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ON THE EXISTENCE OF INTERSECTIONAL LOCAL TIME EXCEPT ON ZERO CAPACITY SET

Dedicated to the memory of Professor Takehiko Miyata

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0. Introduction

Let W be the space of $\boldsymbol{R}^d$-valued continuous functions on [0, 1], where $d \geqq 2$. We shall consider the functionals on W

$$\psi(\alpha, w) = \frac{d-\alpha}{4} \int_0^1 \int_0^1 |w(t)-w(s)|^{-\alpha} \, dsdt\,, \qquad \alpha<2\,, \tag{0.1}$$

which may take infinite value. These functionals play an important role in the investigation of properties of function w: the finiteness of $\psi(\alpha, w)$ implies that the Hausdorff dimension of range $\{w(t);\ 0 \leqq t \leqq 1\}$ is no less than α (cf. Taylor [9]). Let Q be the Wiener measure on W. Since $\psi(\alpha, w)$ is finite Q-almost surely for any $\alpha<2$, the Hausdorff dimension of $\{w(t);\ 0 \leqq t \leqq 1\}$ is no less than 2 Q-almost surely.

Next, let α tend to 2. Though the mean of $\psi(\alpha, \cdot)$ with respect to Q diverges to infinity, the functional

$$\Psi_n(w) = \psi(2-2^{-n}, w)-2^n \tag{0.2}$$

converges Q-almost surely. In case $d=2$, Varadhan studied this limit functional in connection with the quantum field theory and proved its existence (cf. Appendix to Symanzik [8]).

Recently Fukushima [1] showed that various famous properties of sample paths such as Lévy's Hölder continuity hold not only Q-almost surely but also *quasi-everywhere*, *i.e.* except on a set of zero capacity with respect to the Ornstein–Uhlenbeck process on W. On the other hand, Kôno [4] and [5] proved that if $d \leqq 4$, then sample paths are recurrent with positive capacity. Therefore 'quasi-everywhere' is strictly finer than 'Q-almost everywhere'.

The purpose of this paper is to show that $\psi(\alpha, w)$ is finite quasi-everywhere for any $\alpha<2$ and that $\lim \Psi_n(w)=\Psi(w)$ exists quasi-everywhere. The former result implies the theorem in Komatsu and Takashima [3]: the Hausdorff dimension of range $\{w(t);\ 0\leqq t\leqq 1\}$ is 2 quasi-everywhere. Let $(\Omega, \mathcal{F}, P, X_\tau(\cdot))$ be the Ornstein-Uhlenbeck process on W. Since a Borel subset A of W has zero capacity if and only if

$$P[X_\tau(\cdot)\notin A \quad \text{for any } \tau] = 1$$

(cf. Fukushima [1], Kusuoka [6]), it is sufficient to prove the continuity of $\psi(\alpha, X_\tau(\cdot))$ in τ and the uniform convergence of $\Psi_n(X_\tau(\cdot))$ in τ.

In case $d=2$, the limit functional $\lim \Psi_n(w)=\Psi(w)$ is formally expressed by

$$\Psi(w) = \frac{\pi}{2}\int_0^1\int_0^1 \delta(w(t)-w(s))\, dsdt - C$$

(C is an infinite constant), which is similar to the *intersectional local time* considered in Wolpert [11]. Westwater [10] investigated a similar functional in connection with the study of long polymer chains in $\boldsymbol{R}^3$. Finally, we shall mention the relative result of Shigekawa [7]: the 1-dimensional Brownian local time exists quasi-everywhere.

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1. Singular Wiener functional

Let W be the Banach space of all $\boldsymbol{R}^d$-valued continuous functions $w=(w(t))$ on $[0, 1]$ satisfying $w(0)=0$; $\mathcal{W}$, the usual Borel field and Q, the Wiener measure on $(W, \mathcal{W})$. Set, for $\alpha<2$,

$$f_\varepsilon(\alpha, x) = \frac{1}{2-\alpha}\{(|x|^2+\varepsilon^2)^{1-\alpha/2}-1\}, \qquad \varepsilon>0. \tag{1.1}$$

Considering that $\Delta f_\varepsilon(\alpha, x) \to (d-\alpha)|x|^{-\alpha}$ as $\varepsilon\downarrow 0$, we shall define

$$\psi_\varepsilon(\alpha, w) = \iint_{0<s<t<1} \frac{1}{2}\Delta f_\varepsilon(\alpha, w(s, t))\, dsdt, \tag{1.2}$$

where Δ denotes the Laplacian and $w(s, t)=w(t)-w(s)$.

Set $\partial_j=\partial/\partial x^j$ and $\partial=(\partial_1, \cdots, \partial_d)$. From the Ito formula

$$\psi_\varepsilon(\alpha, w)+f_\varepsilon(\alpha, 0) = \int_0^1 \Big(f_\varepsilon(\alpha, w(s, 1))-\int_s^1 \partial f_\varepsilon(\alpha, w(s, t))\, dw(t)\Big)\, ds.$$

Using the Fubini type theorem for the product $ds\cdot dw(t)$ (cf. Ikeda and Watanabe [2] Chap. II Sec. 4 Lemma 4.1), we have

$$\psi_\varepsilon(\alpha, w)+f_\varepsilon(\alpha, 0) = \int_0^1 f_\varepsilon(\alpha, w(s, 1))\, ds - \int_0^1 \Big(\int_0^t \partial f_\varepsilon(\alpha, w(s, t))\, ds\Big)\, dw(t).$$

Let $g_\varepsilon(\alpha, x)$ be the isotropic function satisfying

$$\frac{1}{2}\Delta g_\varepsilon(\alpha, x) = f_\varepsilon(\alpha, x) \quad \text{and} \quad g_\varepsilon(\alpha, 0) = 0 .$$

Then $g_\varepsilon(\alpha, x)$ is given by

$$(1.3) \qquad g_\varepsilon(\alpha, x) = \int_0^{|x|} r^{1-d}\left(\int_0^r 2f_\varepsilon(\alpha, u\xi)\, u^{d-1}\, du\right) dr, \qquad |\xi| = 1 .$$

Let x' denote the transposed vector of x, $x \cdot y = x'y$, the inner product of column vectors x and y, and $\partial'\partial = (\partial_i \partial_j)$: $d \times d$-matrix. Define

$$\int_0^1 h(t, w)\, \hat{d}w(t) = \text{L}^2\text{-}\lim_{n\to\infty} \sum_{i=1}^n h\left(\frac{i}{n}, w\right) w\left(\frac{i-1}{n}, \frac{i}{n}\right)$$

for a process $h(t, w)$ adapted to σ-fields $\sigma(w(u);\ t \leqq u \leqq 1)$. Then we see that

$$\int_0^1 f_\varepsilon(\alpha, w(s, 1))\, ds = g_\varepsilon(\alpha, w(0, 1)) - \int_0^1 \partial g_\varepsilon(\alpha, w(s, 1))\, \hat{d}w(s) ,$$

$$\int_0^t \partial f_\varepsilon(\alpha, w(s, t))\, ds = \partial g_\varepsilon(\alpha, w(0, t)) - \int_0^t (\hat{d}w(s))'\, \partial'\, \partial g_\varepsilon(\alpha, w(s, t)) .$$

