

A NEW CLASS OF TRANSLATION PLANES OF ORDER q^3

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1. Introduction

Let q be an odd prime power, where 2 is a non-square in $GF(q)$. The aim of this paper is to construct a new class of translation planes of order q^3 and to determine their linear translation complements. Their kernels are isomorphic to $GF(q)$. If $q \neq 3$, then the linear translation complement of any plane of this class has exactly two orbits of length 2 and $q^3 - 1$ on the line at infinity and it is of order $3(q-1)(q^3-1)$. If $q=3$, then the plane is the Hering plane of order 27 and the translation complement is isomorphic to $SL(2, 13)$.

The planes also differ from those which are generalized André planes [1] and semifield planes.

2. Preliminaries

We list some results that will be required in the calculations of the linear translation complements.

Let q be a prime power. For $\alpha \in GF(q^3)$ put $\bar{\alpha} = \alpha^q$ and $\overline{\bar{\alpha}} = \alpha^{q^2}$. Set

$$M(3, q^3) = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mid \alpha_{ij} \in GF(q^3) \right\}$$

and

$$\mathfrak{U} = \left\{ \begin{pmatrix} \alpha & \bar{\alpha} & \overline{\bar{\alpha}} \\ \beta & \bar{\beta} & \overline{\bar{\beta}} \\ \gamma & \bar{\gamma} & \overline{\bar{\gamma}} \end{pmatrix} \in GL(3, q^3) \right\}.$$

Then $\varepsilon \in \mathfrak{U}$ if and only if

$$\varepsilon = \begin{pmatrix} \alpha & \bar{\alpha} & \overline{\bar{\alpha}} \\ \beta & \bar{\beta} & \overline{\bar{\beta}} \\ \gamma & \bar{\gamma} & \overline{\bar{\gamma}} \end{pmatrix}$$

and α, β, γ are linearly independent over $GF(q)$. Set

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{pmatrix} \alpha & \bar{\gamma} & \bar{\beta} \\ \beta & \bar{\alpha} & \bar{\gamma} \\ \gamma & \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Lemma 2.1 (T. Oyama [3]). *If $\varepsilon \in \mathfrak{A}$ then*

$$M(3, q)^\varepsilon = \left\{ \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in M(3, q^3) \right\} \text{ and } GL(3, q)^\varepsilon = \left\{ \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in GL(3, q^3) \right\}.$$

Since ε is any element of \mathfrak{A} , we denote $M(3, q)^\varepsilon$ by $M(3, q)^*$ and $GL(3, q)^\varepsilon$ by $GL(3, q)^*$.

Set $GF(q^3)^* = GF(q^3) - \{0\}$. Let t be a generator of the multiplicative group $GF(q^3)^*$. Set $W = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The following statements hold:

Lemma 2.2.

- (i) $o(W)$ (the order of W) $= q^3 - 1$ and $o(T) = 3$.
- (ii) $T^{-1}WT = W^q$.

Lemma 2.3. *If $o(W^i) \nmid q-1$, then $N_{GL(3, q)^*}(\langle W^i \rangle) = \langle W, T \rangle$ and $C_{GL(3, q)^*}(\langle W^i \rangle) = \langle W \rangle$.*

Proof. Let $A \in N_{GL(3, q)^*}(\langle W^i \rangle)$. There exists an integer j with $W^i A = AW^{ij}$. Write $A = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. Then $\alpha t^i = \alpha t^{ij}$, $\beta \bar{t}^i = \beta t^{ij}$ and $\gamma \bar{t}^i = \gamma t^{ij}$. Assume $\alpha \neq 0$. Then $t^i = t^{ij}$. Since $t^i \neq \bar{t}^i$, $\beta = 0$ and $\gamma = 0$. Thus $A \in \langle W \rangle$. Assume $\alpha = 0$. If $\beta \neq 0$ and $\gamma \neq 0$, then $\bar{t}^i = \bar{t}^j$. This is a contradiction. Thus $\beta = 0$ or $\gamma = 0$ and hence $A \in \langle W, T \rangle$. Therefore $N_{GL(3, q)^*}(\langle W^i \rangle) \subseteq \langle W, T \rangle$. On the other hand $N_{GL(3, q)^*}(\langle W^i \rangle) \supseteq \langle W, T \rangle$ by Lemma 2.2 (ii) and thus $N_{GL(3, q)^*}(\langle W^i \rangle) = \langle W, T \rangle$.

Similarly, using $\bar{t}^i \neq t^i$, we obtain $C_{GL(3, q)^*}(\langle W^i \rangle) = \langle W \rangle$.

Lemma 2.4. *Let $A \in GL(3, q)^*$ and $o(A) = q^3 - 1$. Then $\langle A \rangle$ is conjugate to $\langle W \rangle$ in $GL(3, q)^*$.*

Proof. Since $(q-1, q^2+q+1) = 1$ or 3 , there exists a prime r such that $r \nmid q-1$ and that $r \mid q^2+q+1$. From this $r \mid o(W)$ and $r \nmid q^3(q+1)(q-1)^3$ follow.

Since $|GL(3, q)^*| = q^3(q+1)(q-1)^3(q^2+q+1)$, $\langle W \rangle$ includes a Sylow r -subgroup $\langle W^i \rangle$ of $GL(3, q)^*$. Thus there exists $B \in GL(3, q)^*$ with $B^{-1}\langle W^i \rangle B = \langle A^i \rangle$. Since $o(W^i) \nmid q-1$, $C_{GL(3, q)^*}(\langle W^i \rangle) = \langle W \rangle$ by Lemma 2.3. Therefore $\langle A \rangle \subseteq C_{GL(3, q)^*}(\langle A^i \rangle) = B^{-1}C_{GL(3, q)^*}(\langle W^i \rangle)B = B^{-1}\langle W \rangle B$. Considering the order of $\langle A \rangle$, we get $\langle A \rangle = B^{-1}\langle W \rangle B$.

