Sumioka, T.
Osaka J. Math.
21 (1984), 629–648

TACHIKAWA'S THEOREM ON ALGEBRAS OF LEFT COLOCAL TYPE

Dedicated to Professor Hirosi Nagao on his 60th birthday

TAKESHI SUMIOKA

(Received December 14, 1982)

Introduction

Let A be an artinian ring. Then A is said to be of right local type if any finitely generated indecomposable right A-module M is local (i.e. M has a unique maximal submodule) and a ring of left colocal type is defined as the dual notion. We say A is left serial if a left A-module A is a direct sum of uniserial submodules. Tachikawa [4, 5] gave characterizations of algebras of right local (or equivalently of left colocal) type.

Theorem (Tachikawa). *For a finite dimensional algebra A with the Jacobson radical N, the following conditions* (a)–(d) *are equivalent.*

(a) *A is of right local type.*

(b) *A is of left colocal type.*

(c) (c_1) *A is left serial.*

(c_2) *For any uniserial left A-modules L_1 and L_2 with $|L_1| \leq |L_2|$, any isomorphism θ: $S_1(L_1) \to S_1(L_2)$ is (L_1, L_2)-maximal or (L_1, L_2)-extendible (see Section* 1 *for the definitions), where $|L_i|$ is the composition length of L_i and $S_1(L_i)$ is the socle of L_i for $i=1, 2$.*

(c_3) *$|eN/eN^2| \leq 2$ for any primitive idempotent e of A.*

(d) (d_1) *A is left serial.*

(d_2) *$eN = M_1 \oplus M_2$ for any primitive idempotent e of A, where M_i is either zero or a uniserial submodule of the right A-module eN for each $i=1, 2$.*

More precisely Tachikawa [4] gave a proof of the equivalence of (b) and (c) for *any* artinian ring. But in the proof of the implication from (c) to (b), there were two gaps. He himself pointed out one of them, namely [4, Lemma 4.9], and informed Fuller of it and that the lemma holds for any artinian ring under a suitable assumption (D) which is satisfied for any finite dimensional algebra over a field (cf. Section 3 for the definition of (D). See also Fuller [3, Note p. 165].). Now the other one (which is related to [4, Corollary 4, 6]) can be filled with an elementary lemma (i.e. Lemma 1.1 below, which is essentially used in [5, Proposition 4, 2]) under the additional assumption (D).

In Section 3 we shall give a self-contained proof for the above stated implication from (c) to (b). On the other hand we shall point out in Section 4 that the equivalence of (c) and (d) holds for any artinian rings. Unfortunately it remains open whether any ring of colocal type satisfies (D), however in the last Section we shall give an example of an artinian ring which satisfies (c) but not (b) and remark that some simultaneous equations with 6-unknowns are closely related to this problem.

For the sake of completeness we shall also give a proof of the implication from (b) to (c) together with proofs for results which have been shown in [4] and [1].

Throughout this paper A is a left and right artinian ring with unity, N is the Jacobson radical of A and all modules are finitely generated (unitary) left A-module unless otherwise stated. For a module M, we denote the top M/NM of M by $\bar{M}$, the composition length of M by $|M|$. For any integer $i\geq 0$ we define a submodule $S_i(M)$ of a module M inductively as following: $S_0(M)=0$ and $S_i(M)/S_{i-1}(M)$ is the socle of $M/S_{i-1}(M)$. We denote by $p(A)$ the set of primitive idempotents of A. Symbols (a), $\cdots$, (d) always mean the conditions in the theorem above.

The author wishes to express his thanks to Professor H. Tachikawa for his valuable advice.

1. Preliminaries

Let M_1 and M_2 be modules with submodules T_1 and T_2, respectively. If a homomorphism φ: $M_1\rightarrow M_2$ canonically induces a map $T_1\rightarrow T_2$, the map is also denoted by φ: $T_1\rightarrow T_2$. Let θ: $T_1\rightarrow T_2$ be a homomorphism. We say θ is (M_1, M_2)-*extendible* if θ is induced from some homomorphism φ: $M_1\rightarrow M_2$, and in this case φ is an extension of θ. We say θ is (M_1, M_2)-*maximal* if there is no module U such that $T_1\subsetneqq U\subset M_1$ and θ is (U, M_2)-extendible. In case $T=T_1=T_2$ and θ is 1_T the identity map of T, we simply say T is (M_1, M_2)-*extendible* (resp.-*maximal*) if 1_T is (M_1, M_2)-extendible (resp.-maximal).

The following lemma is clear.

Lemma 1.1. *Let M_1, M_2 and T be submodules of a module M such that $M=M_1+M_2$ and $T=M_1\cap M_2$. If T' is a submodule of T and φ: $M_1\rightarrow M_2$ is an extension of $1_{T'}$, then for $M_1'=\{x-x\varphi\,|\,x\in M_1\}$ the following hold.*

(1) $M=M_1'+M_2$.

(2) $M_1'\cap M_2=\{x-x\varphi\,|\,x\in T\}$.

(3) *The epimorphism $M_1\rightarrow M_1'$ defined by $x\rightarrow(x-x\varphi)$; $x\in M_1$, induces epimorphisms $M_1/T'\rightarrow M_1'$ and $T/T'\rightarrow M_1'\cap M_2$, in particular $|M_1'\cap M_2|\leq|T|-|T'|$.*

The following lemmas 1.2 and 1.3 are due to Tachikawa [4, Lemma 1.3 and Lemma 4.4].

Lemma 1.2. *Let M_1, M_2 and T be submodules of a module M such that $M=M_1+M_2$ and $T=M_1\cap M_2$. Then*

(1) *T is (M_1, M_2)-extendible if and only if $M=M_1'\oplus M_2$ for some submodule M_1' of M.*

(2) *T is (M_1, M_2)-maximal if and only if $S_1(M)=S_1(M_2)$.*

Proof. (1) 'Only if' part is immediate from Lemma 1.1. If $M=M_1'\oplus M_2$, then the restriction map $\pi_2\colon M_1\to M_2$ of the projection $\pi_2\colon M_1'\oplus M_2\to M_2$ is clearly an extension of 1_T.

(2) 'Only if' part: Assume $S_1(M)=U'\oplus S_1(M_2)$ for a non-zero module U'. Since $U'\cap M_2=0$ and $U'\oplus M_2\supset M_2$, $U'\oplus M_2=(M_1+M_2)\cap(U'\oplus M_2)=(M_1\cap(U'\oplus M_2))+M_2$. Put $U=M_1\cap(U'+M_2)$. Then we have $U+M_2=U'\oplus M_2$, $T=U\cap M_2$ and $T\subsetneqq U\subset M_1$. Applying (1) to $U+M_2$, T is (U, M_2)-extendible.

'If' part: Assume $\varphi\colon U\to M_2$ is an extension of 1_T with $T\subsetneqq U\subset M_1$. From (1) we have $U+M_2=U'\oplus M_2$ for some module $U'\neq 0$. Thus $S_1(M)\supset S_1(U')\oplus S_1(M_2)\supsetneqq S_1(M_2)$.

Lemma 1.3. *Let M_i $(i=1, 2, 3)$ and T be submodules of a module M such that $M=M_1+(M_2\oplus M_3)$ and $T=M_1\cap(M_2\oplus M_3)$, and $\pi_3\colon T\to M_3$ the restriction map of the projection $M_2\oplus M_3\to M_3$. Then π_3 is (M_1, M_3)-extendible if and only if $M=(M_1'+M_2)\oplus M_3$ for some submodule M_1' of M.*

Proof. This is shown by the method similar to the proof of (1) in Lemma 1.2.

Let M and P_i $(i=1, \cdots, n)$ be modules. Then a map $\varphi\colon M\to\bigoplus_{i=1}^{n}P_i$ has a matrix representation $\varphi=(\varphi_1, \cdots, \varphi_n)$ by the composition maps $\varphi_j\colon M\to P_j$ of $\varphi\colon M\to\bigoplus_{i=1}^{n}P_i$ and the projections $\bigoplus_{i=1}^{n}P_i\to P_j$. Similarly a map $\psi\colon\bigoplus_{i=1}^{n}P_i\to M$ has a matrix representation $\psi=(\psi_1, \cdots, \psi_n)^T$ (the transposed matrix of $(\psi_1, \cdots, \psi_n)$) by the maps $\psi_i\colon P_i\to M$. For idempotents e and f of A, we assume that $u\in t(eN^{r-1}f)$ means $eN^{r-1}f\supsetneqq eN^rf$, $u\in eN^{r-1}f$ and $u\notin eN^rf$.

Let $u_i\in t(eN^{r-1}f_i)$, where $e, f_i\in p(A)$ and $i=1, \cdots, n$. Denote a residue class of $x\in eA$ in eA/eN^r by $\bar{x}$ and that of $y\in Af_i$ in Af_i/N^rf_i by $[y]_i$ or simply by $[y]$.

Lemma 1.4. *Let $u_i\in t(eN^{r-1}f_i)$ and put $P_i=Af_i/N^rf_i$ for an integer $r\geq 1$, where e is an idempotent of A, f_i is a primitive idempotent and $i=1, \cdots, n$. Then under the above notation, the following conditions are equivalent.*

(1) $(\bar{u}_1A+\cdots+\bar{u}_{n-1}A)\cap\bar{u}_nA\neq 0$.

(2) $\bar{u}_n \in \bar{u}_1 A + \cdots + \bar{u}_{n-1} A$.

(3) *There is a homomorphism* $\psi\colon \bigoplus_{i=1}^{n-1} P_i \to P_n$ *such that* $(\sum_{i=1}^{n-1} [u_i]_i)\psi = [u_n]_n$.

