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ON REGULAR SELF-INJECTIVE RINGS

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A ring R is called right bounded if every essential right ideal contains a nonzero two-sided ideal which is an essential ideal as a right ideal. If R is both right and left bounded, then we call R bounded. The boundedness is well studied for noetherian rings ([1]). In this paper, we study (Von Neumann) regular rings which are right bounded. This class includes all regular rings whose primitive factor rings are artinian. We know from S. Page [3] that a regular ring R is a FPF-ring if and only if R is a self-injective ring with bounded index of nilpotence. As a result, it follows that regular FPF-rings are bounded.

In Proposition 1, we shall show that right non-singular right FPF-rings are right bounded. In our main theorem (Theorem 1), we shall show that a regular right self-injective ring R is right bounded if and only if $R\cong\prod M_{n(t)}(R_t)\times\prod T_s$, where each R_t is an abelian regular self-injective ring and each T_s is a right full linear ring. In Theorem 2, as an application of Theorem 1, we shall give a necessary and sufficient condition for the maximal right quotient ring $Q(R)$ of a regular ring R to be Type I_f.

1. Preliminaries

Throughout this paper, R denotes an associative ring with identity element and we assume that all R-modules are unitary. We freely use terminologies and the results in [2].

Let R be a regular, right self-injective ring. R is called Type I if it contains an idempotent e such that eR is faithful right R-module and eRe is an abelian regular ring. R is called Type II if it contains an idempotent e such that eR is faithful right R-module and eRe is a directly finite regular ring, and R contains no nonzero idempotent f such that fRf is an abelian regular ring. R is called Type III if it contains no nonzero idempotent e such that eRe is directly finite.

Goodearl and Boyle have shown that if R is a regular, right self-injective ring, then $R=R_1\times R_2\times R_3$, where R_1 is Type I, R_2 is Type II and R_3 is Type III (Theorem 10.13 [2]). Furthermore, they have shown that if R is a regular, right self-injective ring of Type I_f (Type I and directly finite), then $R\cong\prod M_{n(t)}(R_t)$, where each R_t is an abelian regular, self-injective ring (Theorem 10.24 [2]).

For any ring R, we use $B(R)$ to denote the set of all central idempotents of R. If A and B are modules, then the notation $A\subseteq_e B$ means that A is an essential submodule of B.

Let A be a nonzero right R-module over a regular, right self-injective ring R. Put $X=\{e\in B(R)\,|\,A(1-e)=0\}$. Since $B(R)$ is a complete Boolean algebra by Proposition 9.9 [2], there exists an element e in $B(R)$ such that $e=\wedge X$. We call e the central cover of A and denote it by $cc(A)$.

2. Structure theorem

A ring R is said to be *right bounded* if any essential right ideal I of R contains a two-sided ideal J such that $J\subseteq_e I$ as a right R-module. If R is both *right and left bounded*, R is simply called *bounded*. A ring R is called *right finitely pseudo-Frobenius* (FPF) if every finitely generated faithful right R-module is a *generator* in the category of right R-modules.

S. Page has shown in [3] that, for a given regular ring R, the following conditions are equivalent: 1) R is right FPF. 2) R is left FPF. 3) R is a right self-injective ring with bounded index of nilpotence. Therefore, combining this theorem with [2, Lemma 6.20 and Theorem 7.20] we see that regular right FPF-rings are *bounded* rings. This result suggests the following.

Proposition 1. *If R is a right non-singular right FPF-ring, then R is a right bounded ring.*

Proof. Let I be an essential right ideal of R, and let J be the right annihilator ideal of R/I. Assume that $J=0$. Then R/I is a *generator*, since R is a right FPF-ring. Hence R/I generates R. However this is impossible since R is a non-singular right R-module and R/I is a singular right R-module. Therefore $J\neq 0$. Now, by Proposition 3 [3], we can obtain a central idempotent e of R such that $J\subseteq_e eR_R$. We claim that $e=1$. If $e\neq 1$, then $I\cap(1-e)R\neq 0$. Put $M=((1-e)R/(I\cap(1-e)R))\oplus eR$. Then it is easy to see that M is a faithfyl right R-module. Thus M is a *generator* by the assumption. Hence there exist a positive integer n and an epimorphism f from the direct sum nM of n-copies of M to $(1-e)R$. Since $n((1-e)R/(I\cap(1-e)R))$ is singular and $(1-e)R$ is non-singular, we see that the restriction of f on $n(eR)$ is an epimorphism. But this contradicts that $eR(1-e)=0$.

