EQUIVARIANT POINT THEOREMS FOR FIBRE-PRESERVING MAPS

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1 Introduction

Let \( p: X \to B \) and \( p': X' \to B' \) be local trivial fibre spaces with fibre-preserving involutions \( T: X \to X \) and \( T': X' \to X' \) respectively, and let \( f: X \to X' \) be a fibre-preserving map. Denote by \( A_f \) the set of equivariant points of \( f \):

\[
A_f = \{ x \in X; fT(x) = T'f(x) \},
\]

and by \( \bar{A}_f \) its orbit space under \( T \). In this paper we shall study \( H^*(\bar{A}_f) \) in connection with \( H^*(B) \), where \( H^* \) is the Čech cohomology with coefficients in \( \mathbb{Z}_2 \). Two theorems will be proved by making use of the technique of establishing a transfer homomorphism, which was initiated by Becker and Gottlieb ([1], [2]).

In case \( p: X \to B \) is an \( m \)-sphere bundle with the antipodal involution and \( p': X' \to B \) is an \( R^n \)-bundle with the trivial involution, Jaworowski gave in [4], [5] the following theorem which is a "continuous" version of the Borsuk-Ulam theorem: If \( k = m - n \geq 0 \) and all the Stiefel-Whitney classes of \( p': X' \to B \) are zero then the composition

\[
H^i(B) \xrightarrow{\bar{p}^*} H^i(\bar{A}_f) \xrightarrow{\omega(A_f)^k} H^{i+k}(\bar{A}_f)
\]

is injective for every \( i \), where \( \bar{p}: \bar{A}_f \to B \) is induced by \( p|A_f \), and \( \omega(A_f) \) is the characteristic class of the double covering \( A_f \to \bar{A}_f \). It is seen in this paper that the assumption on the Stiefel-Whitney classes is superfluous in the theorem of Jaworowski.

Throughout this paper we use the Čech cohomology with coefficients in \( \mathbb{Z}_2 \).

2 Equivariant fundamental cohomology class

Let \( M \to X \xrightarrow{\bar{p}} B \) be a local trivial fibre space such that both the fibre \( M \) and the base \( B \) are manifolds without boundary. Suppose that there is given a fibre-preserving involution \( T: X \to X \), that is, an involution satisfying \( pT = T \).
We take the fibre square $X \times X$ of the map $p: X \to B$, and define an involution on it by permutation of factors. Then there is an equivariant imbedding $\Delta: X_\mathbb{Z} \times X_\mathbb{Z}$ defined by $\Delta(x) = (x, Tx)$. Consider now the normal bundle of $\Delta X \subset X_\mathbb{Z} \times X_\mathbb{Z}$ in which the total space $E$ is regarded as an invariant tubular neighborhood of $\Delta X$ in $X \times X$. Then we have an $\mathbb{R}^m$-bundle $\pi: E \to \Delta X$ with involution, where $m = \dim M$. Let $S^\infty$ be the infinite dimensional sphere with the antipodal involution, and consider the orbit spaces $S^\infty \times E$ and $S^\infty \times (\Delta X)$ under the diagonal action. Then we have an $\mathbb{R}^m$-bundle $1 \times \pi: S^\infty \times E \to S^\infty \times (\Delta X)$, so that the Thom class $U(1 \times \pi) \in H^m(S^\infty \times E, S^\infty \times E - S^\infty \times (\Delta X)) = H^m_\mathbb{Z}(E, E - \Delta X)$. We define $\hat{U}(p) \in H^m_\mathbb{Z}(X \times X, X \times X - \Delta X)$ to be the element corresponding to $U(1 \times \pi)$ under the excision isomorphism, and call it the equivariant fundamental cohomology class of $p: X \to B$. The restriction $\hat{U}(p)|X \times X \in H^m_\mathbb{Z}(X \times X)$ is denoted by $\hat{U}'(p)$ and is called the equivariant diagonal cohomology class of $p$. If $B$ is a single point, then $\hat{U}(p) \in H^m_\mathbb{Z}(M \times M, M \times M - \Delta M)$ and $\Delta(p) \in H^m_\mathbb{Z}(M \times M)$ are denoted by $\hat{U}(M)$ and $\hat{U}'(M)$ respectively. If $M$ is a closed manifold, we have $\hat{U}'(M) = \Delta(1)$ for the Gysin homomorphism $\Delta: H^m_\mathbb{Z}(M) \to H^m_\mathbb{Z}(M \times M)$.