(4) *There is a homomorphism* $\varphi\colon \bigoplus_{i=1}^{n} P_i \to P_n$ *such that* $(\sum_{i=1}^{n} [u_i]_i)\varphi = 0$ *and* φ_n *is an identity map, where* $\varphi = (\varphi_1, \cdots, \varphi_n)^T$.

Proof. The equivalences (1)$\Leftrightarrow$(2) and (3)$\Leftrightarrow$(4) are clear since $\bar{u}_n A$ is a simple module.

(2)$\Leftrightarrow$(3). Note that any homomorphism $P_i \to P_n$ is induced from a right multiplication map $\tilde{a}_i\colon Af_i \to Af_n$ by $a_i \in A$ with $a_i = f_i a_i f_n$. The condition (2) is equivalent to one that there are elements $a_i = f_i a_i f_n$ of A, $i = 1, \cdots, n-1$, with $\bar{u}_n = \bar{u}_1 a_1 + \cdots + \bar{u}_{n-1} a_{n-1}$ which is equivalent to $[u_n] = [u_1 a_1] + \cdots + [u_{n-1} a_{n-1}]$. This shows the equivalence of (2) and (3).

We say that a module M is uniserial if M has a unique composition series, and an artinian ring A is left serial if a left A-module A is a direct sum of uniserial submodules.

The following corollaries immediate from Lemma 1.4, noting $A[u_1] \simeq A[u_2] \simeq \overline{Ae}$ in Corollary 1.6.

Corollary 1.5. *Let A be a left serial ring, e a primitive idempotent of A and r and n integers≥ 1. Then the following conditions are equivalent.*

(1) $|\overline{eN^{r-1}}| < n$.

(2) *If* $P_1, \cdots, P_n$ *are uniserial modules with* $|P_i| = r$ *and* $\alpha = (\alpha_1, \cdots, \alpha_n)\colon \overline{Ae} \to \bigoplus_{i=1}^{n} P_i$ *is a map with monomorphism* α_i *for each* $i = 1, \cdots, n$, *then there exists a map* $\varphi = (\varphi_1, \cdots, \varphi_n)^T\colon \bigoplus_{i=1}^{n} P_i \to P_j$ *for some* j $(1 \leq j \leq n)$ *such that* $\alpha\varphi = 0$ *and* φ_j *is an identity map.*

Corollary 1.6. *An artinian ring A is right serial if and only if for any* $u_i \in t(eN^{r-1}f_i)$ $(i = 1, 2)$ *the isomorphism* $\theta\colon A[u_1] \to A[u_2]$ *with* $[u_1]\theta = [u_2]$ *is* (P_1, P_2)-*extendible, where* $e, f_i \in p(A)$, $P_i = Af_i/N^r f_i$ *and* $[u_i] = u_i + N^r f_i \in P_i$. *In particular, a left serial ring A is (left and right) serial if and only if for any uniserial modules L_1 and L_2 with* $|L_1| \leq |L_2|$, *any isomorphism* $\theta\colon S_1(L_1) \to S_1(L_2)$ *is* (L_1, L_2)-*extendible.*

2. The implication from (b) to (c)

The results in this section were essentially delt with in [1] (see [1, Theorem 2.5 and Remark 4]).

Let (E): $0 \to T \xrightarrow{\alpha} \bigoplus_{i=1}^{n} P_i \xrightarrow{\beta} M \to 0$ be an exact sequence of modules with monomorphism $\alpha_i\colon T \to P_i$ for each $i = 1, \cdots, n$, where $n \geq 2$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta =$

$(\beta_1, \cdots, \beta_n)^T$. Put $L_i = P_i\beta$. Let α'_j: $T \to \bigoplus_{i\neq j} P_i$ and β'_j: $\bigoplus_{i\neq j} P_j \to M$ denote maps induced from α and β, respectively. Then as easily seen β_j and β'_j are monomorphisms for each j and in particular $P_i \simeq L_i$ and $\sum_{i\neq j} L_i = \bigoplus_{i\neq j} L_i$. Moreover for any non-trivial partition $I = I_1 \cup I_2$ of $I = \{1, \cdots, n\}$ (i.e. I_1, $I_2 \subsetneqq I$ and $I_1 \cap I_2 = \phi$) we have $\oplus_{I_1} L_i \cap \oplus_{I_2} L_i \simeq T$.

Conversely let T be a module and $M = \sum_{i=1}^{n} L_i$ a sum of submodules L_i of a module M with the following property:

(A) For each $j = 1, \cdots, n$, $\sum_{i\neq j} L_i = \bigoplus_{i\neq j} L_i$ and for some non-trivial partition $\{1, \cdots, n\} = I_1 \cup I_2$, $\oplus_{I_1} L_i \cap \oplus_{I_2} L_i \simeq T$.

Put $P_i = L_i$ and let β: $\bigoplus_{i=1}^{n} P_i \to M = \sum_{i=1}^{n} L_i$ be a canonical map (i.e. $(x_1, \cdots, x_n)\beta = \sum_{i=1}^{n} x_i$; $x_i \in P_i$). Then it is easy to see that we have an exact sequence (E) with monomorphism α_i and $L_i = P_i\beta$ as above. We say a sum $M = \sum_{i=1}^{n} L_i$ of submodules L_i with $n \geq 2$ is a *T-amalgamated sum* (by (E)) if it has the property (A) (and $L_i = P_i\beta$ in the exact sequence (E)).

REMARK 1. Consider the above exact sequence (E) and put $T_j = L_j \cap \bigoplus_{i\neq j} L_j$. Then we have commutative diagrams

$$\begin{array}{ccc} \bigoplus_{i\neq j} P_i & \xrightarrow{\beta'_j} & \bigoplus_{i\neq j} L_i \\ \uparrow & & \uparrow \\ T\alpha'_j & \xrightarrow{\beta'_j} & T_j \end{array} \qquad \begin{array}{ccc} P_j & \xrightarrow{\beta_j} & L_j \\ \uparrow & & \uparrow \\ T\alpha_j & \xrightarrow{\beta_j} & T_j \end{array}$$

with isomorphism rows and inclusion columns. Since α'_j: $T \to \bigoplus_{i\neq j} P_i$ and α_j: $T \to P_j$ are monomorphisms, a map θ: $T\alpha'_j \to T\alpha_j$ defined by $t\alpha'_j\theta = t\alpha_j$ $(t \in T)$ is well-defined and an isomorphism. Moreover we have $(t\alpha'_j)(-\theta)\beta_j = -t\alpha_j\beta_j = (t\alpha'_j)\beta'_j 1_{T_j}$; $t \in T$. Therefore it follows from the above diagrams that θ orequivalently $-\theta$ is $(\bigoplus_{i\neq j} P_i, P_j)$-extendible (resp.-maximal) if and only if T_j is $(\bigoplus_{i\neq j} L_i, L_j)$-extendible (resp.-maximal).

Lemma 2.1. *Let S be a simple module and $L_1, \cdots, L_n$ local submodules of a module M such that $M = \sum_{i=1}^{n} L_i$ is an S-amalgamated sum, where $n \geq 2$ and $|L_i| \geq 2$ for each $i = 1, \cdots, n$. Then M is decomposable if and only if S_j is $(\bigoplus_{i\neq j} L_i, L_j)$-extendible for some j, $1 \leq j \leq n$, where $S_j = (\bigoplus_{i\neq j} L_i) \cap L_j$.*

Proof. Assume M has a non-trivial decomposition $M = M_1 \oplus M_2$. If σ: $M \to \bar{M} = M/NM$ is a canonical epimorphism, $L_i\sigma$ is simple and we have

$\bar{M}=L_1\sigma\oplus\cdots\oplus L_n\sigma=M_1\sigma\oplus M_2\sigma$ by the assumption. Then by [1, Lemma 1.1] there exists a non-trivial partition $\{1, \cdots, n\}=I_1\cup I_2$ such that $\bar{M}=M_1\sigma\oplus(\oplus_{I_2}L_i\sigma)=(\oplus_{I_1}L_i\sigma)\oplus M_2\sigma$. Hence we have $M=M_1+(\oplus_{I_2}L_i)=(\oplus_{I_1}L_i)+M_2$, for NM is small in M. But it holds $2|M|=(\sum_{i=1}^{n}|L_i|-1)+(|M_1|+|M_2|)$ since $|S|=1$. This shows $M=M_1\oplus(\oplus_{I_2}L_i)$ or $M=(\oplus_{I_1}L_i)\oplus M_2$. Thus L_j is a direct summand of M for some j, which implies S_j is $(\underset{i\neq j}{\oplus}L_i, L_j)$-extendible by Lemma 1.2. The converse is also immediate from Lemma 1.2.

REMARK 2. 'Only if part' of Lemma 2.1 is essentially used in Proposition 2.4 for $n=2$ or 3. In the case $n=2$ or 3, Lemma 2.1 is shown by applying the Krull-Schmidt Theorem instead of [1, Lemma 1.1].

Corollary 2.2. *Let S be a simple module and P_i a local module with $|P_i|\geq 2$ for each $i=1, \cdots, n$. Assume* (E): $0\to S\xrightarrow{\alpha}\bigoplus_{i=1}^{n}P_i\xrightarrow{\beta}M\to 0$ *is an exact sequence of modues with monomorphisms α_i, where $\alpha=(\alpha_1, \cdots, \alpha_n)$. Then the following conditions are equivalent.*

(1) *M is decomposable.*

(2) *There is a homomorphism ψ: $\underset{i\neq j}{\oplus}P_i\to P_j$ for some j such that $\alpha'_j\psi=\alpha_j$, where α'_j: $S\to\underset{i\neq j}{\oplus}P_i$ is a map induced from α.*

(3) *There is a homomorphism φ: $\bigoplus_{i=1}^{n}P_i\to P_j$ for some j such that $\alpha\varphi=0$ and φ_j is an identity map, where $\varphi=(\varphi_1, \cdots, \varphi_n)^T$.*

Proof. Each condition of (1), (2) and (3) implies $n\geq 2$. Hence, considering the S-amalgamated sum by the exact sequence (E), the corollary is immediate from Lemma 2.1 (see Remark 1).