Now, we are concerned with regular, right self-injective rings which are right bounded. Let R be a regular, right self-injective right bounded ring. By Theorem 10.21 [2], there exists a decomposition $R=R_1\times R_2$ such that R_1 is directly finite and R_2 is purely infinite, where a regular, right self-injective ring R is called *purely infinite* if it contains no nonzero central idempotent e such that eR is directly finite. Clearly each R_i is a regular, right self-injective and right bounded ring. So we may observe R by dividing into two cases; the directly finite case and the purely infinite one. First we mention the directly finite case.

Proposition 2. *Let R be a directly finite regular, right self-injective ring. Then the following conditions are equivalent.*

(1) *R is right bounded.*

(2) *R is bounded.*

(3) *R is Type I_f.*

Proof. (3)⇒(2). According to Theorem 10.24 [2], R is isomorphic to $\prod M_{n(t)}(R_t)$, where each R_t is an abelian regular, self-injective ring. Since each $M_{n(t)}(R_t)$ has bounded index of nilpotence, it is bounded. Thus R is bounded.

(2)⇒(1). This is clear.

(1)⇒(3). By Theorem 10.22 [2], there exists a decomposition $R=R_1\times R_2$ such that R_1 is Type I_f and R_2 is Type II_f. We shall show that $R_2=0$. Assume that $R_2\neq 0$, and put $R=R_2$ for brevity. For any finitely generated projective right R-module P, there exists a decomposition $P=P_2\oplus P_2$ such that $P_1\cong P_2$ by Lemma 11.19 [2]. We use repeatedly this result. First we shall construct orthogonal idempotents e_1, e_2, $\cdots$ in R such that $R_R\cong 2^n(e_nR)$ and $(1-\sum_{i=1}^{n} e_iR)\cong R/\oplus e_iR\cong e_nR$ for all n. In fact, we can take an idempotent e_1 such that $e_1R\cong(1-e_1)R$. Put $I_2=(1-e_1)R$. Since there exists a decomposition $I_2=J_1\oplus J_2$ such that $J_1\cong J_2$, there exist orthogonal idempotents e_{21}, e_{22} such that $J_1=e_{21}R$ and $J_2=e_{22}R$, and $1-e_1=e_{21}+e_{22}$ by Proposition 2.11 [2]. Put $e_2=e_{21}$. Then $e_1e_2=e_2e_1=0$, $R\cong 2^2(e_2R)$ and $R/(e_1R\oplus e_2R)\cong e_2R$. Continuing this procedure, we have desired idempotents $\{e_n\}$. Now we put $I=\oplus_n e_nR$. First we shall show that I does not contain a nonzero central idempotent. Suppose that there exists a nonzero central idempotent e in I. Then e is in $\oplus_{n=1}^{t} e_nR$ for some positive integer t. This shows that $e\cdot e_n=0$ for all $n>t$. On the other hand, $2^n(e\cdot e_nR)\cong eR$ for all $n>1$. Consequently, we obtain that $e=0$. Therefore I does not contain a nonzero central idempotent of R. Next we shall show that I does not contain a nonzero two-sided ideal of $\boldsymbol{R}$. We assume that $\boldsymbol{R}x\boldsymbol{R}\subseteq I$ for some $x\neq 0$. Since $\boldsymbol{R}$ satisfies *the general comparability* by Theorem 9.14 [2], there exist $f_n\in B(R)$ such that $f_ne_n\lesssim f_nxR$ and $(1-f_n)xR\lesssim(1-f_n)e_nR$ for all n. If $f_n=0$ for all n, then $n(xR)\lesssim R$ for all n. By Corollary 9.23 [2], we have that $xR=0$. Hence there exists t such that $f_t\neq 0$. Since $f_te_tR\lesssim f_txR\subseteq xR$ and $f_tR\cong 2^t(f_te_tR)\lesssim 2^t(xR)$, we obtain that