Put $M_b = p^{-1}(b)$ for $b \in B$. Then the restriction of the normal bundle $\pi: E \to \Delta X$ on $\Delta M_b$ may be regarded as the normal bundle of $\Delta(M_b) \subset M_b \times M_b$. Therefore it follows that

$$\hat{U}(p)|(M_b \times M_b, M_b \times M_b - \Delta M_b) = \hat{U}(M_b),$$

so that

$$\hat{U}'(p)|(M_b \times M_b) = \hat{U}'(M_b).$$

In some cases, the equivariant diagonal cohomology class $\hat{U}'(M)$ of a closed manifold $M$ with an involution $T$ is expressed in terms of cohomology of $M$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ be a homogeneous basis of $H^*(M)$, and let $C = (c_{ij})$ be the inverse of the matrix $Y = (y_{ij})$ with $y_{ij} = \langle \alpha_i, T^* \alpha_j \rangle_{[M]}$. Let $\pi_i: H^*(M \times M) \to H^m_\mathbb{Z}(M \times M)$ denote the transfer homomorphism for the covering $S^\infty \times (M \times M) \to S^\infty \times (M \times M)$. We have

**Proposition 1.** (i) If $T$ is trivial, then

$$\hat{U}'(M) = \sum_{i=0}^{m/2} q^* \omega^{m-2i} \cup P_0(V_i) + \sum_{i < j} (c_{ij} + c_{ij} \pi_i(\alpha_i \times \alpha_j)),$$

where $q: S^\infty \times (M \times M) \to S^\infty / \mathbb{Z}_2$ is the projection, $\omega \in H^1(S^\infty / \mathbb{Z}_2)$ is the generator,
V_i is the i-th Wu class of M, and $P_0$ is the external Steenrod square (See [3], [7], [8]).

(ii) If $T$ is free, then

$$\hat{U}(M) = \sum_{i \in I} c_i \pi_i(\alpha_i \times \alpha_j)$$

(See [7], [8]).

3 Equivariant point theorem of Borsuk-Ulam type

For any space $X$ with a free involution, we denote by $\mathcal{X}$ the orbit space of $X$ under the involution, and by $\omega(X) \in H^1(\mathcal{X})$ the characteristic class of the double covering $X \to \mathcal{X}$.

**Theorem 1.** Let $M \to X \overset{p}{\to} B$ and $M' \to X' \overset{p'}{\to} B'$ be local trivial fibre spaces over ENR's (=Euclidean neighborhood retracts), where the fibre $M$ is a closed $m$-manifold, and $M'$ is a compact $n$-manifold with or without boundary. Let $f: X \to X'$ be a fibre-preserving map covering a map $g: B \to B'$, and let $f_b: M_b = p^{-1}(b) \to M'_{g(b)} = p'^{-1}(g(b)) (b \in B)$ denote the restriction of $f$. Suppose that $X$ provides a fibre-preserving free involution $T$, and put $A_f = \{x \in X; f(x) = f(Tx)\}$. If $k = m - n \geq 0$ and, for some point $b$ of each connected component of $B$,

$$\omega(M_b)^w \neq 0, \quad f^*_b: H^*(M'_{g(b)}) \to H^*(M_b),$$

then the composition

$$H^i(B) \overset{\bar{f}^*}{\to} H^i(\mathcal{A}_f) \overset{\omega(A_f)^w}{\to} H^{i+k}(\mathcal{A}_f)$$

is injective for every $i \geq 0$.

