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EXTENDIBILITY OF G-MAPS TO PSEUDO-EQUIVALENCES TO FINITE G-CW-COMPLEXES WHOSE FUNDAMENTAL GROUPS ARE FINITE

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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0. Introduction

In this paper we let G be a finite group. A. Assadi [2] and R. Oliver-T. Petrie [6] treated the following question. What is a necessary and sufficient condition, for given finite G-CW-complexes X and Y and a G-map f: $X \to Y$, to extend f to a quasi-equivalence f': $X' \to Y$ (with some reservations)? Here a G-map is called a quasi-equivalence if it induces isomorphisms of fundamental groups and of integral homology groups. We apply the Oliver-Petrie theory to covering spaces to give a necessary and sufficient condition so that we may extend above f to a pseudo-equivalence f'': $X'' \to Y$ (with some reservations), when $\pi_1(Y)$ is finite.

We take Oliver-Petrie [6] as our general reference and use their terms and notations.

Let Y be a finite connected G-complex. Then $\tilde{G}=\pi_1(EG \times_G Y)$ acts on the universal covering space $\tilde{Y}$ of Y as is shown in section 1 (compare the action with that of D. Anderson [1]). Assume $\pi_1(Y)$ is finite. Then $\tilde{G}$ is finite, so we have a $\tilde{G}$-poset $\tilde{\Pi}=\Pi(\tilde{Y})$ and a G-poset $\Pi=\Pi(Y)$. In section 3 we give a one to one correspondence T from the set of G-families in Π to the set of $\tilde{G}$-families in $\tilde{\Pi}$, and an isomorphism ν from $\Omega(\tilde{G}, \tilde{\Pi})$ to $\Omega(G, \Pi)$. A subgroup $\Delta_h(G, Y, \mathcal{F})$ of $\Delta(G, \mathcal{F})$ is defined by $\Delta_h(G, Y, \mathcal{F})=\nu(\Delta(\tilde{G}, T(\mathcal{F})))$. Under certain conditions $\Delta_h(G, Y, \mathcal{F})$ agrees with the set

$$\{[M_f]\in\Omega(G, \Pi) \mid f\colon X\to Y \text{ is a pseudo-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}$$

(see Proposition 4.1), where M_f is the mapping cone of f.

Our main results are:

Theorem 1. *Let X be a finite G-complex, Y a finite connected G-complex with finite $\pi_1(Y)$, f: $X \to Y$ a skeletal G-map, and $\mathcal{F} \subset \Pi$ any connected G-*

family containing $\mathcal{F}_f$. *Let* $\mathcal{F}'\subset\mathcal{F}$ *be any subfamily containing* $\hat{\mathcal{F}}$. *Assume* $T(\hat{\mathcal{F}})$ *is simply generated. Then there exist a finite G-complex* $X'\supset X$ *and a pseudo-equivalence* $f'\colon X'\to Y$ *extending* f *with* X'/X *an* $\mathcal{F}'$*-complex, if and only if*

$$[M_f]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}') \ \ in \ \ \Omega(G, \Pi)\,.$$

Corollary 2. *Assume G is not of prime power order. Let Y be a finite connected G-complex with finite* $\pi_1(Y)$, *and* $F_1, \cdots, F_k$ *the connected components of* $F=Y^G$. *Then there is a subgroup* $N_Y\subset \boldsymbol{Z}^k$ *such that given any finite G-complex* F' *and a map* $\hat{f}\colon F'\to F$, *there exist a finite G-complex X with* $X^G=F'$ *and a pseudo-equivalence* $f\colon X\to Y$ *with* $f^G=\hat{f}$ *if and only if*

$$(\chi(F_1)-\chi(F_1'), \cdots, \chi(F_k)-\chi(F_k'))\in N_Y\,,$$
$$(F_i'=\hat{f}^{-1}(F_i))\,.$$

Above N_Y is the image of $\Delta_h(G, Y, \Pi)$ by the homomorphism $\psi\colon \Omega(G, \Pi)\to \boldsymbol{Z}^k$ defined in section 3 of [6]. Thus N_Y is included in n_Y.

Corollary 3. *Let G and Y be as above. Moreover we assume F is connected and G belongs to* $\mathcal{G}^1$, *i.e.* G/P *is cyclic for some normal subgroup P of G of prime power order. Given any finite G-complex* F' *and any map* $\hat{f}\colon F'\to F$, *there exist a finite G-complex X with* $X^G=F'$ *and a pseudo-equivalence* $f\colon X\to Y$ *extending* $\hat{f}$, *if and only if* $\chi(F)=\chi(F')$.

The proofs of Theorem 1 and Corollaries 2 and 3 are given in section 4.

In a subsequent paper we will calculate N_Y in several cases.

In this paper we often omit the adjective skeletal from a skeletal G-map, however, a G-map should be understood to be a skeletal G-map when its mapping cone appeares.

1. A standard action of $\pi_1(EG\times_G Y)$ on the universal covering space of Y

Let Y be a connected G-complex, $p\colon \tilde{Y}\to Y$ the universal covering, EG the universal principal G-bundle. Arbitrarily choose and fix base points a_0 of Y, b_0 of $\tilde{Y}$ with $p(b_0)=a_0$, and c_0 of EG. Let $q\colon EG\times Y\to EG\times_G Y$ be the canonical projection. We use $u_0=(c_0, a_0)$ and $v_0=q(u_0)$ as the base points of $EG\times Y$ and $EG\times_G Y$ respectively. We put $\pi=\pi_1(Y)$ and $\tilde{G}=\pi_1(EG\times_G Y)$ in this section.