Corollary 2.3. *Let $u_i\in t(eN^{r-1}f_i)$ for $r\geq 2$ and put $S=\overline{Ae}$ and $P_i=Af_i/N^rf_i$, where $e, f_i\in p(A)$ and $i=1, \cdots, n$. Let α_i: $S\to P_i$ denote the monomorphism defined by $[ae]\alpha_i=[aeu_i]$; $ae\in Ae$, where $[-]$ is a residue class in S or P_i. If $0\to S\xrightarrow{\alpha}P_1\oplus\cdots\oplus P_n\xrightarrow{\beta}M\to 0$ is an exact sequence with $\alpha=(\alpha_1, \cdots, \alpha_n)$, then the following conditions are equivalent.*

(1) *M is indecomposable.*

(2) *$\bar{u}_1A\oplus\cdots\oplus\bar{u}_nA\subset\overline{eN^{r-1}}$, where $\bar{u}_i\in\overline{eN^{r-1}}$ is a residue class of u_i.*

Proof. This is immediate from Corollary 2.2 and Lemma 1.4.

We say that an artinian ring A is of *left colocal type* if any finitely generated indecomposable left A-module is colocal.

Proposition 2.4. *Let A be an artinian ring of left colocal type. Then A*

satisfies the condition (c) (i.e. (c_1), (c_2) *and* (c_3)).

Proof. (c_1) If $\overline{N^{r-1}f}\neq 0$ for $f\in p(A)$ and an integer $r\geq 1$, then by the assumption an indecomposable module Af/N^rf has a simple socle $S_1(Af/N^rf)$ which contains $\overline{N^{r-1}f}$. This shows $\overline{N^{r-1}f}$ is simple. Thus A is left serial.

(c_2) Let θ: $S_1(L_1)\to S_1L_2)$ be an isomorphism, where L_1 and L_2 are uniserial modules with $|L_1|\leq|L_2|$. Then as easily seen we may assume L_1 and L_2 are submodules of a module M such that $M=L_1+L_2$, $S=L_1\cap L_2$ is simple and θ is the identity map of S (see Remark 1). If θ is not (L_1, L_2)-maximal, then by Lemma 1.2 $S_1(M)\supsetneqq S_1(L_2)=S$. Hence M is not colocal, so M is decomposable by the assumption. Thus θ is (L_1, L_2)-extendible by Lemma 2.1.

(c_3) Suppose $|\overline{eN}|\geq 3$, where $e\in p(A)$. Then $\overline{eN}\supset\bar{u}_1A\oplus\bar{u}_2A\oplus\bar{u}_3A$ for some $u_i\in t(eNf_i)$; $f_i\in p(A)$. Then there exists an indecomposable module M such that $|S_1(M)|\geq 2$ by Corollary 2.3. This is a contradiction. Thus it holds $|\overline{eN}|\leq 2$ for each $e\in p(A)$.

3. The implication from (c) to (b) under a condition (D)

Throughout this section, assume that A is a left serial ring. In this case any local left A-module is quasi-projective. Let L be a uniserial module with $|L|=n$ and put $L_i=S_i(L)$ and $D_i(L)=\mathrm{Hom}(\bar{L}_i, \bar{L}_i)$ for each $i=1, \cdots, n$. Then $D_i(L)$ is a division ring. If $n\geq i\geq j\geq 1$, any element $\bar{\varphi}_i$: $\bar{L}_i\to\bar{L}_i$ of $D_i(L)$ is induced from a map φ_i: $L_i\to L_i$, and moreover φ_i induces an map $\bar{\varphi}_j$: $\bar{L}_j\to\bar{L}_j$. Now we define a map λ_{ij}: $D_i(L)\to D_j(L)$ by $(\bar{\varphi}_i)\lambda_{ij}=\bar{\varphi}_j$. Then as easily seen λ_{ij} are well-defined and ring monomorphisms with equalities $\lambda_{ij}\lambda_{jk}=\lambda_{ik}$ for all i, j and k $(n\geq i\geq j\geq k\geq 1)$. Hence through the maps λ_{ij}, we can regard a sequence $D_1(L)$, $D_2(L)$, $\cdots$, $D_n(L)$ as a descending chain

$$D_1(L)\supset D_2(L)\supset\cdots\supset D_n(L)$$

of division rings (cf. [4, p. 211]).

Lemma 3.1. *Let A be a left serial ring. For a uniserial module L with $|L|=n$ and an integer r with $1\leq r\leq n$, the following conditions are equivalent.*

(1) $D_r(L)=D_n(L)$.

(2) *Any isomorphism θ: $S_1(L)\to S_1(L)$ is (L, L)-extendible whenever θ is $(S_r(L), S_r(L))$-extendible.*

Proof. Put $L_i=S_i(L)$, $i=1, \cdots, n$ and let $\bar{\varphi}_r$: $\bar{L}_r\to\bar{L}_r$ be a map induced from an isomorphism φ_r: $L_r\to L_r$. As easily seen (1) is equivalent to a condition that there is a map ψ_n: $L_n\to L_n$ with $(L_r)(\varphi_r-\psi_n)\subset L_{r-1}$. Since L is uniserial, the last conditions is equivalent to $(L_1)(\varphi_r-\psi_n)=0$ which implies (2).

REMARK 3. For an integer $r\geq 2$, the condition (2) of Lemma 3.1 does not imply that any isomorphism φ_r: $S_r(L)\to S_r(L)$ is (L, L)-extendible (see Example 1).

It is called by *S_r-classes* isomorphism classes of uniserial modules with composition length r. Note that for e and f in $p(A)$ and an integer $r\geq 1$, $\overline{fA}$ is embedded in $\overline{eN^{r-1}}$ if and only if $\overline{Ae}$ is embedded in $\overline{N^{r-1}f}$, since these conditions are equivalent to $eN^{r-1}f/eN^rf\neq 0$.

Lemma 3.2. *Let A be a left serial ring and $e, f_1, \cdots, f_s$ and f be primitive idempotents with $f_iA\not\simeq f_jA$ for $i\neq j$. Then for any integer $r\geq 1$ the following hold.*

(1) *$\overline{f_1A}\oplus\cdots\oplus\overline{f_sA}$ is embedded in $\overline{eN^{r-1}}$ if and only if $L_i=Af_i/N^rf_i$ $(i=1, \cdots, s)$ satisfy $|L_i|=r$ and $S_1(L_i)\simeq\overline{Ae}$. (Thus in this case there are s S_r-classes whose socles are isomorphic to $\overline{Ae}$.)*

(2) *$(\overline{fA})^t$ (i.e. a direct sum of t-copies of $\overline{fA}$) is embedded in $\overline{eN^{r-1}}$ if and only if $\dim D_1(L)/_{D_r(L)}\geq t$ and $S_1(L)=\overline{N^{r-1}f}\simeq\overline{Ae}$, where $L=Af/N^rf$.*

Proof. (1) This is clear by the note above.

(2) Put $\overline{eN^{r-1}f}=eN^{r-1}f/eN^rf$ and $D=fAf/fNf$. Then $(\overline{fA})^t$ is embedded in $\overline{eN^{r-1}}$ if and only if $\dim\overline{eN^{r-1}f}_D\geq t$. By the above note, $S_1(L)=\overline{N^{r-1}f}\simeq\overline{Ae}$ if $\overline{fA}$ i sembedded in $\overline{eN^{r-1}}$. Therefore $D_1(L)=\mathrm{Hom}_A(\overline{N^{r-1}f}, \overline{N^{r-1}f})\simeq\mathrm{Hom}_A(\overline{Ae}, \overline{N^{r-1}f})\simeq\overline{eN^{r-1}f}$ as right D-modules. The restriction maps φ_1: $S_1(L)\to S_1(L)$ of maps φ_r: $L\to L$ coincide with the right multiplication maps by elements of D. Therefore we can identify $D_r(L)$ with D, so the assertion is immediate from the above D-isomorphisms.

Let S be a simple module and L a uniserial module with $|L|\geq 2$. Denote by $c(S)$ the number of S_2-classes whose socles are isomorphic to S and put $m(L)=\dim D_1(L)_{D_2(L)}$. The following lemma is easily seen by Lemma 3.2.

Lemma 3.3. *Let A be a left serial ring and e a primitive idempotent. Then $|\overline{eN}|\leq 2$ if and only if $c(S_1(L))+m(L)\leq 3$ for any uniserial module L with the conditions $|L|\geq 2$ and $S_1(L)\simeq\overline{Ae}$.*

Let S be a simple module. We call S of *first kind* if $m(L)=1$ (i.e. $D_1(L)=D_2(L)$) for any uniserial module L with $S\simeq S_1(L)\subsetneqq L$, and S of *second kind* if S is not if first kind. By Lemma 3.2 $\overline{Ae}$ is of first kind if and only if $\overline{eN}$ is (zero or) square free (i.e. a direct sum of pair-wise non-isomorphic simple modules).