Therefore we have

$$(1.4) \qquad \begin{aligned} \psi_\varepsilon(\alpha, w) + f_\varepsilon(\alpha, 0) &= g_\varepsilon(\alpha, w(0, 1)) \\ &\quad - \int_0^1 \partial g_\varepsilon(\alpha, w(s, 1))\, \hat{d}w(s) - \int_0^1 \partial g_\varepsilon(\alpha, w(0, t))\, dw(t) \\ &\quad + \iint_{0<s<t<1} \hat{d}w(s) \cdot \partial'\, \partial g_\varepsilon(\alpha, w(s, t))\, dw(t) \qquad \text{a.e.} \end{aligned}$$

Define, for $\alpha < 2$,

$$(1.5) \qquad \begin{aligned} g_0(\alpha, x) &= \frac{2}{2-\alpha} \int_0^{|x|}\left(\int_0^r (u^{2-\alpha} - 1)\, u^{d-1}\, du\right) r^{1-d}\, dr \\ &= \frac{2|x|^2}{(4-\alpha)(2+d-\alpha)} \left\{\frac{|x|^{2-\alpha} - 1}{2-\alpha} - \frac{4+d-\alpha}{2d}\right\} . \end{aligned}$$

It is easy to show that, for $|\nu| \leqq 2$,

$$\partial^\nu g_\varepsilon(\alpha, x) \to \partial^\nu g_0(\alpha, x) \qquad \text{as } \varepsilon \downarrow 0 ,$$

$$|\partial^\nu g_\varepsilon(\alpha, x)| \leqq \text{const.}\ (1 + |x|)^{2-|\nu|} ,$$

where $\nu = (\nu_1, \cdots, \nu_d) \in \boldsymbol{Z}_+^d$, $|\nu| = \nu_1 + \cdots + \nu_d$ and

$$\partial^\nu = \partial_1^{\nu_1}\, \partial_2^{\nu_2} \cdots \partial_d^{\nu_d} .$$

Let $\psi(\alpha, w)$ be the functional defined by (0.1). Since $d \geq 2$,

$$\psi_\varepsilon(\alpha, w) \to \psi(\alpha, w) \quad \text{for all } w \text{ as } \varepsilon \downarrow 0 .$$

From (1.4) we have

$$(1.6) \qquad \psi(\alpha, w) - \frac{1}{2-\alpha} = g_0(\alpha, w(0, 1)) - \int_0^1 \partial g_0(\alpha, w(s, 1))\, \hat{d}w(s) - \int_0^1 \partial g_0(\alpha, w(0, t))\, dw(t) + \int\int_{0<s<t<1} \hat{d}w(s)\cdot\partial'\, \partial g_0(\alpha, w(s, t))\, dw(t) \quad \text{a.e.}$$

The following theorem is proved in Section 2.

Theorem 1. *$\{\Psi_n(w)\}$ converge for almost all $w \in W$ and the limit functional* $\lim \Psi_n(w) = \Psi(w)$ *satisfies*

$$(1.7) \qquad \Psi(w) = g(w(0, 1)) - \int_0^1 \partial g(w(s, 1))\, \hat{d}w(s) - \int_0^1 \partial g(w(0, t))\, dw(t) + \int\int_{0<s<t<1} \hat{d}w(s)\cdot\partial'\, \partial g(w(s, t))\, dw(t) \quad a.e.,$$

where

$$(1.8) \qquad g(x) = \frac{1}{d}\, |x|^2 \left\{\log|x| - \frac{d+2}{2d}\right\} .$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space; $B(d\tau \times dt) = (B^i(d\tau \times dt))$, a d-dimensional two-parameter white noise and $B_0(t) = (B_0^i(t))$, a d-dimensional Brownian motion independent of $B(d\tau \times dt)$ satisfying $B_0(0) = 0$. Define

$$(1.9) \qquad X_\tau^i(t) = e^{-\tau/2} \{B_0^i(t) + \int_0^\tau e^{\sigma/2} B^i(d\sigma \times [0, t])\} .$$

The process $X_\tau = (X_\tau^i(\cdot))$ is called the Ornstein–Uhlenbeck process on W. Fix τ and σ. Then the process $t \rightsquigarrow X_\tau(t)$ is a d-dimensional Brownian motion and

$$(1.10) \qquad \langle dX_\tau^i(t), dX_\sigma^j(t)\rangle = \delta_{ij}\, e^{-|\tau-\sigma|/2}\, dt .$$

We shall prove the following theorems.

Theorem 2. *For any* $0 < \alpha < 2$,

$$P[\psi(\alpha, X_\tau) \text{ is continuous in } \tau] = 1 .$$

From the theorem we see that $\psi(\alpha, w) < \infty$ quasi-everywhere, *i.e.* except on a zero capacity set. Especially $\Psi_n(w)<\infty$ quasi-everywhere.

Theorem 3. *The sequence* $\{\Psi_n(w)\}$ *converges quasi-eveiywhere. Let* $\Psi(w)=\lim \Psi_n(w)$ *quasi-everywhere. Then*

$$P[\Psi(X_\tau) \text{ is continuous in } \tau] = 1 .$$

2. Elementary inequalities

Fix $0<\alpha<2$ and set $2\beta=2-\alpha$. We shall consider the functions

$$G_\varepsilon(x) = g_\varepsilon(\alpha, x)-g_{2\varepsilon}(\alpha, x) . \tag{2.1}$$

Let $k_\varepsilon=\beta^{-1}(\varepsilon^{2\beta}-(2\varepsilon)^{2\beta})$ and

$$\phi_\varepsilon(u) = \beta^{-1}((u^2+\varepsilon^2)^\beta-(u^2+4\varepsilon^2)^\beta)-k_\varepsilon .$$

Then we have

$$\begin{cases} G_\varepsilon(x) = \int_0^{|x|} \left(\int_0^1 \phi_\varepsilon(ru)\, u^{d-1}\, du\right) rdr+\frac{1}{2d}\, k_\varepsilon |x|^2 , \\ \partial_j G_\varepsilon(x) = x_j \int_0^1 \phi_\varepsilon(|x|u)\, u^{d-1}\, du+\frac{1}{d}\, k_\varepsilon\, x_j , \\ \partial_i\partial_j G_\varepsilon(x) = x_i\, x_j |x|^{-2}\, \phi_\varepsilon(|x|) \\ \quad +(\delta_{ij}-d\, x_i\, x_j |x|^{-2}) \int_0^1 \phi_\varepsilon(|x|u)\, u^{d-1}\, du+\frac{1}{d}\, \delta_{ij}\, k_\varepsilon . \end{cases} \tag{2.2}$$

In this and the following sections, we shall use the convenient practice of letting $c\cdot$'s stand for unimportant positive constants which may change from line to line.