We define a map $k\colon Y\to EG\times_G Y$ by $k(y)=q(c_0, y)$ for $y\in Y$. The covering $q\colon EG\times Y\to EG\times_G Y$ induces the exact sequence

$$\{1\}\to\pi\xrightarrow{j}\tilde{G}\xrightarrow{\sigma}G\to\{1\}$$

where j is the induced map by k, and σ is the map obtained by identifying $\pi_0(G)$

with G. We regard π as a subgroup of $\tilde{G}$ through j.

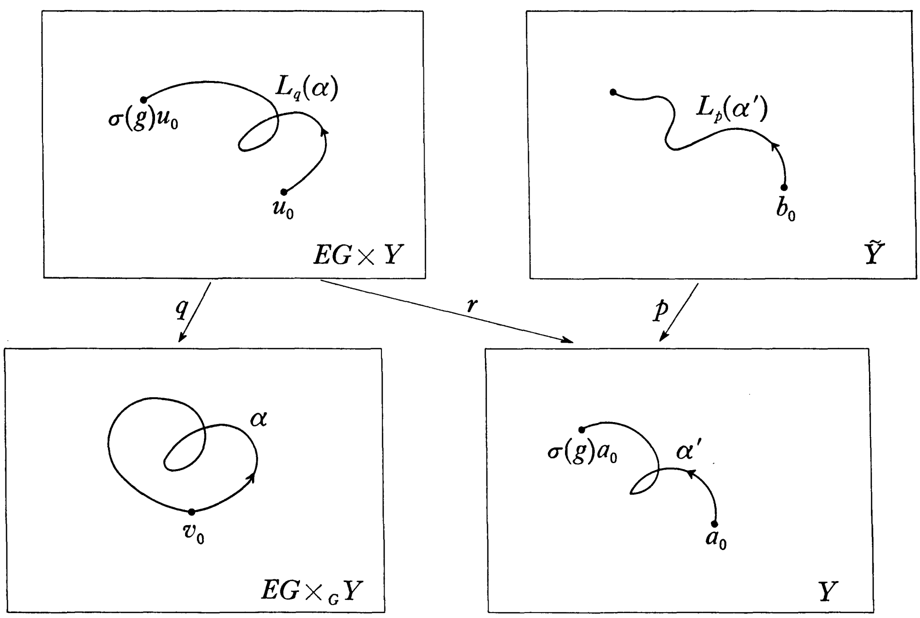
In the following we illustrate a standard action of $\tilde{G}$ on $\tilde{Y}$ such that

(1) p is σ-equivariant, i.e. for $g \in \tilde{G}$ and $b \in \tilde{Y}$ $p(gb) = \sigma(g)p(b)$,

(2) the induced CW-complex structure on $\tilde{Y}$ by p and the $\tilde{G}$-action make $\tilde{Y}$ a $\tilde{G}$-complex,

(3) the restriction of the $\tilde{G}$-action to π agrees with the action given by M. Cohen [3; p. 12].

We denote by r the projection from $EG \times Y$ to the second factor Y. Immediately $r(u_0) = a_0$ follows. We are going to give gb for $g \in \tilde{G}$ and $b \in \tilde{Y}$. An element g of $\tilde{G}$ is represented by a path α: $[0, 1] \to EG \times_G Y$ with $\alpha(0) = \alpha(1) = v_0$. There is a unique lift $L_q(\alpha)$: $[0, 1] \to EG \times Y$ of α (i.e. $q \circ L_q(\alpha) = \alpha$) with $L_q(\alpha)(0) = u_0$. The homomorphism σ: $\tilde{G} \to G$ is given by the relation $\sigma(g)u_0 = L_g(\alpha)(1)$. The path α gives two paths $\alpha' = r \circ L_q(\alpha)$: $[0, 1] \to Y$ and its lift $L_p(\alpha')$: $[0, 1] \to \tilde{Y}$ (i.e. $p \circ L_p(\alpha') = \alpha$) with $L_p(\alpha')(0) = b_0$. We have $\alpha'(0) = a_0$ and $\alpha'(1) = \sigma(g)a_0$.



For given $b \in \tilde{Y}$, choose arbitrarily a path β: $[0, 1] \to \tilde{Y}$ with $\beta(0) = b_0$ and $\beta(1) = b$. β gives two paths $p \circ \beta$: $[0, 1] \to Y$ and β': $[0, 1] \to Y$ defined by $\beta'(t) = \sigma(g)p(\beta(t))$ for $t \in [0, 1]$. We have $\beta'(0) = \sigma(g)a_0 = \alpha'(1)$ and

$$\beta'(1) = \sigma(g)p(b)\,. \tag{1.1}$$

There is a unique lift $L_p(\beta')$: $[0, 1] \to \tilde{Y}$ of β' with $L_p(\beta')(0) = L_p(\alpha')(1)$. We define gb to be the point $L_p(\beta')(1)$.

By (1.1) we have $p(gb)=\sigma(g)p(b)$. That is, p is σ-equivariant. The properties (2) and (3) follow immediately.

2. Remarks on $\mathscr{F}$-complexes

For a finite group G, a *G-poset* is axiomatically defined as follows. Let $\mathscr{S}(G)$ be the set of subgroups of G. By conjugation G acts on $\mathscr{S}(G)$: $(g, H)\mapsto gHg^{-1}$ for $g\in G$, $H\in\mathscr{S}(G)$.