$f_tR \subseteq RxR$ by Corollary 2.23 [2]. Hence $(0 \neq) f_t \in I$, a contradiction. Finally we shall show that I is an essential right ideal of R. Suppose that $I \cap fR = 0$ for some idempotent f of R. Since $fR \cap \bigoplus_{n=1}^{t} e_nR = 0$ for all t, we obtain that $fR \lesssim R / \bigoplus_{n=1}^{t} e_nR \cong e_tR$. Since $t(fR) \lesssim 2^t(fR) \lesssim 2^t(e_tR) \cong R$ for all t, we have that $R=0$ by Corollary 9.23 [2]. This is a contradiction. Therefore a regular, right self-injective ring of Type II_f is not right bounded. Thus R is Type I_f.

We note that *the general comparability* is left-right symmetric, and moreover, for idempotents e, f of a regular ring R, $Rf \lesssim Re$ implies $fR \lesssim eR$. Therefore by Proposition 2 and it's proof, we have the following corollary.

Corollary 1. *Let R be a regular, right self-injective ring of Type II_f. Then R is neither right nor left bounded.*

Corollary 2. *Let R be a regular, right slef-injective and right bounded ring. Then R is directly finite if and only if R is left self-injective.*

Proof. If R is a directly finite, then R is Type I_f by Proposition 2. Hence R is left self-injective by Corollary 10.25 [2]. The converse is clear by Theorem 9.29 [2].

Now we consider the case that R is a purely infinite, right self-injective, regular ring. Let $Z(R)$ be *the center* of R and S be *the socle* of $Z(R)$. There exists a central idempotent e of R such that $S \subseteq_e eZ(R)_{Z(R)}$. Therefore we have a decomposition $R = R_1 \times R_2$ such that $Z(R_1)$ has an essential socle and $Z(R_2)$ has a zero socle.

Proposition 3. *Let R be a purely infinite, regular, right self-injective ring whose socle of $Z(R)$ is zero. Then R is neither right nor left bounded.*

Proof. According to Theorems 9.7 and 10.16 [2], there exist orthogonal idempotents $e_1, e_2, \cdots$ in R such that $\bigoplus_n e_nR$ is an essential right ideal of R and $e_nR \cong R_R$ for all n. First we show that R is not left bounded. Assume that R is left bounded, then since $Re_n \cong {}_RR$ for all n, we set $H = (\bigoplus Re_n) \bigoplus_n I$, where I is a complement left ideal of $\bigoplus_n Re_n$ in R, and take a nonzero two-sided ideal J in H. Choose a nonzero element x in J and set $c = cc(xR)$. Since $Soc(Z(R)) = 0$, we can write $c = \bigvee_n c_n$ for some orthogonal, countably infinite, nonzero central idempotents $c_1, c_2, \cdots$ in R. Since $Re_n \cong {}_RR$, we have a nonzero $\varphi_n \in Hom_R(Rxc_n, Re_n)$ for all n. We define an R-homomorphism φ: $\bigoplus_n Rxc_n \rightarrow \bigoplus_n Re_n$ by $\varphi(\sum_n rxc_n) = \varphi_n(\sum_n r_nxc_n)$. Furthermore, each φ_n can be extended to an R-

homomorphism $\bar{\varphi}=(\bar{\varphi}_n)$: $\prod Rc_n \to \prod Re_n$, and there exists a natural isomorphism θ: $Rc \to \prod Rc_n$ given by $\theta(rc)=(rc_n)$ [2, Proposition 9.10]. Hence $\bar{\varphi}$ is given by the right multiplication by some element a of R. Therefore we obtain that φ is the right multiplication by a. On the other hand, since J is a two-sided ideal of R, $xa \in J \subseteq \bigoplus_n Re_n \oplus I$, so we can write that $xa=\sum_{n=t}^{s} r_n e_n + r$ for some $r_n \in R$ and $r \in I$. But we observe that $xae_{s+1} \in (Re_{s+1} \cap \sum_{n=t}^{s} Re_n \oplus I)$ $(=0)$. This is a contradiction. Consequently H can not contain any nonzero two-sided ideal of R. Therefore R is not left bounded. Similarly we can show that R is not right bounded. Now the proof is complete.

