\end{equation}
has the unique $C^\infty$-solution \( \{ X^j, \Xi \}_{j=1} \).

ii) Setting \( J_j(x, \xi) = \phi_j(x, \xi) - x \cdot \xi \), we assume, furthermore, that the set
\( \{ J_j(\tau) \} \) is bounded in \( S^{k}(k+2) \) with some \( k \geq 0 \). Then, the sets \( \{ X_j \}_{j,v} \) and
\( \{ \Xi \}_{j,v} \) are bounded in \( S^k((k+1)) \) and \( S^k((k+1)) \), respectively.

Remark. In [11] and [8] only the case \( k=0 \) is considered, but we can prove the proposition similarly for the case \( k \geq 1 \).

For any fixed \( v \) we define
\begin{equation}
\Phi_{v+1}(x, \xi) = \sum_{j=1}^{v} (\phi_j(X^j, \Xi_j) - X^j \cdot \Xi_j) + \phi_{v+1}(X_v, \xi) \quad (X_v = x).
\end{equation}
Then, if \( \phi_j \in \mathcal{P}_\rho(\tau_j, l) \), setting \( J_{v+1}(x, \xi) = \Phi_{v+1}(x, \xi) - x \cdot \xi \) it follows that
\begin{equation}
||J_{v+1}||_{l} \leq \epsilon_v \tau_{v+1} \quad (\tau_{v+1} = \tau_1 + \tau_2 + \cdots + \tau_{v+1})
\end{equation}
for a constant \( \epsilon_v \) determined only by \( n, \rho, \tau'_0 \) and \( l \). Taking account of this we have

Proposition 2.6. Let \( \phi_j(x, \xi) \) belong to \( \mathcal{P}_\rho(\tau_j, l) \) and assume that the set
\( \{ J_j(\tau) \} \) \( (J_j(x, \xi) = \phi_j(x, \xi) - x \cdot \xi) \) is bounded in \( S^k((k+2)) \) \( \text{with some } k \geq 0 \). Then, if \( \epsilon_v \tau_{v+1} < 1 \), the function \( \Phi_{v+1}(x, \xi) \) of (2.27) is a phase function in \( \mathcal{P}_\rho(\epsilon_v \tau_{v+1}, l) \) and the set \( \{ J_{v+1}(\tau_{v+1}) \} \) is bounded in \( S^{k}((k+2)) \), where \( J_{v+1} = \Phi_{v+1} - x \cdot \xi \).

Setting \( \epsilon_v = \epsilon_v \rho \), we take a constant \( \tau_v \) satisfying \( 0 \leq \tau_v \leq \tau'_0 \) and \( \epsilon_v \tau_v < 1 \).
Then, applying the above proposition with \( l=0 \), the following is justified.

Definition 2.7. Let \( \tau_v \) be the constant above. For phase functions
\( \phi_j(x, \xi) \in \mathcal{P}_\rho(\tau_j), \quad j = 1, 2, \cdots, v+1, \cdots, \) with \( \sum_{j=1}^{v+1} \tau_j \leq \tau_v \) we define the multi(-\#)-product
\( \Phi_{v+1}(x, \xi) = \phi_1 \# \phi_2 \# \cdots \# \phi_{v+1}(x, \xi) \in \mathcal{P}_\rho(\epsilon_v \tau_{v+1}) \) of \( \phi_1(x, \xi) \), \( \phi_2(x, \xi) \), \( \cdots, \phi_{v+1}(x, \xi) \) by (2.27).

We return to products of Fourier integral operators.

Proposition 2.8. i) Let \( \phi_j(x, \xi) \) belong to \( \mathcal{P}_\rho(\tau_j), \quad j = 1, 2, \quad \tau_1 + \tau_2 \leq \tau_v \) and let \( \{ X, \Xi \} = \{ X^j, \Xi_j \} \) \( (x, \xi) \) be the solution of (2.26) with \( v=1 \). Set
\begin{equation}
\phi(x, \xi) = \phi_1 \# \phi_2(x, \xi) = \phi_1(x, \Xi) - X \cdot \Xi + \phi_2(x, \xi)
\end{equation}
and

\begin{equation}
(2.30) \quad p(x, \xi') = O_\omega \int \int e^{ix' dx' d\xi}
\end{equation}

with

\[ \psi(x, x'; \xi, \xi') = \phi_1(x, \xi) - x' \cdot \xi + \phi_2(x', \xi') - \Phi(x, \xi'). \]

Then, we have \( p(x, \xi) \in S_0^0 \) and

\begin{equation}
(2.31) \quad I_{\phi_1} P_{\phi_2} = P_{\Phi}.
\end{equation}

ii) Let \( \{\phi_1, \gamma\} \gamma \in \Gamma \) and \( \{\phi_2, \gamma\} \gamma \in \Gamma \) be bounded sets in \( Q_\rho(\tau_0) \) and assume that for any \( \gamma \in \Gamma \) the pair \( \{\phi_1, \gamma, \phi_2, \gamma\} \) satisfies the condition in i). Then, for the symbol \( p_\gamma(x, \xi) \) defined from the pair \( \{\phi_1, \gamma, \phi_2, \gamma\} \) the set \( \{P_{\gamma}\} \gamma \in \Gamma \) is bounded in \( S_0^0 \).

**Remark.** In [4] Hörmander gave this proposition in the generalized form. Here, we shall give the simplified version of the proof studied in [10].

**Proof.** We divide the proof into two steps.

I) From the definition of \( I_{\phi_1} \) and \( I_{\phi_2} \) we have

\begin{equation}
(2.32) \quad I_{\phi_1} I_{\phi_2} = O_\omega \int \int e^{i\langle \phi_1(s, \xi) - s' \cdot \xi + \phi_2(s', \xi') \rangle} \hat{u}(\xi') d\xi' dx' d\xi
\end{equation}

Substituting (2.30) into (2.32),

\[ I_{\phi_1} I_{\phi_2} = \int e^{i\Phi(x, \xi')} p(x, \xi') \hat{u}(\xi') d\xi' \]

holds. This is nothing but (2.31).

Now, we set

\begin{equation}
(2.33) \quad \chi_\omega(\xi, \xi') = 1 - \chi(\langle \xi - \xi' \rangle / \langle \xi' \rangle)
\end{equation}

and consider

\begin{equation}
(2.34) \quad p_\omega(x, \xi') = O_\omega \int \int e^{ix' \chi_\omega(\xi, \xi')} dx' d\xi
\end{equation}

\[ = O_\omega \int \int e^{-ix' \cdot \xi} \tilde{p}_\omega(x', \xi; x, \xi') dx' d\xi, \]

where

\[ \tilde{p}_\omega(x', \xi; x, \xi') = e^{i\langle \phi_1(s, \xi) + \phi_2(s', \xi') - \Phi(s, \xi') \rangle} \chi_\omega(\xi, \xi'). \]

Considering \( x \) and \( \xi' \) as parameters, the symbol \( \partial_\xi^m D_{\xi'}^n \tilde{p}_\omega(x', \xi; x, \xi') \) belongs to \( \mathcal{M}_{|I|}^{|J|} \) defined in §6 of Chap. 1 of [8]. Hence, applying Theorem 6.6 of Chap. 1 of [8] we obtain

\begin{equation}
(2.35) \quad p_\omega^{(m,n)}(x, \xi') = O_\omega \int \int e^{-ix' \cdot \xi} \partial_\xi^m D_{\xi'}^n \tilde{p}_\omega(x', \xi; x, \xi') dx' d\xi.
\end{equation}
MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS

Set
\[ \tilde{p}_\omega(y, \xi') = e^{-i\left(\phi_1(x, \xi) + \phi_2(x', \xi') - \phi(x, \xi')\right)} \delta_\xi D_\xi \tilde{p}_\omega(x', \xi; x, \xi') . \]

Then, we have

(2.36) \[ p_{\omega}(\xi, \xi') = O_\varepsilon - \int e^{i\eta\phi}\tilde{p}_{\omega}(y, \xi') dx'd\xi . \]

From \( |\nabla_x \{ \phi_1(x, \xi) + \phi_2(x', \xi') - \Phi(x, \xi') \} | \leq C < x - x' > \) and \( |\nabla_x \{ \phi_1(x, \xi) + \phi_2(x', \xi') - \Phi(x, \xi') \} | \leq C < \xi; \xi' > \) we have

(2.37) \[ |\partial_\xi^\beta D_\xi^\beta \tilde{p}_{\omega}(y, \xi); x, \xi') | \leq C_{\alpha, \beta} |x - x'|^{\alpha} |\xi; \xi'|^{\beta} (1 - p)|\beta'| . \]

Since we have \( |\xi - \xi'| \geq (2/5)|\xi'| \) on supp \( \chi_{\omega}(\xi, \xi') \), we obtain on supp \( \tilde{p}_{\omega}(\xi, \xi') \)

\[ |\nabla_x^\alpha \psi| = | -i \xi + \nabla_x^\alpha \phi_2 (x', \xi') | \]
\[ \geq |\xi - \xi'| - \tau_\varepsilon |\xi'| \]
\[ \geq \frac{1}{6} |\xi - \xi'| \geq \frac{1}{15} |\xi'| . \]

Moreover, we can prove

\[ 1 + |\nabla_x^\alpha \psi| \geq C < x - x' > \]

with some positive constant \( C \). Set

\[ \begin{cases} L_1 = -i |\nabla_x^\alpha \psi|^2 \nabla_x^\alpha \psi \cdot \nabla_x' , \\ L_2 = (1 + |\nabla_x^\alpha \psi|)^2 (1 - i\nabla_x^\alpha \psi \cdot \nabla_x) \end{cases} \]

and write

(2.38) \[ p_{\omega}(\xi, \xi') = \int e^{i\psi(L_1')^t(L_2')^t \tilde{p}_{\omega}(y, \xi') dx'd\xi . \]

for a fixed \( l_2 > n + |\alpha| \) and large \( l_1 \). Then, we get for any \( N \)

(2.39) \[ p_{\omega}(\xi, \xi') \leq C_{N, \omega} |\xi'|^{-N} , \]

that is, we have

(2.40) \[ p_\omega(\xi, \xi') \equiv S^{-\infty} . \]

II. For \( X_\omega(\xi, \xi') \equiv X((\xi - \xi')/|\xi'|) \) we consider

(2.40) \[ p_\omega(\xi, \xi') = O_{\varepsilon} - \int e^{i\psi X_\omega(\xi, \xi') dx'd\xi . \]

Using a change of the variables: \( x' = X(x, \xi') + y, \xi = \Xi(x, \xi') + \eta \), we write

(2.41) \[ p_\omega(\xi, \xi') = O_{\varepsilon} - \int e^{-i\tilde{\psi}(y, \eta) s\xi'}^\Xi (\eta; x, \xi') dyd\eta . \]
Here,

\[(2.42) \quad \mathcal{X}_0(\eta; x, \xi) = \mathcal{X}(\Xi(x, \xi) + \eta - \xi)/\langle \xi \rangle \]

and

\[(2.43) \quad \tilde{\psi} \equiv \tilde{\psi}(y, \eta; x, \xi) = -\psi(x, X(x, \xi) + y; \Xi(x, \xi) + \eta, \xi) = -y \cdot \eta - \{\phi_1(x, \Xi + \eta) - X \cdot \eta - \phi_1(x, \Xi)\} - \{\phi_2(X + y, \xi) - y \cdot \Xi - \phi_2(X, \xi)\}.\]

Since \(\{X, \Xi\}\) is the solution of (2.26) with \(\nu=1\), we have

\[|X - x| \leq \tau_1 \leq \frac{1}{3} \quad \Xi - \xi \leq \tau_2 \langle \xi \rangle \leq \frac{1}{3} \langle \xi \rangle.\]

Hence, we have from the definition of \(\mathcal{X}_0 = \mathcal{X}_0(\eta; x, \xi)\)

\[(2.44) \quad \begin{cases} |\Xi + \theta \eta - \xi| \leq \theta |\Xi + \eta - \xi| + (1 - \theta) |\Xi - \xi| \leq \frac{1}{2} \langle \xi \rangle, \\ \frac{1}{2} \langle \xi \rangle \leq \langle \Xi + \theta \eta \rangle \leq 2 \langle \xi \rangle \quad (0 \leq \theta \leq 1) \quad \text{on supp } \mathcal{X}_0. \end{cases}\]

Taking account of (2.26) for \(\nu=1\) we have

\[\nabla_{\eta} \tilde{\psi} = -\left(\int_{0}^{1} \frac{\partial}{\partial \xi} \nabla_{\xi} J_1(x, \Xi + \theta \eta) d\theta\right) \eta, \quad \nabla_{\xi} \tilde{\psi} = -\left(\int_{0}^{1} \frac{\partial}{\partial x} \nabla_{x} J_2(X + \theta y, \xi) d\theta\right) y.\]

Then, from (2.44) we have

\[\begin{cases} |\nabla_{\eta} \tilde{\psi}| \geq |y| - 2 \tau_1 \langle \xi \rangle^{-1} |\eta| \geq |y| - \frac{2}{3} \langle \xi \rangle^{-1} |\eta|, \\ |\nabla_{\xi} \tilde{\psi}| \geq |\eta| - \tau_2 \langle \xi \rangle |y| \geq |\eta| - \frac{1}{3} \langle \xi \rangle |y| \quad \text{on supp } \mathcal{X}_0 \end{cases}\]

and get

\[(2.45) \quad \langle \xi \rangle^2 |\nabla_{\eta} \tilde{\psi}|^2 + |\nabla_{\xi} \tilde{\psi}|^2 \geq \frac{1}{2} \langle \xi \rangle |\nabla_{\eta} \tilde{\psi}| + |\nabla_{\xi} \tilde{\psi}| \geq \frac{1}{18} \langle \xi \rangle |y| + |\eta| \quad \text{on supp } \mathcal{X}_0.\]

On the other hand, we rewrite \(\tilde{\psi}\) in the form

\[(2.46) \quad \tilde{\psi} = y \cdot \eta - B \eta \cdot \eta - B'y \cdot y\]

with
\[ \begin{align*}
B &= B(\eta; x, \xi) = \int_0^1 (1- \theta) \frac{\partial}{\partial \xi} \nabla_x J_1(x, \Xi + \theta \eta) \, d\theta , \\
B' &= B'(y; x, \xi) = \int_0^1 (1- \theta) \frac{\partial}{\partial x} \nabla_x J_2(x + \theta y, \xi) \, d\theta .
\end{align*} \]