Lemma 2.5. Set $I = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Let $A \in GL(3, q)^*$ and assume that $\det(A - xI) \neq 0$ for any $x \in GF(q)$. The following statements hold:

- (i) $o(A) \mid q^3 - 1$ and $o(A) \nmid q - 1$.
- (ii) There exists $B \in GL(3, q)^*$ with $o(B) = q^3 - 1$ and $A \in \langle B \rangle$.

Proof. Let $\varepsilon \in \mathfrak{A}$. Set $V = V(3, q)$ and $C = A^{\varepsilon^{-1}}$. There exists a 2-dimensional subspace V_1 of V such that $V_1 C \neq V_1$. Let $V_1 \cap V_1 C = \langle VC \rangle$. Since $\det(A - xI) = \det(C - xI) \neq 0$ for any $x \in GF(q)$, v, vC, vC^2 are linearly independent over $GF(q)$. Hence C is conjugate to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{pmatrix}$$

in $GL(3, q)$, where $a_0, a_1, a_2 \in GF(q)$ and $vC^3 = a_0v + a_1vC + a_2vC^2$. Let λ be a root of the characteristic polynomial of C . Then $\lambda \in GF(q^3)$ and $\lambda^3 = a_0 + a_1\lambda + a_2\lambda^2$. Set

$$\mu = \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \bar{\lambda} & \bar{\bar{\lambda}} \\ \lambda^2 & \bar{\lambda}^2 & \bar{\bar{\lambda}}^2 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{pmatrix}^\mu = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix},$$

$A^{\varepsilon^{-1}\mu}$ is conjugate to $\begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}$ in $GL(3, q)^*$. From this (i) and (ii) follow.

3. Description of the class of translation planes

Let q be an odd prime power where 2 is a non-square over $GF(q)$. Any translation plane is defined by a spread. We define the spreads using

the Oyama's Method (T. Oyama [3]). For $\alpha \in GF(q^3)$ put $((\alpha)) = (\alpha, \bar{\alpha}, \bar{\bar{\alpha}})$. $X = \{((\alpha)) \mid \alpha \in GF(q^3)\}$ becomes a vector space of dimension 3 over $GF(q)$, when + and scalar product are defined by $((\alpha)) + ((\beta)) = ((\alpha + \beta))$ and $a((\alpha)) = ((a\alpha))$. We may assume that $V = X \oplus X$ is the outer sum of two copies of X . Set $V(\infty) = \{(0, ((\alpha))) \mid ((\alpha)) \in X\}$. If there exists a subset Σ of $GL(3, q)^* \cup \{0\}$ such that $0 \in \Sigma$, $|\Sigma| = q^3$ and $\det(M_1 - M_2) \neq 0$ for all $M_1 \neq M_2 \in \Sigma$, then we can construct a translation plane $\pi(\Sigma)$ of order q^3 such that its kernel contains $GF(q)$, as follows:

- (a) The points of $\pi(\Sigma)$ are the vectors in V .
- (b) The lines are all cosets of all the components of $\Pi = \{V(M) \mid M \in \Sigma \cup \{\infty\}\}$, where $V(M) = \{((\alpha)), ((\alpha)))M \mid ((\alpha)) \in X\}$ for $M \in \Sigma$.
- (c) Incidence is inclusion.

We call Σ a spread set of degree 3 over $GF(q)$.

Set $S = \{\alpha^2 \mid \alpha \in GF(q^3)^*\}$. For $\alpha \in GF(q^3)$ put $n(\alpha) = \alpha\bar{\alpha}\bar{\bar{\alpha}}$ and $tr(\alpha) = \alpha + \bar{\alpha} + \bar{\bar{\alpha}}$. Clearly $n(\alpha), tr(\alpha) \in GF(q)$ and $\det \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = n(\alpha) + n(\beta) + n(\gamma) - tr(\alpha\bar{\beta}\bar{\gamma})$.

For $\alpha \in S$ put $M(\alpha) = \begin{bmatrix} \bar{\alpha} \\ \alpha \\ 0 \end{bmatrix}$ and $N(\alpha) = \begin{bmatrix} \bar{\alpha} \\ \alpha \\ -\alpha\bar{\alpha}\bar{\bar{\alpha}}^{-1} \end{bmatrix}$.

Theorem 3.1. $\Sigma = \{M(\alpha), N(\alpha) \mid \alpha \in S\} \cup \{0\}$ is a spread set of degree 3 over $GF(q)$.

Proof. Let $\alpha \in S$. Since $M(\alpha) = \begin{bmatrix} \alpha^{-1} \\ 0 \\ 0 \end{bmatrix} M(1) \begin{bmatrix} \alpha\bar{\alpha} \\ 0 \\ 0 \end{bmatrix}$ and $N(\alpha) = \begin{bmatrix} \alpha^{-1} \\ 0 \\ 0 \end{bmatrix} N(1) \begin{bmatrix} \alpha\bar{\alpha} \\ 0 \\ 0 \end{bmatrix}$, $\det(M(\alpha)) = 2 \cdot n(\alpha) \neq 0$ and $\det(N(\alpha)) = 4 \cdot n(\alpha) \neq 0$.

If $\alpha \neq 1$, then $\det(M(\alpha) - M(1)) = 2 \cdot n(\alpha - 1) \neq 0$. Hence $\det(M(\alpha) - M(\beta)) \neq 0$ for any $\alpha \neq \beta \in S$.

If $\alpha \neq 1$, then $\det(N(\alpha) - N(1)) = 4(n(\alpha) + tr(\alpha) - tr(\alpha\bar{\alpha}) - 1) = 4(\alpha - 1) \cdot (\bar{\alpha} - 1)(\bar{\bar{\alpha}} - 1) \neq 0$. Hence $\det(N(\alpha) - N(\beta)) \neq 0$ for any $\alpha \neq \beta \in S$.