2.1. A parcially ordered G-set Π equipped with a G-map $\rho\colon \Pi\to\mathscr{S}(G)$ is called a *G-poset* if the following four conditions are satisfied: for $\alpha\in\Pi$, $\beta\in\Pi$ (i) $\rho(\alpha)\subset G_\alpha$, (ii) if $\alpha\leqq\beta$ then $g\alpha\leqq g\beta$ for $g\in G$, (iii) if $\alpha\lneqq\beta$ then $\rho(\alpha)\supsetneqq\rho(\beta)$, and (iv) for a subgroup H of $\rho(\alpha)$ there exists a unique element γ of Π such that $\gamma\geqq\alpha$ and $\rho(\gamma)=H$.

Typical examples of G-posets are $\Pi(X)$ for G-spaces X (see [4] and [6]). A G-subset of a G-poset Π is called a *G-family* (in Π). A *Π-complex* Z for a G-poset Π is a finite G-complex with base point $*$ and subcomplexes $Z_\alpha\subset Z$, $(*\in Z_\alpha)$, for all $\alpha\in\Pi$ such that $Z_{g\alpha}=gZ_\alpha$ for $g\in G$, $Z_\alpha\subset Z_\beta$ for $\alpha\leqq\beta$, and
$$Z^H=\bigvee_{\rho(\alpha)=H} Z_\alpha \quad\text{for } H\leqq G\,.$$
For a G-family $\mathscr{F}$ in Π a Π-complex Z is called an *$\mathscr{F}$-complex* if
$$Z_\alpha=\{*\}\cup\bigcup\{Z_\beta\,|\,\beta\in\mathscr{F},\ \beta\leqq\alpha\}$$
for any $\alpha\in\Pi$.

2.2. Let Π be a G-poset, $\mathscr{F}$ a G-family in Π, and Z an $\mathscr{F}$-complex. For $\alpha\in\Pi$, $\beta\in\Pi$ and $x\in(Z_\alpha\cap Z_\beta)\setminus\{*\}$, there is a unique element γ of $\mathscr{F}$ such that $\gamma\leqq\alpha$, $\gamma\leqq\beta$, $\rho(\gamma)=G_x$ and $x\in Z_\gamma$.

2.3. Let Π be a G-poset, Z a Π-complex. For each (non-equivariant) cell c in $Z\setminus\{*\}$, there exists a unique element $\alpha(c)\in\Pi$ such that $\rho(\alpha(c))=G_x$, $x\in c$, and $c\subset Z_{\alpha(c)}$. If $Z_\beta\supset c$ for $\beta\in\Pi$, then $\alpha(c)\leqq\beta$. So we call c *of type* $\alpha(c)$.

2.4. Let $\mathscr{F}$ be a G-family in Π. A Π-complex Z is an $\mathscr{F}$-complex if and only if $\mathscr{F}$ contains $\alpha(c)$ for any cell c in $Z\setminus\{*\}$.

Let Π be a G-poset. For each $\alpha\in\Pi$, the Π-complexes (α) is the G-space $\{*\}\amalg G/\rho(\alpha)$ with
$$(\alpha)_\beta=\{*\}\amalg\bigcup\{g\rho(\alpha)\,|\,g\in G,\ g\alpha\leqq\beta\}$$
for $\beta\in\Pi$.

In the rest of this section we let Y be a finite connected G-complex and

$\Pi=\Pi(Y)$.

Let X be another finite G-complex, and f a G-map from X to Y. For $\alpha\in\Pi$ $X_\alpha=X^{\rho(\alpha)}\cap f^{-1}(|\alpha|)$. $X^+=X\coprod\{*\}$ (disjoint union) has a Π-complex structure given by $(X^+)_\alpha=X_\alpha\coprod\{*\}$. We call this Π-complex structure the Π-complex structure *induced by* f.

2.5. Let α be an element of Π. For an arbitrary G-map f from $X=G/\rho(\alpha)$ to Y with $f(\rho(\alpha))\in|\alpha|$, the induced Π-complex X^+ by f agrees with (α) as Π-complex.

2.6. Let F be a finite CW-complex, and α an element of Π. For a G-map f from $X=(G/\rho(\alpha))\times F$ to Y with $f(\rho(\alpha)\times F)\subset|\alpha|$, $[X^+]=\chi(F)[\alpha]$ in $\Omega(G,\Pi(Y))$.

Proposition 2.7. *Let $\mathcal{F}$ be a G-family in $\Pi=\Pi(Y)$ containing $\mathcal{F}(Y)$. Then*

$$\Omega(G,\mathcal{F})=\{[M_f]\in\Omega(G,\Pi)\,|\,f\colon X\to Y \text{ is a } G\text{-map such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}.$$

Proof. Choose integers $z(\alpha)$, $\alpha\in\mathcal{F}$, such that $[Y^+]=\sum z(\alpha)[\alpha]$, where α runs over $\mathcal{F}$. For any $\xi\in\Omega(G,\mathcal{F})$, there are integers $z'(\alpha)$, $\alpha\in\mathcal{F}$, such that

$$\xi=\sum_{\alpha\in\mathcal{F}} z(\alpha)[\alpha]-\sum_{\alpha\in\mathcal{F}} z'(\alpha)[\alpha].$$

Take finite CW-complexes $F(\alpha)$ with $\chi(F(\alpha))=z'(\alpha)$, and put $X=\coprod\{(G/\rho(\alpha))\times F(\alpha)\,|\,\alpha\in\mathcal{F}\}$. There is a G-map $f\colon X\to Y$ with $f(\rho(\alpha)\times F(\alpha))\subset|\alpha|$. We have $[M_f]=[Y^+]-[X^+]=\xi$ by 2.6.