Then, as in the first step, we have

\[ \begin{align*}
p_{\alpha, \beta}^{(2)}(x, \xi) &= O_s - \int e^{-i y \eta} \partial_\xi^\alpha D_x^\beta \{ e^{(\theta y \eta + B' y \eta)} \tilde{\chi}_0 \} \, dy \, d\eta \\
&= O_s - \int e^{-i y \eta} \tilde{p}_{\alpha, \beta}(y, \eta; x, \xi) \, dy \, d\eta
\end{align*} \]

with

\[ \begin{align*}
\tilde{p}_{\alpha, \beta}(y, \eta; x, \xi) &= e^{-i (\theta y \eta + B' y \eta)} \partial_\xi^\alpha D_x^\beta \{ e^{i (\theta y \eta + B' y \eta)} \tilde{\chi}_0 \} \\
&= \sum_{\beta^1 + \cdots + \beta^k = \beta} \sum_{\alpha_1 + \cdots + \alpha_k = \alpha} C_{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \ldots, \beta_k} \\
&\quad \times \prod_{j=1}^k \{ (\partial_\xi^{\beta_j} D_x^{\alpha_j} B) \eta \cdot \eta + (\partial_\xi^{\beta_j} D_x^{\alpha_j} B') y \cdot y \} \, \partial_\xi^\alpha \tilde{\chi}_0 .
\end{align*} \]

This expression (2.49) yields

\[ \begin{align*}
|\partial_\xi^\alpha D_x^\beta \tilde{p}_{\alpha, \beta}(y, \eta; x, \xi)|
&\leq C_{\alpha, \alpha', \beta, \beta'} \langle \xi \rangle^{-|\alpha| + (1-\rho) |\alpha| + |\beta|} \{ 1 + \langle \xi \rangle ^{-1/2} \langle \eta \rangle ^{1/2} \langle \eta \rangle ^{1/2} \} ^{2 (|\alpha| + |\beta|)}
\end{align*} \]

in view of the fact that the symbols $B$ and $B'$ in (2.47) have the orders $-1$ and $1$, respectively, with respect to $\xi$. We set

\[ \begin{align*}
L_3 &= (1 + \langle \xi \rangle ^{-1} \langle \eta \rangle ^{-1} \beta_\eta \nabla_\eta \tilde{\eta} + \nabla_\eta \tilde{\eta})^{-1} \\
&\quad \times (1 + i \langle \xi \rangle ^{-1} \langle \eta \rangle ^{-1} \nabla_\eta \tilde{\eta} \cdot \nabla_\eta + \nabla_\eta \tilde{\eta} \cdot \nabla_\eta)
\end{align*} \]

and write (2.48) in the form

\[ \begin{align*}
p_{\alpha, \beta}^{(2)}(x, \xi) &= \int e^{-i y \eta} \tilde{p}_{\alpha, \beta}(y, \eta; x, \xi) \, dy \, d\eta
\end{align*} \]

for $l=2n+1+2(\langle \alpha \rangle + \langle \beta \rangle)$. Then, we have

\[ \begin{align*}
|p_{\alpha, \beta}^{(2)}(x, \xi)| &\leq C_{\alpha, \beta} \langle \xi \rangle^{-|\alpha| + (1-\rho) |\alpha| + |\beta|} .
\end{align*} \]

Consequently, we have

\[ \begin{align*}
p_{\alpha, \beta} &\in S^0_ho
\end{align*} \]

and combining this with (2.39) we obtain

\[ \begin{align*}
p(x, \xi) &\in S^0_ho
\end{align*} \]
for $p(x, \xi)$ in (2.30). Finally, we get ii) if we go over the proof carefully once again. Q.E.D.

Now, we apply Proposition 2.3-i) to the Fourier integral operator $P_\phi$ in (2.31). Then, we have

**Corollary 2.9.** Let $\phi_j(x, \xi)$ belong to $\mathcal{D}_p(\tau_j), j = 1, 2$, and assume that the phase function $\Phi(x, \xi)$ defined by (2.29) belongs to $\mathcal{D}_p(\tau, \bar{l}_o)$ for the constant $\tau$ and the integer $\bar{l}_o$ in Proposition 2.2. Then, there exist symbols $p_j(x, \xi), j = 1, 2$, in $S^0$ such that for $P_j = p_j(X, D_x)$

\[(2.54) \quad I_{\phi_1} I_{\phi_2} = P_1 I_{\phi_1} = I_{\phi_2} P_2 \]

holds.

**Remark 1.** Let $c_0, T_0$ be the constant defined in Proposition 2.6 with $l = \bar{l}_o$. Then, if $\phi_j(x, \xi)$ in Corollary 2.9 belongs to $\mathcal{D}_p(\tau_j, \bar{l}_o), j = 1, 2$, and $\tau_1 + \tau_2 \leq \tau$ holds, the phase function $\Phi(x, \xi)$ of (2.29) belongs to $\mathcal{D}_p(\tau, \bar{l}_o)$.

**Remark 2.** Let $\{\phi_1, \gamma\} \gamma \in \Gamma$ and $\{\phi_2, \gamma\} \gamma \in \Gamma$ be bounded sets in $\mathcal{D}_p(\tau_j)$ and assume that for any $\gamma \in \Gamma$ the pair $\{\phi_1, \gamma, \phi_2, \gamma\}$ satisfies the condition in the corollary. Then, for the symbols $p_1, \gamma(x, \xi)$ and $p_2, \gamma(x, \xi)$ defined from the pair $\{\phi_1, \gamma, \phi_2, \gamma\}$ the sets $\{p_1, \gamma\} \gamma \in \Gamma$ and $\{p_2, \gamma\} \gamma \in \Gamma$ are bounded in $S^0$.

**Lemma 2.10.** Let $\phi_j(x, \xi)$ belong to $\mathcal{D}_p(\tau_j, \bar{l}_o), j = 1, 2$, with $\tau_1 + \tau_2 \leq \tau$, satisfying $\Phi(x, \xi) \equiv \phi_1, \# \phi_2(x, \xi) \in \mathcal{D}_p(\tau, \bar{l}_o)$ for the constant $\tau$ and the integer $\bar{l}_o$ in Proposition 2.2, and let $p(x, \xi)$ belong to $S^m$. Then, there exist pseudo-differential operators $P'$ and $P''$ in $S^m$ such that

\[(2.55) \quad I_{\phi_1} P_{\phi_2} = P' I_{\phi_1} \]

and

\[(2.56) \quad P_{\phi_1} I_{\phi_2} = I_{\phi_2} P'' \]

Moreover, estimates

\[(2.57) \quad \left\{ \begin{array}{l} |p'|^{(m)} \leq C_1 |p|^{(\gamma)} , \\ |p''|^{(m)} \leq C_1 |p|^{(\gamma)} \end{array} \right. \]

hold for a constant $C_1$ depending only on $m, \rho, l$ and $\{|J_j|^{l_j'}\}_{j=1,2}$ (for some $l'$) and an integer $l'$ depending only on $m, \rho$ and $l$.

**Proof.** We prove (2.55). Then, we can prove (2.56) similarly. From Proposition 2.3-i) there exists a pseudo-differential operator $P_1$ in $S^m$ satisfying

$P_{\phi_1} = P_1 I_{\phi_1}$. 
Next, we apply Proposition 2.3-ii) to find a pseudo-differential operator $P_2$ in $S^{\sigma^m}$ satisfying

$$I_{\phi_1}P_1 = P_2I_{\phi_1}.$$  

Then, we have

$$I_{\phi_1}P_{\phi_2} = P_2I_{\phi_1}I_{\phi_2}.$$  

Use Corollary 2.9 to find a pseudo-differential operator $R^\circ$ in $S^{\sigma^0}$ satisfying

$$I_{\phi_1}I_{\phi_2} = R^\circ I_{\phi_1}.$$  

Then, setting $P' = P_2R^\circ$, we get (2.55). If we go over the proof once again, we can prove the last statement. Q.E.D.

Now, we prove Theorem 1. We take the integer $I_o$ in Proposition 2.2 as the one in Theorem 1. Define

$$(2.58) \quad \tau^0 = \min \{ \tau_o/c_o, \pi/c_o\tau_o \}$$

with the constants $\tau_o$, $c_o (=c_o,0)$, $\phi$ and $c_o\tau_o$ introduced in Definition 2.7, Proposition 2.2 and Proposition 2.6. Then, if phase functions $\phi_j(x, \xi) \in \mathcal{P}_\rho(\tau_j, I_o)$ satisfy (*) in Introduction, we have for multi-products $\Phi_j = \Phi_j \# \cdots \# \Phi_j$

$$\Phi_j \in \mathcal{P}_\rho(\tau_o), \quad \Phi_{j+1} = \Phi_j \# \phi_{j+1} \in \mathcal{P}_\rho(\tau, I_o) \quad (\Phi_1 = \phi_1)$$

from (1.30) of [11] and Proposition 2.6. Using this we prove (9)-i) for the multi-product

$$\Phi_{v+1} = P_{1,\phi_1}P_{2,\phi_2} \cdots P_{v+1,\phi_{v+1}}$$

with $P_{j,\phi_j}(X, D_x)$ for $\phi_j(x, \xi) \in S^{\sigma^m}$. First, we apply Proposition 2.3-i). Then, there exists a pseudo-differential operator $P'_1$ in $S^{\sigma^m}$ such that

$$(2.59) \quad P_{1,\phi_1} = P'_1I_{\phi_1}.$$  

For $j$ with $j \geq 2$ we apply Lemma 2.10. Then, there exists a pseudo-differential operator $P'_j$ in $S^{\sigma^m}$ such that

$$(2.60) \quad I_{\phi_{j-1}}P_{j,\phi_j} = P'_jI_{\phi_j} \qquad (\Phi_1 = \phi_1).$$  

Combining (2.59) and (2.60), we get

$$(2.61) \quad \Phi_{v+1} = P'_1(I_{\phi_1}P_{2,\phi_2})P_{3,\phi_3} \cdots P_{v+1,\phi_{v+1}}$$

$$= P'_1P'_2(I_{\phi_2}P_{3,\phi_3})P_{4,\phi_4} \cdots P_{v+1,\phi_{v+1}}$$

$$= \cdots$$

$$= P'_1P'_2 \cdots P'_v(I_{\phi_v}P_{v+1,\phi_{v+1}})$$

$$= P'_1P'_2 \cdots P'_vP'_{v+1,\phi_{v+1}}.$$
This proves (9)-i). Similarly, we can prove (9)-ii).

From the above discussion the boundedness of the sequence \( \{m_j\} \) implies

\[
(2.62) \quad \left\{ \begin{array}{l}
|\sigma(P_j)||^{(m_j)} \leq C_1 |p_j| |^{(m_j)} , \\
|\sigma(P'_j)||^{(m_j)} \leq C_1 |p_j| |^{(m_j)}
\end{array} \right.
\]

with a constant \( C_i \) and an integer \( l' \) independent of \( j \). This comes from the fact that \( P_j \) and \( P'_j \) are determined only by \( P_j, \Phi_j \) and \( \{\phi_k\}^{k=1}_l \). Combining (9) and (2.62) with Theorem 2 we get Theorem 1. This concludes the proof of Theorem 1.

The asymptotic expansion for the symbol \( q_{+,+1}(x, \xi) \) of multi-products (3) was discussed in the proof of Theorem 2.4 in [10]. Here, we give its well-arranged form, which is not used in the following but which is derived directly from the discussions of the proof of Theorem 1.

**Theorem 2.11.** Let \( \phi_j(x, \xi) \) belong to \( \mathcal{D}_\rho(\tau_j) \) and \( p_j(x, \xi) \) belong to \( S^m_\rho \) for \( 1/2<\rho\leq1 \) verifying the assumptions in Theorem 1. Let \( \{X_j, \Xi_j\}^{j=1}_j \) be the solution of (2.26). Then, the symbol \( q_{+,+1}(x, \xi) \) of the multi-product (3) of Fourier integral operators \( P_j \Phi_j = p_j \Phi_j(X, D_x) \) satisfies

\[
(2.63) \quad q_{+,+1}(x, \xi) \sim \sum \sum r^{+,+1}_{k,\alpha^\gamma,\beta^\gamma}(x, \xi)p_1^{(a)}(x, \Xi_1)p_{\xi_2^{(s)}}(X_1, \Xi_1) \ldots
\]

\[
\times p_1^{(s)}(X_1, \Xi_1)p_{+,+1}(\psi_1)(X_1, \xi)
\]

in the sense of Definition 1.6 of Chap. 2 in [8], where \( r^{+,+1}_{k,\alpha^\gamma,\beta^\gamma}(x, \xi) \) belong to \( S^m_\rho(\alpha^\gamma,\beta^\gamma) \) with \( m(k, \alpha^\gamma, \beta^\gamma) = -(2p-1)k + |\alpha^\gamma| - (1-\rho)(|\beta^\gamma| + |\beta^\gamma|) \) and \( |\alpha^\gamma| = |\alpha^1| + \ldots + |\alpha^v|, \ |\beta^\gamma| = |\beta^1| + \ldots + |\beta^v| \) for \( \alpha^\gamma = (\alpha^1, \ldots, \alpha^v), \beta^\gamma = (\beta^1, \ldots, \beta^v) \).

**Proof.** From the proof of Theorem 1 the pseudo-differential operators \( P_j \) (\( j=1, 2, \ldots, v \)) in (2.59)-(2.60) have the forms

\[
(2.64) \quad \left\{ \begin{array}{l}
P'_j = P_{1,\Phi_1} I_{\Phi_j} R'_j , \\
P_j = I_{\Phi_j} (P_{1,\Phi_1} I_{\Phi_j} R'_j) I_{\Phi_j} R'_j
\end{array} \right. \quad (2 \leq j \leq v)
\]

with some pseudo-differential operators \( R'_j \) and \( R'_j \) in \( S^0_\rho \), where \( \Phi_1 = \Phi_1, \Phi_2 = \Phi_2, \ldots, \Phi_j = \Phi_j \) (\( j \geq 2 \)). For \( j=v+1 \), applying Proposition 2.3-i), we write

\[
(2.65) \quad P_{+,+1,\Phi_{+,+1}} = I_{\Phi_{+,+1}} P'_{+,+1}
\]

with

\[
P'_{+,+1} = R_{+,+1} I_{\Phi_{+,+1}} P_{+,+1,\Phi_{+,+1}} \quad (\sigma(R_{+,+1}) \subseteq S^0_\rho)
\]

Then, with the aid of \( I_{\Phi} I_{\Phi_{+,+1}} = R_\Phi I_{\Phi_{+,+1}} \) by Corollary 2.9 the multi-product \( \tilde{Q}_{+,+1} \) of (3) has the form
Here, $R_0$ is a pseudo-differential operator in $S^0$.