Proposition 4. *Let R be a purely infinite, prime, regular right self-injective and right bounded ring. Then R is a right full linear ring.*

Proof. Since R is a prime, regular, right self-injective ring, R is indecomposable as a ring. Hence $Soc(R)$ is essential or zero. We claim that $Soc(R)$ is essential. If not, then we can take a minimal ideal I of R by Theorem 12.23 [2]. Now there exists nonzero orthogonal, infinite, idempotents $\{f_\beta\}_{\beta \in K}$ in I such that $\bigoplus_\beta f_\beta R \subseteq_e I_R$ since $Soc(R)=0$. If $\bigoplus_\beta f_\beta R = I$, then we can take countably, infinite orthogonal idempotents $\{g_{\beta n}\}$ in $f_\beta R$ for all β, such that $\bigoplus g_{\beta n} R \not\subseteq_e fR$ since $Soc(R)=0$. So we may assume that $\bigoplus_\beta f_\beta R \neq I$. On the other hand, since I is a minimal ideal of R, $\bigoplus_\beta f_\beta R$ can not contain nonzero ideal. This contradicts that R is right bounded. Therefore $Soc(R)$ is an essential ideal of R, as claimed. Then [2, Theorem 9.12] shows that R is a right full linear ring.

Corollary. *Let R be a purely infinite, regular, right self-injective and right bounded such that $Soc(Z(R))$ is an essential ideal of $Z(R)$. Then $R \cong \prod R_\alpha$, where each R_α is a right full linear ring.*

Proof. Let $\{e_\alpha\}_{\alpha \in I}$ be all central primitive idempotents of $Soc(Z(R))$. Evidently, we obtain that $\bigvee_\alpha e_\alpha = 1$ since $Soc(Z(R)) \subseteq_e Z(R)$. Then by Proposition 9.10 [2], $R \cong \prod_\alpha e_\alpha R$, where each $e_\alpha R$ is a prime, regular, right self-injective and right bounded ring. Then Proposition 4 shows that each $e_\alpha R$ is a right full linear ring.

Now we shall prove our main theorem.

Theorem 1. *Let R be a regular, right self-injective ring. Then the following conditions are equivalent.*

(1) *R is right bounded.*

(2) $R \cong \prod M_{n(t)}(R_t) \times \prod T_s$, *where each* R_t *is an abelian regular, self-injective ring and each* T_s *is a right full linear ring.*

Proof. (1)⇒(2). We have a decomposition $R=R_1 \times R_2$ such that R_1 is directly finite and R_2 is purely infinite by Theorem 10.21 [2]. Since each R_i is also right bounded, R_1 is Type I_f by Proposition 2 and R_2 is isomorphic to a direct product of right full linear rings by Propositions 3 and 4. Furthermore, Theorem 10.24 [2] shows that $R \cong \prod M_{n(t)}(R_t)$, where each R_t is an abelian regular, self-injective ring.
(2)⇒(1). It is easily seen that a direct product of right bounded rings is also right bounded. Moreover, we have pointed out that a regular, right self-injective of bounded index of nilpotence is right bounded, and by Theorem 9.12 [2], we see that a right full linear ring is also right bounded. Therefore R is right bounded.

Corollary. *Let R be a regular, right self-injective ring. Then if R is right bounded, then R is bounded.*

Proof. This is clear by Theorem 1.

3. Application

In this section, we apply Theorem 1 to a right bounded regular ring which is not necessarily self-injective.