Lemma 2.1. *There is a constant* C *independent of* ε *such that*

$$|\partial^\nu G_\varepsilon(x)| \leqq C\, |x|^{2-|\nu|}\, \varepsilon^{2\beta} , \tag{2.3}$$

$$|\partial^\nu G_\varepsilon(x+y)-\partial^\nu G_\varepsilon(x)| \leqq C\, \varepsilon^\beta |y|(|x| \vee |y|)^{\beta-1}(1+|x| \vee |y|)^{2-|\nu|} \tag{2.4}$$

for any x, y, $0<\varepsilon<1$ *and* $|\nu|\leqq 2$.

Proof. Since $0\leqq\phi_\varepsilon(u)\leqq -k_\varepsilon\leqq c\cdot\varepsilon^{2\beta}$, inequality (2.3) follows from (2.2). Set $G'_\varepsilon(x)=G_\varepsilon(x)-k_\varepsilon|x|^2/2d$. It suffices for the proof of (2.4) to show that

$$|\partial^\nu G'_\varepsilon(x+y)-\partial^\nu G'_\varepsilon(x)| \leqq c\cdot\varepsilon^\beta |y|(|x| \vee |y|)^{\beta+1-|\nu|} . \tag{2.5}$$

We see that $0\leqq\phi_\varepsilon(u)\leqq c\cdot(\varepsilon u)^\beta$, for $\phi_\varepsilon(u)\leqq c\cdot\varepsilon^{2\beta}$ and

$$\beta\phi_\varepsilon(u) = ((u^2+\varepsilon^2)^\beta-\varepsilon^{2\beta})-((u^2+4\varepsilon^2)^\beta-(2\varepsilon)^{2\beta})\leqq(u^2+\varepsilon^2)^\beta-\varepsilon^{2\beta}\leqq u^{2\beta} .$$

Therefore we have

$$|\partial^\nu G_\varepsilon'(x)| \leqq c \cdot |x|^{2-|\nu|}(\varepsilon|x|)^\beta \qquad (|\nu| \leqq 2).$$

It is easy to see that

$$\partial_i \partial_j \partial_k G_\varepsilon'(x) = h_0(x)|x|^2((|x|^2+\varepsilon^2)^{\beta-1}-(|x|^2+4\varepsilon^2)^{\beta-1}) + h_1(x)\,\phi_\varepsilon(|x|) + h_2(x)\int_0^1 \phi_\varepsilon(|x|u)\,u^{d-1}\,du,$$

where h_0, h_1 and h_2 are homogeneous functions with index -1. Since

$$u^2((u^2+\varepsilon^2)^{\beta-1}-(u^2+4\varepsilon^2)^{\beta-1}) \leqq c \cdot (u\varepsilon)^\beta,$$

we have

$$|\partial_i\partial_j\partial_k G_\varepsilon'(x)| \leqq c \cdot |x|^{-1}(\varepsilon|x|)^\beta.$$

Now, suppose that $|x|<2|y|$. Then, for $|\nu| \leqq 2$,

$$|\partial^\nu G_\varepsilon'(x+y)-\partial^\nu G_\varepsilon'(x)| \leqq |\partial^\nu G_\varepsilon'(x+y)| + |\partial^\nu G_\varepsilon'(x)| \leqq c \cdot |x+y|^{2-|\nu|}(\varepsilon|x+y|)^\beta + c \cdot |x|^{2-|\nu|}(\varepsilon|x|)^\beta \leqq c \cdot |y|^{2-|\nu|}(\varepsilon|y|)^\beta.$$

Suppose that $|x| \geqq 2|y|$. Then, for $|\nu| \leqq 2$,

$$|\partial^\nu G_\varepsilon'(x+y)-\partial^\nu G_\varepsilon'(x)| = \left|\int_0^1 \partial\partial^\nu G_\varepsilon'(x+\theta y)\,y\,d\theta\right| \leqq c \cdot \varepsilon^\beta |y|\,|x|^{1+\beta-|\nu|}.$$

These prove (2.5). q.e.d.

Next, define

$$G^\beta(x) = g_0(2-2\beta, x)-g_0(2-\beta, x) = |x|^2\beta^{-1}\int_0^1\left(\int_0^1 ((|x|ru)^\beta-1)^2\,u^{d-1}\,du\right) r\,dr. \tag{2.6}$$

Then we have

$$\begin{cases} \partial_j G^\beta(x) = x_j\,\beta^{-1}\displaystyle\int_0^1 ((|x|u)^\beta-1)^2\,u^{d-1}\,du, \\ \partial_i\partial_j G^\beta(x) = x_i x_j |x|^{-2}\beta^{-1}(|x|^\beta-1)^2 \\ \qquad +(\delta_{ij}-d\,x_i x_j|x|^{-2})\,\beta^{-1}\displaystyle\int_0^1 ((|x|u)^\beta-1)^2\,u^{d-1}\,du. \end{cases} \tag{2.7}$$

Lemma 2.2. *There is a constant C independent of β such that*

$$|\partial^\nu G^\beta(x)| \leqq C\beta|x|^{2-|\nu|}(1+|\log|x||)^2\,(1+|x|^2) \tag{2.8}$$

for any x and $|\nu| \leqq 2$ and that

(2.9) $$|\partial^\nu G^\beta(x)-\partial^\nu G^\beta(y)|\leqq C\beta\left(|x^\nu|x|^{-2}-y^\nu|y|^{-2}|+|\log\frac{|y|}{|x|}|\right)$$
$$\times(1+|\log|x||)(1+|\log|y||)(1+(|x|\vee|y|)^2)$$

for any x, y and $|\nu|=2$, where x^ν denotes

$$(x^1)^{\nu_1}(x^2)^{\nu_2}\cdots(x^d)^{\nu_d}.$$

Proof. By the inequality

$$\beta^{-1}|u^\beta-1|=(u^\beta+1)|\beta^{-1}\,\mathrm{th}\left(\frac{\beta}{2}\log u\right)|\leqq\frac{1}{2}|\log u|(1+u^\beta),$$

(2.8) is easily proved from (2.6) and (2.7). For example, in case $|\nu|=1$,

$$|\partial_j G^\beta(x)|\leqq\frac{\beta}{4}|x|\int_0^1(\log(|x|u))^2(1+|x|^\beta u^\beta)^2u^{d-1}du$$
$$\leqq c\cdot\beta|x|(1+|\log|x||)^2(1+|x|^2).$$