Lemma 3.4. *Let A be a ring satisfying* (c) *and let L_1 and L_2 be uniserial modules with $|L_1|\leq|L_2|$ and $S=S_1(L_1)\simeq S_1(L_2)$.*

(1) *If $S_2(L_1)\simeq S_2(L_2)$, then L_1 can be embedded in L_2.*

(2) *If S is of first kind and $S_2(L_1)\simeq S_2(L_2)$, then any isomorphism θ: $S_1(L_1)$*

$\to S_1(L_2)$ *is* (L_1, L_2)*-extendible.*

(3) *If* S *is of second kind, then* L_1 *can be embedded in* L_2.

Proof. (1) is clear by (c_2), and (2) follows from Lemma 3.1 and (c_2). Moreover (3) is an immediate consequence of (1) since it holds $c(S)=1$ by Lemma 3.3.

Lemma 3.5. *Let* A *be a ring satisfying* (c) *and let* $P_1, \cdots, P_n$ *be uniserial modules with* $|P_i|\geq 2$ *and* α_i: $S\to P_i$ *a homomorphism for each* $i=1, \cdots, n$, *where* S *is a simple module and* $n\geq 3$. *If* $0\to S\xrightarrow{\alpha} P_1\oplus\cdots\oplus P_n\xrightarrow{\beta} M\to 0$ *is an exact sequence with* $\alpha=(\alpha_1, \cdots, \alpha_n)$, *then* M *is decomposable.*

Proof. We may assume that $|P_1|\leq|P_2|\leq\cdots\leq|P_n|$ and each α_i is a non-zero map. Put $S_i=S_1(P_i)$ and $P_i'=S_2(P_i)$ and consider an exact sequence $0\to S\xrightarrow{\alpha'} P_1'\oplus\cdots\oplus P_n'\xrightarrow{\beta'} M'\to 0$ induced from the above one. Then by (c_3), $n\geq 3$ and Corollary 1.5, there exists a map $\varphi'=(\varphi_1', \cdots, \varphi_n')^T$: $\bigoplus_{i=1}^{n} P_i'\to P_j'$ for some j such that $\alpha'\varphi'=0$ and φ_j' is an identity map. Put $I=\{i\,|\,\varphi_i'$ is an isomorphism (i.e. $(S_i)\varphi_i'\neq 0)\}$. Then we may assume $j=\max_{i\in I} i$ by considering a map $\varphi_i'\varphi_k'^{-1}$ instead of φ_i' for each $i=1, \cdots, n$ if $k>j$ for some $k\in I$. By (c_2) for each $i\in I$, there exists a map φ_i: $P_i\to P_j$ such that $(S_i)(\varphi_i-\varphi_i')$, where we take an identity map as φ_j. For each $k\notin I$, let φ_k: $P_k\to P_j$ be a zero map. Then for $\varphi=(\varphi_1, \cdots, \varphi_n)^T$ we have $\alpha\varphi=0$, and therefore by Corollary 2.2 M is decomposable.

We say that a module M is of I_1*-type* (resp. I_2*-type*) if M is indecomposable and $|\bar{M}|=1$ (resp. $|\bar{M}|=2$), and M is of I*-type* if M is of I_1- or I_2-type. Since A is left serial, the modules of I_1-type coincide with the uniserial modules.

Proposition 3.6. *Let* A *be a left serial ring satisfying* (c_2). *Then a module* M *is of* I_2*-type if and only if there exist uniserial submodules* L_1 *and* L_2 *which satisfy the following conditions.*

(1) $M=L_1+L_2$ *and* $|L_1|$, $|L_2|\geq 2$.

(2) $S=L_1\cap L_2$ *is a simple module and* S *is* (L_1, L_2)*-maximal. Moreover in this case* $S=S_1(M)$, *so* M *is colocal.*

Proof. 'If' part and $S=S_1(M)$ are immediate from Lemma 1.2.

'Only if' part: Let M be an indecomposable module with $|\bar{M}|=2$. Then we have clearly $M=L_1+L_2$ for some uniserial submodules L_1 and L_2 such that $L_1\cap L_2\neq 0$ and $2\leq|L_1|\leq|L_2|$. Assume $L_1\cap L_2$ is not simple. If S' is a simple submodule $L_1\cap L_2$, then S' is not (L_1, L_2)-maximal so S' is (L_1, L_2)-extendible from (c_2). Thus by Lemma 1.1, $M=L_1'+L_2$ for some uniserial submodule L_1' of M such that $|L_1'\cap L_2|<|L_1\cap L_2|$. Iterating this argument, the assertion holds.

Let M, L_1 and L_2 be as the above proposition. If $|L_1| \leq |L_2|$, then $|L_2|$ is equal to the Loewy length t of M (i.e. $N^{t-1}M \neq 0$ and $N^tM=0$) and we have $|L_1| = |M| - |L_2| + 1$. Thus we define an integer $s(M)$ as $\min\{|L_1|, |L_2|\}$ determined by M. Moreover we define $s(L)$ as $|L|$ if L is a uniserial module.

Now we consider the following condition (D) which is always satisfied for finite dimensional algebras over a field.

(D) $\dim_{D_2(L)} D_1(L) = \dim D_1(L)_{D_2(L)}$ for any uniserial left A-module L with $|L| \geq 2$.

Note that the condition (D) is equivalent to the following: $\dim_D \mathrm{Hom}_A(\overline{Nf}, \overline{Nf}) = \dim \mathrm{Hom}_A(\overline{Nf}, \overline{Nf})_D$ for any $f \in p(A)$, where D denotes a division ring fAf/fNf and $\mathrm{Hom}_A(\overline{Nf}, \overline{Nf})$ is canonically regarded as a (D, D)-bimodule.

Lemma 3.7. *Let A be a ring satisfying the conditions* (c) *and* (D). *If M is a module of I_2-type and L is a uniserial module with $|L| \leq s(M)$, then any homomorphism θ: $S_1(L) \to S_1(M)$ is (L, M)-extendible.*

Proof. Put $S=S_1(L)$ and $S'=S_1(M)$. From Proposition 3.6, there exist uniserial submodules L_1 and L_2 of M such that $M=L_1+L_2$, $|L_i| \geq 2$ $(i=1, 2)$, $S'=L_1 \cap L_2$ is simple and (L_1, L_2)-maximal. Then we have $|L| \leq |L_i|$; $i=1, 2$, from the definition of $s(M)$. We may assume θ: $S \to S'$ is an isomorphism, since otherwise θ is a zero map.

(i) In case S is of first kind. Since S' $(\simeq S)$ is of first kind and (L_1, L_2)-maximal, we have $S_2(L_i) \not\simeq S_2(L_2)$ by Lemma 3.4. It follows from $c(S_1(L)) \leq 2$ that $S_2(L) \simeq S_2(L_1)$ or $S_2(L) \simeq S_2(L_2)$. Thus by Lemma 3.4 θ is (L, L_i)-extendible for some $i=1, 2$, and consequently (L, M)-extendible.

(ii) In case S is of second kind. Put $r=|L|$ and $M'=S_r(L_1)+S_r(L_2) \subset M$. It suffices to show that θ: $S \to S_1(M')$ is (L, M')-extendible. Thus we may assume $M=M'$ and $r=|L|=|L_1|=|L_2|$. Since S is of second kind and $S=S_1(L) \simeq S_1(L_1) \simeq S_1(L_2)$, we have isomorphisms β_i: $L \to L_i$ for $i=1, 2$ by Lemma 3.4. Let s be an elements of S. Since the restriction maps β_i: $S \to S_1(L_i)=S'$ are isomorphisms, there is an isomorphism λ: $S \to S$ such that $s\lambda\beta_1 = -s\beta_2$. Define α: $S \to L \oplus L$ and β: $L \oplus L \to M$ as $s\alpha = (s\lambda, s)$ and $\beta = (\beta_1, \beta_2)^T$. Then we have an exact sequence $0 \to S \xrightarrow{\alpha} L \oplus L \xrightarrow{\beta} M \to 0$. Since S' is (L_1, L_2)-maximal, λ is also (L, L)-maximal (see Remark 1). The maps β_1: $S \to S'$ and θ: $S \to S'$ are isomorphisms, so we have an isomorphism μ: $S \to S$ such that $s\theta = s\mu\beta_1$, i.e. $s\theta = s(\mu, 0)\beta$. By Lemma 3.1, Lemma 3.3 and the assumption, it holds that $D_2(L)=D_r(L)$ and $\dim_{D_r(L)} D_1(L) = \dim D_1(L)_{D_r(L)} = 2$. On the other hand λ: $S \to S$ is (L, L)-maximal, so $\lambda \notin D_2L) = D_r(L)$. Consequently $D_1(L) = D_r(L)1_S + D_r(L)\lambda$ and there exist maps φ_i: $L \to L$ $(i=1, 2)$

such that $\mu=\varphi_1 1_S-\varphi_2\lambda_i$ in $D_1(L)$, i.e. $s\mu=s\varphi_1-s\varphi_2\lambda$. Put $\varphi=(\varphi_1, \varphi_2): L\to L\oplus L$. Since $(s\varphi_2\lambda, s\varphi_2)\beta=(s\varphi_2)\alpha\beta=0$, we have $s\varphi\beta=(s\varphi_1, s\varphi_2)\beta=(s\varphi_1-s\varphi_2\lambda, 0)\beta=s(\mu, 0)\beta=s\theta$. This shows $\varphi\beta: L\to M$ is an extension of θ: $S\to S'$.

For any artinian ring A, the condition (b) implies (c) by Proposition 2.4. But its converse does not necessarily hold (see Example 3). The following proposition shows the converse holds under the condition (D).