Denote for a phase function $\phi(x, \xi) \in \mathcal{P}_p(t)$ the inverses of $\xi = \nabla_x \phi(x, \eta)$ and $x = \nabla_\eta \phi(y, \xi)$ by $\eta = \nabla_x \phi^{-1}(x, \xi)$ and $y = \nabla_\eta \phi^{-1}(x, \xi)$, respectively. We note that $\nabla_x \phi^{-1}(x, \xi) = \nabla_x \phi^{-1}(x, \xi, x)$ and $\nabla_\eta \phi^{-1}(x, \xi) = \nabla_\eta \phi^{-1}(\xi, x, \xi)$ hold. Using Theorem 1.6, Theorem 1.7 and Theorem 2.1 of Chap. 10 in [8] we have from (2.64) and (2.65)

\begin{align}
(2.67) & \quad P_\ell \phi(x, \xi) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} r_{j+k, \ell, \alpha} (x, \xi) \phi_\ell(x, \nabla_x \phi^{-1}(x, \xi)) \\
(2.68) & \quad P_\ell \phi(x, \xi) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} r_{j+k, \ell, \alpha} (x, \xi) \phi_\ell(x, \nabla_\xi \phi^{-1}(\nabla_\xi \phi^{-1}(x, \xi), \eta)) \\
(2.69) & \quad P_{\ell+1} \phi(x, \xi) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} r_{j+k+1, \ell, \alpha} (x, \xi) \phi_{\ell+1}(\nabla_\xi \phi^{-1}(x, \xi))
\end{align}

with symbols

\begin{align}
(2.70) & \quad r_{j+k, \ell, \alpha} \in S^{-(2\ell-1)k+\beta[\alpha]} \\
& \quad r_{j+k, \ell, \alpha} \in S^{-(2\ell-1)k+(1-\beta)[\beta]} (2 \leq j \leq \nu), \\
& \quad r_{j+k+1, \ell, \alpha} \in S^{-(2\ell-1)k-(1-\beta)[\beta]} .
\end{align}

On the other hand, we can prove by the same method as the discussions in Section 1 of [11]

\begin{align}
(2.71) & \quad \nabla_x \phi^{-1}(x, \nabla_x \phi_{\ell+1}(x, \xi)) = \Xi_{\ell+1}(x, \xi), \\
(2.72) & \quad \nabla_\xi \phi_{\ell+1}(x, \nabla_\xi \phi_{\ell+1}(x, \xi)) = \Xi_{\ell+1}^{-1}(x, \xi), \\
(2.73) & \quad \nabla_\xi \phi_{\ell+1}(\nabla_\xi \phi_{\ell+1}(x, \xi), \xi) = X_{\ell+1}(x, \xi).
\end{align}

Consequently, applying Theorem 2.5 of Chap. 7 in [8], we can derive (2.63) from (2.66)-(2.69), (2.71)-(2.73) and (2.4-i) and (2.23-i) of Chap. 10 in [8]. Q.E.D.

3. Commutative law for $\#$-products of phase functions. Let $\phi_j(t, s; x, \xi) (j=1, 2)$ be the phase function defined by an eikonal equation

\begin{align}
(3.1) & \quad \frac{\partial \phi}{\partial t} - \lambda(t, x, \nabla_x \phi) = 0 \quad \text{on } [0, T], \\
& \quad \phi_{1|_{s=t}} = x + \xi
\end{align}

for $\lambda(t, x, \xi) = \lambda_j(t, x, \xi)$ (real symbol of order one), and let $I_{\phi_j}(t, s)$ be the Fourier integral operator with phase function $\phi_j(t, s; x, \xi)$ and symbol 1. What we
want to study is the following problem: When do $I_{\phi_1}(t, s)$ and $I_{\phi_2}(t, s)$ commute, or in a wider sense, when is the product $I_{\phi_2}(t, \omega)I_{\phi_1}(\theta, s)R$ for an appropriate constant $\omega$ and a pseudo-differential operator $R$? The positive answer of this problem suggests the possibility of the reduction of the infinite sum expression (2) for the fundamental solution to the finite sum expression (4), that is, the possibility of the proof of Theorem 4. If the Poisson bracket

$$\{\tau - \lambda_1, \tau - \lambda_2\} = \partial \lambda_1/\partial t - \partial \lambda_2/\partial t + \nabla_t \lambda_1 \cdot \nabla_t \lambda_2 - \nabla_x \lambda_1 \cdot \nabla_x \lambda_2$$

of $\tau - \lambda_1$ and $\tau - \lambda_2$ ($\tau$ is the dual variable of $t$) is identically zero, Kumano-go-Taniguchi-Tozaki [11] proved

$$\text{(3.2)} \quad (\phi_2(t, \theta)\#\phi_1(\theta, s))(x, \xi) = (\phi_2(t, t - \theta + s)\#\phi_1(t - \theta + s, s))(x, \xi),$$

which implies

$$\text{(3.3)} \quad I_{\phi_2}(t, \theta)I_{\phi_1}(\theta, s) = I_{\phi_2}(t - \theta + s)I_{\phi_1}(t - \theta + s, s)R$$

on account of (2.54). In this way the above problem is reduced to the problem of the commutative law for phase functions. In the present paper, we shall show their commutative law under the condition

$$\text{(3.4)} \quad \{\tau - \lambda_1, \tau - \lambda_2\} = a(t, x, \xi) (\lambda_1 - \lambda_2) + a'(t, x, \xi)$$

where $a(t, x, \xi)$ and $a'(t, x, \xi)$ are real symbols of order zero.

For the further study, we shall review the properties with additional results for the phase function $\phi(t, s; x, \xi)$ defined by an eikonal equation (3.1). We note that (3.1) corresponds to a hyperbolic operator

$$\text{(3.5)} \quad L_\alpha = D_t - \lambda(t, X, D_x) \quad \text{on } [0, T],$$

where $D_t = -i \partial_t$, $\partial_\tau = \partial/\partial t$. To begin with, we introduce the following definition.

**DEFINITION 3.1.** Let $Z$ be a subset of Euclidean space $\mathbb{R}^n$ and let $F(\subset S^m_{p,\alpha})$ be a Fréchet space of symbol class of pseudo-differential operators (for example, $F = S^m_p$, $S^m_{p,\alpha}$ or $S^m_{p,\alpha}(\langle \tau \rangle)$). We say that a $C^1$-function $p(\tilde{t}, x, \xi)$ in $Z \times \mathbb{R}^n \times \mathbb{R}^n$ belongs to a class $M'(Z; F)$ when $\partial_t p(\tilde{t}, x, \xi)$ is a $C^1$-function for any $\alpha, \beta$, $p(\tilde{t}, x, \xi)$ belongs to $F$ for any $\tilde{t} \in Z$ and the set $\{\partial/\partial \tilde{t} \tilde{p}(\tilde{t}, x, \xi)\}_{\tilde{t} \in Z}$ is a bounded set in $F$ for any $\tilde{p}$ with $|\tilde{p}| \leq l$. We set $M(Z; F) = \cap_{l=0}^M M'(Z; F)$ and use the expression "$\{p_\tilde{p}(\tilde{t}, x, \xi)\}_{\tilde{p} \in \mathbb{R}^n}$ is bounded in $M'(Z; F)$ [resp. in $M(Z; F)$]" if the set $\{\partial/\partial \tilde{t} \tilde{p}(\tilde{t}, x, \xi)\}_{\tilde{t} \in Z, \tilde{p} \in \mathbb{R}^n}$ is a bounded set in $F$ for any $\tilde{p}$ satisfying $|\tilde{p}| \leq l$ [resp. $|\tilde{p}| < \infty$]. For an integer $k$ and $\rho \in [1/2, 1]$ we also set
We consider the Hamilton equation corresponding to (3.1):

\[
\begin{align*}
\frac{dq}{dt} &= -\nabla_x \lambda(t, q, p), \\
\frac{dp}{dt} &= \nabla_x \lambda(t, q, p), \\
\{q, p\}_{t=t^*} &= \{y, \eta\}.
\end{align*}
\]

Then, we have

**Lemma 3.2.** i) Let \( \lambda(t, x, \xi) \) belong to \( M^p([0, T]; S_p^0((k + 2))) \) \((k \geq 0)\).
Then, the solution \( \{q, p\}(t; s, y, \eta) \) of (3.6) satisfies for a small \( T_1(\leq T) \)

\[
\begin{align*}
\{\frac{q-y}{|t-s|}\} & \text{ is bounded in } S_p^0((k+1)), \\
\{\frac{p-\eta}{|t-s|}\} & \text{ is bounded in } S_p^0((k+1)),
\end{align*}
\]

(0 \leq s, t \leq T_1, s \neq t),

and

\[
\begin{align*}
q-y \in M^p(Z(T_1); S_p^0((k+1))(k+1))) \cap M^p(Z(T_1); S_p^0((k)) \\
p \in M^p(Z(T_1); S_p^0((k+1))) \cap M^p(Z(T_1); S_p^0((k))),
\end{align*}
\]

where \( Z(T) = \{(t, s); 0 \leq t, s \leq T\} \).

ii) We assume, furthermore, that \( \lambda(t, x, \xi) \) belongs to \( M([0, T]; S_p^0((k+2))) \).
Then, \( q(t; s, y, \eta) \) \(- y \) belongs to \( \bar{M}(Z(T_1); S_p^0; k+1) \) and \( p(t; s, y, \eta) \) belongs to \( \bar{M}(Z(T_1); S_p^0; k+1) \).

Proof. By the similar way as in the proof of Lemma 3.1 in [7] we can prove (3.7) and

\[
\begin{align*}
\{\frac{q-y}{|t-s|}\} & \in M^p(Z(T_1); S_p^0((k+1))), \\
p & \in M^p(Z(T_1); S_p^0((k+1)))
\end{align*}
\]

for a small \( T_1(\leq T) \). Consider the equation (3.6) and

\[
\begin{align*}
\left[ \frac{\partial q(t, s; y, \eta)}{\partial y} \right] & = \left[ \frac{\partial q(t, s; y, \eta)}{\partial \eta} \right], \\
\left[ \frac{\partial p(t, s; y, \eta)}{\partial y} \right] & = -\nabla_x \lambda(s, y, \eta)
\end{align*}
\]

Then, from (3.8)' we get (3.8). For the proof of ii) we differentiate the equations in (3.6) and (3.9) with respect to \( t \) and \( s \). Then, using (3.8) we get ii) inductively.

Q.E.D.

Let \( \xi_1 \) be \( 0 < \xi_1 \leq 1 \). Then, from (3.7) we can find a constant \( T_2(\leq T_1) \) such that
holds, where $E$ is the identity matrix and $||W||$ is a matrix norm $\sum_{j,k} |w_{jk}|$ of a matrix $W= (w_{jk})$. We fix such a $T_2$. Then, we have

**Lemma 3.3.** Let $\lambda(t, x, \xi) \in \mathcal{M}^p([0, T]; \mathcal{S}_1^p((k+2)))$. Then, for the above $q(t, s; y, \eta)$ the equation $x = q(t, s; y, \xi)$ has the unique solution $y = Y(t, s; x, \xi, \xi)$ satisfying

$$
(3.11) \quad \{Y(t, s; x, \xi) - x\} \in M((Z(T_2)); S_0^p((k+1))) \cap M((Z(T_2)); S_0^p((k)))
$$

$$
\{Y(x) - x\} \text{ is bounded in } S_0^p((k+1)) \quad (0 \leq s, t \leq T_2, s \neq t).
$$

Furthermore, if we assume $\lambda(t, x, \xi) \in \mathcal{M}([0, T]; S_1^p((k+2)))$, $Y(t, s; x, \xi) - x$ belongs to $\mathcal{M}(Z(T_2); S_1^p; k+1)$.

We can prove this lemma by the similar way as the one in Lemma 3.2 of [7].

**Proposition 3.4.** Let $\lambda(t, x, \xi) \in \mathcal{M}^p([0, T]; \mathcal{S}_1^p((k+2)))$ and let $(q, p)(t, s; y, \eta)$ and $Y(t, s; x, \xi)$ be the symbols constructed in Lemma 3.2 and Lemma 3.3. We put

$$
(3.12) \quad u(t, s; y, \eta) = y \cdot \eta + \int_s^t \{\lambda - \xi \cdot \nabla \xi \lambda\} (\sigma, q(\sigma, s; y, \eta), p(\sigma, s; y, \eta)) d\sigma
$$

and define

$$
(3.13) \quad \phi(t, s; x, \xi) = u(t, s; Y(t, s; x, \xi, \xi)).
$$

Then, $\phi(t, s; x, \xi)$ is a solution of (3.1) and satisfies

$$
(3.14) \quad \nabla \phi(t, s; x, \xi) = Y(t, s; x, \xi),
$$

$$
(3.15) \quad \nabla_x \phi(t, s; x, \xi) = p(t, s; Y(t, s; x, \xi, \xi)),
$$

$$
(3.16) \quad \partial_s \phi(t, s; x, \xi) = -\lambda(s, \nabla_x \phi(t, s; x, \xi, \xi)).
$$

For any $l(\geq 0)$ there exists a constant $\varepsilon_{0, l}$ such that, if $\varepsilon_{0, l}T_2 < 1$, $\phi(t, s; x, \xi)$ belongs to $\mathcal{P}_l(\mathbb{C}_{\varepsilon_{0, l}}(1-s, l)$ and $\{J(t, s) / |t-s|\}$ is bounded in $S_1^p((k+2))$ for $0 \leq t, s \leq T_2$, $s \neq t$, where $J(t, s; x, \xi) = \phi(t, s; x, \xi) - x \cdot \xi$. We assume, furthermore, that $\lambda(t, x, \xi)$ belongs to $\mathcal{M}([0, T]; S_1^p((k+2)))$. Then, $J(t, s; x, \xi)$ belongs to $\mathcal{M}(Z(T_2); S_1^p; k+2)$.

If we follow the proofs of Theorem 3.1 in [7] and Proposition 2.2 in [11], we obtain the above proposition.

Take $\lambda_j(t, x, \xi), j = 1, 2, \ldots, \nu + 1, \ldots$, as $\lambda(t, x, \xi)$ of (3.5) and let $\phi_j(t, s) = \phi_j(t, s; x, \xi)$ be the solution of (3.1) corresponding to $\lambda_j$. Assume that $\{\lambda_j(t, x, \xi)\}_{j=1}^{\nu+1}$ is bounded in $\mathcal{M}^p([0, T]; S_1^p((2)))$. Then, by Proposition 3.4
there exists a constant $\bar{c}$ independent of $j$ such that

\[(3.17) \quad \phi_j(t, s; x, \xi) \equiv \mathcal{P}(\bar{c}|t-s|).\]

Take a constant $T_0$ satisfying $T_0 \leq \tau_0/\bar{c}$ for the constant $\tau_0$ in Definition 2.7. Then, the multi-product

\[(3.18) \quad \Phi_{\nu+1}(t_0, T^{\nu+1}; x, \xi) = (\phi_1(t_0, t_1) \circ \phi_2(t_1, t_2) \circ \cdots \circ \phi_{\nu+1}(t_\nu, t_{\nu+1}))(x, \xi)
\]

\[=(T^{\nu+1} = (t_1, t_2, \ldots, t_{\nu+1}))\]

is well-defined for $(t_0, T^{\nu+1}) \in \Delta_{\nu+1}(T_0) \equiv \{(t_0, T^{\nu+1}); 0 \leq t_{\nu+1} \leq t_\nu \leq \cdots \leq t_1 \leq t_0 \leq T_0\}$. In the following, we denote (3.18) simply by $\Phi_{\nu+1}(t_0, T^{\nu+1})$ or $\Phi_{\nu+1}$ unless otherwise specified. Corresponding to (3.18) we denote by $\{X_{\nu}^{(\nu)}; (t_\nu, T^{\nu+1}; x, \xi)\}$ for $(t_0, T^{\nu+1}) \in \Delta_{\nu+1}(T_0)$ the solution of

\[(3.19) \quad \begin{cases} x^j = \nabla x \phi_j(t_{j-1}, t_j; x^{j-1}, \xi^j), \\ \xi^j = \nabla_x \phi_{j+1}(t_j, t_{j+1}, x^j, \xi^{j+1}) \end{cases} \quad j = 1, \ldots, \nu \quad (x^0 = x, \xi^{\nu+1} = \xi)\]

and we also write $X_{\nu}^{(\nu)}(t_\nu, T^{\nu+1}; x, \xi)$ and $\Xi_{\nu}(t_\nu, T^{\nu+1}; x, \xi)$ simply by $X_{\nu}$ and $\Xi_{\nu}$.