We see that

$$\begin{aligned}&|\beta^{-2}(u^\beta-1)^2-\beta^{-2}(v^\beta-1)^2|\\&=\left|\log\frac{u}{v}\right|\left|\frac{u^\beta-v^\beta}{\beta\log(u/v)}\right||\beta^{-1}(u^\beta-1)+\beta^{-1}(v^\beta-1)|\\&\leqq\left|\log\frac{u}{v}\right|(u\vee v)^\beta\frac{1}{2}\{|\log u|(1+u^\beta)+|\log v|(1+v^\beta)\}\\&\leqq c\cdot\left|\log\frac{u}{v}\right|(1+|\log u|+|\log v|)(1+(u\vee v)^2).\end{aligned}$$

Then it is easy to prove (2.9) from (2.7). For example,

$$\begin{aligned}&|x^\nu|x|^{-2}(|x|^\beta-1)^2-y^\nu|y|^{-2}(|y|^\beta-1)^2|\\&\leqq|(|x|^\beta-1)^2-(|y|^\beta-1)^2|+|x^\nu|x|^{-2}-y^\nu|y|^{-2}|\cdot|(|x|^\beta-1)(|y|^\beta-1)|\\&\leqq c\cdot\beta^2\left|\log\frac{|x|}{|y|}\right|(1+|\log|x||+|\log|y||)(1+(|x|\vee|y|)^2)\\&+c\cdot\beta^2|x^\nu|x|^{-2}-y^\nu|y|^{-2}|\cdot|\log|x|\cdot\log|y||(1+(|x|\vee|y|)^2).\end{aligned}$$ q.e.d.

Proof of Theorem 1. Note that

$$g_0(2-2^{-n},x)-g(x)=\sum_{k=n+1}^{\infty}G^{2^{-k}}(x).$$

From (2.8) we have, for $|\nu|\leqq2$,

$$\begin{aligned}&|\partial^\nu g_0(2-2^{-n},x)-\partial^\nu g(x)|\leqq c\cdot\sum_{k=n+1}^{\infty}2^{-k}|x|^{2-|\nu|}(1+|\log|x||)^2(1+|x|^2)\\&\leqq c\cdot2^{-n}|x|^{2-|\nu|}((\log|x|)^2+|x|^3).\end{aligned}$$

Let $\Psi(w)$ denote the right hand side of (1.7). From (0.2), (1.6) and the above inequality, it is easily proved that

$$\mathrm{E}|\Psi_n(\cdot)-\Psi(\cdot)|^2 \leqq c\cdot 2^{-2n}.$$

From

$$\sum_{n=1}^{\infty} \mathrm{E}|\Psi_n(\cdot)-\Psi(\cdot)|^2 < \infty,$$

we know that $\Psi_n(w)\to\Psi(w)$ a.e. as $n\to\infty$. q.e.d.

3. Preliminary estimates

Let $G^\beta(x)$ be the function defined by (2.6). From (1.6) we see that

$$(3.1)\qquad \begin{aligned} &\psi(2-2\beta, X_\tau)-\psi(2-\beta, X_\tau)+\frac{1}{2\beta}=G^\beta(X_\tau(0,1)) \\ &-\int_0^1 \partial G^\beta(X_\tau(s,1))\,\hat{d}X_\tau(s)-\int_0^1 \partial G^\beta(X_\tau(0,t))\,dX_\tau(t) \\ &+\iint_{0<s<t<1} \hat{d}X_\tau(s)\cdot\partial'\partial G^\beta(X_\tau(s,t))\,dX_\tau(t) \qquad \text{a.e. } (P), \end{aligned}$$

where $X_\tau(s,t)=X_\tau(t)-X_\tau(s)$. Hence, for any $0\leqq\tau, \sigma\leqq 1$,

$$\begin{aligned} &|\psi(2-2\beta, X_\tau)-\psi(2-\beta, X_\tau)-\psi(2-2\beta, X_\sigma)+\psi(2-\beta, X_\sigma)| \\ &\leqq |G^\beta(X_\tau(0,1))-G^\beta(X_\sigma(0,1))| \\ &\quad+\Big|\int_0^1 \{\partial G^\beta(X_\tau(s,1))\,\hat{d}X_\tau(s)-\partial G^\beta(X_\sigma(s,1))\,\hat{d}X_\sigma(s)\} \\ &\qquad+\int_0^1 \{\partial G^\beta(X_\tau(0,t))\,dX_\tau(t)-\partial G^\beta(X_\sigma(0,t))\,dX_\sigma(t)\}\Big| \\ &\quad+\Big|\iint_{s<t} \{\hat{d}X_\tau(s)\cdot\partial'\partial G^\beta(X_\tau(s,t))\,dX_\tau(t) \\ &\qquad-\hat{d}X_\sigma(s)\cdot\partial'\partial G^\beta(X_\sigma(s,t))\,dX_\sigma(t)\}\Big| \\ &=|G^\beta(X_\tau(0,1))-G^\beta(X_\sigma(0,1))|+\Xi_1+\Xi_2. \end{aligned}$$

Let $p>2$. Using the Burkholder inequality and (1.10), we have

$$(3.2)\qquad \begin{aligned} \mathrm{E}|\Xi_1|^p &\leqq c\cdot\mathrm{E}\Big|\int_0^1 \{|\partial G^\beta(X_\tau(s,1))-\partial G^\beta(X_\sigma(s,1))|^2 \\ &\quad+2(1-e^{-|\tau-\sigma|/2})\,\partial G^\beta(X_\tau(s,1))\,\partial' G^\beta(X_\sigma(s,1))\}\,ds\Big|^{p/2} \\ &\quad+c\cdot\mathrm{E}\Big|\int_0^1 \{|\partial G^\beta(X_\tau(0,t))-\partial G^\beta(X_\sigma(0,t))|^2 \\ &\quad+2(1-e^{-|\tau-\sigma|/2})\,\partial G^\beta(X_\tau(0,t))\,\partial' G^\beta(X_\sigma(0,t))\}\,ds\Big|^{p/2} \\ &\leqq c\cdot\int_0^1 \mathrm{E}|\partial G^\beta(X_\tau(0,t))-\partial G^\beta(X_\sigma(0,t))|^p\,dt \end{aligned}$$

$$+c\cdot\left(\text{th}\frac{|\tau-\sigma|}{4}\right)^{p/2}\int_0^1 \mathrm{E}\,|\partial G^\beta(X_\tau(0,t))|^p\,dt\,.$$