**Proof.** Case 1: $B$ and $B'$ are manifolds without boundary, and $M'$ is a closed manifold.

By the continuity property of Čech cohomology, it suffices to prove that, for any invariant neighborhood $V$ of $A_f$, the composition

$$\bar{f}^*_b: H^i(B) \overset{\bar{f}^*}{\to} H^i(V) \overset{\omega(V)^w}{\to} H^{i+k}(V)$$

is injective for every $i \geq 0$. To do this we shall establish a transfer homomorphism

$$\tau_b: H^{i+k}(V) \to H^i(B)$$

such that $\tau_b \circ \bar{f}^*_b = id$.

Regard $X'$ as a space with involution by the trivial action. Then we have the equivariant fundamental cohomology class $\hat{U}(p') \in H^*_E(X' \times X', X' \times X' - dX')$ of $p': X' \to B'$, where $dX'$ is the diagonal. There is an equivariant map $\bar{f}: (X, X - A_f) \to (X' \times X', X' \times X' - dX')$ defined by $\bar{f}(x) = (f(x)$,
Consider \( f^*(\hat{\mathcal{U}}(p')) \subseteq H^\sharp_2(X, X - A_f) = H^\sharp(X, X - A_f) \), and define \( \tau_k \) to be the composition

\[
H^{i+k}(\mathcal{V}) \xrightarrow{l^*f^*(\hat{\mathcal{U}}(p'))} H^{i+m}(\mathcal{V}, \mathcal{V} - A_f) \\
\xrightarrow{\tilde{l}^*} H^{i+m}(X, X - A_f) \xrightarrow{j^*} H^{i+m}(X) \xrightarrow{\tilde{\mathcal{P}}_i} H^i(B),
\]

where \( l: (\mathcal{V}, \mathcal{V} - A_f) \subseteq (X, X - A_f) \), \( j: X \subseteq (X, X - A_f) \), and \( \tilde{\mathcal{P}}_i \) is the integration along the fibre ([2]) for the fibre space \( \mathcal{P}: X \to B \). We shall show \( \tau_k \circ \tilde{\mathcal{P}}_k = \text{id} \).

For any \( \beta \in H^i(B) \) we have

\[
\tau_k \tilde{\mathcal{P}}_k(\beta) = \tilde{\mathcal{P}}_i j^* l^* (\tilde{\mathcal{P}}^*(\beta) \omega(\mathcal{V})^k l^* f^*(\hat{\mathcal{U}}(p'))) = \tilde{\mathcal{P}}_i (\tilde{\mathcal{P}}^*(\beta) \omega(\mathcal{X})^k f^*(\hat{\mathcal{U}}'(p'))) = \beta \tilde{\mathcal{P}}_i (\omega(\mathcal{X})^k f^*(\hat{\mathcal{U}}'(p'))).
\]

Therefore it remains to prove

\[
\tilde{\mathcal{P}}_i (\omega(\mathcal{X})^k f^*(\hat{\mathcal{U}}'(p'))) = 1.
\]

We have a commutative diagram:

\[
\begin{array}{ccc}
H^\sharp_2(X' \times X') & \xrightarrow{\hat{f}^*} & H^\sharp_2(X) \\
\downarrow i^* & & \downarrow i^* \\
H^\sharp_2(M'_+(\mathbb{B}) \times M'_+(\mathbb{B})) & \xrightarrow{\hat{f}_+^*} & H^\sharp_2(M_+) \\
\end{array}
\]

where \( i \) are inclusions. Therefore we see

\[
i^* \tilde{\mathcal{P}}_i (\omega(\mathcal{X})^k f^*(\hat{\mathcal{U}}'(p'))) = \tilde{\mathcal{P}}_i (\omega(\mathcal{M}_+)^k f^*(\hat{\mathcal{U}}'(\mathcal{M}_+))).
\]

From (i) of Proposition 1 and our assumption \( f^*_+ = 0 \), it follows that

\[
\hat{f}_+^*(\hat{\mathcal{U}}'(\mathcal{M}_+)) = \omega(\mathcal{M}_+)^k.
\]

Since \( \omega(\mathcal{M}_+) = 0 \) we have \( \mathcal{P}_i(\omega(\mathcal{M}_+)) = 1 \). Thus it holds that

\[
i^* \tilde{\mathcal{P}}_i (\omega(\mathcal{X})^k f^*(\hat{\mathcal{U}}'(p'))) = 1
\]

which shows the desired result.