Proposition 3.8. *Let A be a ring satisfying conditions* (c) *and* (D). *Then A is of left colocal type.*

Proof. Let M be an A-module with $|\bar{M}|=n$. By induction on n, we show that M has a decomposition $M=M_1\oplus\cdots\oplus M_r$ such that each M_i is of I-type. If $n=1$ or 2, then the assertion holds by Proposition 3.6. Assume $n\geq 3$. Then it suffices to show that M is decomposable, for any proper direct summands of M has a decomposition as above by the inductional assumption. From $|\bar{M}|=n$ we have $M=L_1+\cdots+L_n$ for some uniserial modules L_i, $i=1, \cdots, n$, since A is left serial. We may assume $|L_1|\leq|L_i|$ for each $i=1, \cdots, n$. By inductional assumption $L_2+\cdots+L_n=M_2\oplus\cdots\oplus M_r$ for some modules M_i of I-type; $i=2, \cdots, r$. If there is a module M_i, $2\leq i\leq r$, such that $s(M_i)<|L_1|$, then we have $M=L_1'+\cdots+L_n'$ for some uniserial submodules L_i' with $|L_1'|<|L_1|$. Iterating of this argument, we may assume that $M=L_1+(M_2\oplus\cdots\oplus M_r)$ and $|L_1|\leq s(M_i)$ for each i. Put $M'=M_2\oplus\cdots\oplus M_r$ and $T=L_1\cap M'$. If T is a zero module, our assertion is clear. Assume $|T|\geq 2$. Let S be the simple submodule of T and denote by π_i: $T\to M_i$ the restriction map of a projection $M_2\oplus\cdots\oplus M_r\to M_i$ for each i. Then by (c_2) and Lemma 3.7, π_i: $S\to M_i$ is (L_1, M_i)-extendible, for this is clear in case π_i is zero-map. Hence S is (L_1, M')-extendible, so there exists a uniserial submodule L_1' such that $M=L_1'+M'$, $|L_1'|<|L_1|$ and $|L_1'\cap M'|<|T|$ by Lemma 1.1. Iterating this argument, we may assume $M=L_1+(M_2\oplus\cdots\oplus M_r)$, $|L_1|\leq s(M_i)$ for each $i=2, \cdots, r$, and $T=L_1\cap(M_2\oplus\cdots\oplus M_r)$ is simple. If M_j is of I_2-type for some $j\,(2\leq j\leq r)$, then π_j: $T\to M_j$ is (L_1, M_j)-extendible and therefore by Lemma 1.3 M is decomposable. If M_i is of I_1-type for any $i\,(2\leq i\leq r)$, then M is decomposable by Lemma 3.5.

4. The equivalence of (c) and (d)

In this section we study the following condition (Er) (for any integer $r\geq 1$) which is a generalization of (c_2) (i.e. (E2) implies (c_2)).

(Er) For any uniserial modules L_1 and L_2 with $r\leq|L_1|\leq|L_2|$, any isomorphism θ: $S_1(L_1)\to S_1(L_2)$ is (L_1, L_2)-extendible whenever θ is $(S_r(L_1), S_r(L_2))$-extendible, where r is an integer ≥ 1.

In particular the equivalence of (c) and (d) is shown as an immediate consequence of a necessary and sufficient condition for left serial rings to satisfy (Er) (c.f. Corollary 4.4).

For submodules $L_1, \cdots, L_n$ of a module M, we say that $L_1, \cdots, L_n$ are independent if the sum $\sum_{i=1}^{n} L_i$ is direct (i.e. $\sum_{i=1}^{n} L_i = \bigoplus_{i=1}^{n} L_i$).

Lemma 4.1. *Let A be a left serial ring and $u_i \in t(eN^{r-1}f_i)$ for $i=1, \cdots, n$, where r is an integer, e is an idempotent and f_i is a primitive idempotent. Then the following conditions are equivalent.*

(1) *$\bar{u}_1A, \cdots, \bar{u}_nA$ are independent, where $\bar{u}_i$ is a residue class u_i+eN^r of u_i in eN^{r-1}/eN^r.*

(2) *$u_1A, \cdots, u_nA$ are independent and $u_1A\oplus\cdots\oplus u_nA$ is a direct summand of eN^{r-1}.*

Proof. (1)$\Rightarrow$(2). Assume $u_1A, \cdots, u_nA$ are dependent. Then there are elements $a_i=f_ia_ig$ of A, $i=1, \cdots, n$ such that $u_1a_1+\cdots+u_na_n=0$ and $u_ka_k \neq 0$ for some k, $1\leq k\leq n$, where $g\in p(A)$. Since Ag is uniserial by the assumption, there is an integer j, $1\leq j\leq n$, say $j=n$, with $Au_ia_i\subset Au_na_n\subset Ag$ for each $i=1,\cdots, n$. Clearly we have $Au_i=N^{r-1}f_i$ and $Au_na_n=N^{s-1}g$ for some integer s. Consider $\tilde{a}_i$: $Af_i\rightarrow Ag$ a right multiplication map by a_i. Then we have $(N^{r-1}f_n)\tilde{a}_n=Au_na_n=N^{s-1}g$, which shows $s\geq r$ and $\tilde{a}_n$ induces an isomorphism ψ_n: $Af_n/N^rf_n\rightarrow N^{s-r}g/N^sg$. Moreover $(N^rf_i)\tilde{a}_i=Nu_ia_i\subset Nu_no_n=N^sg$ and so $\tilde{a}_i$ induces a homomorphism ψ_i: $Af_i/N^rf_i\rightarrow N^{s-r}g/N^sg$. Put $\psi=(\psi_1, \cdots, \psi_n)^T$ and $\varphi=(\varphi_1, \cdots, \varphi_n)^T=\psi\psi_n^{-1}$. Then from $u_1a_1+\cdots+u_na_n=0$, we have $\sum_{i=1}^{n}[u_i]\varphi_i=(\sum_{i=1}^{n}[u_i]\psi_i)\psi_n^{-1}=[\sum_{i=1}^{n}u_ia_i]\psi_n^{-1}=0$, where $[u_i]$ and $[\sum_{i=1}^{n}u_ia_i]$ denote residue classes $u_i+N^rf_i$ and $\sum_{i=1}^{n}u^ia_i+N^s g$, respectively. Clearly $\varphi_n=\psi_n\psi_n^{-1}$ is an identity map. Hence by Lemma 1.4, $\bar{u}_aA, \cdots, \bar{u}_nA$ are dependent. Thus (1) implies that $u_1A, \cdots, u_nA$ are independent.

Next under the condition (1) we show $u_1A\oplus\cdots\oplus u_nA$ is a direct summand of $\overline{eN^{r-1}}$. Since $\bar{u}_1A, \cdots, \bar{u}_nA$ are independent, there are elements $v_i\in t(eN^{r-1}g_i)$; $g_i\in p(A)$, $i=1, \cdots, n$, such that $\overline{eN^{r-1}}=\bar{u}_1A\oplus\cdots\oplus\bar{u}_nA\oplus\bar{v}_1A\oplus\cdots\oplus\bar{v}_mA$. Therefore it holds $eN^{r-1}=u_1A\oplus\cdots\oplus u_nA\oplus v_1A\oplus\cdots\oplus v_mA$, for $\bar{u}_1A, \cdots, \bar{u}_nA, \bar{v}_1A, \cdots, \bar{v}_mA$ are independent and eN^r is small in eN^{r-1}.

(2)$\Rightarrow$(1). This is clear.

Corollary 4.2. *Let A be a left serial ring, e an idempotent of A and r an integer≥ 1. Assume a right A-module M is a direct summand of eN^{r-1} with $|\bar{M}|=n$. Then we have $M=u_1A\oplus\cdots\oplus u_nA$ for some $u_i\in t(eN^{r-1}f_i)$, where $f_i\in p(A)$ and $i=1, \cdots, n$. Therefore M is a direct sum of local right A-modules.*

Proof. If σ: $eN^{r-1}\to\overline{eN^{r-1}}$ is a canonical map, $\bar{M}\simeq M\sigma$. By the assumption $eN^{r-1}=M\oplus M'$ for some submodule M' of eN^{r-1}. Hence $\overline{eN^{r-1}}=M\sigma\oplus M'\sigma=\bar{u}_1A\oplus\cdots\oplus\bar{u}_nA\oplus M'\sigma$ for some $u_i\in t(eN^{r-1}f_i)$ with $u_i\in M$. Therefore by Lemma 4.1, $eN^{r-1}=u_1A\oplus\cdots\oplus u_nA\oplus M'$. But $u_1A\oplus\cdots\oplus u_nA\subset M$, and so $M=u_1A\oplus\cdots\oplus u_nA$.

Lemma 4.3. *Let A be a left serial ring and e a primitive idempotent of A and r an integer ≥ 1. Then the following conditions are equivalent.*

(1) *For any uniserial modules L_1 and L_2 such that $S_1(L_1)\simeq\overline{Ae}$ and $r\leq|L_1|\leq|L_2|$, any isomorphism θ: $S_1(L_1)\to S_1(L_2)$ is (L_1, L_2)-extendible whenever θ is $(S_r(L_1), S_r(L_2))$-extendible.*

(2) *The right A-module eN^{r-1} is a direct sum of uniserial submodules.*

Proof. Note by Lemma 4.1 and the Krull-Schmidt Theorem the condition (2) is equivalent to the following: For any $v\in t(eN^{r-1}g)$; $g\in p(A)$, vA is a uniserial right A-module.

