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Osaka J. Math.
23 (1986), 217–221

THE GROUP OF NORMALIZED UNITS OF A GROUP RING

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(Received December 17, 1984)

Introduction

Let RG be the group ring of a group G over a commutative ring R with identity and $\Delta_R(G)$ its augmentation ideal. For a normal subgroup N of G, the kernel of the natural homomorphism $RG \to R(G/N)$ will be denoted by $\Delta_R(G, N)$. It is equal to $\Delta_R(N)RG$. Also, we shall denote by $V(RG)$ the group of normalized units of RG, that is, $V(RG)=U(RG)\cap(1+\Delta_R(G))$ where $U(RG)$ is the unit group of RG.

The aim of this paper is to prove the following theorem which generalizes [1, Proposition 1.3].

Theorem 1.3. *Let G be an arbitrary group and R an integral domain of characteristic* 0. *Let I be an ideal of RG and set $J=\bigcap_{n=1}^{\infty}(I+\Delta_R(G)^n)$. Then the factor group*

$$V(RG)\cap(1+J)/V(RG)\cap(1+\Delta_R(G, G\cap(1+J)))$$

is torsion-free.

As an immediate consequence of this result we can weaken the condition on R in [1, Proposition 2.4]. To be more precise, let $D_{n,R}(G)$ be the n-th dimension subgroup of G over R. Then, for two groups G and H with isomorphic group algebras over an integral domain R of characteristic 0, we can show that $D_{n,R}(G)=\{1\}$ if and only if $D_{n,R}(H)=\{1\}$.

Let A be a ring and ${}^0A$ the group of all quasi-regular elements in A. Here we say that A is residually nilpotent if $\bigcap_{n=1}^{\infty}A^n=0$. As another application of Theorem 1.3, we show that if A is a residually nilpotent algebra over an integral domain R of characteristic 0, then the group $G={}^0A$ has a torsion-free normal complement in $V(RG)$. This is proved by D.S. Passman and P.F. Smith [3, Theorem 1.4] for the case where A is a finite nilpotent ring and R is the ring of rational integers.

1. The group of normalized units

We start by making two simple observations. Let G be a group, R a commutative ring with identity. The n-th dimension subgroup $D_{n,R}(G)$ ($n=1, 2,\cdots$) of G over R is defined by $D_{n,R}(G)=G\cap(1+\Delta_R(G)^n)$, where $\Delta_R(G)$ denotes the augmentation ideal of RG. The series $\{D_{n,R}(G)\}_{n\geqq 1}$ forms a descending central series of G.

Lemma 1.1. *Suppose $D_{n,R}(G)=\{1\}$ for some n. Then no element $g\neq 1$ of G has order invertible in R.*

Proof. It suffices to verify that whenever a rational prime p is a unit in R, G is p-torsion-free. It is well-known that the map f: $D_{i,R}(G)\to\Delta_R(G)^i/\Delta_R(G)^{i+1}$ defined by $f(g)=g-1+\Delta_R(G)^{i+1}$ induces a monomorphism $D_{i,R}(G)/D_{i+1,R}(G)\to\Delta_R(G)^i/\Delta_R(G)^{i+1}$ of abelian groups. Therefore, if a rational prime p is a unit in R, then each additive group $\Delta_R(G)^i/\Delta_R(G)^{i+1}$ is clearly p-torsion-free, so is each $D_{i,R}(G)/D_{i+1,R}(G)$. Since $D_{n,R}(G)=\{1\}$ it follows that G is p-torsion-free and thus the lemma is proved.

Lemma 1.2. *Let $H_1\supseteqq H_2\supseteqq\cdots\supseteqq H_i\supseteqq\cdots$ be a decreasing series of subgroups of G. Then*

$$\bigcap_{i=1}^{\infty}\{\Delta_R(H_i)RG\} = \Delta_R(\bigcap_{i=1}^{\infty}H_i)RG\,.$$