Similarly we have

$$\mathrm{E}\,|\Xi_2|^p\leqq c\cdot\int_0^1 \mathrm{E}\,|\int_s^1 \{\partial'\partial G^\beta(X_\tau(s,t))\,dX_\tau(t)-\partial'\partial G^\beta(X_\sigma(s,t))\,dX_\sigma(t)\}\,|^p\,ds$$
$$+c\cdot\left(\text{th}\frac{|\tau-\sigma|}{4}\right)^{p/2}\int_0^1 \mathrm{E}\,|\int_s^1 \partial'\partial G^\beta(X_\tau(s,t))\,dX_\tau(t)\,|^p\,ds\,.$$

From the Burkholder inequality we see that

$$\mathrm{E}\,|\int_s^1 \{\partial'\partial G^\beta(X_\tau(s,t))\,dX_\tau(t)-\partial'\partial G^\beta(X_\sigma(s,t))\,dX_\sigma(t)\}\,|^p$$
$$\leqq c\cdot\mathrm{E}\,|\sum_{j=1}^d\int_s^1 \{|\partial'\partial_j G^\beta(X_\tau(s,t))-\partial'\partial_j G^\beta(X_\sigma(s,t))|^2$$
$$+2(1-e^{-|\tau-\sigma|/2})\,\partial\partial_j G^\beta(X_\tau(s,t))\,\partial'\partial_j G^\beta(X_\sigma(s,t))\}\,dt\,|^{p/2}$$
$$\leqq c\cdot\sum_{|\nu|=2}\mathrm{E}\,|\int_0^{1-s}|\partial^\nu G^\beta(X_\tau(0,t))-\partial^\nu G^\beta(X_\sigma(0,t))|^2\,dt\,|^{p/2}$$
$$+c\cdot\left(\text{th}\frac{|\tau-\sigma|}{4}\right)^{p/2}\sum_{|\nu|=2}\int_0^{1-s} E\,|\partial^\nu G^\beta(X_\tau(0,t))|^p\,dt\,,$$

and

$$E\,|\int_s^1 \partial'\partial G^\beta(X_\tau(s,t))\,dX_\tau(t)\,|^p\leqq c\cdot\sum_{|\nu|=2}\int_0^{1-s} E\,|\partial^\nu G^\beta(X_\tau(0,t))|^p\,dt\,.$$

Combining these inequalities, we have

$$\mathrm{E}\,|\Xi_2|^p\leqq c\cdot\sum_{|\nu|=2}\mathrm{E}\,|\int_0^1|\partial^\nu G^\beta(X_\tau(0,t))-\partial^\nu G^\beta(X_\sigma(0,t))|^2\,dt\,|^{p/2} \tag{3.3}$$
$$+c\cdot\left(\text{th}\frac{|\tau-\sigma|}{4}\right)^{p/2}\sum_{|\nu|=2}\int_0^1 E\,|\partial^\nu G^\beta(X_\tau(0,t))|^p\,dt\,.$$

For $0\leq\tau,\ \sigma\leqq 1$, set

$$a=\left(\text{th}\frac{|\tau-\sigma|}{4}\right)^{1/2},\qquad b=\frac{1}{2}(1+e^{-|\tau-\sigma|/2})\,. \tag{3.4}$$

Let $B_1(t)$ and $B_2(t)$ be independent d-dimensional Brownian motions defined on $(\Omega,\mathcal{F},P)$ satisfying $B_1(0)=B_2(0)=0$. From (1.10) the law of the process

$$t\rightsquigarrow(X_\tau(0,t),\,X_\sigma(0,t))$$

is equal to that of the process

$$t\rightsquigarrow(B_1(bt)+a\,B_2(bt),\,B_1(bt)-a\,B_2(bt))\,.$$

Therefore

$$\begin{aligned}&\mathrm{E}|G^{\beta}(X_{\tau}(0,1))-G^{\beta}(X_{\sigma}(0,1))|^{p}\\&=\mathrm{E}|G^{\beta}(B_1(b)+aB_2(b))-G^{\beta}(B_1(b)-aB_2(b))|^{p}\\&\leqq 2^{p}\sup_{t\leqq 1}\mathrm{E}|G^{\beta}(B_1(t)+aB_2(t))-G^{\beta}(B_1(t))|^{p}.\end{aligned}$$

By a similar argument we have the following lemma from (3.2) and (3.3).

Lemma 3.1. *For $p>2$, it holds that*

$$\begin{aligned}(3.5)\quad &\mathrm{E}|\psi(2-2\beta, X_{\tau})-\psi(2-\beta, X_{\tau})-\psi(2-2\beta, X_{\sigma})+\psi(2-\beta, X_{\sigma})|^{p}\\&\leqq c\cdot a^{p}\sum_{|\nu|=1,2}\int_0^1 E|\partial^{\nu}G^{\beta}(B_1(t))|^{p}\,dt\\&+\sup_{t\leqq 1}\mathrm{E}|G^{\beta}(B_1(t)+aB_2(t))-G^{\beta}(B_1(t))|^{p}\\&+c\cdot\int_0^1\mathrm{E}|\partial G^{\beta}(B_1(t)+aB_2(t))-\partial G^{\beta}(B_1(t))|^{p}\,dt\\&+c\cdot\sum_{|\nu|=2}\mathrm{E}\left|\int_0^1|\partial^{\nu}G^{\beta}(B_1(t)+aB_2(t))-\partial^{\nu}G^{\beta}(B_1(t))|^{2}\,dt\right|^{p/2}.\end{aligned}$$

Fix $0<\alpha<2$ and let $G_{\varepsilon}(x)$ be the function defined by (2.1). Replace the function $G^{\beta}(x)$ by the function $G_{\varepsilon}(x)$ in the above arguments. Then we have the following estimate, which is much simpler than (3.5).

Lemma 3.2. *For fixed $0<\alpha<2$ and $p>2$, it holds that*

$$\begin{aligned}(3.6)\qquad &\mathrm{E}|\psi_{\varepsilon}(\alpha, X_{\tau})-\psi_{2\varepsilon}(\alpha, X_{\tau})-\psi_{\varepsilon}(\alpha, X_{\sigma})+\psi_{2\varepsilon}(\alpha, X_{\sigma})|^{p}\\&\leqq c\cdot a^{p}\sum_{|\nu|=1,2}\sup_{t\leqq 1}\mathrm{E}|\partial^{\nu}G_{\varepsilon}(B_1(t))|^{p}\\&+c\cdot\sum_{|\nu|\leqq 2}\sup_{t\leqq 1}\mathrm{E}|\partial^{\nu}G_{\varepsilon}(B_1(t)+aB_2(t))-\partial^{\nu}G_{\varepsilon}(B_1(t))|^{p}.\end{aligned}$$

4. Moment inequalities

Let $2-\alpha=2\beta>0$. Define a function $\zeta_p(a)$, $0\leqq a<1$, by

$$(4.1)\qquad \zeta_p(a)=\begin{cases}a^{p\wedge(\beta p+d)} & (p\neq\beta p+d)\\ a^{p}(1-\log a) & (p=\beta p+d).\end{cases}$$

Lemma 4.1. *There is a constant C independent of ε such that*

$$\begin{aligned}(4.2)\qquad &\mathrm{E}|\psi_{\varepsilon}(\alpha, X_{\tau})-\psi_{2\varepsilon}(\alpha, X_{\tau})-\psi_{\varepsilon}(\alpha, X_{\sigma})+\psi_{2\varepsilon}(\alpha, X_{\sigma})|^{p}\\&\leqq C\,\varepsilon^{\beta p}\,\zeta_p\left(\left(\mathrm{th}\frac{|\tau-\sigma|}{4}\right)^{1/2}\right)\end{aligned}$$

for any $0\leqq\tau$, $\sigma\leqq 1$.