(1)$\Rightarrow$(2). Let $v\in t(eN^{r-1}g)$; $g\in p(A)$. By Lemma 4.1, $eN^{r-1}=vA\oplus M$ for some submodule M. Assume vA is not uniserial. Then $\overline{vN^s}$ is not simple for some $s\geq 1$. Since $eN^{s+r-1}=vN^s\oplus MN^s$, by Lemma 4.2 $vN^s=u_1A\oplus\cdots\oplus u_mA$ for some $u_i\in t(eN^{s+r-1}f_i)$; $f_i\in p(A)$, where $m\geq 2$ and $i=1, \cdots, m$. Hence we have $u_i=va_i$ for an element a_i of A with $a_i=ga_if_i$, $i=1, 2$. By the assumption $Au_i=N^{s+r-1}f_i$ and $Av=N^{r-1}g$. Put $P=Ag/N^rg$ and $L_i=Af_i/N^{s+r}f_i$. Since $Ava_i=Au_i$ and $N^rga_i=N^{s+r}f_i$, a right multiplication map $\tilde{a}_i$: $Ag\to Af_i$ induces an isomorphism ψ_i: $P\to S_r(L_i)$ with $[v]\psi_i=[u_i]$, where $[v]\in P$ and $[u_i]\in L_i$ are residue classes of v and u_i, respectively. Put $\varphi'=\psi_1^{-1}\psi_2$. We have an isomorphism φ': $S_r(L_1)\to S_r(L_2)$ with $[u_1]\varphi'=[u_2]$, and clearly $A[u_i]=S_1(L_i)$. Then by the condition (1), there is an isomorphism φ: $L_1\to L_2$ with $[u_1]\varphi=[u_2]$. This is a contradiction by Lemma 1.4 and Lemma 4.1.

(2)$\Rightarrow$(1). Assume (2). Let L_1 and L_2 be uniserial modules as in (1) and θ: $S_1(L_1)\to S_1(L_2)$ be an isomorphism which has an extension φ_r: $S_r(L_1)\to S_r(L_2)$. It suffices to show (1) in the case $|L_1|=|L_2|=s+r$ and $L_i=Af_i/N^{s+r}f_i$ where $s\geq 1$, $f_i\in p(A)$ and $i=1, 2$. Since L_i is uniserial, $P\simeq S_r(L_1)\simeq S_rL_2)$ for some $P=Ag/N^rg$; $g\in p(A)$. Hence we have isomorphism ψ_i: $P\to S_r(L_i)$; $i=1, 2$, with $\psi_1\varphi_r=\psi_2$, which are induced from right multiplication maps $\tilde{a}_i$: $Ag\to Af_i$ by $a_i=ga_if_i\in A$. Thus for some $v\in t(eN^{r-1}g)$ and $u_i\in t(eN^{s+r-1}f_i)$; $i=1, 2$, it is satisfied $A[v]=S_1(P)$, $A[u_i]=S_1(L_i)$, $[va_i]=[u_i]$ and $[u_1]\theta=[u_2]$. By the assumption vA is uniserial and hence it holds $va_1A\supset va_2A$ or $va_1A\subset va_2A$. If $va_1A\supset va_2A$, then we have $va_2=va_1c$ for some $c=f_1cf_2\in A$. Hence θ is extended to the right multiplication map $\tilde{c}$: $L_1\to L_2$ since $[u_1]\tilde{c}=[va_1]\tilde{c}=[va_1c]=[va_2]=[u_2]$. The assertion is similarly shown in the case $va_1A\subset va_2A$.

By Lemma 4.3 we have the following corollary. In case $r=1$, the corollary implies the last assertion in Corollary 1.6.

Corollary 4.4. *A left serial ring A satisfies the condition* (Er) *if and only if the right A-module N^{r-1} is a direct sum of uniserial submodules.*

If A is a finite dimensional algebra over a field, A is of right local type if and only if A is of left colocal type by the duality. Thus by Propositions 2.4 and 3.8 and Corollary 4.4, we have the following theorem which is shown by Tachikawa [4, 5] except for the equivalence (c) and (d) for artinian rings (see the introduction).

Theorem 4.5 (Tachikawa). *Let A be an artinian ring and consider the following conditions.*

(a) *A is of right local type.*

(b) *A is of left colocal type.*

(c) (c_1) *A is left serial.*

(c_2) *For any uniserial left A-modules L_1 and L_2 with $|L_1| \leq |L_2|$, any isomorphism θ: $S_1(L_1) \to S_1(L_2)$ is (L_1, L_2)-maximal or (L_1, L_2)-extendible.*

(c_3) *$|eN/eN^2| \leq 2$ for any primitive idempotent e of A.*

(d) (d_1) *A is left serial.*

(d_2) *$eN = M_1 \oplus M_2$, for any primitive idempotent e of A, where M_i is either zero or a uniserial submodule of the right A-module eN for each $i=1, 2$.*

Then (b) *implies* (c), *and* (c) *is equivalent to* (d). *If A satisfies the condition* (D), *then* (c) *implies* (b). *In particular if A is a finite dimensional algebra over a field, then the conditions* (a)–(d) *are equivalent.*

5. Examples

EXAMPLE 1. Let K be a field and A a subalgebra of a full matrix algebra $M_3(K)$ which is defined by the following:

$$A = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| \, a_{22} = a_{33}, \ a_{ij} \in K \right\}.$$

Then A satisfies (d) (and so (c)). Let e_{ij} be the (i, j)-matrix unit of $M_3(K)$ $(1 \leq i, j \leq 3)$ and put $e = e_{11}$. We have $Ne = Ke_{21} + Ke_{31}$, where $N = \operatorname{rad} A$. Define a map φ: $Ne \to Ne$ by $(be_{21} + ce_{31})\varphi = be_{21} + (b+c)e_{31}$; $b, c \in K$. It is easy to see that φ is an automorphism of Ne. Since the restriction map φ_1: $S_1(Ne) \to S_1(Ne)$ of φ is an identity map, φ_1 is (Ae, Ae)-extendible. (More generally $S_1(Ne)$ is of first kind, so any automorphism $S_1(Ne) \to S_1(Ne)$ is (Ae, Ae)-extendible.) But φ: $Ne \to Ne$ is not (Ae, Ae)-extendible, since any automorphism $Ae \to Ae$ is a right multiplication map $\tilde{a}$ by an element a of $eAe \,(\simeq K)$. Thus the condition (c_2) does not necessarily imply the following: Any isomorphism θ': $S_2(L_1)\to S_2(L_2)$ is (L_1, L_2)-extendible for any uniserial module L_1 and L_2 with $2\leq |L_1|\leq |L_2|$. (This example shows the condition II in Introduction of [4] is not equivalent to the condition II in [4, Theorem 5.3].)

EXAMPLE 2. There exists an artinian ring of left colocal type which is not a finite dimensional algebra over a field and moreover is not serial: Let K be a field and F a field of quotients of the polynomial ring $K[x]$ in one indeterminate. Let τ: $F\to F$ be a ring endomorphism extended from an endomorphism $K[x]\to K[x]$ which fixes K and maps x onto x^2. Put $M=F$ and consider an (F, F)-bimodule M defined as $a\cdot m\cdot b=am(b)\tau$; $a, b\in F$, where the multiplication in right side of the equality are those in the field $M(=F)$. Let $A=F\ltimes M$ be a trivial extension of F over M. Then A is an artinian ring with Jacobson radical $N=M$ which satisfies the condition (d). Moreover A satisfies the condition (D) since $\mathrm{Hom}_A(A/N, A/N)\simeq F$ is a field for a unique simple left A-module A/N (up to isomorphism). Hence A is of left colocal type by Theorem 4.5. On the other hand A is not a finite dimensional algebra over a field since the center of A is equal to $(K, 0)$. Clearly A is not serial.

Next we give an example of an artinian ring which satisfies the condition (c) but is not of left colocal type. For modules S, L' and a submodule S' of L' and a homomorphism θ: $S\to S'$, we denote also by θ: $S\to L'$ the composition map $S\to L'$ of the map θ: $S\to S'$ and the inclusion map $S'\to L'$.

The following lemma is due to [1, Proposition 2.6].

Lemma 5.1. *Let* $0\to T\xrightarrow{\alpha}\bigoplus_{i=1}^{n} P_i\xrightarrow{\beta} M\to 0$ *be an exact sequence with colocal modules* P_i *and* $\alpha=(\alpha_1, \cdots, \alpha_n)$. *Assume each map* α_i: $T\to P_i$ *is a monomorphism with* $\mathrm{Im}\,\alpha_i=P_i$ *and* Coker α_1 *is a simple module. Then* M *is decomposable if and only if there exists a map* ψ: $\bigoplus_{i\neq j} P_i\to P_j$ *for some* $j\,(1\leq j\leq n)$ *such that* $\alpha'_j\psi=\alpha_j$, *where* α'_j: $T\to\bigoplus_{i\neq j} P_i$ *is the map induced from* α: $T\to\bigoplus_{i=1}^{n} P_i$.