Suppose $\det(M(\alpha) - N(1)) = 0$. Since $\det(M(\alpha) - N(1)) = 2 \cdot n(\alpha) - (tr(\alpha))^2 + 4 \cdot tr(\alpha) - 4$, $2 = (tr(\alpha) - 2)^2 (n(t^{-i}))^2$, where $\alpha = t^{2i}$, a contradiction. Hence we have $\det(M(\alpha) - N(\beta)) \neq 0$ for any $\alpha, \beta \in S$.

Clearly $|\Sigma| = q^3$ and the results follow.

Let π be the translation plane which corresponds to the spread set Σ of Theorem 3.1 and G its linear translation complement. Set $\Sigma^* = \Sigma \cup \{\infty\}$ and $\Pi = \{V(M) \mid M \in \Sigma^*\}$.

4. The linear translation complement of π

In this section we show the linear translation complement G of π . We describe the Sherk's Theorem in the case $n=3$ using the Oyama's Method.

Lemma 4.1 (F.A. Sherk [2]). *Let $i \in \{1, 2\}$. Let Σ_i be a spread set of degree 3 over $GF(q)$ with $0 \in \Sigma_i$ and π_i be the translation plane of order q^3 which corresponds to the spread set Σ_i . Set $\Pi_i = \{V(M) \mid M \in \Sigma_i \cup \{\infty\}\}$. Then π_1 and π_2 are isomorphic if and only if there exist A, B, C and D in $M(3, q)^*$ and θ in $Aut(GF(q^3))$ with the following properties.*

(a) $det \begin{pmatrix} A & C \\ B & D \end{pmatrix} \neq 0.$

(b) *One of the following holds*

(i) $B=0, det(A) \neq 0$ and $\Sigma_2 = \{A^{-1}(C + M^\theta D) \mid M \in \Sigma_1\}$; or

(ii) $det(B) \neq 0, B^{-1}D \in \Sigma_2$, there is $M_0 \in \Sigma_1$ such that $A + M_0^\theta B = 0$ and for any $M \in \Sigma_1 - \{M_0\}$, $det(A + M^\theta B) \neq 0$ and $(A + M^\theta B)^{-1}(C + M^\theta D) \in \Sigma_2$.

Each $\tau \in G$ induces a permutation on Π which we denote by $\bar{\tau}$.

Theorem 4.2. *If $q=3$, then π is the Hering plane of order 27.*

Proof. Let $t \in GF(27)$ and $t^3 = -1 + t$. Then $GF(27)^* = \langle t \rangle$. Set

$$\varepsilon = \begin{pmatrix} 1 & 1 & 1 \\ t & \bar{t} & \bar{\bar{t}} \\ t^2 & \bar{t}^2 & \bar{\bar{t}}^2 \end{pmatrix} \in GL(3, 27).$$

F.A. Sherk [2] gave a spread set

$$S_H = \left\{ \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} R \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} R = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right.$$

$$\text{or } \left. \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, 1 \leq i \leq 13 \right\} \cup \{0\}$$

defining the Hering plane of order 27. Since

$$\varepsilon^{-1} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \varepsilon = \begin{bmatrix} t^7 \\ t^8 \\ t^{12} \end{bmatrix}, \varepsilon^{-1} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \varepsilon = \begin{bmatrix} t^9 \\ -1 \\ -t^7 \end{bmatrix}$$

$$\text{and } \varepsilon^{-1} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \varepsilon = \begin{bmatrix} t^3 \\ t^2 \\ -t^2 \end{bmatrix}, \varepsilon^{-1} S_H \varepsilon =$$

$$\left\{ \begin{bmatrix} t^7 \\ t^8 \\ t^{12} \end{bmatrix} R \begin{bmatrix} t^7 \\ t^8 \\ t^{12} \end{bmatrix} \middle| R = \begin{bmatrix} t^9 \\ -1 \\ -t^7 \end{bmatrix} \text{ or } \begin{bmatrix} t^3 \\ t^2 \\ -t^2 \end{bmatrix}, 1 \leq i \leq 13 \right\} \cup \{0\}$$

Set $\varepsilon^{-1}S_H\varepsilon = \sum_H$ and $\Pi_H = \{V(M) \mid M \in \sum_H \cup \{\infty\}\}$. Set $\varphi = \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} \in GL(V)$,

where $A = \begin{bmatrix} -t \\ t^6 \\ -t^5 \end{bmatrix}$ and $B = \begin{bmatrix} t \\ -1 \\ t^7 \end{bmatrix}$. By a computation, we get $\Pi^\varphi = \Pi_H$. Thus

Thorem 4.2 is proved.

The following statements hold:

Lemma 4.3.

(i) If $e \in GF(q) \cap S$, then $\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \in G_{V(\infty), V(t)}$, where $E = \begin{bmatrix} e \\ 0 \\ 0 \end{bmatrix}$.

(ii) Set $\tau_0 = \begin{pmatrix} 0 & 2T \\ I & 0 \end{pmatrix}$, where $T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\tau_0 \in G$ and $\bar{\tau}_0 =$

$(V(\infty), V(0))(V(M(1)), V(N(1))) \dots$.

(iii) Set $\Gamma = \{V(M(\alpha)) \mid \alpha \in S\}$, $\Delta = \{V(N(\alpha)) \mid \alpha \in S\}$ and $H = \{\varphi_\alpha \mid \alpha \in S\}$,

where $\varphi_\alpha = \begin{pmatrix} A_\alpha & 0 \\ 0 & B_\alpha \end{pmatrix}$, $A_\alpha = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$ and $B_\alpha = \begin{bmatrix} \alpha\bar{\alpha} \\ 0 \\ 0 \end{bmatrix}$. Then Γ and Δ are H -orbits

on Π and $H \leq G_{V(0), V(\infty)}$.

Lemma 4.4. Set $K = \left\{ \begin{pmatrix} aI & 0 \\ 0 & aI \end{pmatrix} \middle| a \in GF(q) - \{0\} \right\}$ and $L = G_{V(\infty), V(0), V(M(1))}$.

Then

(i) $L = \langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \rangle \cdot K$, where $T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(ii) If $\alpha \in S - GF(q)$, then $L_{V(M(\alpha))} = K$.

Proof.