Proof. From (2.3) we see that, for $|\nu|\leqq 2$,

$$\mathrm{E}|\partial^{\nu}G_{\varepsilon}(B_1(t))|^p \leqq c\cdot\varepsilon^{2\beta p}\int|\sqrt{t}\,x|^{(2+\beta-|\nu|)p}\,e^{-|x|^2/2}\,dx \leqq c\cdot\varepsilon^{2\beta p}.$$

Since $a^p \leqq \zeta_p(a)$, we have

$$a^p\sum_{|\nu|=1,2}\sup_{t\leqq 1}\mathrm{E}|\partial^{\nu}G_{\varepsilon}(B_1(t))|^p \leqq c\cdot\varepsilon^{2\beta p}\,\zeta_p(a).$$

From (2.4)

$$\begin{aligned}
&\mathrm{E}\,[|\partial^{\nu}G_{\varepsilon}(B_1(t)+aB_2(t))-\partial^{\nu}G_{\varepsilon}(B_1(t))|^p;\ |B_1(t)|\leqq 2a|B_2(t)|]\\
&\leqq c\cdot\varepsilon^{\beta p}\iint_{|x|\leqq 2a|y|}(a|y|)^{\beta p}(1+|y|)^{2p}\,e^{-|y|^2/2}\,dxdy\\
&\leqq c\cdot\varepsilon^{\beta p}\int(a|y|)^{\beta p+d}(1+|y|)^{2p}\,e^{-|y|^2/2}\,dy\\
&\leqq c\cdot\varepsilon^{\beta p}\,a^{\beta p+d} \leqq c\cdot\varepsilon^{\beta p}\,\zeta_p(a).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&\mathrm{E}\,[|\partial^{\nu}G_{\varepsilon}(B_1(t)+aB_2(t))-\partial^{\nu}G_{\varepsilon}(B_1(t))|^p;\ |B_1(t)|>2a|B_2(t)|]\\
&\leqq c\cdot\varepsilon^{\beta p}\iint_{|x|>2a|y|}(a|y|\,|x|^{\beta-1}(1+|x|)^2)^p\,e^{-(|x|^2+|y|^2)/2}\,dxdy\\
&\leqq c\cdot\varepsilon^{\beta p}\int\left(\int_{a|y|}^{\infty}r^{(\beta-1)p+d-1}\,e^{-r}\,dr\right)a(|y|)^p\,e^{-|y|^2/2}\,dy,
\end{aligned}$$

for $(1+r)^{2p}\exp(-r^2/2)\leqq c\cdot\exp(-r)$. Using the estimate

$$\int_{a|y|}^{\infty}r^{q-1}\,e^{-r}\,dr \leqq \begin{cases} c\cdot(a|y|)^{q\wedge 0} & (q\neq 0)\\ c\cdot(1-\log a)\,(1+|\log|y||) & (q=0)\end{cases}$$

We obtain the inequality

$$\begin{aligned}
&\mathrm{E}\,[|\partial^{\nu}G_{\varepsilon}(B_1(t)+aB_2(t))-\partial^{\nu}G_{\varepsilon}(B_1(t))|^p;\ |B_1(t)|>2a|B_2(t)|]\\
&\leqq c\cdot\varepsilon^{\beta p}\,\zeta_p(a).
\end{aligned}$$

From (3.6) the proof is completed. q.e.d.

Next, we shall consider the moment inequality with respect to the process $\psi(\alpha, X_\tau)$. Let $G^{\beta}(x)$ be the function defined by (2.6).

Lemma 4.2. *There is a constant C independent of $0<\beta<1$ such that, for $0<a<1$,*

$$\begin{aligned}
(4.3)\qquad &a^p\sum_{|\nu|=1,2}\int_0^1\mathrm{E}|\partial^{\nu}G^{\beta}(B_1(t))|^p\,dt\\
&+\sup_{t\leqq 1}\mathrm{E}|G^{\beta}(B_1(t)+aB_2(t))-G^{\beta}(B_1(t))|^p\\
&+\int_0^1\mathrm{E}|\partial G^{\beta}(B_1(t)+aB_2(t))-\partial G^{\beta}(B_1(t))|^p\,dt \leqq C(a\beta)^p.
\end{aligned}$$

Proof. From (2.8) we see that

$$|\partial_j G^\beta(x)| \leqq c\cdot\beta(1+|x|^4),$$
$$|\partial_i\partial_j G^\beta(x)| \leqq c\cdot\beta((\log|x|)^2+|x|^3).$$

Immediately we have, for $|\nu|=1, 2$,

$$a^p\int_0^1 \mathrm{E}|\partial^\nu G^\beta(B_1(t))|^p\, dt \leqq c\cdot(a\beta)^p.$$

Using the estimate

$$\begin{aligned}&|\partial_j G^\beta(x+ay)-\partial_j G^\beta(x)|^p\\ &\leqq a^p\Big|\int_0^1 \partial\partial_j G^\beta(x+\theta ay)\, y\, d\theta\Big|^p\\ &\leqq c\cdot(a\beta|y|)^p\int_0^1((\log|x+\theta ay|)^2+|x+\theta ay|^3)^p\, d\theta\\ &\leqq c\cdot(a\beta)^p\Big\{\int_0^1(\log|x+\theta ay|)^{4p}\, d\theta+|x|^{6p}+|y|^{6p}+|y|^{2p}\Big\},\end{aligned}$$

we have

$$\int_0^1 \mathrm{E}|\partial G^\beta(B_1(t)+aB_2(t))-\partial G^\beta(B_1(t))|^p\, dt \leqq c\cdot(a\beta)^p.$$

It is much easier to show that

$$\sup_{t\leqq 1}\mathrm{E}|G^\beta(B_1(t)+aB_2(t))-G^\beta(B_1(t))|^p \leqq c\cdot(a\beta)^p.$$

So the proof is completed. q.e.d.