According to Proposition 1.6 of [6],

$$\Delta(G,\mathcal{F})=\{[Z]\in\Omega(G,\Pi)\,|\,Z \text{ is a contractible } \mathcal{F}\text{-complex}\}.$$

Moreover we have the following.

Proposition 2.8. *Let $\mathcal{F}$ be a connected G-family in $\Pi=\Pi(Y)$ such that $\mathcal{F}$ contains $\mathcal{F}(Y)$ and $\hat{\mathcal{F}}$ is simply generated. Then*

$$\Delta(G,\mathcal{F})=\{[M_f]\in\Omega(G,\Pi)\,|\,f\colon X\to Y \text{ is a quasi-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}.$$

Proof. We prove that for given $\xi\in\Delta(G,\mathcal{F})$ there exist a finite G-complex X and a (skeletal) G-map $f\colon X\to Y$ with $[M_f]=\xi$. For $\xi\in\Delta(G,\mathcal{F})$ there are a finite G-complex X_0 and a G-map $f_0\colon X_0\to Y$ with $[M_{f_0}]=\xi$ by Proposition 2.7. By the same argument as Oliver-Petrie used at Steps 2 and 3 of the proof of [6; Proposition 2.9], we get a finite G-complex $X_1\supset X_0$ and a G-map $f_1\colon X_1\to Y$ extending f_0 such that X_1/X_0 is an $\hat{\mathcal{F}}$-complex, M_{f_1} is an $\mathcal{F}$-resolution and $\{M_{f_1}\}=\xi$. Adding free cells to X appropriately if necessary, we may assume $\dim X_1 \geqq 3$. We use the same argument as was used in the proof (1) of [6; Theorem 3.2], and obtain a finite G-complex $X \supset X_1$ and a G-map f: $X \to Y$ extending f_1 such that X/X_1 is a $\hat{\mathcal{F}}$-complex and f is a quasi-equivalence. We have to check $[M_f]=\xi$. Both $[M_f]$ and $[M_{f_0}]$ belong to $\Delta(G, \mathcal{F})$, and $[X/X_0]=[M_{f_0}]-[M_f]$. We have $\chi((X/X_0)_\alpha)=1$ for $\alpha \in \hat{\mathcal{F}}$ by Proposition 2.6 of [6]. Since X/X_0 is an $\hat{\mathcal{F}}$-complex, we have $[X/X_0]=0$. That is $[M_f]=[M_{f_0}]=\xi$

3. Correspondences between the posets of a finite covering space and a base space

In this section we let G and $\tilde{G}$ be finite groups, σ: $\tilde{G} \to G$ a epimorphism, Y a finite connected G-complex, $\tilde{Y}$ a finite connected $\tilde{G}$-complex, and p: $\tilde{Y} \to Y$ a σ-equivariant covering. We put $\pi = \ker \sigma$. Moreover we assume that π acts freely and transitively on each fiber.

The $\tilde{G}$-action on $\tilde{Y}$ gives a $\tilde{G}$-poset $\tilde{\Pi}=\Pi(\tilde{Y})$ and a $\tilde{G}$-map $\tilde{\rho}$: $\tilde{\Pi} \to \mathcal{S}(\tilde{G})$. The set of G-families in Π is denoted by $\boldsymbol{F}$ and that of $\tilde{G}$-families in $\tilde{\Pi}$ is denoted by $\tilde{\boldsymbol{F}}$.

For arbitrarily given $\alpha \in \tilde{\Pi}$, there is a unique element $\beta \in \Pi$ such that $\rho(\beta)=\sigma(\tilde{\rho}(\alpha))$ and $|\beta| \supset p(|\alpha|)$. This correspondence defines a map μ: $\tilde{\Pi} \to \Pi$.

For $\alpha \in \Pi$, we denote the connected components of $p^{-1}(|\alpha|)$ by $A_1, \cdots, A_k$. We have $p(A_i)=|\alpha|$ for any $i=1, \cdots, k$. Each A_i is fixed by a subgroup H_i of $\tilde{G}$ with $\sigma(H_i)=\rho(\alpha)$, since π preserves each fiber. As π acts freely on each fiber, σ: $H_i \to \rho(\alpha)$ is bijective. Each A_i is contained in a connected component B_i of the H_i-fixed point set of $\tilde{Y}$. The projection p is σ-equivariant, so $p(B_i)=|\alpha|$. We have $A_i=B_i$. There is a unique element $\beta_i \in \tilde{\Pi}$ such that $\tilde{\rho}(\beta_i)=H_i$ and $|\beta_i|=A_i$. We define a map τ: $\Pi \to \mathcal{S}(\Pi)$ by $\tau(\alpha)=\{\beta_1, \cdots, \beta_k\}$, where $\mathcal{S}(\tilde{\Pi})$ denotes the set of subsets of $\tilde{\Pi}$.