Concerning the multi-products (3.18) the following is obtained by Kumanogo–Taniguchi–Tozaki [11].

**Proposition 3.5.**

i) $\Phi_{\nu+1} = \Phi_{\nu+1}(t_0, T^{\nu+1}; x, \xi)$ satisfies

\[
\begin{aligned}
\partial_{t_0} \Phi_{\nu+1} &= \lambda_{\nu}(t_0, x, \nabla_x \Phi_{\nu+1}), \\
\partial_{t_j} \Phi_{\nu+1} &= \lambda_{j+1}(t_j, X_{\nu}^{(\nu)}, \Xi_{\nu}^{(\nu)}) - \lambda_j(t_j, X_{\nu}^{(\nu)}, \Xi_{\nu}^{(\nu)}), \quad j = 1, \ldots, \nu, \\
\partial_{x_{\nu+1}} \Phi_{\nu+1} &= - \lambda_{\nu+1}(t_{\nu+1}, \nabla_x \Phi_{\nu+1}, \xi_{\nu+1}).
\end{aligned}
\]

ii) The following holds.

\[(3.21) \quad \begin{aligned}
\phi_1(t, s) \circ \phi_2(s, t) &= \phi_1(t, s), \\
\phi_1(t, t) \circ \phi_2(t, s) &= \phi_2(t, s).
\end{aligned}\]

**Proposition 3.6.** Assume that the set $\{\lambda_j\}_{j=1}^{\nu}$ is bounded in $M([0, T]; S_{\nu}^{(k-2)})$. Then, we have the following:

i) For the solution $\{X_{\nu}^{(\nu)}; (t_\nu, T^{\nu+1}; x, \xi)\}$ of (3.19) we have

\[(3.22) \quad \begin{aligned}
\{\partial_{(t_0, T^{\nu+1})} X_{\nu}^{(\nu)}\}_{j, \nu} \text{ is bounded in } S_{\nu}^{(k+1-l)} &\quad \text{for } |\gamma^{\nu+1}| = l \leq k+1, \\
\{\partial_{(t_0, T^{\nu+1})} X_{\nu}^{(\nu)}\}_{j, \nu} \text{ is bounded in } S_{\nu}^{(l-k-1)} &\quad \text{for } |\gamma^{\nu+1}| = l \geq k+2, \\
\{\partial_{(t_0, T^{\nu+1})} \Xi_{\nu}^{(\nu)}\}_{j, \nu} \text{ is bounded in } S_{\nu}^{(k+1-l)} &\quad \text{for } |\gamma^{\nu+1}| = l \leq k+1, \\
\{\partial_{(t_0, T^{\nu+1})} \Xi_{\nu}^{(\nu)}\}_{j, \nu} \text{ is bounded in } S_{\nu}^{(l-k-1)} &\quad \text{for } |\gamma^{\nu+1}| = l \geq k+2,
\end{aligned}\]

where $\partial_{(t_0, T^{\nu+1})} = \partial_{t_0}^{\nu+1} \cdots \partial_{t_\nu} \partial_{t_{\nu+1}}^{\nu+1}$ and $|\gamma^{\nu+1}| = \gamma_0 + \cdots + \gamma_{\nu+1}$ for $(\nu + 2)$-tuple.
\( \tilde{\gamma}^{v+1} = (\gamma_0, \gamma_1, \ldots, \gamma_{v+1}) \).

ii)

Set

\[
J_{v+1}(t_0, \bar{t}^{v+1}; x, \xi) = \Phi_{v+1}(t_0, \bar{t}^{v+1}; x, \xi) - x \cdot \xi.
\]

Then, we have

\[
(3.23) \quad \begin{cases}
\{q(\bar{t}_0, \tilde{t}_0^{v+1}) J_{v+1}\} \text{ is bounded in } S_{\bar{t}_0}^l((k+2-l)) \quad \text{for } |\tilde{\gamma}^{v+1}| = l \leq k+2, \\
\{p(\bar{t}_0, \tilde{t}_0^{v+1}) J_{v+1}\} \text{ is bounded in } S_{\bar{t}_0}^{i+(1-\rho)(l-k+2)} \quad \text{for } |\tilde{\gamma}^{v+1}| = l \geq k+3.
\end{cases}
\]

Proof. Since \( \{\lambda_i\} \) is bounded in \( M([0, T]; S_{\bar{t}_0}^1((k+2))) \), the following holds by virtue of Proposition 3.4:

\[
\begin{align*}
\{J_j(t, s; x, \xi)/|t-s|\}_{0 \leq s < t \leq T_2} \text{ is bounded in } S_{\bar{t}_0}^l(k+2), \\
\{\partial_t^i \partial_s^j J_j(t, s; x, \xi)\}_{0 \leq s < t \leq T_2} \text{ is bounded in } S_{\bar{t}_0}^{i+(1-\rho)(l-k+2)},
\end{align*}
\]

where \( J_j(t, s; x, \xi) = \Phi_j(t, s; x, \xi) - x \cdot \xi \). Hence, we obtain (3.22) and (3.23) with \( \tilde{\gamma}^{v+1} = 0 \) by Proposition 2.6. Concerning the derivatives of \( \bar{X}^i_j \) and \( \bar{\Xi}^i_j \) with respect to \( (t_0, \bar{t}^{v+1}) \) we follow the proof of Theorem 1.7' of [11]. Then, we get (3.22) for any \( \tilde{\gamma}^{v+1} \). Using this and (3.20) we get (3.23) from the boundedness of \( \{\lambda_i\} \).

Q.E.D.

For the above \( \lambda_1, \lambda_2, \ldots \), we consider the solution \( \{q^j, p^j\}(t, s; y, \eta) \) of the Hamilton equation (3.6) corresponding to \( \lambda_j \), and define for the point \( (y, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \) and \( (t_0, \bar{t}^{v+1}) \in \Delta_{v+1}(T_2) \) the trajectory \( \{q_1, \ldots, \eta, \bar{P}_{1, \ldots, -j}(t_0, t_1, \ldots, t_{j-1}, \sigma; y, \eta)(t_j \leq \sigma \leq t_{j-1}) \) by

\[
(3.24) \quad \begin{cases}
\{q^i, p^i\}(t_0, \sigma; y, \eta) = \{q^j, p^j\}(\sigma, t_0; y, \eta) & (t_1 \leq \sigma \leq t_0) \\
\{q_1, \ldots, \bar{P}_{1, \ldots, -j}(t_0, t_1, \ldots, t_{j-1}, \sigma; y, \eta) & (t_j \leq \sigma \leq t_{j-1}), j \geq 2.
\end{cases}
\]

Proposition 3.7. Let \( \{X^1_j, \bar{\Xi}^1_j\}_{j=1}^v(t_0, \bar{t}^{v+1}; x, \xi) \) be the solution of (3.19). Then, we have

\[
(3.25) \quad \begin{cases}
q^i(t_0, t_0; x, \nabla_x \Phi_{v+1}(t_0, \bar{t}^{v+1}; x, \xi)) = X^1_i(t_0, \bar{t}^{v+1}; x, \xi), \\
p^i(t_0, t_0; x, \nabla_x \Phi_{v+1}(t_0, \bar{t}^{v+1}; x, \xi)) = \bar{\Xi}^1_i(t_0, \bar{t}^{v+1}; x, \xi).
\end{cases}
\]

\[
(3.26) \quad \begin{cases}
q^i(t_j, t_{j-1}; X^i_{j-1}, \bar{\Xi}^i_{j-1}) = X^i_j, \\
p^i(t_j, t_{j-1}; X^i_{j-1}, \bar{\Xi}^i_{j-1}) = \bar{\Xi}^i_j & (2 \leq j \leq v)
\end{cases}
\]

and for any \( j \leq v \)
Proof. From Lemma 3.3, (3.14) and (3.15) we get for any $j$

$$q(t, s; \xi \phi_j(t, s; x, \xi), \xi) = \nabla_x \phi_j(t, s; x, \xi).$$

Hence, by the uniqueness of the initial value problem (3.6) for $\lambda = \lambda_j$ we get

$$q(t, s; x, \nabla_x \phi_j(t, s; x, \xi)) = \nabla_x \phi_j(t, s; x, \xi).$$

From (1.25) of [11] we have $\nabla_x \Phi_{\nu+1} = \nabla_x \phi_j(t, t_1; x, \Xi_i^j)$. Using this with (3.28) and (3.19) we obtain

$$q(t, s; x, \nabla_x \phi_j(t, s; x, \xi)) = \xi.$$

Hence, we get (3.27). Next, we use

$$\nabla_x \phi_j(t_j, t_j; X_v^{j-1}, \Xi_v^j) = \Xi_v^{j-1}$$

in (3.19). Then, we get from (3.28), and (3.19)

$$q(t, t_{j-1}; X_v^{j-1}, \Xi_v^{j-1}) = q(t, t_{j-1}; X_v^{j-1}, \nabla_x \phi_j(t_{j-1}, t_j; X_v^{j-1}, \Xi_v^j)) = \nabla_x \phi_j(t_{j-1}, t_j; X_v^{j-1}, \Xi_v^j) = X_v^j,$$

$$p(t, t_{j-1}; X_v^{j-1}, \Xi_v^{j-1}) = p(t, t_{j-1}; X_v^{j-1}, \nabla_x \phi_j(t_{j-1}, t_j; X_v^{j-1}, \Xi_v^j)) = \Xi_v^j.$$

Hence, we get (3.26).

We prove (3.27) by the induction. Since (3.27) is (3.25), we suppose (3.27) and prove (3.27) for $j+1$. From (3.27), and (3.26) we have

$$\{q_{l_1, \ldots, l_j, \nu}, p_{l_1, \ldots, l_j, \nu}(t_0, t_1, \ldots, t_j, x, \nabla_x \Phi_{\nu+1}) = \{q_{l_1, \ldots, l_j, \nu}^{j+1}(t_{j+1}, t_j; \{q_{l_1, \ldots, l_j, \nu}(t_0, t_1, \ldots, t_j, x, \nabla_x \Phi_{\nu+1})) = \{q_{l_1, \ldots, l_j, \nu}^{j+1}(t_{j+1}, t_j; X_v^{j+1}, \Xi_v^{j+1}) = \{X_v^{j+1}, \Xi_v^{j+1}\}. $$

Hence, we obtain (3.27). Q.E.D.

From Proposition 3.6 and Proposition 3.7 we get the following proposition.

**Proposition 3.8.** Assume that the set $\{\lambda_j\}$ is bounded in $M([0, T];$
Then, we have for the trajectory \( \{ \tilde{g}_1, \ldots, \tilde{g}_{n+1} \} \) defined by (3.24)

\[
\begin{align*}
    \{ \partial_{(t_0, i^{j-1}, x)} \tilde{g}_1, \ldots, \tilde{g}_{n+1} \} \partial_{(t_0, i^{j-1}, x)} & \quad \text{is bounded in } S^0_p((k+1-l)) \text{ for } |\tilde{\tau}| = l \leq k+1, \\
    \{ \partial_{(t_0, i^{j-1}, x)} \tilde{g}_1, \ldots, \tilde{g}_{n+1} \} \partial_{(t_0, i^{j-1}, x)} & \quad \text{is bounded in } S^0_p((k+1-l)) \text{ for } |\tilde{\tau}| = l \geq k+2, \\
    \{ \partial_{(t_0, i^{j-1}, x)} \tilde{g}_1, \ldots, \tilde{g}_{n+1} \} \partial_{(t_0, i^{j-1}, x)} & \quad \text{is bounded in } S^0_p((k+1-l)) \text{ for } |\tilde{\tau}| = l \leq k+1, \\
    \{ \partial_{(t_0, i^{j-1}, x)} \tilde{g}_1, \ldots, \tilde{g}_{n+1} \} \partial_{(t_0, i^{j-1}, x)} & \quad \text{is bounded in } S^0_p((k+1-l)) \text{ for } |\tilde{\tau}| = l \geq k+2.
\end{align*}
\]

Now, we turn to study the commutative law for \#-products of phase functions. Let \( \{ \lambda_j \}_{j=1}^n \) be a bounded set of real symbols \( \lambda_j(t, x, \xi) \) in \( M([0, T]; S^0_p((3))) \) and let \( \phi_j(t, s; x, \xi) \in \mathcal{D}_p(\partial | t-s |) \) be the phase function corresponding to \( \lambda_j(t, x, \xi) \). For the multi-product (3.18) we commute \( \phi_j \) and \( \phi_{j+1} \) and denote

\[
\Phi_{j+1}; x, \xi = (\phi_j(t_0, t_1) \# \cdots \# \phi_{j+1}(t_{j-1}, t_{j-1})) \quad \text{for } (t_0, t^{j+1}) \in \Delta_{j+1}(T_0).
\]

We put an assumption: There exist real symbols \( a_j(t, x, \xi) \) in \( M([0, T]; S^0_p((1))) \) and \( a'_j(t, x, \xi) \) in \( M([0, T]; S^0_p) \) such that

\[
\lambda_j(t_0, t^{j+1}) = a_j(t, x, \xi) \lambda_j(t_0, t^{j+1}) + a'_j(t, x, \xi).
\]

Then, we have

**Theorem 3.9**. Let \( \{ \lambda_j(t, x, \xi) \}_{j=1}^n \) be a bounded set in \( M([0, T]; S^0_p((3))) \) and let \( \phi_j(t, s; x, \xi) \in \mathcal{D}_p(\partial | t-s |) \) (with some \( \partial \)) be the phase function corresponding to \( \lambda_j \). We assume that (3.31) holds and that the sets \( \{ \partial_j \} \) and \( \{ \partial'_j \} \) are bounded in \( M([0, T]; S^0_p((1))) \) and \( M([0, T]; S^0_p) \), respectively. For any \( \nu, j (\leq \nu) \) and \( (t_0, t^{j+1}) \in \Delta_{j+1}(T_0) \) (for some \( T_0 \)) we consider the multi-products \( \Phi_{j+1}(t_0, t^{j+1}) \) and \( \Phi_{j+1}(t_0, t^{j+1}) \) of (3.18) and (3.30). Then, there exists a constant \( T_0 \) independent of \( \nu \) such that the following hold:

1) We can find for any \( \nu \) and \( j (\leq \nu) \) a symbol \( \Omega_{\nu, j}(t_0, t^{j+1}; x, \xi) \) in \( \mathcal{M}(\Delta_{j+1}(T_0); S^0_p; 1) \) such that it satisfies