Step 1. If $\alpha \in S$, then $\{V(M(\alpha)), V(M(\bar{\alpha})), V(M(\overline{\bar{\alpha}}))\}$ is an L -orbit.

Since $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \in L$, there exists an L -orbit containing $\{V(M(\alpha)), V(M(\bar{\alpha})), V(M(\overline{\bar{\alpha}}))\}$.

Let $\tau \in L$. Since $V(\infty)^\tau = V(\infty)$, $V(0)^\tau = V(0)$ and $V(M(1))^\tau = V(M(1))$, $\tau = \begin{pmatrix} A & 0 \\ 0 & M(1)^{-1}AM(1) \end{pmatrix}$ for some $A \in GL(3, q)^*$. Set $B = M(1)^{-1}AM(1)$. Let $\alpha \in S$. Suppose that $A^{-1}M(\alpha)B = N(\beta)$ for some $\beta \in S$. Then $A^{-1}M(\alpha)M(1)^{-1}A = N(\beta)M(1)^{-1}$. Hence $\det(M(\alpha)) = \det(N(\beta))$. From this $n(\alpha\beta^{-1}) = 2$ follows.

This is a contradiction since 2 is a nonsquare. Thus $A^{-1}M(\alpha)B=M(\beta)$ for some $\beta \in S$. Therefore $A^{-1}M(\alpha)M(1)^{-1}A=M(\beta)M(1)^{-1}$. Let $x \in GF(q)$. Since

$$\det(A^{-1}M(\alpha)M(1)^{-1}A-xI) = \det(M(\beta)M(1)^{-1}-xI), \det \begin{bmatrix} \alpha-x \\ \alpha-x \\ 0 \end{bmatrix} = \det \begin{bmatrix} \beta-x \\ \beta-x \\ 0 \end{bmatrix}.$$

Hence $n(\alpha)-x \cdot tr(\alpha\bar{\alpha})+x^2 \cdot tr(\alpha)-x^3=n(\beta)-x \cdot tr(\beta\bar{\beta})+x^2 \cdot tr(\beta)-x^3$. From this $n(\alpha)=n(\beta)$, $tr(\alpha\bar{\alpha})=tr(\beta\bar{\beta})$ and $tr(\alpha)=tr(\beta)$ follow. This implies that $(\beta-\alpha) \cdot (\beta-\bar{\alpha})(\beta-\bar{\bar{\alpha}})=0$. Thus $\beta \in \{\alpha, \bar{\alpha}, \bar{\bar{\alpha}}\}$.

Step 2. $L=L_{V(N(1))}$.

Let $\tau \in L$. Then $\tau = \begin{pmatrix} A & 0 \\ 0 & M(1)^{-1}AM(1) \end{pmatrix}$ for some $A \in GL(3, q)^*$. Set $B=M(1)^{-1}AM(1)$. By Step 1, $A^{-1}N(1)B=N(\alpha)$ for some $\alpha \in S$. Since

$$A^{-1}N(1)M(1)^{-1}A = N(\alpha)M(1)^{-1} \quad \text{and} \quad A^{-1}M(1)N(1)^{-1}A = M(1)N(\alpha)^{-1},$$

$$A^{-1}(N(1)M(1)^{-1}+M(1)N(1)^{-1})A = N(\alpha)M(1)^{-1}+M(1)N(\alpha)^{-1}.$$

From this, since

$$N(1)M(1)^{-1}+M(1)N(1)^{-1} = (5/2)I \quad \text{and} \quad N(\alpha)M(1)^{-1}+M(1)N(\alpha)^{-1}$$

$$= 1/2 \begin{bmatrix} \alpha+\bar{\alpha}+\alpha\bar{\alpha}\alpha^{-1}+\bar{\alpha}^{-1}+\bar{\bar{\alpha}}^{-1} \\ \alpha-\bar{\alpha}-\alpha\bar{\alpha}\bar{\alpha}^{-1}+\bar{\alpha}^{-1} \\ \alpha-\bar{\alpha}-\alpha\bar{\alpha}\bar{\bar{\alpha}}^{-1}+\bar{\alpha}^{-1} \end{bmatrix},$$

$$0 = \alpha-\bar{\alpha}-\alpha\bar{\alpha}\bar{\alpha}^{-1}+\bar{\alpha}^{-1} \tag{4.1}$$

and

$$0 = \alpha-\bar{\alpha}-\alpha\bar{\alpha}\bar{\bar{\alpha}}^{-1}+\bar{\alpha}^{-1} \tag{4.2}$$

follow. From (4.1),

$$0 = \bar{\alpha}-\alpha-\alpha\bar{\alpha}\bar{\alpha}^{-1}+\bar{\alpha}^{-1} \tag{4.3}$$

follows. Subtracting (4.3) from (4.2) we have $0=2\alpha-2\bar{\alpha}$. Therefore $\bar{\alpha}=\alpha$ and so $\alpha \in GF(q)$. From this and (4.3), $\alpha=1$ follows. Thus $V(N(1))^\tau = V(N(1))$.

Step 3. Proof of (ii).

Let $\alpha \in S-GF(q)$ and $\tau \in L_{V(M(\alpha))}$. Then $\tau = \begin{pmatrix} A & 0 \\ 0 & M(1)^{-1}AM(1) \end{pmatrix}$ for some $A \in GL(3, q)^*$. Since $V(M(\alpha))^\tau = V(M(\alpha))$, $A^{-1}M(\alpha)M(1)^{-1}A = M(\alpha)M(1)^{-1}$. Suppose that $\tau \notin K$. Since $\det(M(\alpha)M(1)^{-1}-xI) \neq 0$ for any $x \in GF(q)$, there exists $W \in GL(3, q)^*$ such that $o(W)=q^3-1$ and $\langle W \rangle \ni M(\alpha)M(1)^{-1}$ by Lemma 2.5. By Lemma 2.3 and Lemma 2.5, $C_{GL(3, q)^*}(M(\alpha)M(1)^{-1}) = \langle W \rangle$. Thus we get $A \in \langle W \rangle$ and $C_{GL(3, q)^*}(A) = \langle W \rangle$. By Step 2, $A^{-1}N(1)M(1)^{-1}A = N(1)M(1)^{-1}$, hence $N(1)M(1)^{-1} \in \langle W \rangle$. Clearly $2I \in \langle W \rangle$ and $N(1)M(1)^{-1} \neq 2I$. Therefore $N(1)M(1)^{-1}-2I \in \langle W \rangle$ by Lemma 2.4 and so $\det(N(1)M(1)^{-1}-2I) \neq 0$, a contradiction. Thus we get (ii).