Immediately we have $\mu(\tau(\alpha))=\{\alpha\}$ for $\alpha \in \Pi$. The above argument implies $|\mu(\alpha)|=p(|\alpha|)$ for $\alpha \in \tilde{\Pi}$. The following two diagrams are commutative:

$$
\begin{array}{ccc}
\tilde{\Pi} & \xrightarrow{\mu} & \Pi \\
\downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \rho} \\
\mathcal{S}(\tilde{G}) & \xrightarrow{\sigma} & \mathcal{S}(G)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\tilde{\Pi} & \xrightarrow{\mu} & \Pi \\
\downarrow{\scriptstyle |\,|} & & \downarrow{\scriptstyle |\,|} \\
\mathcal{S}(\tilde{Y}) & \xrightarrow{p} & \mathcal{S}(Y)
\end{array},
$$

where $\mathcal{S}(\tilde{Y})$ and $\mathcal{S}(Y)$ are the sets of subspaces of $\tilde{Y}$ and Y respectively. For $\alpha \in \tilde{\Pi}$, α is an element of $\tau(\mu(\alpha))$.

Proposition and definition 3.1. *The following two equations define maps*

M: $\tilde{\boldsymbol{F}} \to \boldsymbol{F}$ and T: $\boldsymbol{F} \to \tilde{\boldsymbol{F}}$,

$$M(\tilde{\mathcal{F}}) = \{\mu(\alpha) \mid \alpha \in \tilde{\mathcal{F}}\} \quad \textit{for } \tilde{\mathcal{F}} \in \tilde{\boldsymbol{F}},$$
$$T(\mathcal{F}) = \cup \{\tau(\alpha) \mid \alpha \in \mathcal{F}\} \quad \textit{for } \mathcal{F} \in \boldsymbol{F}.$$

We have $M \circ T = id_{\boldsymbol{F}}$ and $T \circ M = id_{\tilde{\boldsymbol{F}}}$.

We omit the proof.

Lemma 3.2. (i) *If $\tilde{\mathcal{F}} \in \tilde{\boldsymbol{F}}$ is connected, then $M(\tilde{\mathcal{F}})$ is connected, and $\widehat{M(\tilde{\mathcal{F}})} = M(\hat{\tilde{\mathcal{F}}})$.*

(ii) *If $\mathcal{F} \in \boldsymbol{F}$ is connected and contains $\mathcal{F}(Y)$, then $T(\mathcal{F})$ is connected, and $\widehat{T(\mathcal{F})} = T(\hat{\mathcal{F}})$.*

Proof. We prove (ii), and let (i) remain to be proved by the reader. We denote the maximal element of $\tilde{\Pi}$ by $\tilde{m}$, so we have $\tau(m) = \{\tilde{m}\}$. Since $m \in \mathcal{F}$, $\tilde{m} \in T(\mathcal{F})$. Assume α is an element of $\tilde{\Pi}$ such that $\tilde{\rho}(\alpha)$ is of prime power order and $\{\beta \in T(\mathcal{F}) \mid \beta \leqq \alpha\}$ is not empty. Since $M \circ T = \mathrm{id}$ and μ preserves the order, $\{\beta \in \mathcal{F} \mid \beta \leqq \mu(\alpha)\}$ is not empty. There is the unique maximal element γ of $\{\beta \in \mathcal{F} \mid \beta \leqq \mu(\alpha)\}$, $(\gamma = \widehat{\mu(\alpha)})$. Since Y^+ is a $\mathcal{F}$-complex, we have $|\gamma| = |\mu(\alpha)|$ by Proposition 1.2 of [6]. There uniquely exists $\delta \in \tau(\gamma)$ with $\delta \leqq \alpha$. For any $\beta \in T(\mathcal{F})$ with $\beta \leqq \alpha$, we have $|\beta| \subset |\alpha| = |\delta|$ and $\mu(\beta) \leqq \gamma$. Thus $\sigma(\tilde{\rho}(\beta)) = \rho(\mu(\beta)) \supset \rho(\gamma) = \sigma(\tilde{\rho}(\delta))$. Since $\pi = \ker \sigma$ acts freely on each fiber, we see $\tilde{\rho}(\beta) \supset \tilde{\rho}(\delta)$. Therefore $\beta \leqq \delta$, that is, δ is the unique maximal element of $\{\beta \in T(\mathcal{F}) \mid \beta \leqq \alpha\}$. $T(\mathcal{F})$ is connected. This argument implies $T(\hat{\mathcal{F}}) = \widehat{T(\mathcal{F})}$.

Let X be another finite G-complex, and f: $X \to Y$ a skeletal G-map. Then f induces the covering f^*p: $\tilde{X} = f^*\tilde{Y} \to X$,

$$\tilde{X} = \{(x, b) \in X \times \tilde{Y} \mid f(x) = p(b)\},$$

$(f^*p)(x, b) = x$ for $(x, b) \in \tilde{X}$. $\tilde{G}$ acts on $\tilde{X}$ by $g(x, b) = (\sigma(g)x, gb)$ for $g \in \tilde{G}$, $(x, b) \in \tilde{X}$. $\tilde{X}$ has the CW-complex structure induced by f^*p, and becomes a $\tilde{G}$-complex. A $\tilde{G}$-map $\tilde{f}$: $\tilde{X} \to \tilde{Y}$ is given by $\tilde{f}(x, b) = b$ for $(x, b) \in \tilde{X}$, and $\tilde{f}$ is skeletal.