Proof. It is trivial that the right-hand side is contained in the left-hand side. To show the reverse inclusion, let $\alpha\in\bigcap_{i=1}^{\infty}\{\Delta_R(H_i)RG\}$ and set $H=\bigcap_{i=1}^{\infty}H_i$. Then, choosing a right transversal T for H in G, we may express α, uniquely, as $\alpha=\sum_{j=1}^{n}\alpha_j t_j$, $\alpha_j\in RH$, $t_j\in T$. We first show that $\alpha_\nu\in\Delta_R(H)$ for a fixed integer ν with $1\leqq\nu\leqq n$. Since the set $\{t_j t_\nu^{-1}\,|\,1\leqq j\leqq n\}$ is finite, we can pick some H_k with $H_k\cap\{t_j t_\nu^{-1}\,|\,1\leqq j\leqq n\}=\{1\}$. Then, under the natural projection map π: $RG\to RH_k$, we have $\pi(\alpha t_\nu^{-1})=\alpha_\nu$ since π is a left RH_k-homomorphism (see [2, p. 6]). On the other hand, we have

$$\pi(\alpha t_\nu^{-1})\in\pi(\Delta_R(H_k)RG) = \Delta_R(H_k)\,,$$

so $\alpha_\nu\in\Delta_R(H_k)\cap RH=\Delta_R(H)$. Thus we see that all α_j's are in $\Delta_R(H)$ so that $\alpha\in\Delta_R(H)RG$. This completes the proof of the lemma.

We are now in a position to prove our main theorem which is a generalization of [1, Proposition 1.3]. Recall that for any (two-sided) ideal I of RG, $V(RG)\cap(1+I)=\{u\in V(RG)\,|\,u-1\in I\}$ forms a normal subgroup of $V(RG)$.

Theorem 1.3. *Let G be an arbitrary group and R an integral domain of characteristic 0. Let I be an ideal of RG and set $J=\bigcap_{n=1}^{\infty}(I+\Delta_R(G)^n)$. Then the factor group*

$$V(RG)\cap(1+J)/V(RG)\cap(1+\Delta_R(G, G\cap(1+J)))$$

is torsion-free.

Proof. For simplicity of notation, the normal subgroup $G\cap(1+K)$ of G determined by an ideal K of RG will be denoted by $D(K)$. Let $I_n=I+\Delta_R(G)^n$. Then, $I_m I_n+I_n I_m\subseteqq I_{m+n}$ for all $m, n\geqq 1$, so we obtain a descending central series $\{D(I_n)\}_{n\geqq 1}$ of G with $D(I_1)=G$. Note that $D_{n,R}(G)\subseteqq D(I_n)$ because $\Delta_R(G)^n\subseteqq I_n$.

We first prove the following:

(*) If $D(I_n)=\{1\}$, then $V(RG)\cap(1+I_n)$ is torsion-free.

We proceed by induction on n, the case $n=1$ being trivial. Let $n\geqq 2$ and assume that (*) holds for $n-1$. Set $\bar{G}=G/D(I_{n-1})$ and let $^-$: $RG\to R\bar{G}$ be the natural homomorphism. Then, since $D(\bar{I}_{n-1})=\bar{G}\cap(1+\bar{I}_{n-1})=\{1\}$, $V(R\bar{G})\cap(1+\bar{I}_{n-1})$ is torsion-free by induction hypothesis. Let $u\in V(RG)\cap(1+I_n)$ have finite order. Then $\bar{u}\in V(R\bar{G})\cap(1+\bar{I}_{n-1})$, and $\bar{u}$ still has finite order, so $\bar{u}=1$, that is, $u-1\in\Delta_R(G, D(I_{n-1}))$. Note here that since $D(I_n)=\{1\}$, G is nilpotent and $D(I_{n-1})$ is central in G. Moreover, as $D_{n,R}(G)=\{1\}$, we know from Lemma 1.1 that no element $g\neq 1$ of G has order invertible in R. Thus, by [1, Lemma 1.2], $u=x$ for some $x\in D(I_{n-1})$. This implies that $x\in D(I_n)=\{1\}$ because $u-1\in I_n$. Hence we have $u=1$, so $V(RG)\cap(1+I_n)$ is torsion-free.