Step 4. Proof of (i).

Let $\alpha \in S - GF(q)$. By Step 3, $L_{V(M(\alpha))} = K$. Furthermore $\{V(M(\alpha)), V(M(\bar{\alpha})), V(M(\overline{\alpha}))\}$ is an orbit of both L and $\langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \rangle$ by Step 1. Hence $L = \langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \rangle \cdot K$.

Lemma 4.5.

(i) $G_{V(\infty), V(0), V(1)} = \langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \rangle \cdot K$, where $T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(ii) If $\alpha \in S - GF(q)$, then $G_{V(\infty), V(0), V(N(1)), V(N(\alpha))} = K$.

Proof.

(i) Since $\tau_0^{-1} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \tau_0 = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$, $G_{V(\infty), V(0), V(N(1))} = (G_{V(0), V(\infty), V(M(1))})^{\tau_0} = \langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \rangle \cdot K$ by Lemma 4.4 (i).

(ii) Let $\alpha \in S - GF(q)$. Since $V(M(\overline{\alpha^{-1}}))^{\tau_0} = V(N(\alpha))$, $G_{V(\infty), V(0), V(N(1)), V(N(\alpha))} = (G_{V(0), V(\infty), V(M(1)), V(M(\overline{\alpha^{-1}}))})^{\tau_0} = K$ by Lemma 4.4 (ii).

Lemma 4.6. $G_{V(\infty)} = G_{V(\infty), V(0)}$.

Proof.

Case (a). $q \neq 3$.

Suppose false. By Lemma 4.3 (iii), there exists $\tau \in G_{V(\infty)}$ with $V(0)^\tau = V(M(1))$ or $V(N(1))$. As $V(\infty)^\tau = V(\infty)$, there exist $A, B, C \in M(3, q)^*$ such that $\tau = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Since $|GF(q) \cap S| = (q-1)/2 \geq 2$, there exists $e \in S \cap GF(q)$

with $e \neq 1$. Since $\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \in G$ where $E = \begin{bmatrix} e \\ 0 \\ 0 \end{bmatrix}$, $\Sigma = \{A^{-1}MB + A^{-1}C \mid M \in \Sigma\} =$

$\{EA^{-1}MB + A^{-1}C \mid M \in \Sigma\}$. Assume that $V(0)^\tau = V(M(1))$. Then $A^{-1}C = M(1)$. There exists $M_0 \in \Sigma$ with $A^{-1}M_0B + M(1) = N(1)$. Therefore $A^{-1}M_0B =$

$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and $EA^{-1}M_0B + M(1) = \begin{bmatrix} 1 \\ 1 \\ -e \end{bmatrix} \in \Sigma$. From this $e=1$ follows, a contradic-

tion.

Next assume that $V(0)^\tau = V(N(1))$. Then by the similar argument above, we have a contradiction again.

Case (b). $q=3$.

Assume that $G_{V(\infty)}$ is transitive on $\Pi - \{V(\infty)\}$. Then G is 2-transitive on Π . Since π has not a Baer subplane, π is a desarguesian plane by Theorem 39.3 of [4]. This is a contradiction.

Assume that $G_{V(\infty)} \neq G_{V(\infty), V(0)}$. Then $\{V(0)\} \cup \Gamma$ or $\{V(0)\} \cup \Delta$ is $G_{V(\infty)}$ -orbit on Π . Write this orbit by Ω . $G_{V(\infty)}$ induces the permutation group $G_{V(\infty)}/K$ on Ω by Lemma 4.4 (ii) and Lemma 4.5 (ii). Since $|\Omega|=14$ and $G_{V(\infty)}/K$ is a 2-transitive permutation group on Ω , $G_{V(\infty)}/K \geq PSL(2, 13)$. Since $|\Omega|=14$, the permutation group $PSL(2,13)$ on Ω contains an involution g which fixes exactly two points of Ω . There exists $\tau \in G_{V(\infty)}$ with $g = \tau K$. Since $|K|=2$, $o(\tau)=2$ or 4 . Suppose that $o(\tau)=2$. As π has not a Baer subplane, τ is a $((0, 0), l_\infty)$ -perspectivity. Therefore τ fixes any component of Ω , a contradiction. Suppose that $o(\tau)=4$. As τ^2 is a $((0, 0), l_\infty)$ -perspectivity, any cycle of τ on $V(\infty) - \{(0, 0), V(\infty) \cap l_\infty\}$ is 4-cycle. Therefore $4|26$, a contradiction.

Lemma 4.7. $G_{V(0)} = G_{V(0), V(\infty)}$.

Proof. $G_{V(0)} = (G_{V(\infty)})^{\tau_0} = (G_{V(\infty), V(0)})^{\tau_0} = G_{V(0), V(\infty)}$.

Lemma 4.8. Set $\Psi = \{V(\infty), V(0)\}$. Then Ψ is a G -block on Π .

Proof. Suppose $\varphi \in G$ and $\Psi^\varphi \cap \Psi \neq \emptyset$. We may assume that $V(\infty)^\varphi = V(\infty)$ or $V(0)^\varphi = V(\infty)$. Assume $V(\infty)^\varphi = V(\infty)$. Then $\varphi \in G_{V(\infty)} = G_{V(\infty), V(0)}$ and so $V(0)^\varphi = V(0)$. Assume $V(0)^\varphi = V(\infty)$. Then $G_{V(0), V(\infty)} = G_{V(\infty)} = G_{V(0)^\varphi} = G_{V(\infty)^\varphi, V(0)^\varphi} = G_{V(\infty), V(0)}$. From this and Lemma 4.3 (iii), $V(\infty)^\varphi = V(0)$ follows. Therefore $\Psi^\varphi = \Psi$.