2) The idea of the proof is found in Section 1 of [10], where the theorem is proved for the case of \( a_{m,k} = a'_{m,k} = 0 \). In [13] Morimoto proved this theorem in the case of \( a_{m,k}(t, x, \xi) \equiv a_{m,k}(t) \) and \( a'_{m,k}(t, x, \xi) \equiv 0 \).
\begin{align}
(3.32) \quad t_{j+1} \leq \Omega_{v,j}(t_0, \tilde{t}^{v+1}; x, \xi) \leq t_{j-1}, \\
(3.33) \quad \Omega_{v,1} = t_{j+1}, \quad \Omega_{v,j} = t_{j-1}, \quad \text{and} \\
(3.34) \quad \Phi_{v+1,j}(t_0, t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_{v+1}; x, \xi) \\
\quad = \Phi_{v+1}(t_0, t_1, \ldots, t_{j-1}, \Omega_{v,j}(t_0, \tilde{t}^{v+1}; x, \xi), t_{j+1}, \ldots, t_{v+1}; x, \xi) \\
\quad + \psi_{v,j}(t_0, \tilde{t}^{v+1}; x, \xi)
\end{align}

with some \(\psi_{v,j}(t_0, \tilde{t}^{v+1}, x, \xi)\) satisfying

\begin{align}
(3.35) \quad \psi_{v,j}(t_0, \tilde{t}^{v+1}; x, \xi) \in M(\Delta_{v+1}(T_0); S_p^0; 0) \\
\text{and} \\
(3.36) \quad \psi_{v,j} \equiv 0 \quad \text{if} \quad \alpha_j = 0.
\end{align}

II) It holds that

\begin{equation}
(3.37) \begin{cases}
\{\Omega_{v,j}(t_0, \tilde{t}^{v+1})\}_{v,j} \text{ is bounded in } S_p^0((1)), \\
\{\partial_{(t_0, \tilde{t}^{v+1})} \Omega_{v,j}\}_{v,j} \text{ is bounded in } S_p^{(1-\rho)(l-1)} \\
\quad \text{for } |\tilde{\gamma}^{v+1}| = l \geq 1, \\
\{\partial_{(t_0, \tilde{t}^{v+1})} \psi_{v,j}\}_{v,j} \text{ is bounded in } S_p^{(1-\rho)l} \\
\quad \text{for } |\tilde{\gamma}^{v+1}| = l.
\end{cases}
\end{equation}

We can find a constant \(A_0\) independent of \(v\) such that we have

\begin{equation}
(3.38) \quad |\partial_{t_j} \Omega_{v,j+1}| \leq A_0(t_0 - t_{v+1}).
\end{equation}

\textbf{Remark.} In [10] and [13] the commutative law for multi-\#-products follows from the commutative law for \#-products between two phase functions, since \(\{\Omega_{v,j}\}\) are determined only by \((t, \tilde{t}^{v+1})\). In our case we emphasize that we cannot apply the above method because \(\{\Omega_{v,j}\}\) depend also on \(x\) and \(\xi\).

We begin the proof with finding \(\Omega_{v,j}(t_0, \tilde{t}^{v+1}; x, \xi)\) satisfying (3.32)-(3.34). To simplify the notation below, we use \((t, \theta, s)\) or \((t, \omega, s)\) instead of \((t_{j-1}, t_j, t_{j+1})\) and write

\begin{align}
\Phi_{v+1}(t, \omega, s) &\equiv \Phi_{v+1}(t, \omega, s; \tilde{t}_j^{v+1}, x, \xi) \\
&= \Phi_{v+1}(t_0, \ldots, t_{j-2}, t, \omega, s, t_{j+2}, \ldots, t_{v+1}; x, \xi), \\
\Phi_{v+1; j}(t, \theta, s) &\equiv \Phi_{v+1; j}(t, \theta, s; \tilde{t}_j^{v+1}, x, \xi) \\
&= \Phi_{v+1; j}(t_0, \ldots, t_{j-2}, t, \theta, s, t_{j+2}, \ldots, t_{v+1}; x, \xi),
\end{align}

where \(\tilde{t}_j^{v+1} = (t_0, \ldots, t_{j-2}, t_{j+2}, \ldots, t_{v+1})\) when \(v \geq 2\). Now, we set
\( \psi = \psi_{v,j}(t, \theta, s) = \Phi_{v+1}; j(t, \theta, s) - \Phi_{v+1}(t, \Omega, s) \)

and seek the symbol \( \Omega = \Omega(t, \theta, s; t_j^{v+1}, x, \xi) \) such that \( \psi \) belongs to \( \overline{M}(S_\rho^0; 0) \) and \( \Omega \) satisfies

\begin{align*}
(3.32)' & \quad s \leq \Omega(t, \theta, s) \leq t, \\
(3.33)' & \quad \Omega(t, t, s) = s, \quad \Omega(t, s, s) = t.
\end{align*}

Here, we suppress the domain of \( (t_0, \ldots, t_{j-2}, t, \theta, s, t_{j+2}, \ldots, t_{v+1}) \) and write \( \overline{M}(S_\rho^0; 0) \) instead of writing \( \overline{M}( \{ 0 \leq t_{v+1} \leq \cdots \leq t_{j+2} \leq s \leq \theta \leq t_{j-2} \leq \cdots \leq t_0 \leq T \} ; S_\rho^0, 0) \). In the following we also suppress domains of \( t_0, \ldots, t_{j-2}, t, \theta, \omega, s, t_{j+2}, \ldots, t_{v+1} \) and use the notation \( \overline{M}(S_\rho^n; k) \) if no confusion occurs.

Let \( \{ X^k_v, \Xi^k_v \}_{k=1}^v = \{ X^k_v, \Xi^k_v \}_{k=1}^v(t, \theta, s; t_j^{v+1}, x, \xi) \) be the solution of

\[
\begin{cases}
    x^k = \nabla_x \phi_k(t_{k-1}, t_k; x^{k-1}, \xi^k), \\
    \xi^k = \nabla_x \phi_{k+1}(t_k, t_{k+1}; x^k, \xi^{k+1}), & k = 1, \ldots, v,
\end{cases}
\]

and let \( \{ \bar{X}^k_v, \bar{\Xi}^k_v \}_{k=1}^v = \{ \bar{X}^k_v, \bar{\Xi}^k_v \}_{k=1}^v(t, \theta, s; t_j^{v+1}, x, \xi) \) be the solution of

\[
\begin{cases}
    x^k = \nabla_x \phi_k(t_{k-1}, t_k; x^{k-1}, \xi^k), & 1 \leq k \leq v, k \neq j, j+1, \\
    x^j = \nabla_x \phi_{j+1}(t_j, t_{j+1}; x^{j-1}, \xi^j), \\
    \xi^j = \nabla_x \phi_j(t_j, t_{j+1}; x^j, \xi^{j+1}), \\
    \xi^{j-1} = \nabla_x \phi_{j+1}(t_j, t_{j+1}; x^{j-1}, \xi^j), \\
    \xi^j = \nabla_x \phi_j(t_j, t_{j+1}; x^j, \xi^{j+1}),
\end{cases}
\]

For convenience, we set

\[
\begin{align*}
\lambda_0(t, x, \xi) &= 0, \\
X_v^0 &= \bar{X}_v^0 = x, \quad \Xi_v^0 = \nabla_x \Phi_{v+1}(t, \omega, s), \quad \bar{\Xi}_v^0 = \nabla_x \Phi_{v+1}(t, \theta, s).
\end{align*}
\]

Then, we have from (3.20)

\[
\partial_t \psi = (\partial_{t_{j-1}} \Phi_{v+1}; j)(t, \theta, s) - (\partial_{t_{j-1}} \Phi_{v+1})(t, \Omega, s) - \partial_{t} \Phi_{v+1}(t, \Omega, s) \partial_{t} \Omega
\]

\[
= \lambda_{j-1}(t, X_v^{j-1}, \Xi_v^{j-1}) - \lambda_{j-1}(t, X_v^{j-1}, \Xi_v^{j-1})
\]

\[
- \{ \lambda_j(t, X_v^{j-1}, \Xi_v^{j-1}) - \lambda_{j-1}(t, X_v^{j-1}, \Xi_v^{j-1}) \}_{j=1}^\infty
\]

\[
- \partial_{t0} \Phi_{v+1}(t, \Omega, s) \partial_{t} \Omega.
\]

When \( j \geq 2 \), we use the trajectory \( \{ \bar{q}_1, \ldots, j-1, \bar{h}_1, \ldots, j-1 \}(t_0, t_1, \ldots, t_{j-2}, t, y, \eta) \) defined by (3.24). Then, we have from Proposition 3.7

\[
\begin{align*}
\{ \bar{q}_1, \ldots, j-1, \bar{h}_1, \ldots, j-1 \}(t_0, \bar{t}_{j-2}^1, t, x, \nabla_x \Phi_{v+1}) &= \{ X_v^{j-1}, \Xi_v^{j-1} \}, \\
\{ \bar{q}_1, \ldots, j-1, \bar{h}_1, \ldots, j-1 \}(t_0, \bar{t}_{j-2}^1, t, x, \nabla_x \Phi_{v+1}; j) &= \{ X_v^{j-1}, \bar{\Xi}_v^{j-1} \}.
\end{align*}
\]
Hence, if we set
\[
\begin{align*}
\tilde{\lambda}_j(t; z, \xi) &= \lambda_j(t, z, \xi), \\
\tilde{\lambda}_j(t; t_0, \tilde{t}^j-2, z, \xi) &= \{\lambda_{j+1} - \lambda_{j-1}\} (t, \{\tilde{t}_1, \ldots, \tilde{t}_{j-1}, \tilde{t}_{j+1}\}(t_0, \tilde{t}^j-2, z, \xi)) \quad (j \geq 2)
\end{align*}
\]
\(\tilde{\lambda}_j\) belongs to \(\tilde{M}(S^1_r; 2)\) from Proposition 3.8 and satisfies
\[
(3.44) \quad \int_{t_0}^{t} \Phi(t; \cdot; \tilde{t}^j-2, t, z, \xi) = \{\lambda_{j+1} - \lambda_{j-1}\} (t, \{\tilde{t}_1, \ldots, \tilde{t}_{j-1}, \tilde{t}_{j+1}\}(t_0, \tilde{t}^j-2, z, \xi)).
\]
Define the symbol \(\Lambda_j(\omega) = \Lambda_j(\omega; \tau, \theta, s; \tilde{t}_j^{0, \nu+1}, x, \xi)\) in \(\tilde{M}(S^1_r; 1)\) by
\[
(3.45) \quad \Lambda_j(\omega) = \int_{0}^{1} \Phi_j(t; t_0, \tilde{t}^j-2, x, \sigma \Phi_{\nu+1}; j(t, \theta, s; \tilde{t}_j^{0, \nu+1}, x, \xi)) d\sigma \quad + (1-\sigma) \Phi_{\nu+1}(t, \omega, \tilde{t}_j^{0, \nu+1}, x, \xi) d\sigma.
\]
Then, we can write
\[
(3.46) \quad \Lambda_j(t; t_0, \tilde{t}^j-2, x, \sigma \Phi_{\nu+1}; j(t, \theta, s; \tilde{t}_j^{0, \nu+1}, x, \xi)) = \Lambda_j(\omega; \sigma \Phi_{\nu+1}; j(t, \theta, s; \tilde{t}_j^{0, \nu+1}, x, \xi)) d\sigma.
\]
From (3.39) we have
\[
(3.47) \quad \Phi_{\nu+1} = \Phi_{\nu+1; j(t, \theta, s; \tilde{t}_j^{0, \nu+1}, x, \xi)) d\sigma.
\]
Hence, from (3.42), (3.44), (3.46) and (3.47) we have
\[
(3.48) \quad \partial_{\nu} \Phi = \Lambda_j(t; t_0, \tilde{t}^j-2, x, \sigma \Phi_{\nu+1}; j(t, \theta, s)) - \Lambda_j(t; t_0, \tilde{t}^j-2, x, \sigma \Phi_{\nu+1}; j(t, \theta, s)) d\sigma
\]
\[
= \Lambda_j(\omega; \sigma \Phi_{\nu+1}; j(t, \theta, s; \tilde{t}_j^{0, \nu+1}, x, \xi)) d\sigma
\]
\[
- \{\lambda_j - \lambda_{j+1}\} (t, \tilde{t}_j^{0, \nu+1}, \tilde{t}_j^{0, \nu+1}, \omega) \quad (j \geq 1).
\]
Let \(\{q^i, p^i\} (t, s; y, \eta)\) be the solution of (3.6) with \(\lambda\) replaced by \(\lambda_j\) and set \(\lambda^0(\sigma, t; y, \eta) = \lambda_j(\sigma, t; y, \eta)\) \(\{\lambda_j - \lambda_{j+1}\} (\sigma, \{q^i, p^i\} (\sigma, t; y, \eta)).\) Then, as the proof of Corollary of Theorem 2.3 in [11] we get from (3.31)
\[
(3.49) \quad \frac{d\lambda^0}{d\sigma} = \{\tau - \lambda_j, \tau - \lambda_{j+1}\} (\sigma, \{q^i, p^i\} (\sigma, t; y, \eta))
\]
\[
= a_j(\sigma, \{q^i, p^i\} (\sigma, t; y, \eta) + a_j(\sigma, \{q^i, p^i\} (\sigma, t; y, \eta))
\]
and the solution \(\lambda^0(\sigma, t; y, \eta)\) of (3.49) has the form
\[
(3.50) \quad \lambda^0(\sigma, t; y, \eta) = \lambda^0(\omega, t; y, \eta) \exp \int_{\sigma}^{\sigma'} a_j(\sigma', \{q^i, p^i\} (\sigma', t; y, \eta)) d\sigma'
\]
From (3.25)-(3.26) and (3.20) we get

\[
\begin{align*}
q'(t_j, t_{j-1}; X_{j-1}^{j-1}, \Xi_{j-1}^{j-1}) &= X_j^{j-1}, \\
p'(t_j, t_{j-1}; X_{j-1}^{j-1}, \Xi_{j-1}^{j-1}) &= \Xi_j^{j-1}
\end{align*}
\]

and

\[
(3.51) \quad \lambda^\circ(t_j, t_{j-1}; X_{j-1}^{j-1}, \Xi_{j-1}^{j-1}) = \{\lambda_j - \lambda_{j+1}\} (t_j, X_j^{j-1}, \Xi_j^{j-1}) = -\partial_\omega \Phi_{\nu+1}
\]

with \( t_j = \omega \) and \( t_{j-1} = t \). Set

\[
(3.52) \quad \alpha_j(\omega) \equiv \alpha_j(\omega; t, s, \bar{f}_j^{\nu+1}, x, \xi)
\]

\[
= \exp \int_{\omega}^{t} a_j(\sigma', \{q^i, p^j\} (\sigma', t; X_{\nu}^{\nu-1}, \Xi_{\nu}^{\nu-1}))d\sigma' \quad (\in \widetilde{M}(S_\nu^0; 1))
\]

and

\[
(3.53) \quad \alpha_j'(\omega) = \alpha_j'(\omega; t, s, \bar{f}_j^{\nu+1}, x, \xi)
\]

\[
= \int_{\omega}^{t} \left( \exp \int_{\omega}^{\sigma} a_j(\sigma'', \{q^i, p^j\} (\sigma'', t; X_{\nu}^{\nu-1}, \Xi_{\nu}^{\nu-1}))d\sigma'' \right)
\]

\[
\times a_j(\sigma', \{q^i, p^j\} (\sigma', t; X_{\nu}^{\nu-1}, \Xi_{\nu}^{\nu-1}))d\sigma' \quad (\in \widetilde{M}(S_\nu^0, 0)).
\]