Case 2: \( B \) and \( B' \) are manifolds without boundary, and \( M' \) is a compact manifold with boundary.

In this case \( X' \) is a manifold with boundary, and a local trivial fibre space
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DM' \to DX' \to B'

is defined naturally, where DX' and DM' are the doubles of X' and M' respectively. Put \( \tilde{f} = i \circ f : X \to DX' \) where \( i : X' \subseteq DX' \). Obviously \( A_f = A_{f'} \).

Therefore, by applying Case 1 to \( p, \tilde{p}' \) and \( \tilde{f} \), we have the result.

Case 3: \( B \) and \( B' \) are ENR's, and \( M' \) is a compact manifold.

There are continuous maps

\[
B \overset{i}{\rightarrow} W \overset{r}{\rightarrow} B, \quad B' \overset{i'}{\rightarrow} W' \overset{r'}{\rightarrow} B'
\]

such that \( r \circ i = id, r' \circ i' = id \), where \( W \) and \( W' \) are open sets in Euclidean spaces. Let \( q : Z \to W \) and \( q' : Z' \to W' \) be the induced fibre spaces of \( p : X \to B \) and \( p' : X' \to B' \) under \( r \) and \( r' \) respectively. Define \( \tilde{r} : Z \to X, \tilde{i}' : X' \to Z' \) and \( S : Z \to Z \) by

\[
\tilde{r}(w, x) = x, \quad \tilde{i}'(x') = (i'p'(x'), x'),
\]

\[
S(w, x) = (w, T(x)), \quad (x \in X, x' \in X', w \in W).
\]

Then \( h = \tilde{i}' \circ r \circ \tilde{r} : Z \to Z' \) is a fibre-preserving map, and \( S \) is a fibre-preserving free involution. We see \( \tilde{r}(A_f) \subseteq A_{f'} \). In a commutative diagram

\[
\begin{array}{ccc}
H^i(W) & \xrightarrow{\tilde{p}^*} & H^i(A_f) \xrightarrow{\omega(A_f)^k} H^{i+k}(A_f) \\
\downarrow \rho^* & & \downarrow \rho^* \\
H^i(B) & \xrightarrow{\tilde{p}^*} & H^i(\tilde{A}_f) \xrightarrow{\omega(\tilde{A}_f)^k} H^{i+k}(\tilde{A}_f)
\end{array}
\]

\( \rho^* \) is injective and the lower composition is injective by Cases 1 and 2. Therefore the upper composition is injective.

**Corollary 1.** Let \( f : X \to X' \) be a fibre-preserving map of an m-sphere bundle \( p : X \to B \) with the antipodal involution into an \( R^n \)-bundle \( p' : X' \to B' \), where \( B \) and \( B' \) are ENR's. Then if \( k = m - n \geq 0 \) the composition

\[
H^i(B) \overset{\tilde{p}^*}{\rightarrow} H^i(A_f) \xrightarrow{\omega(A_f)^k} H^{i+k}(A_f)
\]

is injective for every \( i \).

**Proof.** Taking one point compactification of each fibre, \( p' : X' \to B' \) may be regarded as a subbundle of an \( n \)-sphere bundle. Regard \( f \) as a fibre-preserving map between the sphere bundles, and apply Theorem 1. Then we get the corollary.

**Corollary 2.** Let \( M \to X \overset{p}{\rightarrow} B \) be a local trivial fibre space with a fibre-preserving free involution, where \( B \) is a connected ENR, and \( M \) is a closed m-mani-
fold. Then, if \( \omega(M_b)^m \neq 0 \) for some \( b \in B \), the composition

\[
H^i(B) \xrightarrow{\bar{p}^*} H^i(X) \xrightarrow{\omega(X)^k} H^{i+k}(X)
\]

is injective for every \( i \geq 0 \) and \( k = 0, 1, \ldots, m \).