Lemma 4.9. Γ and Δ are $G_{V(\infty)}$ -orbits on Π .

Proof. Suppose false. By Lemma 4.3 (iii) there exists $\tau \in G_{V(\infty)}$ with $V(M(1))^\tau = V(N(1))$. Since $V(0)^\tau = V(0)$, $\tau = \begin{pmatrix} A & 0 \\ 0 & M(1)^{-1}AN(1) \end{pmatrix}$ for some $A \in GL(3, q)^*$. Set $B = M(1)^{-1}AN(1)$. Assume $A^{-1}N(1)B = N(\alpha)$ for some $\alpha \in S$. From $\det(A^{-1}N(1)B) = \det(N(\alpha))$, $n(\alpha) = 2$ follows, a contradiction. Thus $A^{-1}N(1)B = M(\alpha)$ for some $\alpha \in S$. Let $q = p^n$ with p a prime.

Step 1. $p = 3$ or $p = 5$. If $p = 3$, then $A^{-1}N(1)B = M(1)$. If $p = 5$, then $A^{-1}N(1)B = M(-1)$.

Set $\rho = \tau_0^2 = \begin{pmatrix} M(1)N(1) & 0 \\ 0 & M(1)N(1) \end{pmatrix}$. Since $\rho^\tau \in G_{V(\infty), V(0), V(N(1))} = \langle \rho \rangle \cdot K$, $\rho^\tau = b\rho$ or $b\rho^2$ for some $b \in GF(q)$. Therefore τ fixes $\{V(M) \mid V(M)^\rho = V(M)\} - \{V(\infty), V(0)\} = \{V(M(x)), V(N(x)) \mid x \in S \cap GF(q)\}$ as a set. Thus $\alpha \in S \cap GF(q)$. Set $\alpha = a$. Clearly $V(N(1))^\tau = V(M(a))$.

Let $x \in GF(q)$. Clearly $A^{-1}N(1)M(1)^{-1}A = M(a)N(1)^{-1}$. Since $\det(A^{-1}N(1)M(1)^{-1}A - xI) = \det(M(a)N(1)^{-1} - xI)$, $\det((N(1) - M(x))N(1)) = \det((M(a) - N(x))M(1))$. From this $(12a - 18)x^2 + (24 - 9a^2)x + 2a^3 - 8 = 0$. Therefore $3a = 9/2$ and $a^3 = 4$. If $p = 3$, then $a = 1$. If $p \neq 3$, then $p = 5$ and

$a = -1$.

Step 2. $3 \nmid q - 1$.

If $p = 3$, then $3 \nmid q - 1$.

Assume $p = 5$. Suppose that n is even. Then 2 is a square in $GF(5^n)$, a contradiction. Therefore n is odd. Then $5^n - 1 \equiv (-1)^n - 1 \equiv -2 \pmod{3}$. Thus $3 \nmid q - 1$.

Step 3. Set $T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}^\tau = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$. If $p = 3$ then $\tau^2 \in$

$$\left\langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\rangle \cdot K. \text{ If } p = 5, \text{ then } \tau^2 \in \left\langle \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \right\rangle \cdot K.$$

If $p = 3$, then $\tau^2 \in G_{V(\infty), V(M(1))} = \left\langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\rangle \cdot K$. If $p = 5$, then $\tau^2 \in \left\langle \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \right\rangle \cdot K$ as $V(\infty)^{\tau^2} = V(\infty)$ and $V(M(1))^{\tau^2} = V(M(-1))$. Since $(G_{V(\infty), V(M(1))})^\tau = G_{V(\infty), V(M(1))} = \left\langle \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\rangle \cdot K$, $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}^\tau = a \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ or $a \begin{pmatrix} T^2 & 0 \\ 0 & T^2 \end{pmatrix}$ for

some $a \in GF(q)$. From this $a^3 = 1$ follows as $o\left(\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}^\tau\right) = 3$. Thus since $3 \nmid q - 1$, we have $a = 1$.

Suppose $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}^\tau = \begin{pmatrix} T^2 & 0 \\ 0 & T^2 \end{pmatrix}$. Then $\begin{pmatrix} A^{-1}TA & 0 \\ 0 & B^{-1}TB \end{pmatrix} = \begin{pmatrix} T^2 & 0 \\ 0 & T^2 \end{pmatrix}$. Since

$$A^{-1}TA = T^2, A = \begin{bmatrix} \alpha \\ \bar{\alpha} \\ \alpha \end{bmatrix} \text{ for some } \alpha \in GF(q^3) - GF(q). \text{ If } \alpha \notin S, \text{ then we take } e\tau$$

instead of τ with $e \in GF(q) - GF(q)^2$. Thus we may assume that $\alpha \in S$. Since

$$A^2 = \begin{bmatrix} tr(\alpha^2) \\ tr(\alpha\bar{\alpha}) \\ tr(\alpha\bar{\alpha}) \end{bmatrix} \text{ and } \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \in \left\langle \begin{pmatrix} T & 0 \\ 0 & \pm T \end{pmatrix} \right\rangle \cdot K, tr(\alpha\bar{\alpha}) = 0 \text{ and } tr(\alpha^2) \neq 0. \text{ Set } tr(\alpha) = b.$$

Since $b^2 = tr(\alpha^2) + 2 \cdot tr(\alpha\bar{\alpha}) = tr(\alpha^2)$, $b \neq 0$. Set $V(M(\alpha))^\tau = V\left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}\right)$.