In consideration of (2.9) we shall define, for $|\nu|=2$,

$$\text{(4.4)}\quad \lambda_\nu(a, x, y)=\Big(|(x+ay)^\nu|x+ay|^{-2}-x^\nu|x|^{-2}|+\Big|\log\frac{|x+ay|}{|x|}\Big|\Big)$$
$$\times(1+|\log|x+ay||)(1+|\log|x||)(1+|x|^2+|y|^2).$$

Lemma 4.3. *Let $p=4+8\delta>4$. There is a constant C such that, for $0<a<1$,*

$$\text{(4.5)}\qquad \mathrm{E}\Big|\int_0^1 \lambda_\nu(a, B_1(t), B_2(t))^2\, dt\Big|^{p/2} \leqq C\, a^{2(1+\delta)}.$$

Proof. Divide the space $\boldsymbol{R}^{2d}$ into three domains:

$$D(0, a)=\{(x, y);\ |x|\vee|y|>-\log a\},$$
$$D(1, a)=\{(x, y);\ |x|\vee|y|\leqq-\log a,\ |x|>2a|y|\},$$
$$D(2, a)=\{(x, y);\ |x|\vee|y|\leqq-\log a,\ |x|\leqq 2a|y|\},$$

and define

$$\rho_k(a, x, y) = I_{D(k,a)}(x, y)\,.$$

Let $B(t)=(B_1(t), B_2(t))$. First, we have

$$\begin{aligned}
&\mathrm{E}\left|\int_0^1 (\lambda_\nu^2 \rho_0)(a, B(t))\,dt\right|^{p/2} \\
&\leqq (\mathrm{E}\int_0^1 \lambda_\nu^{2p}(a, B(t))\,dt)^{1/2}\,(\mathrm{E}\int_0^1 \rho_0(a, B(t))\,dt)^{1/2} \\
&\leqq c\cdot(\mathrm{E}\int_0^1 \rho_0(a, B(t))\,dt)^{1/2} \\
&\leqq c\cdot(\int_0^1 P[\,|B_1(t)|>-\log a]\,dt)^{1/2} \\
&\leqq c\cdot(P[\,|B_1(1)|>-\log a])^{1/2} \\
&\leqq c\cdot(|\log a|^{d-2}\,e^{-(\log a)^2/2})^{1/2} \\
&= c\cdot|\log a|^{d/2-1}\,a^{(\log(1/a))/4} \leqq c\cdot a^{2(1+\delta)}\,.
\end{aligned}$$

Since

$$\int_0^1 |B_1(t)|^{-1}\,dt = \frac{2}{d-1}\{|B_1(1)|-\int_0^1 |B_1(t)|^{-1} B_1(t)\cdot dB_1(t)\}\,,$$

it holds that

$$\mathrm{E}\left|\int_0^1 |B_1(t)|^{-1}\,dt\right|^{q}<\infty \qquad \text{for any } q>0\,.$$

We see that, for any $0<\mu<1$, using the mean value theorem,

$$\begin{aligned}
&(\lambda_\nu^2 \rho_1)(a, x, y) \\
&\leqq c\cdot\{a\frac{|y|}{|x|}(1+(\log|x|)^2)(1+|x|^2)\}^2 \rho_1(a, x, y) \\
&\leqq c\cdot\left(a\frac{|y|}{|x|}\right)^{\mu}(1+(\log|x|)^4)(1+|x|^4)\,\rho_1(a, x, y) \\
&\leqq c\cdot(a\log(1/a))^{\mu}(|x|^{-1}+|x|^4)\,.
\end{aligned}$$

Therefore we have, setting $\mu=(4+6\delta)/p$,

$$\begin{aligned}
&\mathrm{E}\left|\int_0^1 (\lambda_\nu^2 \rho_1)(a, B(t))\,dt\right|^{p/2} \\
&\leqq c\cdot(a\log(1/a))^{p\mu/2}(1+\mathrm{E}\left|\int_0^1 |B_1(t)|^{-1}\,dt\right|^{p/2}) \\
&\leqq c\cdot a^{2(1+\delta)}.
\end{aligned}$$

Set $r=(2+4\delta)/\delta$. Since $(r-1)p/r=4+6\delta$,

$$
\begin{aligned}
&\mathrm{E}\left|\int_0^1 (\lambda_\nu^2\, \rho_2)\,(a, B(t))\, dt\right|^{p/2} \\
&\leqq \mathrm{E}\left[\left(\int_0^1 \lambda_\nu^{2r}(a, B(t))\, dt\right)^{p/2r} \left(\int_0^1 \rho_2(a, B(t))\, dt\right)^{(r-1)p/2r}\right] \\
&\leqq \left(\mathrm{E}\left|\int_0^1 \lambda_\nu^{2r}(a, B(t))\, dt\right|^{p/r}\right)^{1/2} \left(\mathrm{E}\left|\int_0^1 \rho_2(a, B(t))\, dt\right|^{4+6\delta}\right)^{1/2} \\
&\leqq c\cdot\left(\mathrm{E}\left|\int_0^1 \rho_2(a, B(t))\, dt\right|^{4+6\delta}\right)^{1/2} \\
&\leqq c\cdot\left(\mathrm{E}\left|\int_0^1 \frac{2a\cdot\log(1/a)}{|B_1(t)|}\, dt\right|^{4+6\delta}\right)^{1/2} \\
&= c\cdot(a\log(1/a))^{2+3\delta}\left(\mathrm{E}\left|\int_0^1 |B_1(t)|^{-1}\, dt\right|^{4+6\delta}\right)^{1/2} \\
&\leqq c\cdot(a\log(1/a))^{2+3\delta} \leqq c\cdot a^{2(1+\delta)}\,.
\end{aligned}
$$

These prove (4.5). q.e.d.

From (2.9) and (4.5) we know that

$$
(4.6)\qquad \sum_{|\nu|=2} \mathrm{E}\left|\int_0^1 |\partial^\nu G^\beta(B_1(t)+aB_2(t))-\partial^\nu G^\beta(B_1(t))|^2\, dt\right|^{p/2} \leqq c\cdot\beta^p\, a^{2(1+\delta)}\,.
$$

Combining (3.5), (4.3) and (4.6), we obtain the following lemma.

Lemma 4.4. *There is a constant C independent of $0<\beta<1$ such that*

$$
\begin{aligned}
(4.7)\qquad &\mathrm{E}\,|\psi(2-2\beta, X_\tau)-\psi(2-\beta, X_\tau)-\psi(2-2\beta, X_\sigma)+\psi(2-\beta, X_\sigma)|^p \\
&\leqq C\,\beta^p\,|\tau-\sigma|^{1+\delta}\,,
\end{aligned}
$$

for any $0\leqq\tau, \sigma\leqq 1$, where $p=4+8\delta>4$.