Lemma 1. *Let R be a regular, right bounded ring such that every nonzero two-sided ideal of R contains a nonzero central idempotent. Then the maximal right quotient ring Q(R) of R is also right bounded.*

Proof. Let I be an essential right ideal of $Q(R)$. Then clearly $I \cap R \subseteq_e R_R$. Since R is right bounded, there exists a nonzero two-sided ideal J of R such that $J \subseteq_e I \cap R$. From the assumption, J contains a nonzero central idempotent e of R. Note that e is also central in $Q(R)$. Thus I contains a nonzero central idempotent of $Q(R)$. Let H be the ideal generated by all central idempotents in I. We claim that $H \subseteq_e Q(R)_{Q(R)}$. If not, then $l_{Q(R)}(H)$, *the left annihilator ideal* of H in $Q(R)$, is not zero and $H \cap l_{Q(R)}(H)=0$ since $Q(R)$ is semi-prime. Hence there exists a nonzero central idempotent f in $J \cap l_{Q(R)}(H)$ since $J \cap l_{Q(R)}(H)$ is a nonzero two-sided ideal of R. On the other hand, f is in H since $f \in J \subseteq I$. But this contradicts that $H \cap l_{Q(R)}(H)=0$. Consequently, H is an essential right ideal of $Q(R)$, as claimed. Therefore $Q(R)$ is right bounded.

Theorem 2. *Let R be a regular ring and Q be the maximal right quotient ring of R. Then the following conditions are equivalent.*

(1) $Q \cong \prod M_{n(t)}(S_t)$, *where each* S_t *is an abelian regular, self-injective ring.*

(2) *There exist orthogonal central idempotents* $\{e_{t\alpha}\}$ *such that* $e_{t\alpha}R \cong M_{n(t)}(A_{t\alpha})$, *where each* A_t *is an abelian regular ring and* $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R \subseteq_e R_R$.

(3) *For any nonzero two-sided ideal* J *of* R, J *contains a nonzero central idempotent* e *such that* eR *is isomorphic to a full matrix ring over an abelian regular ring.*

(4) R *is right bounded and every nonzero two-sided ideal of* R *contains a nonzero central idempotent of* R.

Proof. (1)$\Rightarrow$(2). Since $Q \cong \prod M_{n(t)}(S_t)$, there exist orthogonal central idempotents $f_1, f_2, \cdots$ in Q such that $f_tQ = M_{n(t)}(S_t)$. Clearly, f_tQ has bounded index $n(t)$, so $R \cap f_tQ$ has also bounded index $n(t)$ by Corollary 7.4 [2]. Thus in view of Theorem 7.2 and Lemma 7.17 [2], we have a central idempotent $e_{t\alpha}$ in $R \cap f_tQ$ such that $e_{t\alpha}R \cong M_{n(t)}(A_{t\alpha})$ for some abelian regular ring $A_{t\alpha}$. It is easy to see that $e_{t\alpha}$ is a central idempotent of R. For a maximal family $\{e_{t\alpha}\}_{\alpha\in K_t}$ of orthogonal central idempotents of $f_tQ \cap R$ as above, we note that $\sum_{\alpha\in K_t} \oplus e_{t\alpha}R \subseteq_e f_tQ \cap R_{(f_tQ\cap R)}$. Next we show that $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R \subseteq_e R_R$. Let I be a right ideal of R such that $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R \cap I = 0$. Since $\oplus f_tQ$ is an essential right R-submodule of Q, if I is not zero, then $f_tQ \cap I \neq 0$ for some t. But, then since $\sum_{\alpha\in K_t} \oplus e_{t\alpha}R$ is an essential R-submodule of $f_tQ \cap R$, this is impossible by our assumption. Hence I must be zero, so $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R$ is an essential right ideal of R.

(2)$\Rightarrow$(3). Let J be a nonzero two-sided ideal of R. Since $\sum_{t\,;\,\alpha\in K_t} \oplus e_{t\alpha}R$ is an essential right ideal of R, there exists a positive integer t_α such that $e_{t\alpha}R \cap J \neq 0$. Now since $e_{t\alpha}R \cong M_{n(t)}(A_{t\alpha})$ for some abelian regular ring $A_{t\alpha}$, Theorem 6.6 [2] shows that $J \cap e_{t\alpha}R$ contains a nonzero central idempotent e of R such that eR is isomorphic to a full matrix ring over an abelian regular ring.