Proof. Take the disc \( D^{m-k} \), and regard a constant map \( f : X \to D^{m-k} \) as a fibre-preserving map of \( p : X \to B \) to \( p' : D^{m-k} \to \text{pt} \). Then \( A_f = X \), and we get the result by Theorem 1.

4 Equivariant point theorem of Lefschetz type

We shall first recall from [7], [8] the definition of equivariant Lefschetz number \( \hat{L}(f) \) for a continuous map \( f : M \to N \), where \( M \) and \( N \) are closed \( n \)-manifolds with free involutions \( S \) and \( T \) respectively. There exists a homogeneous basis \( \{ \alpha_1, \ldots, \alpha_r, \alpha'_1, \ldots, \alpha'_r \} \) of \( H^*(N) \) such that

\[
\langle \alpha_i \sim T^* \alpha_i, [N] \rangle = 0, \langle \alpha'_i \sim T^* \alpha'_i, [N] \rangle = 0
\]

where \( [N] \) is the fundamental homology class of \( N \). Then the number

\[
\sum_{i=1}^r \langle f^* \alpha_i \sim S^* f^* \alpha'_i, [M] \rangle \in \mathbb{Z}_2
\]

is independent of the choice of \( \{ \alpha_1, \ldots, \alpha_r, \alpha'_1, \ldots, \alpha'_r \} \). This number is \( \hat{L}(f) \) by definition. If \( M = N, S = T \) and \( f^* = \text{id} \), \( \hat{L}(f) \) coincides with the mod 2 semi-characteristic \( \hat{x}(M) \) of \( M \).

**Theorem 2.** Let \( M \to X \to B \) and \( M' \to X' \to B' \) be local trivial fibre spaces over ENR's such that the fibres are closed \( n \)-manifolds, and let \( f : X \to X' \) be a fibre-preserving map. Suppose there are given fibre-preserving free involutions \( T : X \to X \) and \( T' : X' \to X' \), and put \( A_f = \{ x \in X | fT(x) = T'f(x) \} \). If the equivariant Lefschetz number \( \hat{L}(f_b) \) is not zero for some point \( b \) of each connected component of \( B \), then

\[
\bar{p}^* : H^*(B) \to H^*(\bar{A}_f)
\]

is injective.

Proof. If we use (ii) of Proposition 1, Theorem 2 can be proved similarly to the proof of Theorem 1.

**Corollary.** Let \( M \to X \to B \) be a local trivial fibre space with a fibre preserving free involution, where \( B \) is an ENR and \( M \) is a closed manifold. If the mod 2 semi-characteristic \( \hat{x}(M) \neq 0 \) then

\[
\bar{p}^* : H^*(B) \to H^*(X)
\]
is injective.

Proof. Take \( f = id \) in Theorem 2. This corollary can be applied to prove

**Theorem 3** ([7], [8], [9]). If a closed manifold \( M \) admits a free action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( \hat{X}(M) = 0 \).

Proof. Let \( T_1 \) and \( T_2 \) generate \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \). Take an \( n \)-sphere \( S^n \) for sufficiently large \( n \), and consider the orbit space \( X = S^n \times M \) under the diagonal action of the antipodal involution on \( S^n \) and the involution \( T_1 \) on \( M \).

A fibre space \( M \rightarrow X \rightarrow S^n/\mathbb{Z}_2 \) and a fibre-preserving free involution \( T \) on \( X \) are given by \( p(z, x) = x \) and \( T(z, x) = (z, T_2(x)) \), where \( z \in S^n \) and \( x \in M \). We have also a fibre bundle \( S^n \rightarrow X \rightarrow M/\mathbb{Z}_2 \), so that \( H^i(X) = 0 \) if \( m < i < n \), where \( m = \dim M \). Therefore \( \bar{p} : H^i(S^n/\mathbb{Z}_2) \rightarrow H^i(X) \) is not injective if \( m < i < n \). Thus \( \hat{X}(M) = 0 \) by the above corollary.

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**References**


