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AN APPLICATION ON NAGAO'S LEMMA

Dedicated to Professor Hirosi Nagao on his sixtieth birthday

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With time, the importance of Nagao's lemma has grown in modular representation theory of finite groups. In this note, we add another application.

Let G be a finite group, and let F be a field of characteristic $p>0$.

For a subgroup H of G and a (right) FG-module V, we denote V^H the fixed-point-set of H in V, so that V^H is an $FN_G(H)$-module. The trace map $Tr_H^G\colon V^H \to V^G$ is defined by $Tr_H^G(v)=\sum_g vg$, where g runs over a complete set of representatives of $H\backslash G$.

Main Theorem. *Let V be an indecomposable FG-module in a block B, and let P be a p-subgroup of G. Then each composition factor of the $FN_G(P)$-module*

$$V(P) := V^P/\sum_{A<P} Tr_A^P(V^A)\,,$$

where A runs over proper subgroups of P, belongs to a block b such that $b^G=B$.

REMARK. If $V(P)\neq 0$, then P is contained in a defect group of B.

Proof. of the theorem. Set $N=N_G(P)$. Let e be the centrally primitive idempotent of FG corresponding to B. Let $s\colon Z(FG)\to Z(FN)$ be the Brauer homomorphism. Then Nagao's lemma ([2], Chapter III, Theorem 7.5) states that

$$V_N = V_N s(e)\oplus W_1\oplus\cdots\oplus W_n$$

as FN-modules, where each W_i is Q_i-projective FN-module for some p-subgroup Q_i of N with $P\not\subseteq Q_i$. Thus in order to prove the theorem, it will suffice to show that

$$W_i^P \subseteq \sum_{A<P} Tr_A^P(V^A)\,,$$

where A runs over proper subgroups of P. But this follows directly from the following lemma, and so the theorem is proved.

Lemma. *Let N be a finite group with a normal p-subgroup P. Let W be a Q-projective FN-module, where $Q\not\supseteq P$. Then*

$$W^P = \sum_{A<P} Tr_A^P(W^A),$$

where A runs over proper subgroups of P.

Proof. In order to prove this lemma, we may assume that for some FQ-module U,

$$W = \mathrm{Ind}_Q^N(U).$$

Then by Mackey decomposition, we have that

$$W_P = \oplus_n \mathrm{Ind}_{P\cap Q^n}^P(U_{P\cap Q^n}^n),$$

where n runs over a complete set of representatives of $Q\backslash N/P$ and $Q^n=n^{-1}Qn$. Let n be an element of N and set $R=P\cap Q^n$, $X=U_R^n$. Since P is normal in N and Q is not contained in P, we have that R is a proper subgroup of P. Thus, in order to prove the lemma, it will suffice to show that

$$(\mathrm{Ind}_R^P(X))^P \subseteq Tr_R^P(\mathrm{Ind}_R^P(X)^R).$$

But this follows directly from an easy calculation (eq. [2] Chapter II Lemma 3.4). The lemma is proved.

Remark. The main theorem can be proved also by the Brauer homomorphism of modules, which is defined by Broue and Puig [1]. Let B be a block of G and e a corresponding central primitive idempotent of FG. We define the Brauer homomorphism Br_P^V with respect to P by the canonical homomorphism $V^P \to V(P)$. Now le $s_P: Z(FG) \to Z(FC_G(P))$ be the classical Brauer homomorphism with respect to P. Then we can prove that $Br_P^V(ve)=Br_P^V(v)s_P(e)$ for the element v of V^P. The main theorem is immediate from this fact.

References

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[2] W. Feit: The representation theory of finite groups, North-Holland, 1982.

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