By an easy computation, $B = M(1)^{-1}AN(1) = -1/2 \begin{bmatrix} b - 4\bar{\alpha} \\ b - 4\alpha \end{bmatrix}$ follows. Since

$$A^{-1}M(\alpha)B = -1/2b^{-2} \begin{bmatrix} \alpha \\ \bar{\alpha} \\ \alpha \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \alpha \\ 0 \end{bmatrix} \begin{bmatrix} b - 4\bar{\alpha} \\ b - 4\alpha \\ b - 4\alpha \end{bmatrix}, \quad -2b^2\alpha_1 = b(\bar{\alpha}^2 + \alpha^2 + 2\alpha\bar{\alpha} + 2\alpha\bar{\alpha}) -$$

$4(n(\alpha) + \bar{\alpha}^3 + \alpha\bar{\alpha}^2 + \bar{\alpha}\bar{\alpha}^2 + \bar{\alpha}\alpha^2 + \bar{\alpha}\alpha^2)$ and $-2b^2\alpha_2 = b(\alpha^2 + \bar{\alpha}^2 + 2\alpha\bar{\alpha} + 2\alpha\bar{\alpha}) - 4(n(\alpha) + \alpha^3 + \bar{\alpha}\bar{\alpha}^2 + \bar{\alpha}\bar{\alpha}^2 + \alpha\bar{\alpha}^2 + \alpha\bar{\alpha}^2)$. Since $-2b^2\alpha_1 = -2b^2\alpha_2$, we get $b(\bar{\alpha}^2 + 2\alpha\bar{\alpha}) = b(\alpha^2 + 2\bar{\alpha}\bar{\alpha})$. Since $b \neq 0$, $\bar{\alpha}^2 + 2\alpha\bar{\alpha} = \alpha^2 + 2\bar{\alpha}\bar{\alpha}$. Therefore $(\bar{\alpha} - \alpha)(\bar{\alpha} + \alpha - 2\bar{\alpha}) = 0$ and so $\bar{\alpha} + \alpha = 2\bar{\alpha}$. From this $b = 3\bar{\alpha}$ follows. Thus $\alpha \in GF(q)$, a contradiction.

Step 4. Contradiction.

Since $A^{-1}TA=T$, $A=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some $a, b, c \in GF(q)$. Assume $p=3$. Since

$A^{-1}N(1)B=M(1)$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} A=A \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. From this $c=-(a+b)$ follows. But

$\det(A)=\det \begin{bmatrix} a \\ b \\ -(a+b) \end{bmatrix} = a^3+b^3-(a+b)^3=0$, a contradiction. Assume $p=5$.

Since $A^2=\begin{bmatrix} a^2+2bc \\ c^2+2ab \\ b^2+2ac \end{bmatrix}$ and $\begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \in \langle \langle \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \rangle \cdot K$, $A^2=\begin{bmatrix} a^2+2bc \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ c^2+2ab \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 0 \\ 0 \\ b^2+2ac \end{bmatrix}$. Let $\lambda \in S-GF(q)$. We consider the case $A^2=\begin{bmatrix} a^2+2bc \\ 0 \\ 0 \end{bmatrix}$. Then

$c^2+2ab=b^2+2ac=0$. Suppose $b=0$. Then $c=0$, $A=aI$ and $B=M(1)^{-1}AN(1)$
 $=a \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. Therefore $V(M(\lambda))^\tau=V\left(\begin{bmatrix} \bar{\lambda} \\ \lambda \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\right)=V\left(\begin{bmatrix} -\bar{\lambda}+2\bar{\lambda} \\ -\lambda+2\bar{\lambda} \\ \lambda+2\bar{\lambda} \end{bmatrix}\right)$. Thus

$-\bar{\lambda}+2\bar{\lambda}=\overline{-\lambda+2\bar{\lambda}}$. This implies $\lambda \in GF(q)$, a contradiction. Suppose $b \neq 0$. Substituting $a=-c^2/2b$ in $b^2+2ac=0$, we get $b=c$ and $a=2b$. From this

$A=b \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $B=M(1)^{-1}AN(1)=bI$ follow. Therefore $V(M(\lambda))^\tau=$

$V\left(\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} M(\lambda)\right)=V\left(\begin{bmatrix} 2\lambda-\bar{\lambda} \\ -\lambda+2\bar{\lambda} \\ 2\lambda+2\bar{\lambda} \end{bmatrix}\right)$. Thus $2\lambda-\bar{\lambda}=\overline{-\lambda+2\bar{\lambda}}$. This implies $\lambda \in$

$GF(q)$, a contradiction. Also when $A^2=\begin{bmatrix} 0 \\ c^2+2ab \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ b^2+2ac \end{bmatrix}$, similarly we have a contradiction.

Theorem 4.10. *If $q \neq 3$, then G has two orbits of length 2 and length q^3-1 on Π .*

Proof. Let $q \neq 3$. Suppose false. Then G is transitive on Π . Since $\{V(\infty), V(0)\}$ is a G -block by Lemma 4.8, there exists $V(M) \in \Pi - \{V(\infty), V(0), V(M(1))\}$ such that $\Lambda = \{V(M(1)), V(M)\}$ is a G -block. Since $V(M(1))^{\tau_0^2} = V(M(1))$, $\Lambda^{\tau_0^2} = \Lambda$ and so $V(M)^{\tau_0^2} = V(M)$. Therefore $M=M(a)$ or $N(a)$ for some $a \in S \cap GF(q)$.

Assume that $\Lambda = \{V(M(1)), V(M(a))\}$ is a G -block. Set $\rho = \begin{pmatrix} I & 0 \\ 0 & aI \end{pmatrix} \in G$. Then $\Lambda^\rho = \{V(M(a)), V(M(a^2))\} = \Lambda$. Therefore $M(a^2) = M(1)$ and so $a = -1$ as $a \neq 1$. Since $\langle \{\varphi_\alpha | \alpha \in S\}, \tau_0 \rangle \leq G$ by Lemma 4.3, $\{V(M), V(-M)\}$ is a G -block for any $V(M) \in \Pi - \{V(\infty), V(0)\}$. Set $\sigma_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \in G$. Now σ_1 fixes exactly two components $V(\infty)$ and $V(0)$ in Π . Furthermore σ_1 fixes any G -block on Π . Since G is transitive on Π , there exists σ_2 such that σ_2 is conjugate to σ_1 and fixes exactly two components $V(M(1)), V(M(-1))$ in Π and all G -blocks on Π . Therefore $\sigma_1 \sigma_2 \tau_0 \in G_{V(\infty), V(0)}$. But $V(M(1))^{\sigma_1 \sigma_2 \tau_0} = V(N(-1))$. This is contrary to Lemma 4.9.