5. Proof of theorems

We shall prove Theorem 2 and 3 applying the following lemma. The basic idea of the lemma is communicated by Prof. S. Kusuoka.

Lemma 5.1. *Let $\{\Phi_n(\tau)\}$, $0\leqq\tau\leqq 1$, be a sequence of real valued continuous processes. If there are positive constants C, p, q and δ such that, for all τ, σ and n,*

$$
(5.1)\qquad \mathrm{E}\,|\Phi_n(\tau)-\Phi_n(\sigma)|^p \leqq C\, 2^{-nq}\,|\tau-\sigma|^{1+\delta}\,,
$$

then

$$
(5.2)\qquad P\left[\sum_{n=1}^{\infty} \sup_\tau |\Phi_n(\tau)-\Phi_n(0)|<\infty\right] = 1\,.
$$

Proof. Choose γ, $0<\gamma<\delta$, and η, $0<\eta<1$, so small that $(1-\eta)\,(1+\delta-\gamma)>1$. Set

$$
J(m) = \{(i, j)\in \boldsymbol{Z}_+^2\,;\ 0\leqq i<j\leqq 2^m,\ j-i<2^{m\eta}\}\,.
$$

Then, from (5.1) we know that

$$\begin{aligned}&P[|\Phi_n(j2^{-m})-\Phi_n(i2^{-m})|^p>2^{-nq/2}((j-i)2^{-m})^\gamma \quad \text{for any } (i,j)\in J(m)]\\&\leqq c\cdot 2^{-nq/2}\sum_{(i,j)\in J(m)}((j-i)2^{-m})^{1+\delta-\gamma}\\&\leqq c\cdot 2^{-nq/2}\, 2^{-m((1-\eta)(1+\delta-\gamma)-1)}\,.\end{aligned}$$

Let $A(M, N)$ denote the set

$$\begin{aligned}A(M,N)=\{&|\Phi_n(j2^{-m})-\Phi_n(i2^{-m})|^p\leqq 2^{-nq/2}((j-i)\,2^{-m})^\gamma\\&\text{for all } n\geqq N \text{ and } (i,j)\in J(m) \text{ with } m\geqq M\}\,.\end{aligned}$$

Then we have $P[A(M, N)]\uparrow 1$ as $M, N\uparrow\infty$.

For a moment, we shall consider paths of processes $\{\Phi_n(\tau);\ n\geqq N\}$ on the set $A(M, N)$. Pick $0\leqq\sigma<\sigma'\leqq 1$ so close that $\sigma'-\sigma<2^{-M(1-\eta)}$. Choose m such that

$$2^{-(m+1)(1-\eta)}\leqq\sigma'-\sigma<2^{-m(1-\eta)}\,,$$

and expand σ and σ' as follows:

$$\begin{aligned}\sigma &= i2^{-m}+2^{-m(1)}+2^{-m(2)}+\cdots\,,\\ \sigma' &= j2^{-m}-2^{-m'(1)}-2^{-m'(2)}-\cdots\,,\end{aligned}$$

where $m<m(1)<m(2)<\cdots$ and $m<m'(1)<m'(2)<\cdots$. Since $\Phi_n(\tau)$ is continuous in τ, we have

$$\begin{aligned}&|\Phi_n(\sigma')-\Phi_n(\sigma)|\\&\leqq|\Phi_n(\sigma')-\Phi_n(j2^{-m})|+|\Phi_n(i2^{-m})-\Phi_n(\sigma)|+|\Phi_n(j2^{-m})-\Phi_n(i2^{-m})|\\&\leqq 2^{-nq/2p}\,\{2\sum_{k\geqq m}2^{-k\gamma/p}+(j2^{-m}-i2^{-m})^{\gamma/p}\}\\&\leqq c\cdot 2^{-nq/2p}\,\{2^{-m\gamma/p}+(\sigma'-\sigma)^{\gamma/p}\}\\&\leqq c\cdot 2^{-nq/2p}\,(\sigma'-\sigma)^{\gamma/p}\,.\end{aligned}$$

Therefore

$$\sup_\tau|\Phi_n(\tau)-\Phi_n(0)|\leqq c\cdot 2^{-nq/2p}\,2^{M(1-\eta)}\,2^{-M(1-\eta)\gamma/p}\,.$$

This implies (5.2), for $P[A(M, N)]\uparrow 1$ as $M, N\uparrow\infty$. q.e.d.

Proof of Theorem 2. Let $p>2$ and $2-\alpha=2\beta>0$. Then there is a positive constant δ such that $\zeta_p(a)\leqq c\cdot a^{2(1+\delta)}$, where $\zeta_p(a)$ is the function defined by (4.1). Set $\varepsilon(n)=2^{-n}$ and

$$\Phi_n(\tau)=\psi_{\varepsilon(n)}(\alpha, X_\tau)-\psi_{2\varepsilon(n)}(\alpha, X_\tau)\,.$$

From Lemma 4.1, the function $\Phi_n(\tau)$ satisfies condition (5.1) for $q=\beta p$. From

Lemma 5.1 we have

$$\lim_{n\to\infty} \sup_{\tau} |\psi_{\varepsilon(n)}(\alpha, X_\tau)-\psi(\alpha, X_\tau)-\psi_{\varepsilon(n)}(\alpha, X_0)+\psi(\alpha, X_0)| = 0 \qquad \text{a.e.},$$

for

$$\psi(\alpha, X_\tau)-\psi_{\varepsilon(n)}(\alpha, X_\tau) = \sum_{k>n} \Phi_k(\tau)\,.$$

Since $\psi_{\varepsilon(n)}(\alpha, X_0)\to\psi(\alpha, X_0)$ $n\to\infty$, and since $\psi_{\varepsilon(n)}(\alpha, X_\tau)$ is continuous in τ, we conclude that

$$P\,[\psi(\alpha, X_\tau) \text{ is continuous in } \tau] = 1\,. \qquad \text{q.e.d.}$$

Proof of Theorem 3. Set

$$\Phi_n(\tau) = \Psi_n(X_\tau)-\Psi_{n-1}(X_\tau)\,.$$

From Lemma 4.4, the function $\Phi_n(\tau)$ satisfies condition (5.1) for $p=q=4+8\delta$. Therefore $\Psi_n(X_\tau)-\Psi_n(X_0)$ converges uniformly in τ as $n\to\infty$ almost everywhere. By Theorem 1, $\Psi_n(X_0)$ converges to $\Psi(X_0)$ a.e. as $n\to\infty$. Hence

$$P\,[\Psi_n(X_\tau) \text{ converges uniformly in } \tau \text{ as } n\to\infty] = 1\,.$$

This implies that $\{\Psi_n(w)\}$ converges quasi-everywhere and

$$P\,[\lim \Psi_n(X_\tau) \text{ is continuous in } \tau] = 1\,. \qquad \text{q.e.d.}$$

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