(3)$\Rightarrow$(4). We shall show that R is right bounded. Let I be an essential right ideal of R. First we claim that I contains a nonzero central idempotent e of R. Choose a nonzero element x of R. Then RxR contains a nonzero central idempotent f of R such that fR is isomorphic to a full matrix ring over an abelian regular ring. Since I is essential, $I \cap fR \subseteq_e fR_{fR}$. Now since an essential right ideal of a full matrix ring over an abelian regular ring contains a central idempotent, it is easy to see that I contains a nonzero central idempotent e of fR. Clearly, e is also central in R, so I contains a nonzero central idempotent e of R, as claimed. Let H be the ideal generated by all central idempotents in I. We show that H is an essential right ideal of R. If not, then $l_R(H)$, the left annihilator ideal of H in R, is nonzero and $H \cap l_R(H) = 0$. On the other hand, since I is an essential right ideal of R, $I \cap l_R(H) \neq 0$, so $I \cap l_R(H)$ contains a nonzero central idempotent h of R. But this is a contradiction. Therefore R

is right bounded.
(4)$\Rightarrow$(1). By Lemma 1, Q is also right bounded. Hence we have that $Q=Q_1\times Q_2$, where Q_1 is Type I_f and Q_2 is a direct product of right full linear rings by Theorem 1. By the assumption, every nonzero two-sided ideal of R contains a nonzero central idempotent of R. Clearly, the same property holds for Q. But it is easily seen that a right full linear ring does not satisfy this one. Hence we obtain that $Q_2=0$. Therefore Q is isomorphic to a direct product of full matrix rings over abelian regular, self-injective rings.

Corollary. *Let R be a regular, right self-injective ring. Then R is isomorphic to a finite direct product of full matrix rings over abelian regular, self-injective rings if and only if*

(1) *R is right bounded.*

(2) *All prime ideals of R are maximal.*

Proof. Since R is right self-injective, the condition (2) is equivalent to the condition that, for each $x\in R$, the two-sided ideal RxR is generated by a central idempotent. Therefore Corollary is an immediate consequence of Theorems 1 and 2.

REMARK 1. Without (1) or (2), Corollary can fail, as the following examples show.

There exists a prime regular, right self-injective ring which is right bounded, but not simple and not directly finite.

For example, choose a field F, let V be a countable-infinite dimensional vector space over F and set $R=End_R(V)$ and $M=\{x\in R \mid dim_F(xV)<\infty\}$. Then clearly, R is a prime regular, right self-injective ring. Given any $x\in R-M$, we have that $dim_F(xV)=dim_F(V)$ and so $xV\cong V$, whence $xR\cong R_R$. This shows that R is not directly finite. On the other hand, R is right bounded by Theorem 1.

Furthermore, there exists a simple regular, right self-injective ring, which is not right bounded and not Type I_f.

For example, choose fields F_1, F_1, $\cdots$, set $R_n=M_n(F_n)$ for all $n=1, 2, \cdots$, and set $R=\prod R_n$. Let M be a maximal ideal of R which contains $\oplus R_n$. Then R/M be a simple right and left self-injective ring of Type II_f by Example 10.7 [2]. Therefore by Proposition 2, R/M is not right bounded.

REMARK 2. Let R be a regular ring whose maximal right quotient ring $Q(R)$ is right bounded. Then according to Theorem 1, R is also right bounded. But we don't know whether $Q(R)$ is right bounded when R is right bounded.

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References

[1] A.W. Chatter and C.R. Hajarnavis: Rings with chain conditions, Pitman Press, London, 1980.
[2] K.R. Goodearl: Von Neumann regular rings, Pitman Press, London, 1979.
[3] S.S. Page: *Regular FPF-rings,* Pacific J. Math. **10** (1978), 169–176.

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