Then, we obtain from (3.48), (3.51) and (3.50) with \( \sigma = t, y = X_j^{j-1} \) and \( \eta = \Xi_j^{j-1} \)

\[
(3.54) \quad \partial_\nu \psi = \Lambda_j(\Omega) \cdot \nabla_\nu \psi - \alpha_j'(\Omega)
\]

\[-\partial_\omega \Phi_{\nu+1}(t, \Omega, s)[\partial_\nu \Omega - \Lambda_j(\Omega) \cdot \nabla_\nu \Omega - \alpha_j(\Omega)].
\]

Consider the equation with respect to \( \Omega \):

\[
(3.55) \quad \partial_\nu \Omega - \Lambda_j(\Omega) \cdot \nabla_\nu \Omega - \alpha_j(\Omega) = 0
\]

with the initial condition

\[
(3.56) \quad \Omega|_{t=t_0} = s.
\]

Since (3.55) is a quasi-linear equation, we may solve the ordinary differential equation

\[
\begin{align*}
\frac{d\bar{\psi}}{dt} &= -\Lambda_j(\bar{\psi}; t, \theta, s, \bar{\bar{f}}_{\nu}^{\nu+1}, \bar{\psi}, \xi), \\
\frac{d\bar{\chi}}{dt} &= \alpha_j(\bar{\chi}; t, s, \bar{\bar{f}}_{\nu}^{\nu+1}, \bar{\psi}, \xi), \\
\bar{\psi}|_{t=t_0} &= y, \quad \bar{\chi}|_{t=t_0} = s.
\end{align*}
\]
Lemma 3.10. There exists a constant $T'_0$ such that the equation (3.57) has a solution $\{F, 2\} (t, \theta, s; \tilde{t}_j^{\nu+1}, x, \xi) \in \tilde{M}(S^\nu_\nu; 1)$ for $(t, \theta, s; \tilde{t}_j^{\nu+1})$ with $0 \leq s \leq \theta \leq t \leq T'_0$ (and $t_0 \leq T'_0$) and $\{F, 2\}$ satisfy

\begin{align}
\begin{cases}
    s \leq \tilde{z}(t, \theta, s; \tilde{t}_j^{\nu+1}, y, \xi) \leq t, \\
    z(t, s, s) = t
\end{cases}
\end{align}

and

\begin{equation}
\frac{\partial}{\partial y} F - E \leq A_1(t-s)
\end{equation}

with a constant $A_1$ independent of $\nu$.

Admitting this lemma for a moment, we continue the proof of the theorem. Take $T'_0(\leq T'_0')$ such that $T'_0A_1 < 1$. Then, from (3.59) the equation

\begin{equation}
\tilde{F}(t, \theta, s; \tilde{t}_j^{\nu+1}, \tilde{Y}, \xi) = \chi
\end{equation}

has a solution $\tilde{Y}(t, \theta, s; \tilde{t}_j^{\nu+1}, x, \xi)$ satisfying

\begin{equation}
\tilde{Y}(t, \theta, s; \tilde{t}_j^{\nu+1}, x, \xi) - x \in \tilde{M}(S^\nu_\nu; 1)
\end{equation}

when $0 \leq s \leq \theta \leq t \leq T'_0$. In the following the inequality $0 \leq s \leq \theta \leq t \leq T'_0$ always holds. Set

\begin{equation}
\Omega(t, \theta, s; \tilde{t}_j^{\nu+1}, x, \xi) = \tilde{z}(t, \theta, s; \tilde{t}_j^{\nu+1}, \tilde{Y}(t, \theta, s; \tilde{t}_j^{\nu+1}, x, \xi)), \xi).
\end{equation}

Then, $\Omega(t, \theta, s)$ is a solution of (3.55)-(3.56). From (3.56) and (3.58) $\Omega(t, \theta, s)$ satisfies (3.32)'-(3.33)'.

For the solution $\Omega(t, \theta, s)$ of (3.55)-(3.56) the equation (3.54) is reduced to the equation

\begin{equation}
\partial_\nu \psi = \Lambda_1(\Omega) \cdot \nabla \psi - \alpha'(\Omega).
\end{equation}

On the other hand, the equation

\begin{equation}
\psi_{t_0+\theta} = 0
\end{equation}

holds, since we have from (3.33)' and (3.21)

\begin{align}
\psi(\theta, \theta, s) &= \Phi_{\nu+1} ; f(\theta, \theta, s) - \Phi_{\nu+1}(\theta, s, s) \\
&= \Phi_{\nu+1} (t_0, \ldots, t_{j-2}, \theta, \theta, s, t_{j+2}, \ldots, t_{\nu+1}) \\
&\quad - \Phi_{\nu+1} (t_0, \ldots, t_{j-2}, \theta, s, s, t_{j+2}, \ldots, t_{\nu+1}) \\
&= \phi_1(t_0, t_1) \# \cdots \# \phi_{j-1}(t_{j-2}, \theta) \# \{\phi_{j+1}(\theta, \theta) \# \phi_j(\theta, s)\} \\
&\quad \# \phi_{j+2}(s, t_{j+2}) \# \cdots \# \phi_{\nu+1}(t_\nu, t_{\nu+1}) \\
&\quad - \phi_1(t_0, t_1) \# \cdots \# \phi_{j-1}(t_{j-2}, \theta) \# \{\phi_j(\theta, s) \# \phi_{j+1}(s, s)\} \\
&\quad \# \phi_{j+2}(s, t_{j+2}) \# \cdots \# \phi_{\nu+1}(t_\nu, t_{\nu+1}) \\
&= 0.
\end{align}
Hence, if we set

$$\beta_j(t, \theta, s; \bar{\Omega}_j^{0, v+1}, x, \xi) = \alpha_j'(\Omega(t, \theta, s; \bar{\Omega}_j^{0, v+1}, x, \xi); t, s, \bar{\Omega}_j^{0, v+1}, x, \xi),$$

the symbol $\psi$ can be written in the form

$$\psi = -\int_0^t \beta_j(\sigma, \theta, s; \bar{\Omega}_j^{0, v+1}, \bar{\Omega}(\sigma, \theta, s; \bar{\Omega}_j^{0, v+1}, x, \xi), \xi)d\sigma,$$

where $\bar{\Omega}$ and $\bar{\Omega}$ are defined by Lemma 3.10 and (3.60). Hence, $\psi$ belongs to $\mathcal{M}(S^1_{p, 0})$ and is identically zero when $a_j = 0$. Consequently, we have proved I) in the theorem. From the above discussions we also get (3.37).

For the proof of (3.38) we set $\Omega_0(t, \theta, s) = \partial_\theta \Omega(t, \theta, s; \bar{\Omega}_j^{0, v+1}, x, \xi)$. Then, from (3.55)

$$\partial_\theta \Omega_0(t, \theta, s) = \Lambda_j(\Omega) \cdot \nabla_x \Omega + \beta_j(t, \theta, s)$$

holds with

$$\beta_j(t, \theta, s) = \beta_j(t, \theta, s; \bar{\Omega}_j^{0, v+1}, x, \xi) = \partial_\theta (\Lambda_j(\Omega)) \cdot \nabla_x \Omega + \partial_\theta (\alpha_j(\Lambda_j(\Omega))).$$

On the other hand, writing

$$\Omega(t, \theta, s) = \Omega_0(t, \theta, s) + \int_0^t \{\Lambda_j(\Omega(\sigma, \theta, s); \sigma, \theta, s, \bar{\Omega}_j^{0, v+1}, x, \xi) \cdot \nabla_x \Omega(\sigma, \theta, s) + \alpha_j(\Omega(\sigma, \theta, s); \sigma, s, \bar{\Omega}_j^{0, v+1}, x, \xi)\}d\sigma,$$

we have

$$\Omega_0(\theta, \theta, s) = -\alpha_j(s; \theta, s, \bar{\Omega}_j^{0, v+1}, x, \xi) = -\exp \int_0^\theta a_j d\sigma,$$

since $\nabla_x \Omega(\theta, \theta, s) = 0$ from (3.67). Hence, as in (3.65) we can write

$$\Omega_0(t, \theta, s) = \Omega_0(\theta, \theta, s) + \int_0^t \beta_j(\sigma, \theta, s; \bar{\Omega}_j^{0, v+1}, \bar{\Omega}(\sigma, \theta, s; \bar{\Omega}_j^{0, v+1}, x, \xi), \xi)d\sigma,$$

and get (3.38) from (3.68)–(3.69). This completes the proof of Theorem 3.9.

**Remark.** If $a_j(t, x, \xi)$ is identically zero, the solution $\Omega$ of (3.55)–(3.56) is

$$\Omega = t - \theta + s.$$ 

This corresponds to the result in Theorem 1.10 of [10].

Proof of Lemma 3.10. We solve (3.57) by the Picard's method of successive approximation. For simplicity we suppress the dependence of $\bar{\Omega}_j^{0, v+1}$ and $j$ and write $\Lambda(\omega; t, \theta, s, x, \xi)$ and $\alpha(\omega; t, s, x, \xi)$ instead of writing $\Lambda_j(\omega; t, \theta, s, \bar{\Omega}_j^{0, v+1}, x, \xi)$ and $\alpha_j(\omega; t, s, \bar{\Omega}_j^{0, v+1}, x, \xi)$. Define $\{\bar{\Omega}^{(N)}, \bar{z}^{(N)}\} \equiv \{\bar{\Omega}^{(N)}, \bar{z}^{(N)}\} \equiv \{\bar{\Omega}^{(N)}, \bar{z}^{(N)}\} (t, \theta, s; x, \xi), N = 0, 1, 2, \ldots, \text{by}

$$\bar{\Omega}^{(0)}(t) = \gamma, \quad \bar{z}^{(0)}(t) = t - \theta + s,$$

(3.70)
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\[
\begin{aligned}
\frac{\tilde{\varphi}^N(t)}{\varphi^N(t)} = y - \int_0^t \Lambda(\varphi_\sigma(\sigma); \sigma, \theta, s) \varphi^N(\sigma, \xi) \, d\sigma,
\end{aligned}
\]

(3.71)

(3.72)

\[s \leq \varphi^N(t, \theta, s; x, \xi) \leq t.\]

In order for \(\{\varphi^{N+1}, \varphi^{N+1}\}(t)\) to be well-defined, we must prove

(3.72) \(s \leq \varphi^N(t, \theta, s; x, \xi) \leq t.\)

But, (3.72) is derived from \(\varphi^N(t, t, s) = s,\)

(3.73)

\(-2 \leq \partial_\sigma \varphi^N \leq 0,\)

and

(3.74)

\(\varphi^N(t, s, s) = t.\)

Hence, we shall prove (3.73), (3.74) and

(3.75) \(|\partial_\sigma \varphi^N| \leq A_\delta \) with some \(A_\delta > 0 \) (independent of \(\nu\))

by the induction. From (3.52) \(\alpha(t; t, s, x, \xi) = 1\) holds. Hence, using (3.74) we get (3.74) in fact, we have

\[\varphi^N(t, s, s) = s + \int_0^t \alpha(\varphi^{N-1}(\sigma, s, s); \sigma, s) \varphi^{N-1}(\sigma, \xi) \, d\sigma = t.\]

Now, we prove (3.73). Since \(\Lambda, \alpha \in \mathcal{M}(S^0; 1)\) and \(|\alpha(\sigma; t, s, x, \xi) - 1| \leq C_1 \times (t - \omega),\) we have

\[|\partial_\sigma \varphi^N + 1| = | - \{\alpha(s; \theta, s, y, \xi) - 1\} + \int_0^t \partial_\sigma \{\alpha(\varphi^{N-1}(\sigma, \theta, s); \sigma, s) \varphi^{N-1}(\sigma, \xi)\} \, d\sigma| \leq C_1 (\theta - s) + C_2 (t - \theta),\]

by using (3.73) and (3.75). Hence, if \(T'_0\) is small enough, we obtain

\[|\partial_\sigma \varphi^N + 1| \leq 1\]

and (3.73) when \(0 \leq s \leq \theta \leq t \leq T'_0.\) Similarly, we can prove (3.75) by using (3.73) and (3.75). Consequently, by the induction the functions \(\{\varphi^{N}, \varphi^{N}\}(t, \theta, s; y, \xi)\) are well-defined and satisfy (3.72) and (3.73) for \(0 \leq s \leq \theta \leq t \leq T'_0\) if \(T'_0\) is small enough.

As usual we can prove

(3.76)

\[|\varphi^{N+1} - \varphi^N| \leq C^N(t - \theta)^N/N!,\]

\[|\varphi^{N+1} - \varphi^N| \leq C^N(t - \theta)^N/N!.\]
with a constant $C$ independent of $N$. Hence, we obtain the desired symbols $\mathcal{S}(t, \theta, s; y, \xi)$ and $\mathcal{R}(t, \theta, s; y, \xi)$ as limits of $\{\mathcal{S}^{(N)}\}$ and $\{\mathcal{R}^{(N)}\}$. From (3.72) and (3.74), we get (3.58). Moreover, we can easily prove $\mathcal{S} \in \mathcal{M}(S^0_1)$, $\mathcal{R} \in \mathcal{M}(S^0_1)$, and (3.59).

**Remark.** If $a_j(t, x, \xi)$ in (3.31) are functions of only $t$, we can relax the conditions in Theorem 3.9 as the following: Assume $\lambda_j(t, x, \xi) \in M^0([0, T]; S^0_1(\mathbb{R})) \cap C^1([0, T] \times \mathbb{R}^n)$ and

$$\{\tau - \lambda_j, \tau - \lambda_{j+1}\} = a_j(t)(\lambda_{j-\lambda_{j+1}} + a_j(t, x, \xi))$$

with $C^0$-functions $a_j(t)$ and symbols $a_j^\prime(t, x, \xi)$ in $M^0([0, T]; S^0_1)$. Then, for the function $\Omega_{\nu,j}$ determined by

$$\Omega_{\nu,j} = a_j^{-1}(a_j(t_{j-1}) - a_j(t_j) + a_j(t_{j+1})) \quad (c.f. (2.20) of [13])$$

with $a_j(t) = \int (\exp \phi_j(t) dt) dt$ the results (3.34)–(3.35) holds. In fact, we first prove (3.34)–(3.35) for $\nu=1$ by the method of proving Theorem 3.9. Then, the result (3.34)–(3.35) for any $\nu$ will be derived by the method of proving Theorem 1.7 and Theorem 1.8 in [11]. It seems to us that we cannot prove (3.35) directly from (3.63)–(3.64) when $\nu \geq 2$ and $\lambda_j \in M^0([0, T]; S^0_1((2)))$, since $\Lambda_j(\omega)$ of (3.45) may not belong to $\mathcal{M}(S^0_1(1))$ when $\nu \geq 2$.