Next assume that $\{V(M(1)), V(N(a))\}$ is a G -block. Since $\{V(M(1)), V(N(a))\}^{\varphi_\alpha} = \{V(M(\alpha)), V(N(a\alpha))\}$ is a G -block for any $\alpha \in S$, G is 2-transitive on the set of G -blocks. Therefore there exists $\varphi \in G$ such that φ interchanges $\{V(\infty), V(0)\}$ and $\{V(M(1)), V(N(a))\}$. Let $\varphi = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$. Suppose that $\tilde{\varphi} = (V(\infty), V(M(1)))(V(0), V(N(a))) \cdots$ on Π . Then

$$\varphi = \begin{pmatrix} A & AN(a) \\ -M(1)^{-1}A & -M(1)^{-1}AM(1) \end{pmatrix}.$$

Let $b \in S \cap GF(q)$ with $b \neq 1$. Then

$$\begin{aligned} & V(M(b))^\varphi \\ &= V((A - M(b)M(1)^{-1}A)^{-1}(AN(a) - M(b)M(1)^{-1}AM(1))) \\ &= V((A - bA)^{-1}(AN(a) - bAM(1))) \\ &= V((1 - b)^{-1}(N(a) - M(b))). \end{aligned}$$

Hence $(1 - b)^{-1}(N(a) - M(b)) = (1 - b)^{-1} \begin{bmatrix} a - b \\ a - b \\ -a \end{bmatrix} \in \Sigma$. Since $a \neq 0$, $a - b = a$ and

so $b = 0$, a contradiction. Suppose that $\tilde{\varphi} = (V(\infty), V(N(a)))(V(0), V(M(1))) \cdots$ on Π . Set $\tau = \tau_0 \begin{pmatrix} I & 0 \\ 0 & aI \end{pmatrix}$. Since $\tilde{\tau} = (V(\infty), V(0))(V(M(1)), V(N(a))) \cdots$, $\tilde{\varphi} \tau = (V(\infty), V(M(1)))(V(0), V(N(a))) \cdots$. This is the above case, a contradiction. Suppose that $\tilde{\varphi} = (V(\infty), V(M(1)), V(0), V(N(a))) \cdots$ on Π . Then $\varphi = \begin{pmatrix} A & AN(a) \\ -N(a)^{-1}A & -N(a)^{-1}AM(1) \end{pmatrix}$. Let $b \in S \cap GF(q)$ with $b \neq a$. Then $V(N(b))^\varphi = V((1 - ba^{-1})^{-1}(N(a) - M(ba^{-1})))$. Thus $b = 0$ by the similar argument above, a contradiction. Suppose that $\tilde{\varphi} = (V(\infty), V(N(a)), V(0), V(M(1))) \cdots$ on Π . Then $\tilde{\varphi}^3 = (V(\infty), V(M(1)), V(0), V(N(a))) \cdots$. This is a contradiction.

Theorem 4.11. *If $q \neq 3$, then $|G| = 3(q-1)(q^3-1)$.*

Proof. By Lemma 4.4, Lemma 4.6, Lemma 4.9 and Theorem 4.10, $|G| = |V(\infty)^G| |G_{V(\infty)}| = 2|G_{V(\infty)}| = 2|G_{V(\infty), V(M(1))}| |V(M(1))^{G_{V(\infty)}}| = 3(q-1)(q^3-1)$.

Theorem 4.12. *If $q=3$, then $G \cong SL(2, 13)$.*

Proof. Since G is transitive on Π [5], $|G| = 28|G_{V(\infty)}|$. By Lemma 4.4, Lemma 4.6 and Lemma 4.9, $|G_{V(\infty)}| = |G_{V(\infty), V(M(1))}| |V(M(1))^{G_{V(\infty)}}| = 6 \cdot 13$. Therefore $|G| = 2^3 \cdot 3 \cdot 7 \cdot 13$. On the other hand since $|SL(2, 13)| = |G|$ and $G \geq SL(2, 13)$ by [5], $G \cong SL(2, 13)$.

Theorem 4.13. *π is not a generalized André plane.*

Proof. Assume that π is a generalized André plane. Then there exist $\Sigma_1 \subseteq \left\{ \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \mid \alpha \in GF(q^3)^* \right\}$, $\Sigma_2 \subseteq \left\{ \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} \mid \alpha \in GF(q^3)^* \right\}$ and $\Sigma_3 \subseteq \left\{ \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \mid \alpha \in GF(q^3)^* \right\}$ such that $\Sigma_1 \ni I$ and $\Sigma_A = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \{0\}$ is the spread set defining π . If $\alpha \neq \beta \in GF(q^3)^*$ and $n(\alpha) = n(\beta)$, then $\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} \in \Sigma_1, \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \in \Sigma_2$ or $\begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix} \in \Sigma_3$. Let $\pi(\Sigma_A)$ is the translation plane π which is constructed by Σ_A . Now since the order of π is q^3 , π is an André plane by Corollary 12.5 of [4]. Let $G(\Sigma_A)$ is the linear translation complement of $\pi(\Sigma_A)$.

Set $\tau = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$ where $W = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$. Then $\langle \tau \rangle \leq G(\Sigma_A)_{V(0), V(\infty), V(I)}$ and $\langle \tau \rangle$ is

transitive on $V(0) - \{(0, 0), V(0) \cap l_\infty\}$. This is contrary to Theorem 12.1 of [4].

Let $q \neq 3$. Since the translation complement of any proper semifield plane have an orbit of length 1 on l_∞ , π differs from any semifield plane.

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