Lemma 3.3. *In the above situation, $\mathcal{F}_{\tilde{f}} = T(\mathcal{F}_f)$ and $\mathcal{F}_f = M(\mathcal{F}_{\tilde{f}})$.*

Proof. Firstly we show $M(\mathcal{F}_{\tilde{f}}) \subset \mathcal{F}_f$. For $\alpha \in \mathcal{F}_{\tilde{f}}$, (i) $\tilde{\rho}(\alpha) \in \mathrm{Iso}\,(|\alpha|)$ or (ii) $\tilde{\rho}(\alpha) \in \mathrm{Iso}\,(\tilde{X}_\alpha)$. Assume the case (i). There exists a point $b \in |\alpha|$ with $\tilde{G}_b = \tilde{\rho}(\alpha)$. We have $G_{p(b)} = \sigma(\tilde{G}_b) = \sigma(\tilde{\rho}(\alpha)) = \rho(\mu(\alpha))$, and $\rho(\mu(\alpha)) \in \mathrm{Iso}\,(|\mu(\alpha)|)$. Thus $\mu(\alpha) \in \mathcal{F}_f$. Assume the case (ii). There exists a point $(x, b) \in \tilde{X}_\alpha$ with

$\tilde{G}_{(x,b)}=\tilde{\rho}(\alpha)$. By definition $\tilde{X}_\alpha=\tilde{X}^{\tilde{\rho}(\alpha)}\cap\tilde{f}^{-1}(|\alpha|)=\{(x',b')\in X\times\tilde{Y}\,|\,f(x')=p(b'),$ $x'\in X^{\sigma(\tilde{\rho}(\alpha))}, b'\in|\alpha|\}$. We have $\tilde{G}_{(x,b)}=\sigma^{-1}(G_x)\cap\tilde{G}_b$, and $\rho(\mu(\alpha))=\sigma(\tilde{\rho}(\alpha))=$ $\sigma(\tilde{G}_{(x,b)})=G_x$. Since $f^{-1}(|\mu(\alpha)|)=f^{-1}(p(|\alpha|))$, $x\in X^{\rho(\mu(\alpha))}\cap f^{-1}(p(|\alpha|))=X_{\mu(\alpha)}$. Thus $\mu(\alpha)\in\mathcal{F}_f$. We have $M(\mathcal{F}_{\tilde{f}})\subset\mathcal{F}_f$.

Secondly we show $T(\mathcal{F}_f)\subset\mathcal{F}_{\tilde{f}}$. For $\alpha\in\mathcal{F}_f$, (iii) $\rho(\alpha)\in\mathrm{Iso}(|\alpha|)$ or (iv) $\rho(\alpha)\in\mathrm{Iso}(X_\alpha)$. Assume the case (iii). There exists a point $a\in|\alpha|$ with $\rho(\alpha)=G_a$. Fix $\beta\in\tau(\alpha)$ and $b\in|\beta|\cap p^{-1}(a)$. $\tilde{G}_b$ contains $\tilde{\rho}(\beta)$. Since $\sigma\colon\tilde{G}_b\to G_a$ is bijective, $\sigma(\tilde{\rho}(\beta))=\rho(\alpha)=G_a$ implies $\tilde{G}_b=\rho(\beta)$. Thus $\tilde{\rho}(\beta)\in\mathrm{Iso}(|\beta|)$, and $\beta\in\mathcal{F}_{\tilde{f}}$. We have $\tau(\alpha)\subset\mathcal{F}_{\tilde{f}}$. Assume the case (iv). There exists a point $x\in X_\alpha$ with $G_x=\rho(\alpha)$. Fix $\beta\in\tau(\alpha)$ and $b\in|\beta|$ with $f(x)=p(b)$. Then $(x,b)\in\tilde{X}_\beta$. The isomorphism $\sigma\colon\tilde{G}_b\to G_{p(b)}$ maps both $\tilde{\rho}(\beta)$ and $\tilde{G}_{(x,b)}=\sigma^{-1}(G_x)\cap\tilde{G}_b$ to G_x. We get $\tilde{\rho}(\beta)=\tilde{G}_{(x,b)}$, and $\beta\in\mathcal{F}_{\tilde{f}}$. Thus $\tau(\alpha)\subset\mathcal{F}_{\tilde{f}}$. We have $T(\mathcal{F}_f)\subset\mathcal{F}_{\tilde{f}}$.

By Proposition and definition 3.1, we have $\mathcal{F}_{\tilde{f}}=T(\mathcal{F}_f)$ and $\mathcal{F}_f=M(\mathcal{F}_{\tilde{f}})$.

Let $\tilde{\mathcal{F}}$ be a $\tilde{G}$-family in $\tilde{\Pi}$, $\tilde{Z}$ an $\tilde{\mathcal{F}}$-complex. The quotient space $Z=\tilde{Z}/\pi$ has a Π-complex structure given by

$$Z_\alpha=\Big(\bigcup_{\beta\in\tau(\alpha)}\tilde{Z}_\alpha\Big)/\pi,\ \alpha\in\Pi\,.$$

Moreover Z becomes a $M(\tilde{\mathcal{F}})$-complex. For $\alpha\in\Pi$ we have

$$\chi(Z_\alpha)-1=(\chi(\tilde{Z}_\beta)-1)/|\pi_\beta|\,, \tag{3.4}$$

where β is an arbitrary element of $\tau(\alpha)$.

The correspondence $\tilde{Z}\to Z$ defines a homomorphism $\nu\colon\Omega(\tilde{G},\tilde{\mathcal{F}})\to\Omega(G,M(\tilde{\mathcal{F}}))$. By (3.4) ν is injective. If $\tilde{\mathcal{F}}'\subset\tilde{\mathcal{F}}$ then the following diagram is commutative:

$$\begin{tikzcd}
\Omega(\tilde{G},\tilde{\mathcal{F}}') \arrow[r] \arrow[d, "\nu"] & \Omega(\tilde{G},\tilde{\mathcal{F}}) \arrow[d, "\nu"] \\
\Omega(G,M(\tilde{\mathcal{F}}')) \arrow[r] & \Omega(G,M(\tilde{\mathcal{F}}))\,,
\end{tikzcd}$$

where the horizontal arrows are the canonical maps.