Turning the proof of the theorem, let $u\in V(RG)\cap(1+J)$ and suppose $u^l\in V(RG)\cap(1+\Delta_R(G, D(J)))$ for some integer l. If $\bar{G}=G/D(I_n)$, then $D(\bar{I}_n)=\bar{G}\cap(1+\bar{I}_n)=\{1\}$ under the natural homomorphism $^-$: $RG\to R\bar{G}$, and so (*) shows that each factor group

$$V(RG)\cap(1+I_n)/V(RG)\cap(1+\Delta_R(G, D(I_n)))$$

is torsion-free. Since $u-1\in I_n$ and $u^l-1\in\Delta_R(G, D(I_n))$ for all $n\geqq 1$, it follows that $u-1\in\bigcap_{n=1}^{\infty}\Delta_R(G, D(I_n))$. Furthermore, by Lemma 1.2,

$$\bigcap_{n=1}^{\infty}\Delta_R(G, D(I_n))=\Delta_R(G, \bigcap_{n=1}^{\infty}D(I_n))=\Delta_R(G, D(J)),$$

and hence we conclude that $u\in V(RG)\cap(1+\Delta_R(G, D(J)))$. This completes the proof.

A ring A is said to be residually nilpotent if $\bigcap_{n=1}^{\infty}A^n=0$. In the context of the preceding theorem, the factor ring $(\Delta_R(G)+I)/I=\Delta_R(G)/(\Delta_R(G)\cap I)$ is residually nilpotent if and only if $J=I$, so we note the following

Corollary 1.4. *Let G, R and I be as in Theorem* 1.3. *If $\Delta_R(G)/(\Delta_R(G)\cap I)$ is residually nilpotent, then*

$$V(RG)\cap(1+I)/V(RG)\cap(1+\Delta_R(G, G\cap(1+I)))$$

is torsion-free.

By taking $I=\Delta_R(G)^n$ in this corollary we see that if $D_{n,R}(G)=\{1\}$, then $V(RG)\cap(1+\Delta_R(G)^n)$ is torsion-free. Thus the same argument as in Proposition 2.4 of [1] gives us the following result, whose proof will be omitted.

Proposition 1.5. *Let G and H be two groups with $RG\cong RH$ as R-algebras, where R is an integral domain of characteristic* 0. *Then*

$$D_{n,R}(G)=\{1\} \quad \text{if and only if} \quad D_{n,R}(H)=\{1\}\ .$$

2. Quasi-regular groups

Let A be a ring, and let ${}^0A$ denote the group of all quasi-regular elements of A, that is, ${}^0A$ is the set of those elements of A which are invertible under the circle operation $a\circ b=a+b+ab$. In case ${}^0A=A$, A is a Jacobson radical ring and ${}^0A$ is called the circle group of A. It has been shown in [3] that if G is the circle group of a finite nilpotent ring, then G has a torsion-free normal complement in $V(\boldsymbol{Z}G)$. We shall extend this result as follows.

Proposition 2.1. *Let R be an integral domain of characteristic* 0 *and let A be a residually nilpotent R-algebra. Then the group $G={}^0A$ has a torsion-free normal complement in $V(RG)$.*

Proof. Since any R-algebra can be embedded in an R-algebra with identity, we can regard A as an ideal of some R-algebra A_1 with identity. Then $G={}^0A$ is isomorphic to $U(A_1)\cap(1+A)$ where $U(A_1)$ is the unit group of A_1, so we may suppose that $G=U(A_1)\cap(1+A)$. Then the inclusion map $G\to A_1$ can be extended to the R-algebra homomorphism $RG\to A_1$, which is denoted by f, and we set $F=V(RG)\cap(1+Ker\ f)$ so that $F\triangleleft V(RG)$. Since $f(g-1)=g-1$ for $g\in G$, we have $f(\Delta_R(G))\subseteq A$, and hence $f(V(RG))\subseteq U(A_1)\cap(1+A)=f(G)$ which implies that $V(RG)=GF$. Observe that the factor ring $\Delta_R(G)/(\Delta_R(G)\cap Ker\ f)$ is isomorphic to a subring of A and so is residually nilpotent. Since $G\cap F=G\cap(1+Ker\ f)=\{1\}$, we conclude by Corollary 1.4 that F is a torsion-free normal complement for G in $V(RG)$.

REMARK. As seen in the above proof, the group $G={}^0A$ of any R-algebra A has a normal complement in $V(RG)$.

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