### 4. Fundamental solutions for hyperbolic systems.

In this section we prove Theorem 4 by using Theorem 1 and Theorem 3.9. First, we construct the fundamental solution $E(t, s)$ of the Cauchy problem

$$\begin{cases}
L U(t) = 0 & \text{on } [0, T], \\
U(0) = U_0
\end{cases}$$

for the hyperbolic operator $L$ of (1). Let $\phi_m(t, s) \equiv \phi_m(t, s; x, \xi)$ be the phase function corresponding to $\lambda_m(t, x, \xi)$. Set $M_\nu = \{\mu = (m_1, \ldots, m_\nu); m_j = 1, \ldots, \ell\}$ ($\nu = 1, 2, \ldots$) and denote

$$\Phi_{\nu, \mu}(t, t_1, \ldots, t_{\nu-1}, s; x, \xi) = (\phi_{m_1}(t_1) \# \phi_{m_2}(t_2) \# \cdots \# \phi_{m_{\nu-1}}(t_{\nu-2}, t_{\nu-1}) \# \phi_{m_\nu}(t_{\nu-1}, s))(x, \xi)$$

for $\mu = (m_1, \ldots, m_\nu) \in M_\nu$ when $\nu \geq 2$. Set

$$I_{\varphi}(t, s) = \begin{bmatrix}
I_{\phi_1}(t, s) & 0 \\
0 & I_{\phi_2}(t, s)
\end{bmatrix} \quad (\sigma(I_{\phi_\mu}(t, s)) = 1).$$

Then, we have
Proposition 4.1. Let $1/2 \leq \rho \leq 1$. Assume that $\lambda_m(t, x, \xi)$ in (1) belong to $M^0([0, T]; S^0_\beta(2))$ and $b_m(t, x, \xi)$ in (1) belong to $M^0([0, T]; S^0_\beta)$. Then, the fundamental solution $E(t, s)$ of the Cauchy problem (4.1) for the hyperbolic system (1) can be represented in the form

$$E(t, s) = I_\emptyset(t, s) + \left\{ \sum_{\nu=1}^{\infty} W_{\nu}(\theta, t, s) + \sum_{\nu=1}^{\infty} \sum_{\nu' \in \mathbb{Z}} \int_{s}^{t} dt_{\nu-1}, \ldots, dt_{\nu-1}, dt \right\} d\theta$$

$(t_0 = \theta; 0 \leq s \leq t \leq T_0)$

for some $T_0$, and $w_m(t, s; x, \xi) = \sigma(W_m, \phi_m(t, s))$ and $w_{\nu, (\mu)}(t, \tilde{t}^{\nu-1}, s; x, \xi) = \sigma(W_{\nu, (\mu)}, \phi_{\nu, (\mu)}(t, \tilde{t}^{\nu-1}, s))$ satisfy the following: There exists a constant $C_0$ independent of $\nu$ such that the set $\{w_m\} \cup \{C_0^{-\nu}w_{\nu, (\mu)}\}$ is bounded in $S^0_\rho$. Moreover, if $\lambda_m(t, x, \xi) \in M([0, T]; S^0_\rho)$ and $b_m(t, x, \xi)$ belong to $M([0, T]; S^0_\rho)$, then, setting $\hat{M}(Z; \sigma) = \bigcap_{k=0}^{\infty} M^\emptyset(Z; S^{(1-p)k}_\sigma) = (\hat{M}(Z; S^0_\rho))$, the symbols $w_m(t, s; x, \xi)$ and $w_{\nu, (\mu)}(t, \tilde{t}^{\nu-1}, s; x, \xi)$ satisfy (4.4) belong to $\hat{M}(\Delta_0(T_0); S^0_\rho)$ and $\hat{M}(\Delta_{\nu-1}(T_0); S^0_\rho)$, respectively, and there exists a constant $C_0$ independent of $\nu$ such that for any $\tilde{t}^\nu = (\gamma_0, \gamma_1, \cdots, \gamma_k)$ with $\gamma_0 + \cdots + \gamma_k = k$ the set $\{C_0^{-\nu} \partial_{[\tilde{t}^{\nu-1}, s]} w_{\nu, (\mu)}\}$ is bounded in $S^{(1-p)k}_\sigma$. Here, $\Delta_0(T_0) = \{(t, s); 0 \leq s \leq t \leq T_0\}$ and $\Delta_{\nu}(T_0) = \{(t, \tilde{t}^{\nu}, s); 0 \leq s \leq t_1 \leq \cdots \leq t_\nu \leq T_0\}$. We have

**Remark.** By virtue of Theorem 1 we can take smoothing operators in Sobolev spaces away from the expression (3.17) in [10].

Proof. We fix a constant $T_0$ such that $T_0 \leq T_2$ and $T_2 \leq \tau^0/\partial_{\tilde{t}, \overline{\tau}_0}$. Here, $T_2$ is the constant in Section 3, $\tau^0$ is the constant defined by (2.58) and $\partial_{\tilde{t}, \overline{\tau}_0}$ is the constant in Proposition 3.4 for $l = \overline{\tau}_0$ (the integer defined in Proposition 2.2). Operate $\mathcal{L}$ to (4.3). Then, we have

$$\mathcal{L}I_\emptyset(t, s) = R_\emptyset(t, s)$$

for

$$R_\emptyset(t, s) = \sum_{m=1}^{l} R_{m, \phi_m}(t, s),$$

where $R_{m, \phi_m}(t, s)$ is a matrix of Fourier integral operators with phase function $\phi_m(t, s; x, \xi)$ and its symbol $r_m(t, s; x, \xi)$ belongs to $M^0(\Delta_0(T_0); S^0_\rho)$ (c.f. Theorem 2.2 of Chap. 10 in [8]). From (4.5) we see that the fundamental solution $E(t, s)$ for $\mathcal{L}$, as the continuous operator from the Sobolev space $H_\sigma$ into itself for any fixed real $\sigma$, is constructed in the form

$$E(t, s) = I_\emptyset(t, s) + \left\{ \sum_{\nu=1}^{\infty} W_{\emptyset}(\theta, t, s) \right\} d\theta.$$
Here, \( \{ W_v(t, s) \}_{v=1}^{\infty} \) are defined by

\[
W_v(t, s) = -iR \Phi(t, s),
\]

and

\[
W_{v+1}(t, s) = \int_s W_1(t, \theta) W_v(\theta, s) d\theta, \quad v = 1, 2, \ldots .
\]

Set

\[
\omega_m(t, s; x, \xi) = -ir_m(t, s; x, \xi) \quad (m = 1, \ldots, l).
\]

Then, \( W_v(t, s) \) for \( v \geq 2 \) can be written in the form

\[
W_v(t, s) = \int_s^{s_1} \cdots \int_s^{s_{v-2}} W^{(v)}(t, t_1, \ldots, t_{v-1}, s) dt_{v-1} \cdots dt_1 \quad (t_0 = t),
\]

with \( W_{m, \phi_m}(t, s) = \omega_{m, \phi_m}(t, s; x, X, D_x) \). Consequently, we get the first assertion of the theorem by applying Theorem 1.

Next, we assume \( \lambda_m \in M([0, T]; S^2(\mathbb{R})) \) and \( b_{mk} \in M([0, T]; S^0) \). Then, the symbols \( r_m(t, s; x, \xi) \) in (4.6) belong to \( \tilde{M}(\Delta_0(T_0); S^0) \). Consequently, we get the second assertion of the theorem, when we use the expression (4.9), Proposition 3.6 and the discussions in Section 2. Q.E.D.

Now, we prove Theorem 4. First, we assume the condition (I). Since the expression (4.4) holds including the case \( \rho = 1/2 \), by the method of proving Theorem 3 in [13] we can prove Theorem 4 under the condition (I) not only in the case of \( 1/2 < \rho \leq 1 \) but also in the case of \( \rho = 1/2 \). We note that in Theorem 3 of [13] only the case \( \rho = 1 \) was treated. Next, we consider Theorem 4 under the condition (II). For the proof we prepare the following three propositions.

First, we shall restate Theorem 3.9 in our use. For \( \mu = (m_1, \ldots, m_j, m_{j+1}, \ldots, m_v) \in M_v \) we change the order of \( m_j \) and \( m_{j+1} \) and set \( \mu(j) = (m_1, \ldots, m_{j-1}, m_{j+1}, m_j, m_{j+2}, \ldots, m_v) \). We note that \( \Phi_{\nu, (\mu)}(t, \tilde{t}^{-1}, s) = \Phi_{\nu, (\mu)}(t, \tilde{t}^{-1}, s; x, \xi) \) and \( \Phi_{\nu, (\mu(j))}(t, \tilde{t}^{-1}, s) = \Phi_{\nu, (\mu(j))}(t, \tilde{t}^{-1}, s; x, \xi) \) have the forms \( \Phi_{\nu, (\mu)}(t, \tilde{t}^{-1}, s) \) and \( \Phi_{\nu, (\mu(j))}(t, \tilde{t}^{-1}, s) \).

Then, from Theorem 3.9 we obtain

**Proposition 4.2.** Let \( \lambda_m(t, x, \xi), m=1, \ldots, l, \) belong to \( M([0, T]; S^0_m(\mathbb{R})) \). Assume for any \( m \) and \( k \) the equation (18) holds with \( a_{m,k}(t, x, \xi) \) and \( \alpha_{m,k}(t, x, \xi) \) in (17). Then, if we take a sufficiently small constant \( T_0 \) (independent of \( \nu \), there exist symbols \( \Omega_{\nu, (\mu)}(t, \tilde{t}^{-1}, s; x, \xi) \) and \( \psi_{\nu, (\mu)}(t, \tilde{t}^{-1}, s; x, \xi) \) in \( \tilde{M}(\Delta_{v-1}(T_0); S^0) \) for any \( \nu \geq 2, j \leq \nu - 1 \) and \( \mu \in M_v \) such that they satisfy

\[
t_{j+1} \leq \Omega_{\nu, (\mu)}(t, \tilde{t}^{-1}, s; x, \xi) \leq t_{j-1} \quad (t_0 = t, t_0 = s),
\]
\[(4.11)\] \( \Omega_{\nu, (\mu), j} |_{t_j = t_{j-1}} = t_{j+1}, \quad \Omega_{\nu, (\mu), j} |_{t_j = t_{j+1}} = t_{j-1}, \]

\[(4.12)\] \[
\Phi_{\nu, (\mu), j}(t, t_1, \ldots, t_{j-1}, t_{j+1}, t_j, \ldots, t_{v-1}, s; x, \xi) \\
= \Phi_{\nu, (\mu), j}(t, t_1, \ldots, t_{j-1}, \Omega_{\nu, (\mu), j}(t, \tilde{t}_{j-1}, s; x, \xi), t_{j+1}, \ldots, t_{v-1}, s; x, \xi) \\
+ \psi_{\nu, (\mu), j}(t, \tilde{t}_{j-1}, s; x, \xi),
\]

\[(4.13)\] \[
\begin{cases}
\{\Omega_{\nu, (\mu), j}\} \text{ is bounded in } S^0_p, \\
\{\tilde{t}_{j-1, s} \Omega_{\nu, (\mu), j}\} \text{ is bounded in } S^p_1 \text{ for } |\tilde{\gamma}| = k \geq 1, \\
\{\tilde{t}_{j-1, s} \psi_{\nu, (\mu), j}\} \text{ is bounded in } S^p_1 \text{ for } |\tilde{\gamma}| = k
\end{cases}
\]

and

\[(4.14)\] \[|\partial_{t_j} \Omega_{\nu, (\mu), j+1}| \leq A_0(t-s)
\]

for a constant \(A_0\) independent of \(\nu\).

**Proposition 4.3.** In Proposition 4.2 we assume, furthermore, that the constant \(T_0\) satisfies \(A_0 T_0 \leq 1/2\). Then, the equation

\[(4.15)\] \[\omega = \Omega_{\nu, (\mu), j}(t, t_1, \ldots, t_{j-1}, t_{j+1}, t_{v-1}, s; x, \xi) \quad (1 \leq j \leq \nu-1)
\]

has the inverse \(\theta = \Theta_{\nu, (\mu), j}(t, t_1, \ldots, t_{j-1}, \omega, t_{j+1}, \ldots, t_{v-1}, s; x, \xi)\) satisfying \(t_{j+1} \leq \Theta_{\nu, (\mu), j} \leq t_{j-1}\) and

\[(4.16)\] \[
\begin{cases}
\{\Theta_{\nu, (\mu), j}\} \text{ is bounded in } S^0_p((1)), \\
\{\tilde{t}_{j-1, s} \Theta_{\nu, (\mu), j}\} \text{ is bounded in } S^p_1 \text{ for } |\tilde{\gamma}| = k \geq 1.
\end{cases}
\]

**Proof.** Set \[\mathcal{A} = \{\Theta(t, t_1, \ldots, t_{j-1}, \ldots, s; x, \xi) \in C^\infty, \\
t_{j+1} \leq \Theta \leq t_{j-1}, \Theta_{|u=t_{j+1}} = t_{j+1}, \Theta_{|u=t_{j-1}} = t_{j-1}, -2 \leq \partial_u \Theta \leq 0\}
\]

and consider a mapping \(\Gamma = \Gamma_{\nu, (\mu), j} : \mathcal{A} \ni \Theta \rightarrow G = \Gamma(\Theta) \in \mathcal{A}\) defined by

\[(4.17)\] \[
G = G(t, t_1, \ldots, t_{j-1}, \omega, t_{j+1}, \ldots, t_{v-1}, s; x, \xi) \\
= -\omega + \Omega_{\nu, (\mu), j}(t, t_1, \ldots, t_{j-1}, \Theta, t_{j+1}, \ldots, t_{v-1}, s; x, \xi) + \Theta \\
(t_0 = t, t_v = s).
\]

Since \(t_{j+1} \leq \Theta \leq t_{j-1}\) for \(\Theta \in \mathcal{A}\) the mapping \(\Gamma\) is well-defined. From (4.11), (4.14) and \(A_0 T_0 \leq 1/2\) we get for \(G = \Gamma(\Theta)\) with \(\Theta \in \mathcal{A}\)

\[
G_{|u=t_{j-1}} = -t_{j+1} + \Omega_{\nu, (\mu), j}(t, \ldots, t_{j-1}, t_{j+1}, \ldots, s) + \Theta_{|u=t_{j+1}} \\
= -t_{j+1} + \Omega_{\nu, (\mu), j}(t, \ldots, t_{j-1}, t_{j+1}, \ldots, s) + t_{j+1}.
\]
\[ G_{i_0-t_{j+1}} = -t_{j+1} + \Omega_{\nu,(\nu)_j}(t, \cdots, t_{j-1}, \Theta_{i_0-t_{j+1}, \cdots, \nu}, \cdots, s) + \Theta_{i_0-t_{j+1}} \]
\[ = -t_{j+1} + \Omega_{\nu,(\nu)_j}(t, \cdots, t_{j-1}, t_{j+1}, \cdots, \nu), t_{j-1} \]
\[ = t_{j-1}, \]
\[ |\partial_a G + 1| = |\{\partial_{t_j} \Omega_{\nu,(\nu)_j} + 1\} \partial_a \Theta| \]
\[ \leq A_0 T'_0 \cdot 2 \leq \frac{1}{2} \cdot 2 = 1 \]

and
\[ t_{j+1} = G_{i_0-t_{j-1}} \leq G \leq G_{i_0-t_{j+1}} = t_{j-1} \]

by \( \partial_a G \leq 0 \). This shows that the mapping \( \Gamma: A \to A \) is into. We define a sequence \( \{\Theta^{(N)}\}_{N=0}^{\infty} \) in \( A \) by
\[ \left\{ \begin{array}{l}
\Theta^{(0)} = t_{j-1} + \omega + t_{j+1}, \\
\Theta^{(N+1)} = \Gamma(\Theta^{(N)}).
\end{array} \right. \]

Then, from (4.14) and \( A_0 T'_0 \leq 1/2 \) we get for some constant \( C \) independent of \( N \)
\[ |\Theta^{(N+1)} - \Theta^{(N)}| \leq C 2^{-N}. \]

Consequently, we can find the desired solution \( \Theta = \Theta_{\nu,(\nu)_j} \) of (4.15) as the limit of the sequence \( \{\Theta^{(N)}\}_{N=0}^{\infty} \). Consider the equation (4.15). Then, we get (4.16) by the usual method.