Proposition 3.5. *Let $\mathcal{F}$ be a G-family in Π, and put $\tilde{\mathcal{F}}=T(\mathcal{F})$. Then $\nu\colon\Omega(\tilde{G},\tilde{\mathcal{F}})\to\Omega(G,\mathcal{F})$ is an isomorphism.*

Proof. It is sufficient to show that ν is surjective. Arbitrarily fix $\alpha\in\mathcal{F}$. Put $X=G/H$, $H=\rho(\alpha)$. There is a G-map $f\colon X\to Y$ with $f(1\cdot H)\in Y_\alpha$. Let $\tilde{f}\colon\tilde{X}=f^*\tilde{Y}\to\tilde{Y}$ be the induced $\tilde{G}$-map. $(\tilde{X})^+$ has a $\tilde{\Pi}$-complex structure induced by f. Take a point $(1\cdot H,b)\in\tilde{X}$, so $f(1\cdot H)=p(b)$, and put $\beta=\min.\{\gamma\in\tilde{\Pi}\,|\,\tilde{X}_\gamma\ni(1\cdot H,b)\}$, (that is, $(1\cdot H,b)$ is a point of a cell of type β). Since $\tilde{G}$ acts transitively on $\tilde{X}$, $(\tilde{X})^+$ is a $\{g\beta\,|\,g\in\tilde{G}\}$-complex. Since $\mu(\beta)=\alpha$, $(\tilde{X})^+$

is an $\tilde{\mathcal{F}}$-complex. An easy calculation shows $((\tilde{X})^+/\pi)_\gamma \subset (X^+)_\gamma$ for any $\gamma \in \Pi$. Observing $((\tilde{X})^+/\pi)^K$ for $K \leqq G$, we have $((\tilde{X})^+/\pi)_\gamma = (X^+)_\gamma$ for any $\gamma \in \Pi$. Since $X^+ = (\alpha)$ by 2.5, we have $(\tilde{X})^+/\pi = (\alpha)$ as a Π-complex. Since $\Omega(G, \mathcal{F})$ is generated by (α)'s, ν is surjective.

Proposition 3.6. *Let $\mathcal{F}$ be a G-family in Π, and $\tilde{\mathcal{F}} = T(\mathcal{F})$. Then we have $\nu(\Delta(\tilde{G}, \tilde{\mathcal{F}})) \subset \Delta(G, \mathcal{F})$.*

Proof. Let $\tilde{Z}$ be a contractible $\tilde{\mathcal{F}}$-complex. Then $(\tilde{Z}, *)$ is a π-cofibering pair and $\tilde{Z} \backslash \{*\}$ is a numerable π-space. $\tilde{Z}$ is π-contractible, and $\tilde{Z}/\pi$ is contractible. By Proposition 1.6 of [6] we have $\nu(\Delta(\tilde{G}, \tilde{\mathcal{F}})) \subset \Delta(G, \mathcal{F})$.

4. Proofs of the main results

In this section we let Y be a finite connected G-complex with finite $\pi_1(Y)$, $p: \tilde{Y} \to Y$ the universal covering, and put $\tilde{G} = \pi_1(EG \times_G Y)$ and $\pi = \pi_1(Y)$. As was described in section 1, $\tilde{Y}$ has the standard action of $\tilde{G}$. We use the notations in section 3 for this situation.

For a G-family $\mathcal{F}$ in $\Pi = \Pi(Y)$, we define a subgroup $\Delta_h(G, Y, \mathcal{F})$ of $\Omega(G, \mathcal{F})$ by

$$\Delta_h(G, Y, \mathcal{F}) = \nu(\Delta(\tilde{G}, T(\mathcal{F}))) .$$

By Proposition 3.6 $\Delta_h(G, Y, \mathcal{F})$ is a subgroup of $\Delta(G, \mathcal{F})$.

Proposition 4.1. *Let $\mathcal{F}$ be a connected G-family in Π containing $\mathcal{F}(Y)$. Assume $T(\hat{\mathcal{F}})$ is simply generated, then*

$$\Delta_h(G, Y, \mathcal{F}) = \{[M_f] \in \Omega(G, \Pi) \mid f: X \to Y \text{ is a pseudo-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\} .$$

Proof. By Lemma 3.3 we have $\mathcal{F}(\tilde{Y}) = T(\mathcal{F}(Y)) \subset T(\mathcal{F})$. By Lemma 3.2 (ii) and Proposition 2.8 we have

$$\Delta(\tilde{G}, T(\mathcal{F})) = \{[M_{\tilde{f}}] \in \Omega(\tilde{G}, \tilde{\Pi}) \mid \tilde{f}: \tilde{X} \to \tilde{Y} \text{ is a quasi-equivalence such that } (\tilde{X})^+ \text{ is a } T(\mathcal{F})\text{-complex}\} .$$

Since $\tilde{Y}$ is a numerable π-space, $\tilde{f}$ is a π-homotopy equivalence. Thus the induced map $f: X = \tilde{X}/\pi \to Y$ is a homotopy equivalence. On the other hand $\nu([M_{\tilde{f}}]) = [M_f]$. Through the map ν we have the consequence of the above proposition.

For a moment we assume Theorem 1 and prove the corollaries.