Proof. Put $L_i=P_i\beta$ and $T_i=T\alpha_i\beta$. Then it follows from the assumption that $M=L_1+\cdots+L_n$, $T_j=L_j\cap(\bigoplus_{i\neq j} L_i)$ and L_1/T_1 is simple. If suffices to prove that M is decomposable if and only if T_j is $(\bigoplus_{i\neq j} L_i, L_j)$-extendible for some j (see Remark 1). 'If' part is immediate from Lemma 1.2. Assume M has a non-trivial decomposition $M=M_1\oplus M_2$. Since $L_2+\cdots+L_n=L_2\oplus\cdots\oplus L_n$ and L_i is colocal for each $i=2, \cdots, n$, we have $S_1(L_2)\oplus\cdots\oplus S_1(L_n)\subset S_1(M_1)\oplus S_1(M_2)=S_1(M)$ and $S_1(L_i)$ is simple. Then by [1, Lemma 1.1] there exist a partition $\{2, \cdots, n\}=I_1\cup I_2$ and submodules K_1 and K_2 of $S_1(M)$ such that $S_1(M)=(\oplus_{I_1}S_1(L_i))\oplus S_1(M_2)\oplus K_1=S_1(M_1)\oplus(\oplus_{I_2}S_1(L_i))\oplus K_2$, which shows

$M\supset(\oplus_{I_1}L_i)\oplus M_2$ and $M\supset M_1\oplus(\oplus_{I_2}L_i)$. But $2|M|=\sum_{i=2}^{n}|L_i|+|M_1|+|M_2|+1$ since L_1/T_1 is simple and $T_1=L_1\cap(\bigoplus_{i=2}^{n}L_i)$. This shows that $M=(\oplus_I L_i)\oplus M_2$ (so $I_1\neq\phi$) or $M=M_1\oplus(\oplus_{I_2}L_i)$ (so $I_2\neq\phi$). Thus L_j is a direct summand of M for some j, so T_j is $(\bigoplus_{i\neq j}L_i, L_j)$-extendible by Lemma 1.2.

Lemma 5.2. *Let A be a left serial ring satisfying* (c_2) *and L a uniserial module such that $|L|\geq 2$ and $D_1(L)\supsetneqq D_2(L)$. Then the following statements hold.*

(1) *L is projective.*

(2) *Let M be a module such that $M=L_1+\cdots+L_n$ is a sum of uniserial submodules L_i. If $|L|\leq|L_i|$ for each $i=1,\cdots,n$, then any homomorphism θ: $S_1(L)\rightarrow S_1(M)$ is (L, M)-maximal or (L, M)-extendible.*

Proof. (1) Assume $L\simeq Af/N^rf$; $f\in p(A)$, and $N^rf\neq 0$. Then for $L'=Af/N^{r+1}f$ we have $D_1(L')\supset D_2(L')\supsetneqq D_3(L')$ since we may assume $D_2(L')=D_1(L)$ and $D_3(L')=D_2(L)$. This contradicts (c_2) by Lemma 3.1. Therefore $N^rf=0$, so L is projective.

(2) We may clearly assume θ is a monomorphism. Put $P_i=L_i$ and let $P_1\oplus\cdots\oplus P_n$ be the outer direct sum of $P_1,\cdots,P_n$. We have an epimorphism β: $P_1\oplus\cdots\oplus P_n\rightarrow M$. Suppose θ is not (L, M)-maximal. Then θ is extended to a map θ': $S_2(L)\rightarrow M$. Since $S_2(L)$ is projective by (1), there is a map $\varphi'=(\varphi_1',\cdots,\varphi_n')$: $S_2(L)\rightarrow P_1\oplus\cdots\oplus P_n$ with $\varphi'\beta=\theta'$. Hence the restriction map φ_i': $S_1(L)\rightarrow P_i$ of φ_i': $S_2(L)\rightarrow P_i$ is not (L, P_i)-maximal and so is extended to a map φ_i: $L\rightarrow P_i$ for each $i=1,\cdots,n$ by (c_2). As easily seen $\varphi\beta$: $L\rightarrow M$ is an extension of θ for the map $\varphi=(\varphi_1,\cdots,\varphi_n)$: $L\rightarrow P_1\oplus\cdots\oplus P_n$.

Let A be a ring satisfying the conditions (c). Let S be a simple module of second kind and L, L_1 and L_2 uniserial modules such that $S\subsetneqq L\subset L_1\subset L_2$ (see Lemma 3.4 (3)). Consider an exact sequence $0\rightarrow S\xrightarrow{\alpha}L_1\oplus L_2\xrightarrow{\beta}M\rightarrow 0$ with $\alpha=(\lambda, 1_S)$; λ, $1_S\in D_1(L)$, where for a map γ: $S\rightarrow S$ we denote also by γ: $S\rightarrow L_i$ the composition map $S\rightarrow L_i$ of γ: $S\rightarrow S$ and the inclusion map $S\rightarrow L_i$. Then by Lemma 2.1 and (c_2), M is indecomposable if and only if λ: $S\rightarrow S$ is (L, L)-maximal, i.e. $\lambda\notin D_2(L)$. Assume M is indecomposable. In this case M is of I_2-type and so colocal. Let θ: $S_1(L)\rightarrow S_1(M)$ be an isomorphism. Then we have $\theta=\mu\beta_1=(\mu, 0)\beta$ for some $\mu\in D_1(L)$, where $\beta=(\beta_1, \beta_2)^T$ (see the proof of Lemma 3.7). Since by Lemma 5.2 L is projective, θ: $S_1(L)\rightarrow S_1(M)$ is (L, M)-extendible if and only if there exists a map $\varphi=(\varphi_1, \varphi_2)$: $L\rightarrow L_1\oplus L_2$ such that $\varphi\beta$: $L\rightarrow M$ is an extension of θ. By the same argument in proof of Lemma 3.7 and the fact that β_1: $L_1\rightarrow M$ is a monomorphism, it is easily seen that $\varphi\beta$: $L\rightarrow M$ is an extension of θ if and only if an equality $\mu=\varphi_1 1_S-\varphi_2\lambda$ holds in $D_1(L)$, where we regard φ_i as a map φ_i: $L\rightarrow L$ $(\subset L_i)$; $i=1, 2$. Thus by

Lemma 5.2 we have

Lemma 5.3. *Let A be a ring satisfying* (c) *and M be a module of I_2-type such that $S_1(M)$ is a simple module of second kind. Assume L is a uniserial module with $|L| \leq s(M)$ and $\theta: S_1(L) \to S_1(M)$ is an isomorphism. Then under the above notation, θ is (L, M)-maximal if and only if $D_2(L)1_S \oplus D_2(L)\lambda \oplus D_2(L)\mu \subset D_1(L)$.*

EXAMPLE 3. There exists an artinian ring which satisfies the condition (c) (or equivalently (d)) but is not of left colocal type: Let F and G be division rings such that G is a subring of F and $\dim F_G = 2$, $\dim_G F \geq 3$. There exist these rings by Cohn [2]. Let

$$A = \begin{pmatrix} G & & & \\ G & G & & \\ G & G & G & \\ F & F & F & F \end{pmatrix}$$

be a subring of a full matrix ring $M_4(F)$. Then as easily seen A satisfies the condition (d) and consequently (c). But A does not satisfy (b) (i.e. A is not of left colocal type). In order to show it, we construct an indecomposable module which is not colocal. Put $L = Ae_{11}$, $L_i = N^{4-i}L$ $(1 \leq i \leq 4)$ and $S = L_1$, where e_{11} is a (1, 1)-matrix unit of $M_4(F)$, and $N = \operatorname{rad} A$. Then L_i is uniserial with $|L_i| = i$. We can identify $D_1(L)$ and $D_2(L)$ with F and G, respectively. Let λ and μ be elements of $D_1(L)$ such that $D_2(L)1_S \oplus D_2(L)\lambda \oplus D_2(L)\mu \subset D_1(L)$ and let $\alpha' = (\lambda, 1_S): S \to L_2 \oplus L_4$ and $\alpha'' = (\lambda, 1_S): S \to L_3 \oplus L_3$ be monomorphisms. Then by Lemma 1.2 we have the following exact sequences with colocal modules P_2 and P_3:

$$0 \longrightarrow S \xrightarrow{\alpha'} L_2 \oplus L_4 \xrightarrow{\beta'} P_2 \longrightarrow 0$$
$$0 \longrightarrow S \xrightarrow{\alpha''} L_3 \oplus L_3 \xrightarrow{\beta''} P_3 \longrightarrow 0\,.$$

Let $\alpha_2: S \to P_2$ be the composition map of $(\mu, 0): S \to L_2 \oplus L_4$ and β', and let $\alpha_3: S \to P_3$ be the composition map of $(\mu, 0): S \to L_3 \oplus L_3$ and β''. Define a monomorphism $\alpha: S \to P_1 \oplus P_2 \oplus P_3$ by $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where $P_1 = L$ and $\alpha_1 = 1_S$, and consider an exact sequence $0 \to S \xrightarrow{\alpha} P_1 \oplus P_2 \oplus P_3 \xrightarrow{\beta} M \to 0$. Suppose M is decomposable. Then by Lemma 5.1 there exists a map $\varphi_k = (\varphi_{ik}, \varphi_{jk})^T$: $P_i \oplus P_j \to P_k$ with $\alpha_i\varphi_{ik} + \alpha_j\varphi_{jk} = (\alpha_i, \alpha_j)\varphi_k = \alpha_k$, where (i, j, k) is a permutation of $(1, 2, 3)$. Since the Loewy lengths of P_2 and P_3 are 4 and 3, respectively, P_2 is not isomorphic to P_3. But it holds $|P_1| < |P_2| = |P_3|$. This shows that there are no monomorphisms $P_i \to P_j$ for any i and j with $i \neq j$ and $i \neq 1$. Therefore $\alpha_i\varphi_{ij} = 0$ if $i \neq j$ and $i \neq 1$. Thus for some k $(k = 2$ or $3)$, we have $s\varphi_{1k} = s\alpha_1\varphi_{1k} = s\alpha_k$ $(s \in S)$, which implies $\alpha_k: S \to P_k$ is (P_1, P_k)-extendible. This is a contradiction by Lemma 5.3. Hence M is indecomposable. But the map $(\beta_2, \beta_3)\colon P_2 \oplus P_3 \to M$ induced from β is a monomorphism. Therefore M is not colocal.

From Theorem 4.5 and Example 3, the following question arises.

Question: *Whether does any ring of left colocal type satisfy the condition* (D)?