**Proposition 4.4.** Let \( p(t, \tilde{\nu}^{-1}, s; x, \xi) \) belong to \( \overline{M}(\Delta_{\nu-1}(T); S_0^\nu) \) and let \( \{\Theta_N(t, \tilde{\nu}^{-1}, s; x, \xi)\}_{N=1}^{\infty} \) and \( \{g_N(t, \tilde{\nu}^{-1}, s; x, \xi)\}_{N=1}^{\infty} \) be sequences in \( \overline{M}(\Delta_{\nu-1}(T); S_0^\nu; 1) \) and \( \overline{M}(\Delta_{\nu-1}(T); S_0^\nu) \), respectively. Assume
\[ \{\Theta_N\} \text{ is bounded in } S_0^\nu((1)), \]
\[ \{\partial_{(t, \tilde{\nu}^{-1}, s)} \Theta_N\} \text{ is bounded in } S_0^{(1-\rho)(k-1)} \text{ for } |\tilde{\nu}'| = k \geq 1, \]
\[ \{\partial_{(t, \tilde{\nu}^{-1}, s)} g_N\} \text{ is bounded in } S_0^{(1-\rho)k} \text{ for } |\tilde{\nu}'| = k. \]

For a fixed sequence \( \{j_N\}_{N=1}^{\infty} \) \( (1 \leq j_N \leq \nu - 1) \), we set inductively
\[ p_N(t, \tilde{\nu}^{-1}, s; x, \xi) = p_{N-1}(t, \tilde{\nu}^{-1}, s; x, \xi), \Theta_N(t, \tilde{\nu}^{-1}, s; x, \xi), t_{j_N+1}, \cdots, t_{\nu-1}, \]
\[ s; x, \xi) g_N(t, \tilde{\nu}^{-1}, s; x, \xi) \quad (t_0 = t; t_0 = t, t_\nu = s). \]

Then, for any \( k \) there exists a constant \( C_k \) independent of \( N \) and \( v \) such that
\[ \|p_N\|_{(0)} \leq C_k^N \|p\|_{(0)}, \]
where \( \|p\|_{(0)} = \max_{0 \leq k' \leq k'} \|\partial_{(t, \tilde{\nu}^{-1}, s)} p\|_{k'-(1-\rho)k'} \).
We can prove this proposition by the induction.

Using these propositions and the discussions for the proof of Theorem 3 in [13] we can reduce (4.4) to the finite sum expression

\[ E(t, s) = \sum_{\mathbf{m}} \sum_{\mathbf{m}} \sum_{\mathbf{m}} \sum_{\mathbf{m}} W^0_{\mathbf{m}}(t, s) \]

\[ + \sum_{v=1}^{l} \sum_{\mu \in M_v} \left( \int_{s}^{t} \cdots \int_{s}^{t-v+1} W^0_{\nu, \mu}(t, s) \cdot dt_{v-1} \cdots dt_1 \right) \]

\[ (t_0 = t, 0 \leq s \leq t \leq T_0) \]

with some \( T_0 (\leq T_0') \) and symbols \( \sigma(W_{\nu, \mu}, \phi_{\nu, \mu}) \) in \( \tilde{M}(\Delta_{v-1}(T_0); S^0_\mu) (1 \leq \nu \leq l) \), where \( M_v = \{ \mu = (m_1, \ldots, m_v) \in M_v; m_1 < m_2 < \cdots < m_v \} (2 \leq v \leq l) \). This proves Theorem 4.

**Remark.** When we use the remark at the end of Section 3, we can prove Theorem 4 under the condition (I) with (16) replaced by

\[ \{\tau - \lambda_m, \tau - \lambda_k\} = a_{\mathbf{m}}(t) (\lambda_m - \lambda_k) + a'_{\mathbf{m}}(t, x, \xi) \]

Here, \( a_{\mathbf{m}}(t) \) are continuous functions of \( t \) and \( a'_{\mathbf{m}}(t, x, \xi) \) are symbols in \( M^0 ([0, T]; S^0_\mu) \). This result contains the one studied in [5]. In [5] Ichinose proved this when \( a_{\mathbf{m}}(t) = 0 \) and \( l = 2 \). But, he did not discuss the convergence of the symbols \( \sigma(W_{\nu, \mu}, \phi_{\nu, \mu}) \) derived from the successive approximation.

As a corollary of Theorem 4 we get immediately from the expression (4.23) and Theorem 3.14 of Chap. 10 of [8]

**Corollary 4.5.** In Theorem 4 we assume, furthermore, that the symbols \( \lambda_m(t, x, \xi) \) are homogeneous for large \( |\xi| \). Then, for the solution \( U(t) \) of the Cauchy problem (4.1) we have

\[ WF(U(t)) \subseteq \text{Conic hull of } \Gamma_t \]

for the wave front set \( WF(U(t)) = \bigcup_{m=1}^{l} WF(U_m(t)) \) of \( U(t) = (u_1(t), \cdots, u_l(t)) \), which is defined in [4].

In (4.24) \( \Gamma_t \) is defined by

\[ \Gamma_t = \{ q_{m_1, \ldots, m_v}, p_{m_1, \ldots, m_v} \} (t, t_1, \cdots, t_{v-1}; y, \eta); (m_1, \cdots, m_v) \in M^0_v, \]

\[ v = 1, \cdots, l, \quad 0 \leq t_{v-1} \leq \cdots \leq t_1 \leq t, (y, \eta) \in WF(U_0) \]

\[ \text{for large } |\eta| \} \quad (t_0 = t) \]

for the trajectory \( \{ q_{m_1, \ldots, m_v}, p_{m_1, \ldots, m_v} \} (t, t_1, \cdots, t_{v-1}; y, \eta) \) of \( (m_1, \cdots, m_v) \in \mathbb{M}^0_v \) determined by the following: Let \( \{ q_{m_0}, p_{m_0} \} (t; y, \eta) \) be the solution of
\[
\frac{dq}{dt} = -\nabla_\xi \lambda_m(t, q, p), \quad \frac{dp}{dt} = \nabla_\lambda \lambda_m(t, q, p), \quad \{q, p\}_{t=0} = \{y, \eta\}.
\]

Then, \(\{q_{m_k, \cdots, m_v}, p_{m_k, \cdots, m_v}\}(t, t_k, \cdots, t_{v-1}; y, \eta)\) is defined as the solution of
\[
\begin{cases}
\frac{dq}{dt} = -\nabla_\xi \lambda_m(t, q, p), \\
\frac{dp}{dt} = \nabla_\lambda \lambda_m(t, q, p),
\end{cases}
\]
\[
\{q, p\}_{t=t_k} = \{q_{m_k+1, \cdots, m_v}, p_{m_k+1, \cdots, m_v}\}(t_k, \cdots, t_{v-1}; y, \eta)
\]
\((1 \leq k \leq v-1)\).

In [13] Morimoto has obtained Corollary 4.5 by a different method. In the condition of Corollary 4.5 the symbols \(\lambda_m(t, x, \xi)\) belong to \(M([0, T]; S_{1,0}^1)\). But using the discussions in Section 3 of [5] we can prove the property: Assume that there exist continuous functions \(\lambda^*_m(t, x, \xi)\), \(m=1, \cdots, l\), which have Lipschitz continuous derivatives with respect to \(x\) and \(\xi\) for \(|\xi| \geq 1\), are homogeneous of order 1 with respect to \(\xi\) and satisfy for some \(\kappa<1\)
\[
|\delta_x^\alpha D^\beta_\xi (\lambda^*_m - \lambda_m) | \leq C_\xi^\alpha \xi^{-\kappa} \quad (|\alpha + \beta| \leq 1, |\xi| \geq 1).
\]
Then, the property (4.25) holds with \(\lambda_m(t, x, \xi)\) replaced by \(\lambda^*_m(t, x, \xi)\) in (4.26)–(4.27). Here, we need not assume the homogeneity of \(\lambda_m(t, x, \xi)\).

Finally, we shall study examples in Introduction. First, we consider (19). The characteristic roots are \(\lambda_+(x, \xi) = \pm \sqrt{a_0(x)|\xi|}\), which are \(C^{4-1}\)-class with Lipschitz continuous derivatives of \((k-1)\)-st order for \(|\xi| \geq 1\). We approximate \(\lambda_+(x, \xi)\) and \(\lambda_-(x, \xi)\) by \(\lambda^+_m(x, \xi) = (a_m(x)|\xi|^2 + 1)^{1/2}\) and \(\lambda^-_m(x, \xi) = -(a_m(x)|\xi|^2 + 1)^{1/2}\), respectively. Then, setting \(\rho = 1-1/k\), \(\lambda_0(x, \xi)\) and \(\lambda_\pm(x, \xi)\) belong to \(S^1_{\rho}(k)\) (c.f. §4 of [3]) and we can find pseudo-differential operators \(B\) and \(B'\) in \(S^s_{\rho}\) such that (19) has the form
\[
L_1 = (D_1 - \lambda_2(X, D_x) + B) (D_1 - \lambda_1(X, D_x) - B) + B'.
\]
Hence, the study for the operator (19) is reduced to the study for the system \(\mathcal{L}_0\) of the form
\[
\mathcal{L}_0 = D_1 - \begin{bmatrix}
\lambda_1(X, D_x) & 0 \\
0 & \lambda_2(X, D_x)
\end{bmatrix} + \begin{bmatrix}
-B & -1 \\
B' & B
\end{bmatrix}.
\]
Since the system (4.30) is involutive, that is,
\[
\{\tau - \lambda_1, \tau - \lambda_2\} = 0
\]
holds, the fundamental solution of (19) is constructed in the form
\[
W_{1,\phi_1}(t, s) + W_{2,\phi_2}(t, s) + \int_s^t W_{1,\phi_1}(t, \theta, s) d\theta.
\]
We note that in the case of \(k \geq 3\) we can apply the approximation theory in
Next, we consider (20). The characteristic roots are 
\[ \lambda_{\pm}(x, \xi) = \pm a(x) \mu(x, \xi) \]
and approximate \( \lambda_{+}(x, \xi) \) and \( \lambda_{-}(x, \xi) \) by \( \lambda_{0}(x, \xi) = a(x) \mu(x, \xi) \) and \( \lambda_{2}(x, \xi) = -a(x) \mu(x, \xi) \). Then, we can prove by the method of [15] that \( \lambda_{m}(x, \xi), m = 1, 2 \), belong to \( \mathcal{S}_{1/2}^{1/2}(2) \) and that (20) can be reduced to the system (4.30) with appropriate pseudo-differential operators \( B \) and \( B' \) in \( \mathcal{S}_{1/2}^{1/2} \). In this case we also have (4.31) and the fundamental solution of (20) can be constructed in the form (4.32). For the operator (21) its characteristic roots are \( \lambda_{\pm}(x, \xi) = a(x) \mu(x, \xi) \) of class \( C^{2} \). Hence, if we approximate \( \lambda_{+}(x, \xi) \) and \( \lambda_{-}(x, \xi) \) by \( \lambda_{0}(x, \xi) = a(x) \mu(x, \xi) \) and \( \lambda_{2}(x, \xi) = -a(x) \mu(x, \xi) \) with the aid of \( \mu(x, \xi) \) in (4.33), the symbols \( \lambda_{m}(x, \xi) \) belong to \( \mathcal{S}_{1/2}^{1/2}(3) \) and satisfy

\[ \{ \tau - \lambda_{+}, \tau - \lambda_{-} \} = a_{1,2}(x, \xi) (\lambda_{+} - \lambda_{-}) \]

with a symbol \( a_{1,2}(x, \xi) \) in \( \mathcal{S}_{1/2}^{1/2}(1) \). The operator (21) can also be reduced to a system (4.30) with pseudo-differential operators \( B \) and \( B' \) in \( \mathcal{S}_{1/2}^{1/2} \). Hence, the fundamental solution of (21) can be obtained in the form (4.32).

In three examples (19), (20) and (21) the characteristic roots \( \lambda_{\pm}(x, \xi) \) and the corresponding approximated symbols \( \lambda_{m}(x, \xi), m = 1, 2 \), satisfy (4.28) with \( \lambda_{+} = \lambda_{+}, \lambda_{-} = \lambda_{-} \) and \( \kappa = 1/2 \). Hence, from the statement after Corollary 4.5 we get

\[ \text{WF}(u(t)) \subset \text{Conic hull of } \Gamma_{t} \]

for the solution \( u(t) \) of

\[
\begin{align*}
L_{\mu}u(t) & = 0, \\
u(0) & = u_{0}, \\
\partial_{\nu}u(0) & = u_{1},
\end{align*}
\]

where \( \Gamma(t) = \{ \{Q, P\} (t, s; y, \eta) ; 0 \leq s \leq t, (y, \eta) \in \text{WF}(u_{0}) \cup \text{WF}(u_{1}) \} \) for large \( |\eta| \} \) for the trajectory \( \{Q, P\} (t, s; y, \eta) \) defined by the following: Let \( \{q, p\} (t; y, \eta) \) be the solution of

\[
\begin{align*}
\frac{dq}{dt} & = -\nabla_{\xi} \lambda_{-}(t, q, p), \\
\frac{dp}{dt} & = \nabla_{\xi} \lambda_{+}(t, q, p), \\
\{q, p\} |_{t=0} & = \{y, \eta\}.
\end{align*}
\]

Then, \( \{Q, P\} (t, s; y, \eta) \) is defined as the solution of

\[
\begin{align*}
\frac{dQ}{dt} & = -\nabla_{\xi} \lambda_{+}(t, Q, P), \\
\frac{dP}{dt} & = \nabla_{\xi} \lambda_{+}(t, Q, P), \\
\{Q, P\} |_{t=s} & = \{q, p\} (s, y, \eta).
\end{align*}
\]
References


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