Proof of Corollary 2. We may assume F is not empty. In this case $\tilde{G}$ is a semi-direct product of G by π as is well known. Let $\alpha_1, \cdots, \alpha_k$ be the elements of Π such that $|\alpha_i|=F_i$ and $\rho(\alpha_i)=G$, $i=1, \cdots, k$. Oliver-Petrie defined a homomorphism ψ: $\Omega(G, \Pi)\to \boldsymbol{Z}^k$ by

$$\psi([Z]) = (\chi(Z_{\alpha_1})-1, \cdots, \chi(Z_{\alpha_k})-1)$$

for a Π-complex Z. The image of $\Delta(G, \Pi)$ by ψ is denoted by n_Y. We define N_Y as the image of $\Delta_h(G, Y, \Pi)$ by ψ. Thus N_Y is a subgroup of n_Y. Put $\mathcal{F}=\Pi$ and $\mathcal{F}'=\{\alpha\in\Pi \mid \rho(\alpha)\neq G\}$. Then for $\alpha\in\hat{\mathcal{F}}$ $\rho(\alpha)$ is of prime power order. For $\alpha\in T(\hat{\mathcal{F}})$ $\tilde{\rho}(\alpha)$ is isomorphic to $\rho(\mu(\alpha))$, so $\tilde{\rho}(\alpha)$ is of prime power order. By Corollary 4.14 of [6] $T(\hat{\mathcal{F}})$ is simply generated. Put $f'=\mathrm{incl}\circ\hat{f}$: $F'\to Y$. Since $\ker\psi$ is $\Omega(G, \mathcal{F}')$, we have $[M_{f'}]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}')$ if and only if $\psi([M_{f'}])\in N_Y$. On the other hand $\psi([M_{f'}])=(\chi(F_1)-\chi(F_1'), \cdots, \chi(F_k)-\chi(F_k'))$. Thus we have the conclusion of Corollary 2.

Proof of Corollary 3. Since F is connected, $n_Y=n_G\boldsymbol{Z}$. By the assumption $G\in\mathcal{G}^1$, $n_Y=\{0\}$ (see [5; p. 171]). We obtain $N_Y=\{0\}$. Corollary 2 yields Corollary 3.

Proof of Theorem 1. Let $q=f*p$: $\tilde{X}=f*\tilde{Y}\to X$ be the induced covering and $\tilde{f}$: $\tilde{X}\to\tilde{Y}$ the induced map by f. Since $\mathcal{F}\supset\mathcal{F}_f\supset\mathcal{F}(Y)$, $T(\mathcal{F})$ is connected.

Firstly we assume f is extendible to f': $X'\to Y$ as was mentioned in Theorem 1, (we may assume f' is skeletal without loss of generality). Let $\tilde{f}'$: $\tilde{X}'=f'\tilde{Y}\to\tilde{Y}$ be the induced map by f'. Since f' is a homotopy equivalence, $\tilde{f}'$ is a π-homotopy equivalence. If we show $\tilde{X}'/\tilde{X}$ is a $T(\mathcal{F}')$-complex, we have $[M_{\tilde{f}}]\in\Delta(\tilde{G}, T(\mathcal{F}))+\Omega(\tilde{G}, T(\mathcal{F}'))$ by Theorem 3.2 of [6]. Through the map ν we have $[M_f]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}')$. So we prove $\tilde{X}'/\tilde{X}$ is a $T(\mathcal{F}')$-complex. For a cell c in $\tilde{X}'\backslash\tilde{X}$, let $\alpha\in\tilde{\Pi}$ be the type of c. The isotropy group on c is $\tilde{\rho}(\alpha)$ and that on $q'(c)$ is $\sigma(\tilde{\rho}(\alpha))$, where $q'=f'*p$: $\tilde{X}'\to X'$. Since $\tilde{f}'(c)\subset|\alpha|, f'(q'(c))\subset|\mu(\alpha)|$. The type of $q'(c)$ is $\mu(\alpha)$. By the assumption X'/X is a $\mathcal{F}'$-complex, and so $\mu(\alpha)\in\mathcal{F}'$. Thus $\alpha\in T(\mathcal{F}')$. This means that $\tilde{X}'\backslash\tilde{X}$ is a $T(\mathcal{F}')$-complex.

Secondly we assume $[M_f]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}')$. Since ν: $\Omega(\tilde{G}, T(\mathcal{F}))\to\Omega(G, \mathcal{F})$ is injective and $\nu(\Omega(\tilde{G}, T(\mathcal{F}')))=\Omega(G, \mathcal{F}')$ by Proposition 3.5, we have $[M_{\tilde{f}}]\in\Delta(\tilde{G}, T(\mathcal{F}))+\Omega(\tilde{G}, T(\mathcal{F}'))$. By Theorem 3.2 of [6] there exist a finite $\tilde{G}$-complex $\tilde{X}'\supset\tilde{X}$ and a (skeletal) pseudo-equivalence $\tilde{f}'$: $\tilde{X}\to\tilde{Y}$ extending $\tilde{f}$ such that $\tilde{X}'/\tilde{X}$ is a $T(\mathcal{F}')$-complex. Since $\tilde{Y}$ is a numerable π-space, $\tilde{f}'$ is a π-homotopy equivalence. Put $X'=\tilde{X}'/\pi$. Then $X'\supset X$ and the induced map f': $X'\to Y$ by $\tilde{f}'$ is a homotopy equivalence. Moreover X'/X is an $\mathcal{F}'$-complex by the similar argument used in the first part. This completes the proof.

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