Though we can not answer the question, we study it in relation to simultaneous equations over a division ring. Let A be a ring which satisfies (c) but not (D). Then there is a uniserial module L with $|L|=2$ and $\dim_{D_2(L)} D_1(L) \geq 3$. Put $S=S_1(L)$ and let λ_i and μ_i $(i=1, 2)$ be elements of $D_1(L)$ with $D_2(L)1_S \oplus D_2(L)\lambda_i \oplus D_2(L)\mu_i \subset D_1(L)$. Then as in Example 3, consider the following exact sequences:

$$0 \longrightarrow S \xrightarrow{(\lambda_i,\, 1_S)} L \oplus L \xrightarrow{\beta_i} M_i \longrightarrow 0$$

$$0 \longrightarrow S \xrightarrow{(1_S,\, \theta_1,\, \theta_2)} L \oplus M_1 \oplus M_2 \longrightarrow M \longrightarrow 0\,,$$

where $\theta_i\colon S \to M_i$ is a map with $\theta_i = (\mu_i, 0)\beta_i$ for a map $(\mu_i, 0)\colon S \to L \oplus L$ $(i=1, 2)$. Then M_i is a module of I_2-type and θ_i is (L, M_i)-maximal by Lemma 3.6 and 5.3. Moreover M is not a colocal module. On the other hand by Lemma 5.1 M is decomposable if and only if there exists a map $\psi\colon L \oplus M_i \to M_j$ with $(1_S, \theta_i)\psi = \theta_j$ for some permutation (i, j) of $(1, 2)$ (see the proof of Example 3). Next we give a necessary and sufficient condition in order that there exists a map $\psi\colon L \oplus M_1 \to M_2$ with $(1_S, \theta_1)\psi = \theta_2$. Consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "{(0,\, \lambda_1,\, 1_S)}"] & L \oplus (L \oplus L) \arrow[rr, "{(1_L,\, \beta_1)^T}"] \arrow[dd, dashed, "\varphi"'] & & L \oplus M_1 \arrow[r] \arrow[dd, dashed, "\psi"] & 0 \\
& & & S \arrow[ul, "{(1_S,\, \mu_1,\, 0)}"'] \arrow[ur, "{(1_S,\, \theta_1)}"] \arrow[dl, "{(\mu_2,\, 0)}"] \arrow[dr, "\theta_2"'] & & \\
0 \arrow[r] & S \arrow[r, "{(\lambda_2,\, 1_S)}"] & L \oplus L \arrow[rr, "\beta_2"] & & M_2 \arrow[r] & 0\,.
\end{tikzcd}$$

As easily seen there exists a map $\psi\colon L \oplus M_1 \to M_2$ with $(1_S, \theta_1)\psi = \theta_2$ if and only if there exists a map $\varphi\colon L \oplus (L \oplus L) \to L \oplus L$ with $(\mathrm{Im}(0, \lambda_1, 1_S))\varphi (=(\mathrm{Ker}\,(1, \beta_1)^T)\varphi) \subset \mathrm{Ker}\,\beta_2$ and $(1, \mu_1, 0)\varphi\beta_2 = (\mu_2, 0)\beta_2$, that is for any $s \in S$ it holds $(0, s\lambda_1, s)\varphi\beta_2 = 0$ and $(s, s\mu_1, 0)\varphi\beta_2 = (s\mu_2, 0)\beta_2$. Put $\beta_2 = (\beta_{12}, \beta_{22})^T\colon L \oplus L \to M_2$ and $\varphi = (\varphi_{ij})\colon L \oplus L \oplus L \to L \oplus L$, where (φ_{ij}) is a matrix of type $(3, 2)$ with coefficients φ_{ij}. Since $s'\lambda_2\beta_{12} + s'\beta_{22} = s'(\lambda_2, 1_S)\beta_2 = 0$, by using maps φ_{ij}, we can rewrite the above equalities as following:

$$((s\varphi_{31}+s\lambda_1\varphi_{21})-(s\varphi_{32}+s\lambda_1\varphi_{22})\lambda_2)\beta_{12}=0 \quad \text{and}$$
$$((s\varphi_{11}+s\mu_1\varphi_{21})-(s\varphi_{12}+s\mu_1\varphi_{22})\lambda_2)\beta_{12}=s\mu_2\beta_{12}\,.$$

But β_{12}: $L\to M_2$ is a monomorphism and s is any element of S. Hence the equalities are equivalent to the system of equalities

$$(\varphi_{31}+\lambda_1\varphi_{21})-(\varphi_{32}+\lambda_1\varphi_2)_2\lambda_2=0 \quad \text{and}$$
$$(\varphi_{11}+\mu_1\varphi_{21})-(\varphi_{22}+\mu_1\varphi_{22})\lambda_2=\mu_{21} \quad \text{in} \quad D_1(L)\,.$$

Thus we have

Proposition 5.4. *Under the above notation, there exists a map ψ: $L\oplus M_1\to M_2$ with $(1_S, \theta_1)\psi=\theta_2$ if and only if the following simultaneous equations with 6-unknowns have a solution in $D_2(L)$:*

$$\text{(SE)} \quad \begin{cases} (x+\lambda_1 y)+(z+\lambda_1 u)\lambda_2=0 \\ (v+\mu_1 y)+(w+\mu_1 u)\lambda_2=\mu_2\,. \end{cases}$$

If $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$, then the above simultaneous equations (SE) have a solution in $D_2(L)$ since $M_1=M_2$ and $\theta_1=\theta_2$. Moreover note that $D_2(L)\oplus\lambda_1 D_2(L)=D_2(L)\oplus\mu_1 D_2(L)=D_1(L)$ because λ_1, $\mu_1\notin D_2(L)$ and $\dim D_1(L)_{D_2(L)}$ $(=m(L))=2$ by Lemma 3.3.

Let (SE)$'$ denote the simultaneous equations obtained by exchanging 1 and 2 each other in the indices of (SE) above. Assume that for any division rings $F\supset G$ with $\dim_G F\geq 3$ and $\dim F_G=2$ there exist elements λ_i, μ_i of F with $G1\oplus G\lambda_i\oplus G\mu_i\subset F\,(i=1, 2)$ such that both (SE) and (SE)$'$ have no solution in G. Then the following conditions would be equivalent.

(1) A is of left colocal type.

(2) A is a ring satisfying (c) and (D).

EXAMPLE 4. An artinian ring which satisfies (c) but is not of right local type: Let $F\supset G$ be division rings as in Example 3 and put

$$A=\begin{pmatrix} G & 0 \\ F & F \end{pmatrix} \quad \text{and} \quad L=\begin{pmatrix} G \\ F \end{pmatrix}.$$

Then A is a ring which satisfies (c) but does not (D) and L is a unique non-simple projective module. Moreover we can regard division rings $D_1(L)\supset D_2(L)$ as $F\supset G$. It is an open problem whether A is of left colocal type or not. If there exist elements λ_i, μ_i as above, then A would be not of left colocal type. On the other hand we can show A is of not right local type.

Let λ_1 and λ_2 be elements of F with $G1\oplus G\lambda_1^{-1}\oplus G\lambda_2^{-1}\subset F$ and put $S=(G, 0)$, $S_i=(\lambda_i G, 0)$, $P=(F, F)$ and $L_i=P/S_i$. Denote by θ_i: $S\to L_i$ the composition map of the inclusion $S \to P$ and the canonical epimorphism σ_i: $P \to L_i$. (We write homomorphisms between right modules on the left side.) Let $0 \to S \xrightarrow{(\theta_1, \theta_2)^T} L_1 \oplus L_2 \to M \to 0$ be an exact sequence of right modules. We show M is indecomposable. Suppose M is decomposable. Then there exists an isomorphism ψ: $L_1 \to L_2$ with $\psi\theta_1 = \theta_2$ by Corollary 2.2. Since P is a projective, the map ψ can be lifted to a map $\tilde{\mu}$: $P \to P$ which is a left multiplication map by $\mu \in F$, so $\psi\sigma_1 = \sigma_2\tilde{\mu}$. This shows $\mu(\lambda_1 G, 0) = \mu(\mathrm{Ker}\,\sigma_1) \subset \mathrm{Ker}\,\sigma_2 = (\lambda_2 G, 0)$ and $\sigma_2(1, 0) = \theta_2(1, 0) = \psi\theta_1(1, 0) = \psi\sigma_1(1, 0) = \sigma_2\tilde{\mu}(1, 0) = \sigma_2(\mu, 0)$. Hence $\mu\lambda_1 = \lambda_2 a$ and $\mu = 1 + \lambda_2 b$ for some a and b in G, so $b - a\lambda_1^{-1} + \lambda_2^{-1} = 0$, which contradicts $G1 \oplus G\lambda_1^{-1} \oplus G\lambda_2^{-1} \subset F$. Thus M is indecomposable. But clearly M is not local.

References

[1] H. Asashiba and T. Sumioka: *On Krull-Schmidt's theorem and the indecomposability of amalgamated sums*, Osaka J. Math. **20** (1983), 321–329.

[2] P.M. Cohn: *Quadratic extensions of skew fields*, Proc. London Math. Soc. **11** (1961), 531–556.

[3] K.R. Fuller: *Rings of left invariant module type*, Comm. Algebra **6** (1978), 153–167.

[4] H. Tachikawa: *On rings for which every indecomposable right module has a unique maximal submodule*, Math. Z. **71** (1959), 200–222.

[5] H. Tachikawa: *On algebras of which every indecomposable representation has an irreducible one as the top or the bottom Loewy constituent*, Math. Z. **75** (1961), 215–227.

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558, Japan